Introduction to the Fast Marching Method

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Abstract

In these notes, we present an introduction to the classical *Fast Marching Method* (FMM). The FMM has been first introduced to find numerical approximations of the solutions to the stationary eikonal equation. More generally we show that this method allows to find exactly the solutions of numerical schemes that satisfy a certain causality assumption.
In order to motivate this method, we present the *shape from shading* problem, introduce the notion of viscosity solutions to the stationary eikonal equation and consider finite differences schemes. We also provide various comparison principles for the eikonal equation and for the schemes. We give an error estimate between the numerical solution and the viscosity solution. We finally indicate some extensions of the FMM to more general equations than the stationary eikonal equation.

Keywords: Fast Marching method, Level Set method, viscosity solutions, eikonal equation, finite differences scheme, causality, comparison principle, error estimate, shape from shading.

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1 Introduction

The goal of these notes is to give an introduction to the classical Fast Marching Method and some of its applications. These notes are written such that it should be easy to read at least up to Section 7. Sections 8, 9 and the appendix 10 are more technical and difficult to read. In a first reading of the manuscript, they can be seen by the reader as a complement.

In Section 2, we start with the classical *shape from shading* problem in image processing, in order to introduce easily the eikonal equation, as an example of Hamilton-Jacobi equation. We also discuss the non-uniqueness of solution and the necessity to introduce a new notion of solution, namely the notion of viscosity solution. But the precise notion of viscosity solution is not yet introduced.

In Section 3, we go directly to finite differences schemes in order to discretize the eikonal equation. It is quite easy to write such a scheme and to list some of its general properties. Using these properties, we show in particular the uniqueness of solution to the numerical scheme. We also underline general properties of the scheme:

(H1) Monotonicity,

(H2) Invariance by addition of constants,

(H3) Causality,

(H4) Finite homogeneous antisymmetric stencil,

(H5) Strong monotonicity.

Under these five assumptions, we indicate that there exists a unique numerical solution to the scheme.

In Section 4, using the previous general assumptions, we describe the Fast Marching Method (FMM) in this framework. The FMM appears here as a method to solve the non linear numerical scheme, using in an essential way the causality assumption (H3).

In Section 5, we introduce the reader to the precise definition of viscosity solution, starting from the example of the eikonal equation in dimension 1, when we add vanishing viscosity.

In Section 6, we relate the eikonal equation to the Level Set formulation. We interprete in particular the numerical front (or narrow band) of the FMM as the discrete analogue of the level set of the solution to an evolution equation. It is indicated that this level set moves with a certain normal velocity. We also indicate an application of these notions of moving curves to a problem of image segmentation: we present a simplified version of the model of Chan and Vese.

In Section 7, we indicate references related to further developments of the Fast Marching Method.

In Section 8, we prove a comparison principle for viscosity solutions and give uniqueness results as a corollary.

In Section 9, we prove an error estimate (of Crandall-Lions type) to compare the solution of the FMM and the continuous solution of the eikonal equation.

Finally, in the appendix (Section 10), we show the uniqueness of the solution to the scheme, under assumptions (H1),(H2),(H3),(H4) and (H5). Under the same assumptions we also check that the sequence of times given by the FMM is an increasing sequence, as it is expected (for causal schemes).

Even if the whole material of these notes is contained in the common knowledge (and is indeed an easy adaptation of classical results), we are not aware of written proofs of several of the presented results. For this reason, we think that these notes also include several original proofs and results (like for instance Proposition 3.4, Proposition 4.3, Theorem 8.1 and Theorem 9.1).

2 Shape from shading

In this section, we introduce the reader to a particular problem: the problem of recovering the shape of an object from its shadow. This problem is an example which motivates the introduction of the eikonal equation and the question of the uniqueness of the solution to this equation. This example will also motivate the other sections of this course and in particular the Fast Marching Method.

2.1 The axioms

Let us consider that you shed some light on an three-dimensional object (like for instance a ball) and take a picture in black and white. Then you will see a certain shadow on the object in the picture. From that shadow, the human eye is able to guess what is the full three-dimensional shape of the object. This is called the **shape from shading** procedure.

Then the basic question is:

$\boxed{\textit{what is the physical law that describes the observed brightness/shadow?}}$

We now present a simple mathematical model to compute the brightness (and then the shadow) of an object.

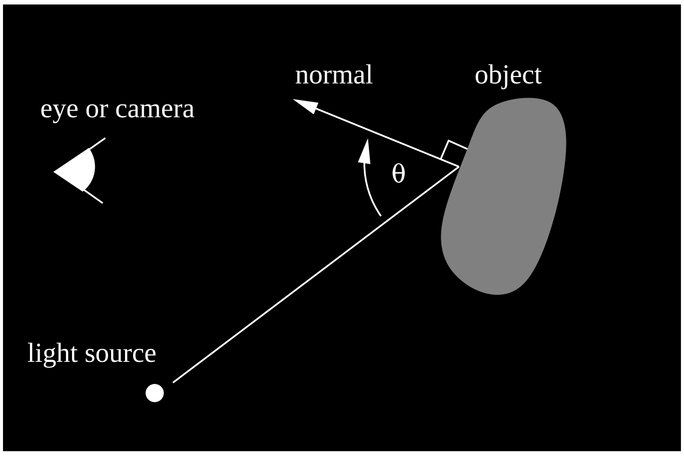

Figure 1: object, direction of light and camera

Axioms for the brightness of an object
We recall that we consider an object, the direction of the light and the position of the camera which takes the picture (see Fig. 1).
(A0) [No distorsions]
We assume that both the source of light and the camera are far from the observed object (with respect to the diameter of the object).
An extreme example of such a situation is the case of a camera on the earth observing the moon which is enlighted by the sun. But even for more common pictures, we will make these assumptions, just in order to avoid distorsions of the pictures.
(A1) [Brightness between 0 and 1]
The maximal brightness is equal to 1 and the minimal brightness is 0 (see Fig. 2 and Fig. 3 for an illustration).
(A2) [The Lambertian case]
The brightness does not depend on the position of the camera (but only depends on the direction of the source of light with respect to the orientation of the surface of the object).
More precisely, we consider the *Lambertian case*: we assume that the local brightness per surface of the picture is given by the following formula:

$$\text{brightness} = \frac{\text{flux of light}}{\text{surface of the object}} = \cos\theta \tag{2.1}$$

where $\theta \in [0, \pi/2]$ is the angle between the direction of the source and the normal to the surface of the object (see Fig. 1).

We see in particular that axiom (A1) can be deduced from axiom (A2).

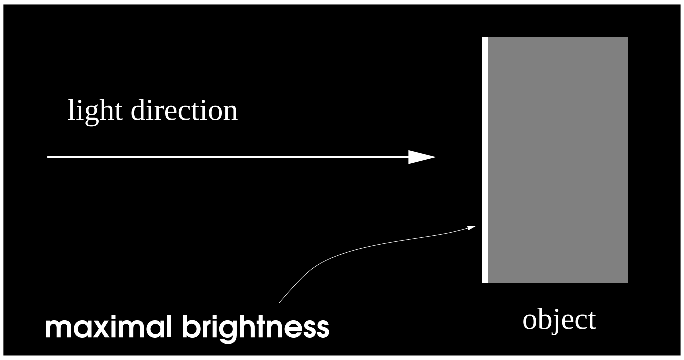

Figure 2: Full reflection of light with maximal brightness

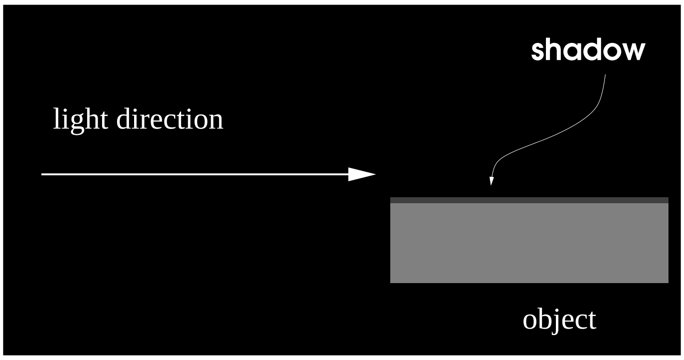

Figure 3: No reflexion of light with no brightness (i.e. shadow)

An other way to understand the axiom (A2) is to consider a flat object (see the triangle in Fig 4) such that the angle between the direction of light and the normal to the object is constant and equal to θ).

Then we can decompose artificially the flat surface (i.e. the long segment of the triangle of Fig. 4) in a succesion of small vertical and horizontal segments (see Fig. 5). In particular the flat surface can be seen in some sense as a limit of this succession of small vertical and horizontal segments S_i, when the size of the segments goes to zero. On the one hand, when the segment is vertical, the reflexion is maximal and the brightness is equal to 1. On the other hand, when the segment is horizontal, there is no light reflected by the segment and the brightness is minimal and equal to 0. Then we see that the total brightness of the surface is:

$$\text{total brightness} = \sum_{\text{vertical } S_i} 1 \times |S_i| + \sum_{\text{horizontal } S_i} 0 \times |S_i|$$

where $|S_i|$ is the length of the segment S_i. Finally we find that the total brightness is equal to the high of the triangle, i.e. equal to $\cos\theta$. This is true before to pass to the limit when the length of the small segments goes to zero. Finally we see that the Lambertian case is nothing else that saying that this is still true after passing to the limit and we recover the formula (2.1).

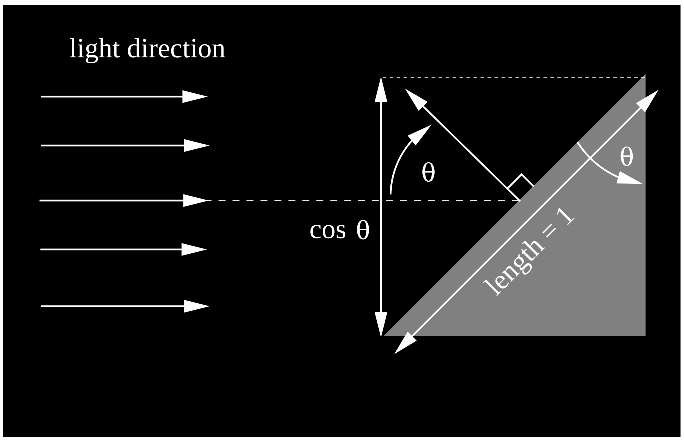

Figure 4: Angle θ between the direction of light and the normal

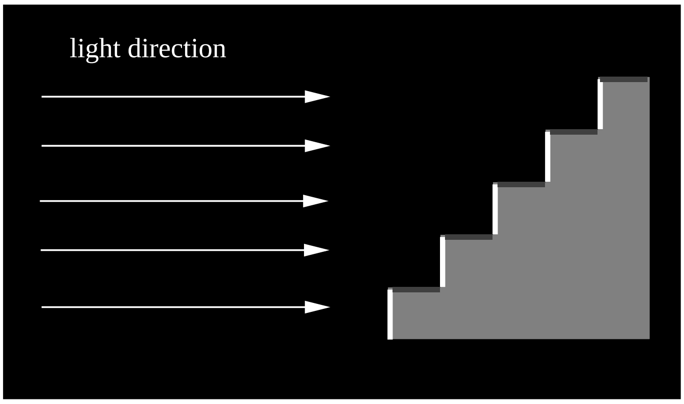

Figure 5: Succession of small vertical and horizontal segments

2.2 Mathematical modeling

To simplify the situation, we consider an object that is posed on a table. We assume that the table is at the level $z = 0$. We set $x = (x_1, x_2)$, and assume that the object fills the volume of $\mathbb{R}^3$ defined by

$$\big\{(x, z) \in \mathbb{R}^2 \times \mathbb{R}, \quad 0 < z < u(x)\big\}$$

where u is a continuous function such that $\Omega = \{x \in \mathbb{R}^2, \quad u(x) > 0\}$ is a non empty bounded set. An example of such an object is a half ball like in Fig. 6.

We assume that the direction of light is vertical (i.e. the object is enlighted from the top). We recall that by axiom (A2), we have

$$\text{brightness} = I(x) = \cos\widehat{(n, e_z)} \tag{2.2}$$

where $\widehat{(n, e_z)}$ is the angle between the outward normal n to the object and the vertical unit vector e_z (see Fig. 6). Here $I(x)$ stands for the intensity of light mesured on the picture. Assuming the function u smooth enough on Ω, we can compute

$$n = \frac{1}{\sqrt{1 + |\nabla u|^2}} \begin{pmatrix} -\nabla u \\ 1 \end{pmatrix}, \qquad e_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

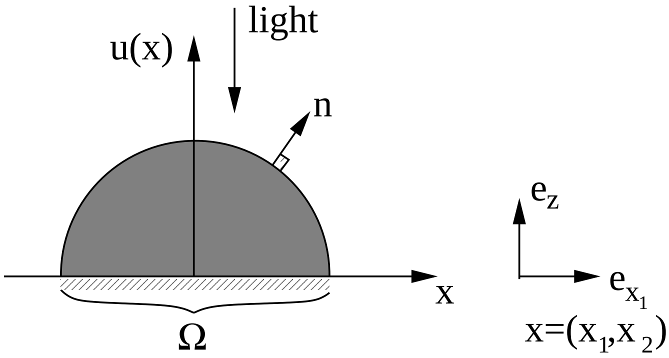

Figure 6: A half ball posed on a table

which shows that

$$I(x) = \frac{1}{\sqrt{1+|\nabla u|^2}}$$

and then

$$|\nabla u(x)| = a(x) \qquad \text{with} \quad a(x) = \sqrt{\frac{1}{I^2(x)} - 1}.$$

The intensity of light $I(x)$ is the observed data (for instance that you can see on the picture). This shows that the function $a(x)$ is also given, and we are looking for a function u solution of the following **eikonal equation**

$$\text{(2.3)} \qquad \left\{ \begin{array}{lll} |\nabla u(x)| = a(x) & \text{for} & x \in \Omega \subset \mathbb{R}^2 \\ \\ u(x) = 0 & \text{for} & x \in \partial\Omega. \end{array} \right.$$

2.3 Uniqueness of the solution ?

In order to understand the uniqueness or non-uniqueness of the solution, we focus in this subsection on problem (2.3) for the particular example

$$\Omega = (-1, 1) \subset \mathbb{R} \quad \text{and} \quad a(x) = 1.$$

This means that we want to solve

$$\left\{ \begin{array}{lll} |u'(x)| = 1 & \text{for} & x \in (-1, 1) \\ \\ u(x) = 0 & \text{for} & x = \pm 1, \end{array} \right.$$

It is not difficult to realize that there is no C^1 solutions to this equation. Indeed there is at least an optimum (maximum or minimum) in the interior of the segment, and at that point, we have $u' = 0$ for a C^1 function, which is in contradiction with the equation. So the next step is to try to look for Lipschitz-continuous solutions which are only solutions almost everywhere in $(-1, 1)$. We show in Fig 7 several such solutions. We should not be surprised by the fact that we can get several solutions. This is a well-known fact. Looking at pictures of a sculpture of a face of somebody in the rock, there is a natural ambiguity. We can either think that this sculpture is essentially convex (like for the subgraph of the first picture of Fig. 7) or convave (like for the subgraph of the second picture of Fig. 7). See further discussion in [15].

We said that there are several solutions, and this is even not difficult to construct an infinite number of such solutions.

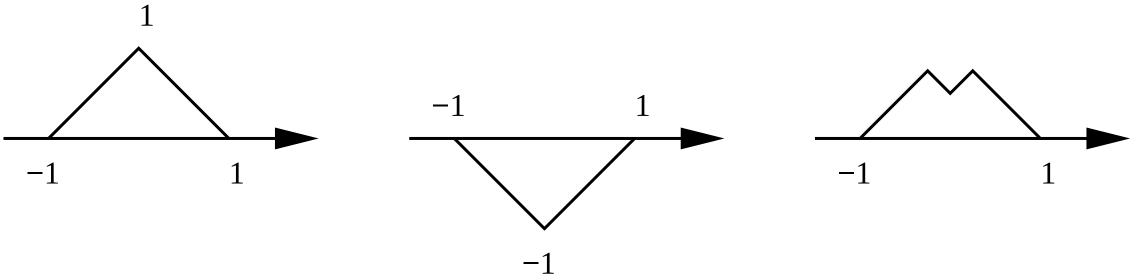

Figure 7: Several Lipschitz-continuous functions which are almost everywhere solutions on $(-1,1)$

Then the next question is: which solution to choose?
One answer would be that we want to choose the one corresponding to the real object that we are observing. But the point is that if we do not know already the object that we are observing, then this does not help at all.

The good news is that there is at least a mathematical idea which can help us to distinguish a particular solution among all the solutions. The idea is to add artificially a small viscosity term in the equation. For $\varepsilon > 0$ small, we can for instance consider the solutions u^ε to the equation

$$\left\{\begin{array}{lll} -\varepsilon(u^\varepsilon)'' + |(u^\varepsilon)'| = 1 & \text{on} & \Omega \\ u^\varepsilon = 0 & \text{on} & \partial\Omega, \end{array}\right. \tag{2.4}$$

Of course equation (2.4) is exactly equation (2.3) in the particular case $\varepsilon = 0$. The interesting property of equation (2.4) is that we can show that the solution is unique and moreover is given by

$$\left\{\begin{array}{lll} u^\varepsilon(-x) = u^\varepsilon(x) & \text{for} & x \in [-1,1] \\ u^\varepsilon(x) = x + 1 - \varepsilon(e^{\frac{x}{\varepsilon}} - e^{-\frac{1}{\varepsilon}}) & \text{if} & -1 \le x \le 0, \end{array}\right.$$

The graph of the function u^ε is sketched on Fig. 8.

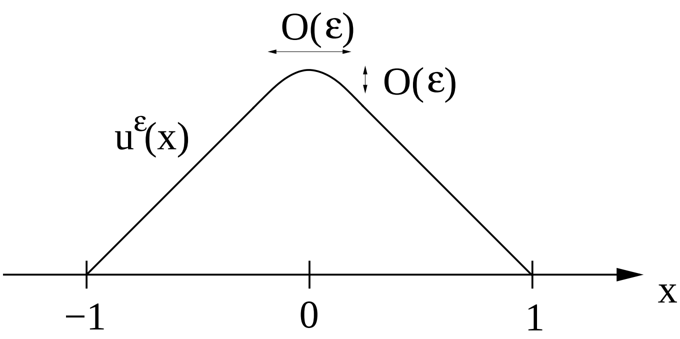

Figure 8: Graph of u^ε

On the other hand, we have

$$u^\varepsilon(x) \longrightarrow u^0(x) = 1 - |x| \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Here we show that the passage to the limit as the viscosity parameter ε vanishes, selects a particular solution of equation (2.3), namely the function u^0. This function u^0 is called the viscosity solution of equation (2.3).

We will see in Section 5 that this particular solution can be characterized directly on problem (2.3) without using the ε-perturbed problem (2.4). We will also see that for this caracterization, the solution is unique.

3 Finite differences scheme

In this section, we discuss numerical schemes to compute discrete approximations of the (viscosity) solutions of equation (2.3). We also underline some properties of the schemes. Some of these properties will be important to apply later the Fast Marching Method.

3.1 The 1D example

We want to approximate the (viscosity) solutions (see Fig. 9) of the following problem (with the function $a(x) > 0$)

$$\begin{cases} |u'| = a & \text{on} \quad \Omega = (-1, 1) \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

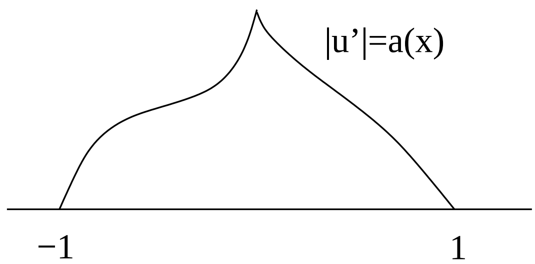

Figure 9: Example of a continuous solution in 1D

We discretize the variable x as $x_i = i\Delta x$ for $i \in \mathbb{Z}$ and a small mesh size $\Delta x > 0$. Then we approximate the value $u(x_i)$ by the quantity u_i that will be a solution of a scheme that we have to write. To this end we approximate the derivative $u'(x_i)$ as

$$\frac{u_{i+1} - u_i}{\Delta x} \quad \text{or} \quad \frac{u_i - u_{i-1}}{\Delta x}$$

We want to discretize the equation

$$|u'(x)| - a(x) = 0$$

as a scheme

$$S_i(u_{i-1}, u_i, u_{i+1}) = 0$$

Here we propose the following scheme for $u = (u_j)_j$:

$$S_i[u] := S_i(u_{i-1}, u_i, u_{i+1}) := -a(x_i) + \max\left(0, \frac{u_i - u_{i-1}}{\Delta x}, \frac{u_i - u_{i+1}}{\Delta x}\right) \tag{3.1}$$

At the stage, it is not clear for the reader why this scheme is a good candidate to compute the (viscosity) solution. Nevertheless, we will extract some general good properties of this scheme and deduce further consequences.

For $u = (u_i)_i$ and any constant $M \in \mathbb{R}$, we set $u + M = (u_i + M)_i$. Then we can easily check that the scheme $S[u] = (S_i[u])_i$ given in (3.1) satisfies the following properties for all u:

(H1) Monotonicity

$$\frac{\partial S_i[u]}{\partial u_j} \leq 0 \quad \text{for all} \quad j \neq i \tag{3.2}$$

(H2) Invariance by addition of a constant

$$S_i[u + M] = S_i[u] \quad \text{for all} \quad M \in \mathbb{R}, \quad \text{for all} \quad i \tag{3.3}$$

The monotonicity property is an important property which will insure the uniqueness of the solution of the scheme. Here our scheme satisfies also a had hoc property which is the following:

(H3') Multiplication by a constant

$$S_i[\lambda u] = \lambda S_i[u] + (\lambda - 1)a(x_i) \quad \text{for all} \quad \lambda \geq 0, \quad \text{for all} \quad i \tag{3.4}$$

This property (H3') will allow us to give a simple proof of uniqueness of the solution to the scheme. Then we have

Proposition 3.1 (Uniqueness of the solution to the scheme)
Let $a > 0$. Assume that $u = (u_i)_{i \in \mathbb{Z}}$ solves

$$\begin{cases} u_i = 0 & \text{if } x_i \notin \Omega = (-1, 1) \\ \\ S_i[u] = 0 & \text{if } x_i \in \Omega. \end{cases} \tag{3.5}$$

where the scheme $S[\cdot]$ satisfies assumptions (H1),(H2) and (H3'). Then u is unique.

Proof of Proposition 3.1
Assume that $(u_i)_i$ and $(v_i)_i$ are two different solutions of the scheme (3.5) and that

$$\sup_{i \in \mathbb{Z}} (u_i - v_i) > 0. \qquad \text{(otherwise we exchange } u \text{ and } v\text{)}$$

Then, for some $\lambda > 1$ (λ close enough to 1), we have

$$\begin{aligned} M_\lambda &= \sup_{i \in \mathbb{Z}} (u_i - \lambda v_i) > 0 \\ &= u_{i_0} - \lambda v_{i_0} \qquad\qquad \text{for some } \; x_{i_0} \in \Omega. \end{aligned}$$

We deduce that

$$\begin{cases} u_i \le w_i := M_\lambda + \lambda v_i \\ \\ u_{i_0} = w_{i_0}. \end{cases}$$

This implies

$$\begin{aligned} 0 &= S_{i_0}[u] \\ &\geq S_{i_0}[w] && \text{(using the monotonicity (H1) of the scheme for } j \neq i_0) \\ &= S_{i_0}[\lambda v] && \text{(using the invariance (H2) by addition of a constant)} \\ &= \lambda \underbrace{S_{i_0}[v]}_{=0} + \underbrace{(\lambda - 1) a(x_{i_0})}_{>0} && \text{(using (H3') for the multiplication by a constant)} \\ &> 0 \end{aligned}$$

Contradiction. Therefore $u_i = v_i$ for all i and this ends the proof. □

Before to close this subsection, let us give the following result that will be used in the next subsection:

Lemma 3.2 (Monotonicity in u_i)
Under assumptions (H1) and (H2), the scheme $S[\cdot]$ also satisfies for all u:

$$\frac{\partial S_i[u]}{\partial u_i} \geq 0 \quad \textit{for all} \quad i \tag{3.6}$$

Proof of Lemma 3.2
For any $h \in \mathbb{R}$, let us define

$$u_j^h = \begin{cases} u_i + h & \text{if} \quad j = i \\ \\ u_j & \text{if} \quad j \neq i \end{cases}$$

and

$$(u_h)_j = \begin{cases} u_i & \text{if} \quad j = i \\ \\ u_j - h & \text{if} \quad j \neq i \end{cases}$$

Then from (H2), we deduce that for $h > 0$

$$S_i[u^h] = S_i[u_h] \geq S_i[u]$$

where the inequality follows from (H1). This implies (3.6) and ends the proof of the lemma.

3.2 The 2D case

We want to approximate the solutions of the following problem

$$\begin{cases} |\nabla u| = a & \text{on} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

where Ω is bounded domain in $\mathbb{R}^2$. Similarly to the 1D case, we discretize the variable x as $x_I = I\Delta x$ for $I = (I_1, I_2) \in \mathbb{Z}^2$ and a small mesh size $\Delta x > 0$. Then we approximate the value $u(x_I)$ by the quantity u_I that will be a solution of a scheme that we have to write. We discretize

$$|\nabla u(x)| - a(x) = 0$$

as a scheme

$$S_I(\{u_J\}_{J\in V(I)}) = 0$$

where the stencil

$$V(I) = \left\{J \in \mathbb{Z}^2, \quad |J - I| \le 1\right\} \tag{3.7}$$

is the "five points" neighborhood of the point I on the grid $\mathbb{Z}^2$ (see Fig. 10). Indeed we have

$$V(I) = \left\{I, I^{1,+}, I^{1,-}, I^{2,+}, I^{2,-}\right\} \quad \text{with} \quad I^{\alpha,\pm} = I \pm e_\alpha \quad \text{for} \quad \alpha = 1, 2$$

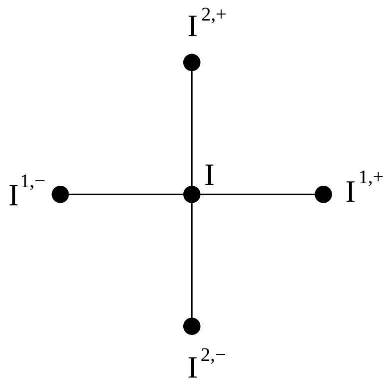

Figure 10: The "five points" neighborhood of the point I on the grid $\mathbb{Z}^2$

Here we propose the following Rouy-Tourin scheme (see [13])

(3.8)

$$-a(x_I) + \sqrt{\left(\max\left(0, \frac{u_I - u_{I^{1,-}}}{\Delta x}, \frac{u_I - u_{I^{1,+}}}{\Delta x}\right)\right)^2 + \left(\max\left(0, \frac{u_I - u_{I^{2,-}}}{\Delta x}, \frac{u_I - u_{I^{2,+}}}{\Delta x}\right)\right)^2}$$

$$:= S_I(u_I, u_{I^{1,-}}, u_{I^{1,+}}, u_{I^{2,-}}, u_{I^{2,+}}) := S_I[u]$$

Again it is easy to check that this scheme $S[u] = (S_I[u])_I$ satisfies assumptions (H1),(H2) and (H3') and we have also a result of uniqueness of the solution of the scheme similar to Proposition 3.1 (with a proof identical word for word).

Indeed it is more interesting to replace the ad hoc property (H3') by three natural properties (H3), (H4) and (H5) that will also be used later for the Fast Marching Method in Section 4. We now give these assumptions.

(H3) Causality

$$S_I[u] = S_I[\hat{u}] \quad \text{with} \quad \hat{u}_J = \begin{cases} u_J & \text{if} \quad u_J < u_I \quad \text{or} \quad J = I \\ +\infty & \text{otherwise} \end{cases} \tag{3.9}$$

This assumption (H3) means that the value u_I only depends on the values of u_J for points J with values $u_J < u_I$. This assumption is quite restrictive in general and does not apply to certain schemes (especially for anisotropic equations in general). Nevertheless, we will see that this assumption is essential to apply later the Fast Marching Method (see Section 4).

(H4) Finite homogeneous antisymmetric stencil

$$S_I[u] = S_I(\{u_J\}_{J\in V(I)}) \quad \text{with} \quad V(I) = I + V_0 \quad \text{and} \quad -V_0 = V_0 \quad \text{is bounded} \tag{3.10}$$

In the special case of our scheme, we have the stencil $V_0 = \{0, e_1, -e_1, e_2, -e_2\}$, which satisfies the antisymmetry property:

$$-V_0 = V_0$$

What is particularly important in practice is to have a bounded stencil. The assumption that V_0 is antisymmetric is not really a constraint in general, because we can always antisymmetrize the stencil, if necessary adding artificial points that are not used by the scheme. Notice that it is very convenient (but not fundamental) to have a size of the stencil $V(I)$ which is homogeneous, i.e. independent on the point I.

Before to introduce our last property, let us recall (see Lemma 3.2) that properties (H1) and (H2) imply in particular

$$\frac{\partial S_I[u]}{\partial u_I} \geq 0 \quad \text{for all} \quad I \tag{3.11}$$

We now write the following strengthened property:

(H5) Strong monotonicity

$$\begin{cases} \dfrac{\partial S_I[u]}{\partial u_I} \geq \delta > 0 \quad \text{if} \quad u_I > \inf\limits_{J\in V(I)\setminus\{I\}} u_J \\ \\ S_I[0] < 0 \\ \\ \text{the map} \quad u_I \mapsto S_I[u] \quad \text{is continuous} \end{cases} \tag{3.12}$$

Notice that the bound from below on $\dfrac{\partial S_I[u]}{\partial u_I}$ could be more general (like any positive constant also depending on I).

This last property (H5) is natural and allows us to find the unique solution u_I of the equation

$$S_I(u_I, \{u_J^*\}_{J\in V(I)\setminus\{I\}}) = 0 \tag{3.13}$$

Proposition 3.3 (Existence and uniqueness of u_I solution of (3.13))
Assume (H1),(H2),(H4) and (H5). Given the numbers u_J^* for $J \neq I$, there exists a unique solution u_I of (3.13). Moreover we have

$$u_I > \inf_{J\in V(I)\setminus\{I\}} u_J^*$$

Proof of Proposition 3.3
Let
$$m = \inf_{J \in V(I) \setminus \{I\}} u^*_J$$
If $z \leq m$, then

$$\begin{array}{ll} S_I(z, \{u^*_J\}_{J\in V(I)\setminus\{I\}}) & \\ \leq S_I(z, \{z\}_{J\in V(I)\setminus\{I\}}) & \text{(using the monotonicity (H1) and } u^*_J \geq z \text{ for all } J \in V(I)\setminus\{I\}) \\ = S_I[z] & \text{(using (H4) for the stencil)} \\ = S_I[0] & \text{(using the invariance (H2) by addition of a constant)} \\ < 0 & \text{(using the second line of (H5))} \end{array}$$

We set
$$f(z) := S_I(z, \{u^*_J\}_{J\in V(I)\setminus\{I\}})$$
and we want to find the solution $u_I = z$ of $f(z) = 0$. Then from the previous computation and from (H5), we deduce that
$$\left\{ \begin{array}{l} f(z) \leq S_I[0] < 0 \quad \text{for} \quad z \in (-\infty, m] \\ \\ f \quad \text{is continuous increasing on} \quad [m, +\infty) \\ \\ f(z) \to +\infty \quad \text{as} \quad z \to +\infty \end{array} \right.$$
Therefore, we conclude that there exists a unique $u_I = z$ such that $f(z) = 0$. Moreover $u_I > m$. This ends the proof of the Proposition.

It is easy to check that our scheme (3.8) satisfies the properties (H1),(H2),(H3),(H4) and (H5) (using in particular the assumption $a > 0$ to check the second line of (H5)).
Then we have the following result:

Proposition 3.4 (Comparison principle for the scheme)
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Let us consider $u = (u_I)_I$ and $v = (v_I)_I$ satisfying

$$\left\{ \begin{array}{lll} u_I \leq v_I & & \text{if } x_I \notin \Omega \\ \\ S_I[u] \leq S_I[v] & \text{and} \quad S_I[v] \geq 0 & \text{if } x_I \in \Omega. \end{array} \right. \tag{3.14}$$

where $S[\cdot]$ is a scheme satisfying assumption (H1),(H2),(H3),(H4) and (H5). Then $u \leq v$.

Remark 3.5 *Notice that we assume that $S_I[v] \geq 0$ for $x_I \in \Omega$. This condition is usually not assumed in the statement of classical comparison results. We use this condition in the proof, but do not know if the result still holds or not without this condition.*

The proof of Proposition 3.4 will be given in the appendix (see subsection 10.1).

As a consequence of Proposition 3.4, we have:

Corollary 3.6 (Uniqueness of the solution to the scheme)
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Assume that $u = (u_I)_I$ solves

$$\left\{ \begin{array}{ll} u_I = 0 & \text{if } x_I \notin \Omega \\ \\ S_I[u] = 0 & \text{if } x_I \in \Omega. \end{array} \right. \tag{3.15}$$

where the scheme $S[\cdot]$ satisfies assumptions (H1),(H2),(H3),(H4) and (H5). Then the solution u of the scheme is unique.

Proof of Corollary 3.6
We consider two solutions u and v and apply Proposition 3.4 to u,v and then exchange the role of u and v.

4 The Fast Marching Method

In practice it is not so easy in general to solve non linear schemes like the scheme in (3.8). Using in a fundamental way the causality property, there is a nice method to solve the scheme. This method is called the **Fast Marching Method (FMM)** and has been introduced by Sethian in 1996. The idea is to solve exactly the scheme starting from the known region (here this is the set of points $x_I \in \mathbb{R}^2 \backslash \Omega$ where $u_I = 0$) and going progressively inside the domain Ω (see Fig. 11). Somehow there is a progressive propagation of information from the boundary of the domain Ω, going inside the domain Ω. This will make appear a discrete front propagating inside the domain.

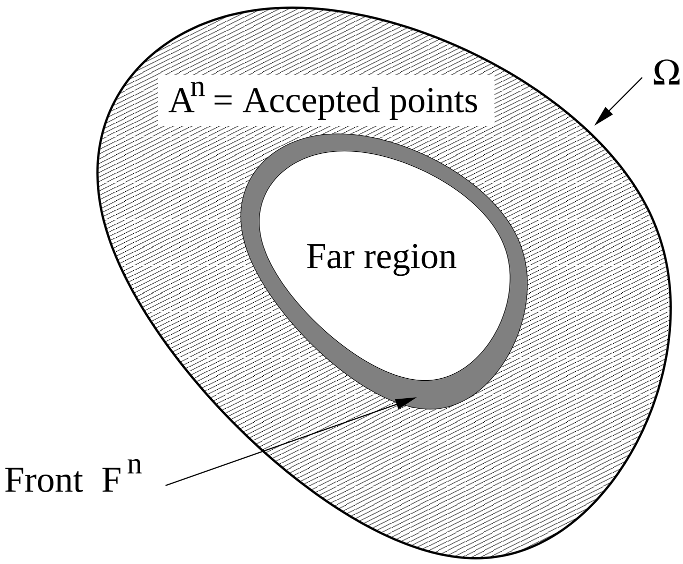

Figure 11: Sets for the FMM: accepted points A^n, front F^n and the far region

4.1 A first description of what the FMM does

General structure of the FMM
We will define an increasing sequence of times $(t_n)_n$ with $t_0 = 0$, an increasing sequence of

sets $(A^n)_n$ and a non increasing sequence of functions $(u^n)_n$ where each $u^n = (u_I^n)_I$ is defined for I on the grid $\mathbb{Z}^2$. Everything is done such that the unique solution u of the scheme (3.14) is given by

$$u_I = u_I^n \quad \text{for all} \quad I \in A^n$$

In other words the values of u^n computed on the sets A^n are the exact values of the solution u of the scheme (3.14). This is the reason why the set A^n is called the set of accepted points (which means the set of points where the value of u is definitely computed and accepted). Because the values of u^n are of no interest for points outside A^n, we simply set

$$u_I^n = +\infty \quad \text{if} \quad I \in \mathbb{Z}^2 \backslash A^n$$

On the other hand the values t_n are such that

$$u_I^n = t_n \quad \text{for all} \quad I \in A^n \backslash A^{n-1}$$

i.e. t_n is the common value of the new points accepted in the set A^n that were not contained in the previous set A^{n-1}. The FMM algorithm stops at the first step N such that the set of accepted points is the full grid, i.e.

$$A^N = \mathbb{Z}^2$$

and then

$$u_I = u_I^N \quad \text{for all} \quad I \in \mathbb{Z}^2$$

We also have that

$$\{t_0, ..., t_N\} = \left\{u_I, I \in \mathbb{Z}^2\right\}$$

Therefore the times $(t_n)_n$ appear as an increasing ordering of the values $\{u_I\}$ of the solution, and we also have

$$A^n \backslash A^{n-1} = \left\{I \in \mathbb{Z}^2, \quad u_I = t_n\right\}$$

The FMM algorithm simply gives a decomposition of the grid in subsets where u_I is constant (and equal to t_n):

$$\mathbb{Z}^2 = \bigcup_{n=0}^{N} \left(A^n \backslash A^{n-1}\right)$$

with $A^{-1} = \emptyset$.

Further details of the FMM

It is natural to define the front F^n as the discrete boundary of the accepted points A^n, i.e.

$$F^n = \left\{I \in \mathbb{Z}^2 \backslash A^n, \quad \text{such that} \quad V(I) \cap A^n \neq \emptyset\right\}$$

Using property (H4) on finite homogeneous antisymmetic stencil, it is easy to see that we can rewrite this front with a new expression that is more convenient for the classical FMM:

$$F^n = \left(\bigcup_{I \in A^n} V(I)\right) \backslash A^n$$

This notion of discrete boundary is modeled on the notion of discrete neighborhood $V(I)$ of a point I, introduced in (3.7). The front F^n is a set of grid points close to A^n, but outside A^n (see Fig. 12 and Fig. 13). For this reason, the front is also called the “narrow band”.

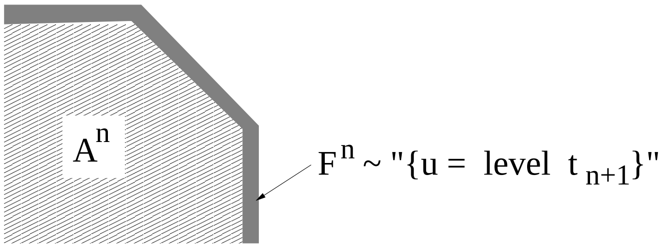

Figure 12: A schematic vue of the front F^n

The front F^n can be seen as a discrete analogue of the level set $\{u = t_{n+1}\}$ for the continuous solution u of the PDE. The main difference is that the discrete function $(u_I)_I$ is not constant on the front F^n. The values

$$\{u_I, \quad I \in F^n\}$$

are essentially close to each other but not the same in general. This means that the discrete function $(u_I)_I$ is only almost constant on the front F^n.

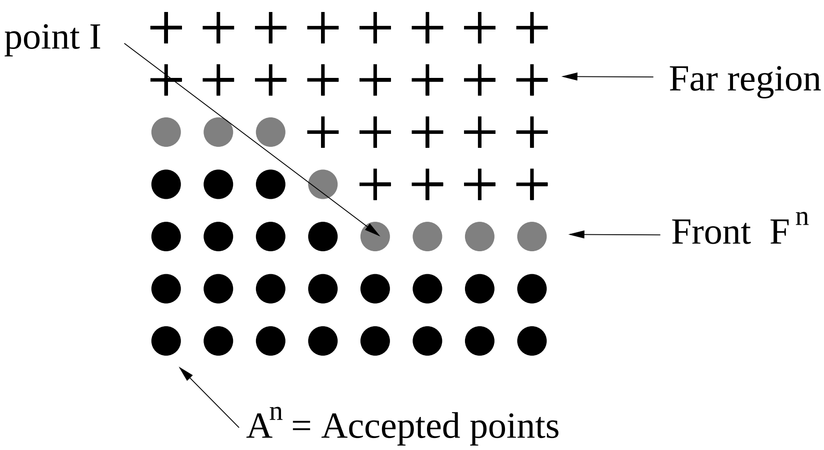

Figure 13: Vue of the grid with the sets A^n and F^n.

The set F^n is particularly interesting, because this is the set where we will look for the new points of $A^{n+1}\backslash A^n$, i.e. we will have

$$A^{n+1}\backslash A^n \subset F^n$$

The complementar of the set $A^n \cup F^n$ will be called the far region (also depending on n), because the points of this region will not be used going from step n to step $n+1$ (but will be used later).

What is important is to decide which points of the front F^n will be the new one in $A^{n+1}\backslash A^n$. To this end, we compute for each point I of the front F^n, a value $\tilde{u}_I^n$. This value is a candidate (or guess) of what should be the value u_I (but it can be bigger than u_I). From those $\tilde{u}_I^n$, we will only keep the minimal value

$$t_{n+1} = \min_{I \in F^n} \tilde{u}_I^n$$

and then we will choose as the new accepted points, the ones which realize this minimum, i.e.

$$A^{n+1} \backslash A^n = \{I \in F^n, \quad \tilde{u}_I^n = t_{n+1}\}$$

(This definition of the new accepted points is indeed related to an interpretation of the value of the function u as the minimal time needed to reach the point x_I, for fronts starting from the boundary of Ω at time $t = 0$ (see Subsection 6.1).) We will naturally define the new function u^{n+1} such that

$$u_I^{n+1} = \begin{cases} t_{n+1} & \text{if} \quad I \in A^{n+1} \backslash A^n \\ \\ u_I^n & \text{otherwise} \end{cases}$$

How to compute the candidate $\tilde{u}_I^n$?

The last thing to explain is now the way we compute the candidate value $\tilde{u}_I^n$. To this end, we obviously have to use the explicit form of the scheme (3.14). Given the function u^n, we simply find the solution $\tilde{u}_I^n$ of the equation

$$S_I(\tilde{u}_I^n, \{u_J^n\}_{J \in V(I) \backslash \{I\}}) = 0 \tag{4.1}$$

We see that equation (4.1) is simply the same as

$$S_I(\{u_J\}_{J \in V(I)}) = 0$$

where the values u_J have been replaced by u_J^n for $J \neq I$ and the value u_I has been replaced by the unknown $\tilde{u}_I^n$. Moreover there is a unique solution $\tilde{u}_I^n$ of (4.1). Notice that if a value $u_J^n = +\infty$ appears in equation (4.1) for some neighbors J of the point I, this means that the scheme will not use the point J. Indeed there is no information carried by the point J in that case, because the point J is then in the complementar of the set A^n of accepted points. Only the points J from the set A^n will be used to compute $\tilde{u}_I^n$. More precisely, the computation of the whole values $\tilde{u}_I^n$ for $I \in F^n$, only requires the knowledge of some "useful points": the points of the discrete boundary of $\mathbb{Z}^2 \backslash A^n$ (i.e. of an inner front or "narrow band" in A^n).

As an example, see Fig. 14 for a closer at point I of Fig. 13. Here we see that points A, B are in A^n, while C, D are not in A^n (C is on the front F^n and D is in the far region). This means that points C and D will not be used in the computation of $\tilde{u}_I^n$. Only the values of $u_J^n = u_J$ for $J = A, B$ will be used to compute $\tilde{u}_I^n$.

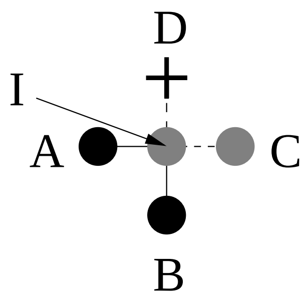

Figure 14: Closer at point I of Fig. 13

4.2 Statement of the FMM

We give here the algorithm describing the FMM.

Initialization

$$\begin{cases} t_0 = 0 \\ \\ A^0 = \{I \in \mathbb{Z}^2, \quad x_I \notin \Omega\} \\ \\ u_I^0 = \begin{cases} 0 & \text{if} \quad I \in A^0 \\ +\infty & \text{if} \quad I \notin A^0 \end{cases} \end{cases}$$

From step n to step $n+1$

We assume that t_n, A^n, u^n are known. We set

$$F^n = \left(\bigcup_{I \in A^n} V(I) \right) \backslash A^n$$

$i)$ Definition of the candidate times: $\tilde{u}_I^n$:

For each $I \in F^n$, we find the unique solution $\tilde{u}_I^n$ of

$$0 = S_I(\tilde{u}_I^n, \{u_J^n\}_{J \in V^*(I)}) \quad \text{with} \quad V^*(I) = V(I) \backslash \{I\}\,.$$

$ii)$ Minimizing the candidate times:

$$t_{n+1} = \inf_{I \in F^n} \tilde{u}_I^n.$$

$iii)$ Redefining the new sets and functions:

Define the new accepted points

$$NA^{n+1} = \{I \in F^n, \quad \tilde{u}_I^n = t_n\}$$

We get

$$\begin{cases} t_{n+1}, \\ \\ A^{n+1} = A^n \cup NA^{n+1}, \\ \\ u_I^{n+1} = \begin{cases} t_{n+1} & \text{if } I \in NA^n \\ \\ u_I^n & \text{otherwise} \end{cases} \end{cases}$$

End of the algorithm

The algorithm stops for the first integer N such that $A^N = \mathbb{Z}^2$.

Remark 4.1 *Notice that by Proposition 3.3, the solution $\tilde{u}_I^n$ is well defined under assumptions (H1),(H2),(H4) and (H5).*

4.3 Properties of the FMM

Proposition 4.2 (Properties of the FMM)
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Assume also (H1),(H2),(H4) and (H5). If $A^0 \neq \mathbb{Z}^2$, then $N \geq 1$ and for all $n \in \{0, ..., N-1\}$, we have

$$\left\{ \begin{array}{l} A^n \subset A^{n+1}, \\ \\ u^{n+1} \leq u^n. \end{array} \right.$$

Proof of Proposition 4.2
If $A^0 \neq \mathbb{Z}^2$, then the front F^0 is non empty and then $N \geq 1$. From the definition of the set A^{n+1}, we have $A^n \subset A^{n+1}$. From the definition of u^{n+1}, we have $u_I^{n+1} \neq u_I^n$ only for the new accepted points I satisfying $I \in NA^{n+1} = A^{n+1} \backslash A^n$. But we have

$$u_I^{n+1} = t_{n+1} < +\infty = u_I^n \quad \text{for all} \quad I \in NA^{n+1}$$

The fact that t_{n+1} is finite and well defined follows from Proposition 3.3. Therefore this implies that

$$u_I^{n+1} \leq u_I^n \quad \text{for all} \quad I \in \mathbb{Z}^2$$

$\square$

Proposition 4.3 (Properties of times of the FMM)
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Assume also that the scheme satisfies (H1),(H2),(H3), (H4) and (H5). If $A^0 \neq \mathbb{Z}^2$, then $N \geq 1$ and for all $n \in \{0, ..., N-1\}$, we have

$$t_{n+1} > t_n.$$

This proposition is admitted and will be proven in the appendix (see Subsection 10.2). Notice that the proof of the monotonicity of the times $(t_n)_n$ strongly uses the causality assumption (H3).

Theorem 4.4 (Existence of a solution to the scheme, using the FMM)
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain. Assume also that the scheme satisfies (H1),(H2),(H3), (H4) and (H5). Then the function $u = u^N$ of the FMM algorithm is a solution of the scheme (3.15).

Proof of Theorem 4.4
We assume that $A^0 \neq \mathbb{Z}^2$, otherwise there is nothing to prove. We consider $n \in \{0, ..., N-1\}$ and a point $I \in NA^{n+1}$. Then we have $u_I^N = \tilde{u}_I^n = t_{n+1}$. We also have

$$\left\{ \begin{array}{lll} u_J^n = u_J^N & \text{if} & u_J^n < +\infty \\ \\ u_J^N \geq t_{n+1} & \text{if} & u_J^n = +\infty \end{array} \right.$$

Combining assumptions (H3) and (H4), we see that the scheme has the following property:
(4.2)

$$S_I(\tilde{u}_I^n, \{u_J^n\}_{J \in V^*(I)}) = S_I(\tilde{u}_I^n, \{\hat{u}_J^n\}_{J \in V^*(I)}) \quad \text{with} \quad \hat{u}_J^n = \left\{ \begin{array}{lll} u_J^n & \text{if} & u_J^n < \tilde{u}_I^n = t_{n+1} \\ \\ +\infty & \text{if} & u_J^n \geq \tilde{u}_I^n = t_{n+1} \end{array} \right.$$

Similarly we have

$$S_I(u_I^N, \{u_J^N\}_{J\in V^*(I)}) = S_I(u_I^N, \{\hat{u}_J^N\}_{J\in V^*(I)}) \quad \text{with} \quad \hat{u}_J^N = \begin{cases} u_J^N & \text{if} \quad u_J^N < u_I^N = t_{n+1} \\ +\infty & \text{if} \quad u_J^N \geq u_I^N = t_{n+1} \end{cases}$$

Because $u_I^N = t_{n+1} = \tilde{u}_I^n$ and $\hat{u}_J^N = \hat{u}_J^n$ for $J \in V^*(I)$, we deduce that

$$S_I(t_{n+1}, \{u_J^N\}_{J\in V^*(I)}) = S_I(u_I^N, \{\hat{u}_J^N\}_{J\in V^*(I)}) = S_I(\tilde{u}_I^n, \{\hat{u}_J^n\}_{J\in V^*(I)}) = S_I(\tilde{u}_I^n, \{u_J^n\}_{J\in V^*(I)})$$

Using the fact that

$$S_I(\tilde{u}_I^n, \{u_J^n\}_{J\in V^*(I)}) = 0,$$

we deduce that

$$S_I(u_I^N, \{u_J^N\}_{J\in V^*(I)}) = 0.$$

We have shown that this last equation is true for any point $I \in \mathbb{Z}^2\backslash A^0$, which ends the proof of the theorem. $\square$

5 Introduction to viscosity solutions

We recall that subsection 2.3 was a kind of pre-introduction to viscosity solutions. We were studying the solutions of the following equation for $\varepsilon > 0$

$$(5.1) \qquad \begin{cases} -\varepsilon(u^\varepsilon)'' + |(u^\varepsilon)'| = 1 & \text{on} \quad \Omega = (-1,1) \\ u^\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where we have added artificially the ε term to the original equation. The function $u(x) = 1-|x|$ was both a solution of the original equation for $\varepsilon = 0$ and also the limit of the solutions u^ε as ε goes to zero.

5.1 Properties of u^ε and u

In order to go further, we ask the question: what are the properties of u^ε, and u?

The answer is given by the following result:

Proposition 5.1 (Properties of u^ε)
i) **(test from above)**
If $\varphi \in C^2(\Omega)$ satisfies

$$\begin{cases} u^\varepsilon \leq \varphi & \textit{on} \;\; \Omega \\ u^\varepsilon = \varphi & \textit{at} \quad x_0 \in \Omega, \end{cases}$$

then

$$-\varepsilon\varphi'' + |\varphi'| \leq 1 \qquad \textit{at} \quad x_0 \in \Omega.$$

ii) **(test from below)**
If $\varphi \in C^2(\Omega)$ satisfies

$$\begin{cases} u^\varepsilon \geq \varphi \quad on \ \ \Omega \\ \\ u^\varepsilon = \varphi \quad at \quad x_0 \in \Omega, \end{cases}$$

then

$$-\varepsilon\varphi'' + |\varphi'| \geq 1 \qquad at \quad x_0 \in \Omega.$$

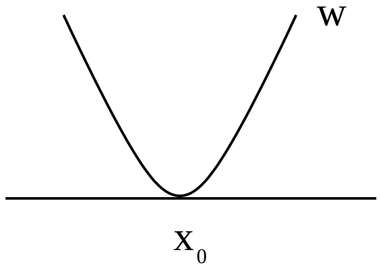

Figure 15: Graph of w

Proof of Proposition 5.1
We only show the i) (the proof of ii) beeing similar).
By assumptions,

$$w = \varphi - u^\varepsilon \geq 0 = w(x_0)$$

From the fact that w is minimal at x_0 (see Fig. 15), we deduce that

$$\begin{cases} w'(x_0) = 0 \\ w''(x_0) \geq 0 \end{cases}$$

i.e.

$$\begin{cases} \varphi'(x_0) = (u^\varepsilon)'(x_0) \\ \varphi''(x_0) \geq (u^\varepsilon)''(x_0) \end{cases}$$

This implies that

$$-\varepsilon\varphi'' + |\varphi'| \leq -\varepsilon(u^\varepsilon)'' + |(u^\varepsilon)'| = 1 \qquad \text{at} \quad x_0$$

which ends the proof of case i). □

5.2 Definition of viscosity solutions

The idea to define what are viscosity solutions of problem (2.3), is simply to use the property satisfied by u^ε for $\varepsilon > 0$ as it is given in Proposition 5.1, and to require that the solution u of (2.3) satisfies the same condition but for $\varepsilon = 0$. We recall problem (2.3), namely

(5.2)
$$\begin{cases} |\nabla u| = a \qquad \text{on} \quad \Omega \\ \\ u = 0 \qquad\quad\ \ \text{on} \quad \partial\Omega. \end{cases}$$

Definition 5.2 **(Viscosity sub/super/solution)**
Let $u \in C(\overline{\Omega})$.

- *We say that* u *is a* *viscosity subsolution* *of* (5.2) *if and only if*

$$u \leq 0 \qquad on \;\; \partial\Omega$$

  and if for any $\varphi \in C^2(\Omega)$ *satisfying (see Fig. 16)*

$$\begin{cases} u \leq \varphi & on \quad \Omega \\ \\ u = \varphi & at \quad x_0 \in \Omega, \end{cases}$$

  we have

$$|\nabla\varphi(x_0)| \leq a(x_0).$$

- *We say that* u *is a* *viscosity supersolution* *of* (5.2) *if and only if*

$$u \geq 0 \qquad on \;\; \partial\Omega$$

  and if for any $\varphi \in C^2(\Omega)$ *satisfying (see Fig. 17)*

$$\begin{cases} u \geq \varphi & on \quad \Omega \\ \\ u = \varphi & at \quad x_0 \in \Omega, \end{cases}$$

  we have

$$|\nabla\varphi(x_0)| \geq a(x_0).$$

- *We say that* u *is a* *viscosity solution* *of* (5.2) *if and only if* u *is a viscosity subsolution and supersolution.*

If you hesitate to know from which side you have to test, just keep in mind the following:
REMINDER: The subsolutions are tested from above.

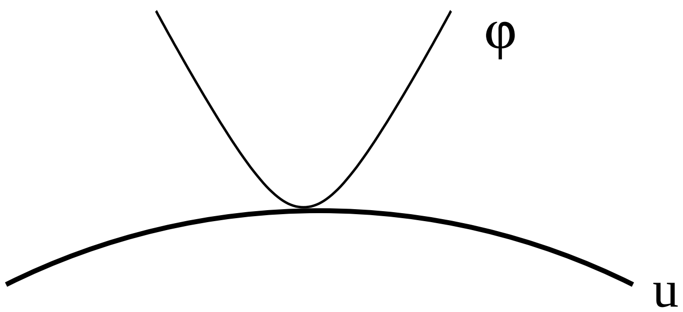

Figure 16: Testing a subsolution from above

Then we have

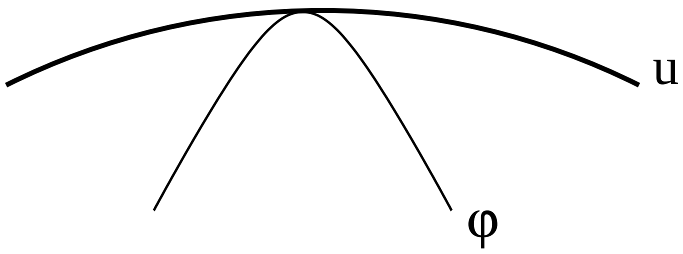

Figure 17: Testing a supersolution from below

Proposition 5.3 (Checking a viscosity solution)
The function

$$u(x) = 1 - |x|$$

is a viscosity solution of

$$\begin{cases} |u'| = 1 & \textit{on} \quad \Omega = (-1,1) \\ u = 0 & \textit{on} \quad \partial\Omega \end{cases}$$

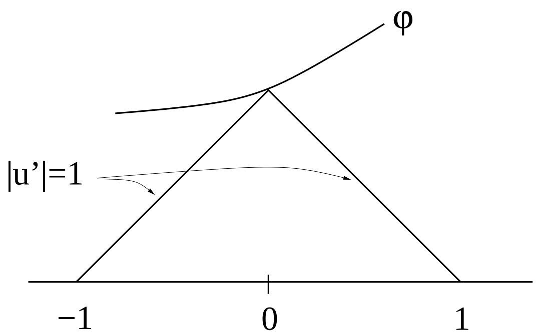

Figure 18: Cheking u from above

Proof of Proposition 5.3
If a test function φ touches the function u at a point x_0 where u is C^1, then φ and u have the same derivatives, i.e. $\varphi'(x_0) = u'(x_0)$, and it is straightforward to check the definition because we have $|\varphi'(x_0)| = |u'(x_0)| = 1$. Therefore we only have to check from above and from below at the point $x_0 = 0$ which is the only point of Ω where u is not C^1.
i) Checking from above at $x_0 = 0$
Let us assume that $\varphi \in C^2(\Omega)$ satisfies

$$\begin{cases} \varphi \geq u & \text{on} \quad (-1,1) \\ \varphi = u & \text{at} \quad x_0 = 0, \end{cases}$$

Then this implies a limitation on the slopes of φ at $x_0 = 0$. Precisely we have (see Fig. 18):

$$-1 = u'(0^+) \leq \varphi'(0) \leq u'(0^-) = 1$$

This implies $|\varphi'(0)| \leq 1$ and then u is a subsolution at $x_0 = 0$.
ii) Checking from below at $x_0 = 0$

If φ satisfies

$$\begin{cases} \varphi \leq u & \text{on} \quad (-1,1) \\ \\ \varphi = u & \text{at} \quad x_0 = 0, \end{cases}$$

then it is easy to realize that φ can not be C^2 (neither C^1) at x_0, because of the shape of u in the neighborhood of $x_0 = 0$. Therefore there is nothing to check in the definition of supersolution at x_0, and this shows that u is a viscosity supersolution at $x_0 = 0$.
This ends the proof of the proposition. □

Then we have the following result that is admitted.

Theorem 5.4 (Existence and uniqueness of the viscosity solution)
Let Ω be a bounded open set and $a > 0$ be a Lipschitz-continuous function on Ω. Then there exists a unique viscosity solution of

$$\begin{cases} |\nabla u| = a & \text{on} \quad \Omega \\ \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

To read a proof of the uniqueness of the solution, see Section 8.

Remark 5.5 (Non uniqueness when $a \geq 0$)
Notice that the condition $a > 0$ is essential for the uniqueness of the solution in Theorem 5.4. We give below an example with at least two solutions when this condition is not satisfied. Consider $\Omega = (-4,4)$ with u defined on $[-4,0]$ by

$$u(x) = \begin{cases} 0 & \text{if} \quad -4 \leq x \leq -3 \\ \\ (x+3)^2 & \text{if} \quad -3 \leq x \leq -2 \\ \\ 2-(x+1)^2 & \text{if} \quad -2 \leq x \leq -1 \\ \\ 2 & \text{if} \quad -1 \leq x \leq 0 \end{cases}$$

with u symmetric, i.e. satisfying

$$u(-x) = u(x).$$

Then we see that u solves (5.3) with a defined on $[-4,0]$ by

$$a(x) = \begin{cases} 0 & \text{if} \quad -4 \leq x \leq -3 \\ \\ 2(x+3) & \text{if} \quad -3 \leq x \leq -2 \\ \\ -2(x+1) & \text{if} \quad -2 \leq x \leq -1 \\ \\ 0 & \text{if} \quad -1 \leq x \leq 0 \end{cases}$$

and a symmetric, i.e. satisfying

$$a(-x) = a(x).$$

In particular, we see that a is Lipschitz continuous and satisfies $a \geq 0$. In this simple example, it is easy to check that both u and $-u$ are viscosity solutions of (5.3), which shows the lack of uniqueness of the solution in that case (see Figure 19).

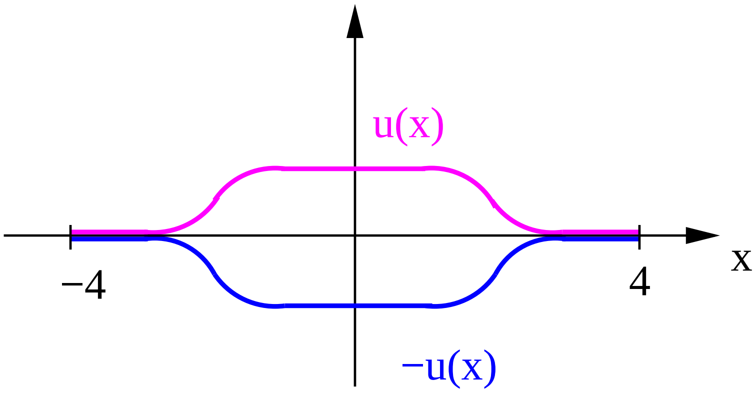

Figure 19: Example with two viscosity solutions u and $-u$ when $a \geq 0$

6 Further remarks and applications of the eikonal equation

In this section, we consider solutions to the eikonal equation. We give an interpretation of the level set of the solution in terms of a front moving with prescribed velocity (this is indeed an analogy with the numerical front apprearing in the FMM). We also give an application of this notion of moving fronts to a model for image segmentation.

6.1 The level set interpretation of the eikonal equation

Let us consider the solution u of the eikonal equation

$$\begin{cases} |\nabla u| = a(x) & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let us now define the function

$$v(x,t) = u(x) - t$$

Then we can check (at least formally) that v solves the following equation

$$\frac{\partial v}{\partial t} = c(x)|\nabla v| \tag{6.1}$$

with

$$c(x) = -\frac{1}{a(x)}.$$

It is possible to see that the levet set

$$\Gamma_t = \{x \in \Omega, \quad v(x,t) = 0\} = \{x \in \Omega, \quad u(x) = t\}$$

moves with normal velocity equal to $c(x)$ (see Fig. 20 and 21). Moreover, equation (6.1) is called a Level Set formulation of the geometric motion of the curve Γ_t.

Notice that the front Γ_t is the continuous analogue of the discrete front F^n that appears in the FMM. And the value $u(x) = t$ is then the time at which the front Γ_t reaches the point x. This is also why the values $\tilde{u}_I^n = t_{n+1}$ in the FMM are interpreted as a time of arrival of the numerical front at the point x_I.

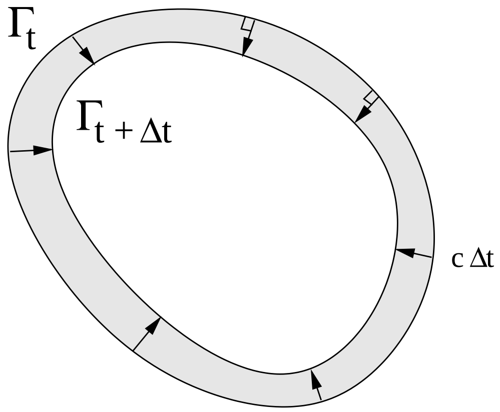

Figure 20: The case with negative velocity c

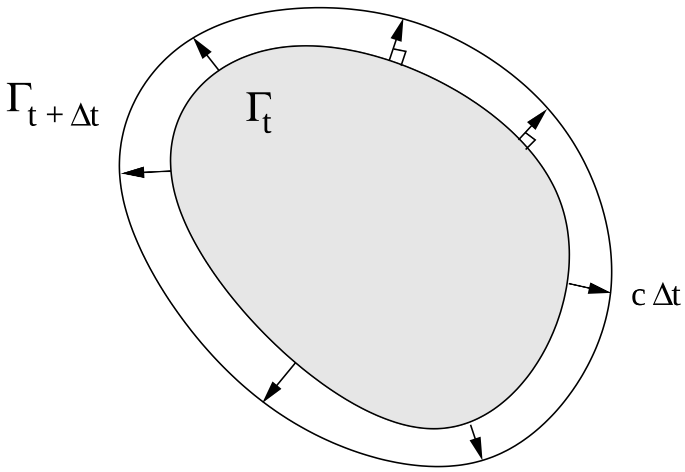

Figure 21: The case with positive velocity c

The case where the velocity $c(x, t)$ depends on the space variable x and on the time variable t is more delicate, especially if the velocity changes sign in space and time. In that case a generalisation of the Fast Marching Method has been given and convergence results for the method have been proven. See Carlini, Falcone, Forcadel, Monneau [4] for a reference on the subject.

6.2 Application to image segmentation

In this subsection, we want to give an example of application of the motion of an curve by normal velocity (and then this can be seen as an application of the FMM with general velocity). We present a simplified version of the **model of Chan, Vese (2001)** [6], used for segmentation of images.

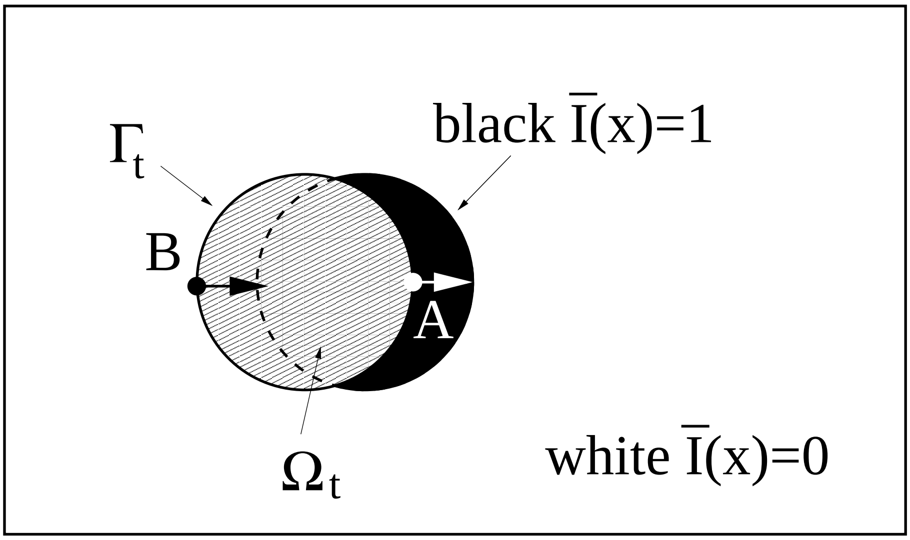

Figure 22: Model of Chan and Vese to move a curve Γ_t towards the boundary of disk

To simplify, we consider an image which is a white rectangle Q containing a black disk, as in Fig. 22. The intensity of the picture is given by a function $\overline{I}(x)$, with the convention that

$$\overline{I}(x) = \begin{cases} 1 & \text{if the point } x \text{ is black} \\ \\ 0 & \text{if the point } x \text{ is white} \end{cases}$$

Notice here that the convention to define the intensity $\overline{I}(x)$ has been reversed with respect to the one introduced in (2.2). We consider an open set Ω_t whose boundary is a curve Γ_t. The goal of the method is to move the curve Γ_t towards the boundary of the black disk. To this end, we define two intermediate quantities which are respectively the mean intensity of the pixels enclosed by the curve Γ_t and the mean intensity of the pixels in the complementar:

$$\begin{cases} c_1(t) = \dfrac{\int_{\Omega_t} \overline{I}(x)\, dx}{\int_{\Omega_t} 1\, dx} \\ \\ c_2(t) = \dfrac{\int_{Q\setminus\Omega_t} \overline{I}(x)\, dx}{\int_{Q\setminus\Omega_t} 1\, dx}. \end{cases}$$

We then define the normal velocity of the curve Γ_t:

$$c(x,t) = (\overline{I}(x) - c_2(t))^2 - (\overline{I}(x) - c_1(t))^2.$$

To understand the mechanism of this motion, let us assume that the curve Γ_t is a circle like on Fig. 22. Because the set Ω_t is quite close to the disk, we see that we have

$$c_1(t) \simeq 1 \quad \text{and} \quad c_2(t) \simeq 0$$

Let us now focus on the points A and B of Fig. 22.

Point A

At the point A, we have

$$c \simeq (1-0)^2 - (1-1)^2 = 1 > 0$$

and then the point A moves with positive normal velocity (for the outward normal to Ω_t), i.e. the point A moves to the right.

Point B

At the point B, we have

$$c \simeq (0-0)^2 - (0-1)^2 = -1 < 0$$

and then the point B moves with negative normal velocity (for the outward normal to Ω_t), i.e. the point B also moves to the right.

We see that this model is able to move the set Ω_t to the right (which is what we wanted to do) and then to move the curve Γ_t in the direction of the boundary of the disk.

We present below in Figure 23 an application of this model to the segmentation of an image of a brain. These results have been obtained by Forcadel, Le Guyader and Gout [12], based on a generalized FMM introduced in [4].

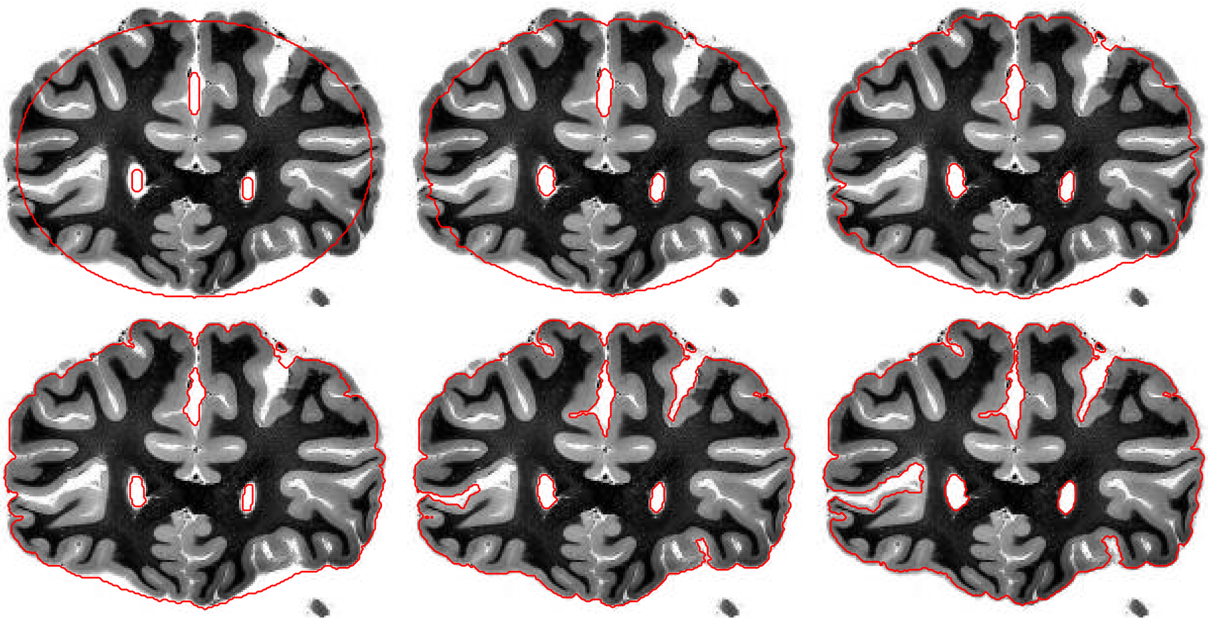

Figure 23: Evolution of the curve (from left to right and top to bottom), taken from [12]

7 More on the FMM

In this section, we give some indications in the litterature to learn further.

The Fast Marching Method

See the original papers [22] of Tsitsiklis and [18] of Sethian. See also the review paper [16] of Sethian.

The Fast Marching Method for sign changing velocities
For special cases of motion, see [7] Chopp (non-signed velocities but time independent), [23] Vladimirsky (case of positive time-dependent velocity). For general velocities, see the paper [4] of Carlini, Falcone, Forcadel, Monneau, where a Generalised Fast Marching Method (GFMM) is presented. For an improved method with a proof of a comparison principle, see also [11] Forcadel. For an application to convergence results for non-local dynamics, see [5].

The Fast Marching Method for anisotropic and more general equations
For an application of the FMM to more general Hamilton-Jacobi equations than the eikonal equation, see [19, 20] Sethian, Vladimirsky. See in particular [21] for a method to solve the eikonal equation on surfaces. For FMM devoted to axis-aligned anisotropy, see [1] Alton, Mitchelld.

The Level Set Method
See the seminal paper [14] of Osher, Sethian for the introduction of the method. See also the book [17] of Sethian, and the book [10] of Fedkiw, Osher.

Viscosity solutions
We refer to the book of Barles [2] for an introduction to viscosity solutions to first order equations. See also the User's Guide of Crandall, Ishii, Lions [8] for a general overview on the subject (including second order equations).

Convergence of finite differences schemes
We refer the reader to the general convergence result of Barles and Souganidis [3] and to the original paper of Crandall, Lions [9] for error estimates. For a proof of convergence of the Generalised FMM, see [4].

8 More on viscosity solutions

In this section we present comparison results and a result of uniqueness for viscosity solutions.

We start with the

Theorem 8.1 (Comparaison principle)
Let $a > 0$ be a Lipschitz continuous function defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Let u be a subsolution of (5.2) and v be a supersolution of (5.2). Then we have

$$u \leq v \quad on \quad \overline{\Omega}.$$

As a consequence we have the

Corollary 8.2 (Uniqueness of the viscosity solution)
Let $a > 0$ be a Lipschitz continuous function defined on a bounded domain $\Omega \subset \mathbb{R}^2$. Then the solution of (5.2) is unique.

Proof of Corollary 8.2
Let u_1, u_2 be two solutions.
From the comparison principle we have

$$u_1 \le u_2$$

and similarly

$$u_2 \le u_1$$

Therefore

$$u_1 = u_2.$$

$\square$

Proof of Theorem 8.1 in the case $a \equiv 1$.
Let us assume by contradiction that

$$M^* = \sup_{x \in \overline{\Omega}} (u(x) - v(x)) > 0.$$

Then for some $\lambda > 1$ (close enough to 1), we have

$$M_0 = \sup_{x \in \overline{\Omega}} (u(x) - \lambda v(x)) > 0.$$

For $\varepsilon > 0$, let us define (introducing the doubling of variables)

$$\begin{aligned} M_\varepsilon &= \sup_{x,y \in \overline{\Omega}} \left(u(x) - \lambda v(y) - \frac{|x-y|^2}{2\varepsilon} \right) \\ &= u(x_\varepsilon) - \lambda v(y_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}. \end{aligned}$$

where the suppremum is reached for $x_\varepsilon, y_\varepsilon \in \overline{\Omega}$. Notice that we have in particular

$$M_\varepsilon \ge \sup_{x=y \in \overline{\Omega}} (...) = M_0 > 0$$

We now distinguish four cases and conclude to a contradiction in each one of these cases.

<u>**Case 1: $x_\varepsilon, y_\varepsilon \in \Omega$.**</u>

• <u>subsolution inequality</u>
From the definition of M_ε, we deduce that

$$\begin{cases} u(x) \le M_\varepsilon + \lambda v(y_\varepsilon) + \dfrac{|x - y_\varepsilon|^2}{2\varepsilon} =: \varphi(x) \\ \\ u(x_\varepsilon) = \varphi(x_\varepsilon). \end{cases}$$

From the subsolution property of u, we deduce that

$$|\nabla \varphi(x_\varepsilon)| \le 1 \quad \text{with} \quad \nabla \varphi(x_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} =: p$$

i.e.

$$(8.1) \qquad |p| \leq 1.$$

• <u>supersolution inequality</u>
Similarly from the definition of M_ε, we have

$$\begin{cases} v(y) \geq \dfrac{1}{\lambda}\left\{-M_\varepsilon + u(x_\varepsilon) - \dfrac{|x_\varepsilon - y|^2}{2\varepsilon}\right\} =: \psi(y) \\ \\ v(y_\varepsilon) = \psi(y_\varepsilon). \end{cases}$$

From the supersolution property of v, we deduce that

$$|\nabla\psi(y_\varepsilon)| \geq 1 \quad \text{with} \quad \nabla\psi(y_\varepsilon) = \frac{1}{\lambda}p,$$

i.e.

$$(8.2) \qquad \frac{1}{\lambda}|p| \geq 1.$$

• <u>Difference of the viscosity inequalities (8.1)-(8.2):</u>
We get

$$(1 - \frac{1}{\lambda})|p| \leq 0 \qquad \text{with } \lambda > 1,$$

This implies

$$p = 0;$$

which gives a contradiction with (8.2).

<u>**Case 2:** $x_\varepsilon, y_\varepsilon \in \partial\Omega$**.**</u>
In that case we have

$$0 < M_0 \leq M_\varepsilon \leq -\frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon} \leq 0$$

which is impossible.

<u>**Case 3:** $x_\varepsilon \in \Omega$, $y_\varepsilon \in \partial\Omega$**.**</u>

Then we have

$$(8.3) \qquad 0 < M_0 \leq M_\varepsilon \leq u(x_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2}{2\varepsilon}.$$

Therefore, up to extract a convergent subsequence, we can assume that

$$x_\varepsilon \to x_0 \in \overline{\Omega} \quad \text{as} \quad \varepsilon \to 0$$

Moreover from (8.3), we have

$$|x_\varepsilon - y_\varepsilon| \leq \left(2\varepsilon \sup_{\overline{\Omega}} |u|\right)^{\frac{1}{2}} \to 0$$

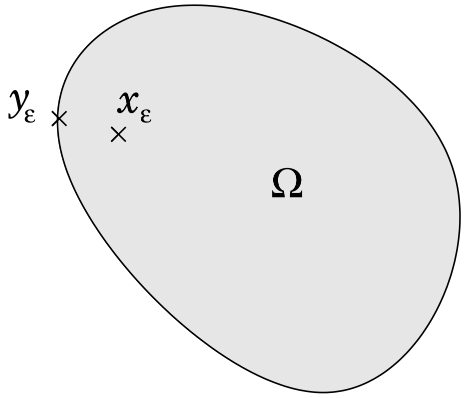

Figure 24: The case x_ε inside Ω and y_ε on the boundary of Ω.

Therefore

$$y_\varepsilon \to x_0 \in \partial\Omega$$

This implies in particular that

$$u(x_\varepsilon) \to u(x_0) \leq 0$$

and then

$$\limsup_{\varepsilon\to 0} M_\varepsilon \leq 0$$

which is in contradiction with $M_\varepsilon \geq M_0 > 0$.

Case 4: $x_\varepsilon \in \partial\Omega$, $y_\varepsilon \in \Omega$.
We get a contradiction similarly as in case 3.

Conclusion
Finally, we have not $M^* = \sup\,(u-v) > 0$, and then

$$u \leq v.$$

This ends the proof of the theorem. □

9 An error estimate for the FMM

In this section, we show how to get an error estimate between the viscosity solution and the solution of the scheme (and then the solution constructed by the FMM).

For a continuous function $a > 0$, we consider a viscosity **subsolution of the equation**:

$$(9.1)\qquad \begin{cases} |\nabla u| = a & \text{on} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

We consider a solution $v = (v_I)_I$ of the scheme, i.e. satisfying

$$(9.2)\qquad \begin{cases} v_I = 0 & \text{if } \ x_I \notin \Omega \\ S_I[v] = 0 & \text{if } \ x_I \in \Omega. \end{cases}$$

where $S[\cdot]$ is a scheme satisfying assumption (H1).

More generally we will say that v is a **supersolution of the scheme** if and only if it satisfies

$$\left\{\begin{array}{ll} v_I \geq 0 & \text{if } \ x_I \notin \Omega \\ \\ S_I[v] \geq 0 & \text{if } \ x_I \in \Omega. \end{array}\right. \tag{9.3}$$

where $S[\cdot]$ is a scheme satisfying assumption (H1).

We recall that

$$x_I = I \cdot \Delta x \quad \text{for} \quad I \in \mathbb{Z}^2$$

In order to prove an error estimate between the solution to the continuous problem and the solution of the scheme, we need a consistency assumption:

(H6) Consistency

$$\left|S_I\left[\{\psi(x_J)\}_{J\in\mathbb{Z}^2}\right] - (|\nabla\psi(x_I)| - a(x_I))\right| \leq C_1|D^2\psi|_{L^\infty}\Delta x \quad \text{for all} \quad I \in \mathbb{Z}^2 \quad \text{and} \quad \psi \in C^2$$

Then we have the following result:

Theorem 9.1 (Discrete-continuous error estimate for sub/supersolutions)
Assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain, that $a \geq 1$ and that a is Lipschitz-continuous. Let us consider a continuous viscosity subsolution u of (9.1) and a supersolution of the scheme, i.e. a function v satisfying (9.3). Let us also assume that there exists a constant $C_0 > 0$ such that

$$|u|, |v| \leq C_0 \tag{9.4}$$

and

$$|u(x) - u(y)| \leq C_0|x - y| \quad \text{and} \quad |v_I - v_J| \leq C_0|x_I - x_J| \tag{9.5}$$

We also assume that there exists a constant $K \geq 1$ such that

(9.6)

$$\text{for all} \quad x \in \partial\Omega, \quad \text{there exists} \quad x_J \in (\mathbb{R}^2\backslash\Omega) \cap ((\Delta x)\mathbb{Z})^2 \quad \text{such that} \quad |x - x_J| \leq K\Delta x$$

If the scheme $S[\cdot]$ satisfies (H1) and (H6), then there exists a constant C (depending on C_0, K, on the scheme and on a, but independent on Δx) such that for $\Delta x \leq 1/C$ we have

$$\sup_{I\in\mathbb{Z}^2} \ (u(x_I) - v_I) \leq C\sqrt{\Delta x}. \tag{9.7}$$

We also have a similar result exchanging (u and v) with a similar proof. As a consequence we also have the following result:

Corollary 9.2 (Discrete-continuous error estimate)
Under the assumptions of Theorem 9.1, if moreover u is a viscosity solution of (9.1) and v a solution of the scheme (9.2), then there exists a constant C (depending on C_0, K, on the scheme and on a, but independent on Δx) such that for $\Delta x \leq 1/C$ we have

$$\sup_{I\in\mathbb{Z}^2} \ |u(x_I) - v_I| \leq C\sqrt{\Delta x}. \tag{9.8}$$

Remark 9.3 (Comment on the existence of the constants C_0 and K)
Notice that for solutions u of the PDE (9.1) and v of the scheme (9.2), it is possible to show the existence of a constant C_0 such that u and v satisfy (9.4) and (9.5). Roughly speaking, the solutions are Lipschitz because of the equation (which implies at least formally an upper bound on the gradient). And the Lipschitz bound and the boundedness of the open set Ω imply the boundedness of the solutions themself.

On the other hand, the existence of a constant K in the technical condition (9.6) is a way to quantify a minimal regularity of the boundary $\partial\Omega$. This condition is for instance satisfied if $\partial\Omega$ is C^1 (for instance for $\Delta x \le 1$).

Remark 9.4 (Error estimate on the level sets)
Notice that the error estimate (Corollary 9.2) on the continuous function u and the discrete function v implies a certain error estimate between a continuous level set of u (i.e. the boundary of $\{x \in \mathbb{R}^2, \quad u(x) \le t\}$) and a "discrete level set" of v (i.e. points x_J with J in the discrete boundary of $\{I \in \mathbb{Z}^2, \quad v_I \le t\}$, which is a front F^n of the FMM for a certain n).
From a bound on $|u - v|$, we can estimate in general the distance between the level sets of u and of v, if the gradients of u and v are not degenerate. This estimate depends strongly on a positive bound from below on the modulus of the gradient of the continuous solution u and of the discrete solution v. For instance, when $a \ge 1$, this bound is formally a consequence of the equation (9.1) satisfied by u and of the scheme (9.2) satisfied by v (keep in mind the scheme (3.8)).
Notive also that all the results hold if the condition $a \ge 1$ is replaced by $a \ge \delta$ for some $\delta > 0$.

As we will see, the proof of Theorem 9.1 can be understood as a variation of the proof of the comparison principle for viscosity sub/supersolutions of the PDE (Theorem 8.1).
Proof of Theorem 9.1 in the case $a \equiv 1$
We introduce the notation

$$v(x) = v_I \quad \text{for} \quad x = x_I \in ((\Delta x)\mathbb{Z})^2 =: G_{\Delta x}$$

where $G_{\Delta x}$ is the grid of mesh size Δx. For $\lambda \in (0,1)$ and $\varepsilon > 0$, let us consider the function

$$\Phi(x,y) = \lambda u(x) - v(y) - \frac{|x-y|^2}{2\varepsilon}$$

and we set

$$M_\varepsilon = \sup_{x \in \overline{\Omega},\ y \in G_{\Delta x}} \Phi(x,y)$$

In particular for $y = x \in G_{\Delta x}$, we deduce that

$$u(x) - v(x) \le (1-\lambda)u(x) + M_\varepsilon \le C_0(1-\lambda) + M_\varepsilon \tag{9.9}$$

i) If $M_\varepsilon \le 0$, then we see that (9.9) implies in particular (9.7) for $1 - \lambda = O(\sqrt{\Delta x})$.
ii) We now assume that $M_\varepsilon > 0$. Because for $x \in \partial\Omega$ and $y \in (\mathbb{R}^2 \backslash \Omega) \cap G_{\Delta x}$, we have $u(x) \le 0$ and $v(y) \ge 0$, we deduce in particular that we can write

$$M_\varepsilon = \Phi(x_\varepsilon, y_\varepsilon)$$

with some points

$$x_\varepsilon \in \overline{\Omega}, y_\varepsilon \in G_{\Delta x} \quad \text{with either} \quad x_\varepsilon \in \Omega \quad \text{or} \quad y_\varepsilon \in \Omega$$

We now distinguish three cases.

Case 1: $x_\varepsilon, y_\varepsilon \in \Omega$.

• subsolution inequality
From the definition of M_ε, we deduce that

$$\begin{cases} u(x) \le \dfrac{1}{\lambda}\left\{M_\varepsilon + v(y_\varepsilon) + \dfrac{|x-y_\varepsilon|^2}{2\varepsilon}\right\} =: \varphi(x) \\ \\ u(x_\varepsilon) = \varphi(x_\varepsilon). \end{cases}$$

From the subsolution property of u, we deduce that

$$|\nabla\varphi(x_\varepsilon)| \le 1 \quad \text{with} \quad \nabla\varphi(x_\varepsilon) = \frac{p}{\lambda} \quad \text{and} \quad p := \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}$$

i.e.

$$|p| \le \lambda. \tag{9.10}$$

• supersolution inequality
Similarly from the definition of M_ε, we have

$$\begin{cases} v(y) \ge -M_\varepsilon + \lambda u(x_\varepsilon) - \dfrac{|x_\varepsilon - y|^2}{2\varepsilon} =: \psi(y) \\ \\ v(y_\varepsilon) = \psi(y_\varepsilon). \end{cases}$$

We now use the fact that v is a supersolution for the scheme, i.e. for $y_\varepsilon = x_{I_\varepsilon}$

$$S_{I_\varepsilon}[v] \ge 0$$

From the monotonicity property (H1) of the scheme, we deduce that

$$S_{I_\varepsilon}[\{\psi(x_I)\}_{I\in\mathbb{Z}^2}] \ge S_{I_\varepsilon}[v] \ge 0$$

From assumption (H6), we deduce that

$$|\nabla\psi(y_\varepsilon)| - 1 \ge S_{I_\varepsilon}[\{\psi(x_I)\}_{I\in\mathbb{Z}^2}] - \frac{C_1}{\varepsilon}\Delta x \qquad \text{with} \quad \nabla\psi(y_\varepsilon) = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} = p$$

This implies in particular that

$$|p| \ge 1 - \frac{C_1}{\varepsilon}\Delta x \tag{9.11}$$

• Difference of the viscosity inequalities (9.10)-(9.11):
We get

$$\frac{C}{\varepsilon}\Delta x \ge 1 - \lambda$$

Therefore we get a contradiction if we choose $\lambda \in (0,1)$ such that

$$1-\lambda > \frac{C_1}{\varepsilon}\Delta x \tag{9.12}$$

.

<u>**Case 2:** $x_\varepsilon \in \Omega,\ y_\varepsilon \notin \Omega$.</u>
Let us consider a point $\bar{y}_\varepsilon \in [x_\varepsilon, y_\varepsilon] \cap \partial\Omega$. Then we have

$$\begin{aligned}
M_\varepsilon &= \lambda u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \\
&\leq \lambda u(x_\varepsilon) - \lambda u(\bar{y}_\varepsilon) - \frac{(x_\varepsilon - \bar{y}_\varepsilon)^2}{2\varepsilon} \qquad \left(\begin{array}{l}\text{using } u(\bar{y}_\varepsilon) \leq 0 \leq v(y_\varepsilon) \text{ for } y_\varepsilon \in (\mathbb{R}^2\backslash\Omega)\cap G_{\Delta x} \\ \text{and } |x_\varepsilon - \bar{y}_\varepsilon| \leq |x_\varepsilon - y_\varepsilon|\end{array}\right) \\
&\leq C_0|x_\varepsilon - \bar{y}_\varepsilon| - \frac{(x_\varepsilon - \bar{y}_\varepsilon)^2}{2\varepsilon} \qquad \text{(using the Lipschitz estimate (9.5))} \\
&\leq \varepsilon\frac{C_0^2}{2} \qquad \text{(doing an optimization on } |x_\varepsilon - \bar{y}_\varepsilon|)
\end{aligned}$$

Therefore (9.9) implies

$$u(x) - v(x) \leq C_0(1-\lambda) + \varepsilon\frac{C_0^2}{2} \tag{9.13}$$

<u>**Case 3:** $x_\varepsilon \in \partial\Omega,\ y_\varepsilon \in \Omega$.</u>
From condition (9.6), we know that there exists $\bar{x}_\varepsilon \in G_{\Delta x} \cap (\mathbb{R}^2\backslash\Omega)$ such that

$$|x_\varepsilon - \bar{x}_\varepsilon| \leq K\Delta x \tag{9.14}$$

Then as in case 2, we have

$$\begin{aligned}
&M_\varepsilon \\
&= \lambda u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \\
&\leq v(\bar{x}_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \qquad (\text{using } u(x_\varepsilon) \leq 0 \leq v(\bar{x}_\varepsilon) \text{ for } x_\varepsilon \in \partial\Omega, \quad \bar{x}_\varepsilon \in (\mathbb{R}^2\backslash\Omega)\cap G_{\Delta x}) \\
&\leq C_0|\bar{x}_\varepsilon - y_\varepsilon| - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \qquad \text{(using the Lipschitz estimate (9.5))} \\
&\leq C_0K\Delta x + C_0|x_\varepsilon - y_\varepsilon| - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \qquad \text{(using estimate (9.14))} \\
&\leq C_0K\Delta x + \varepsilon\frac{C_0^2}{2} \qquad \text{(doing again the optimization on } |x_\varepsilon - y_\varepsilon|)
\end{aligned}$$

Conclusion
For all the cases, we can choose λ such that

$$1-\lambda = 2\frac{C_1}{\varepsilon}\Delta x. \tag{9.15}$$

Then (9.12) is satisfied, and we have (9.9), which implies in all cases

$$\begin{aligned}
u(x)-v(x) &\le C_0(1-\lambda)+\varepsilon\frac{C_0^2}{2}+C_0K\Delta x \\
&\le 2C_0\frac{C_1}{\varepsilon}\Delta x+\varepsilon\frac{C_0^2}{2}+C_0K\Delta x && \text{(using (9.15))} \\
&\le \frac{C_2}{2}\left(\frac{\Delta x}{\varepsilon}+\varepsilon\right)+C_0K\Delta x && \text{(with } C_2=4(\max(C_0,C_1))^2\text{)} \\
&\le C_2\sqrt{\Delta x}+C_0K\Delta x && \text{(optimizing the value of } \varepsilon \text{ with } \varepsilon=\sqrt{\Delta x}\text{)} \\
&\le C\sqrt{\Delta x} && \text{(with } C=C_2+C_0K \text{ for } \Delta x\le 1\text{)}
\end{aligned}$$

In particular we recover $\lambda\in(0,1)$ for $\Delta x<\frac{1}{4C_1^2}$. This ends the proof of the Theorem. □

Lemma 9.5 (Checking the consistency assumption)
The scheme given in (3.8) satisfies the consistency assumption (H6).

Proof of Lemma 9.5
We consider the expression

$$A:=S_I[\{\psi(x_J)\}_{J\in\mathbb{Z}^2}]-(|\nabla\psi(x_I)|-a(x_I))$$

with the scheme

$$S_I[u]=-a(x_I)+F\left(x_I,\frac{u_{I^{1,-}}-u_I}{\Delta x},\frac{u_{I^{1,+}}-u_I}{\Delta x},\frac{u_{I^{2,-}}-u_I}{\Delta x},\frac{u_{I^{2,+}}-u_I}{\Delta x}\right)$$

with

$$F(p^{1,-},p^{1,+},p^{2,-},p^{2,+})=\sqrt{\left(\max\left(0,-p^{1,-},-p^{1,+}\right)\right)^2+\left(\max\left(0,-p^{2,-},-p^{2,+}\right)\right)^2}$$

To evaluate this expression, let us compute

$$\psi(y)-\psi(y_0)=\psi'(y_0)\cdot(y-y_0)+\int_0^1 dt\int_0^t ds\ D^2\psi(y_0+s(y-y_0))\cdot(y-y_0)^2$$

This implies for $p=\psi'(y_0)$ and $y_e=y_0+(\Delta x)e$ with $e\in\mathbb{Z}^2$ and $|e|\le 1$:

$$\left|\frac{\psi(y_e)-\psi(y_0)}{\Delta x}-p\cdot e\right|\le\frac{1}{2}|D^2\psi|_{L^\infty}\Delta x$$

Using the fact that for $P=(p^{1,-},p^{1,+},p^{2,-},p^{2,+})$, the function $F(P)$ is Lipschitz in P, i.e. there exists a constant L (that can indeed be taken equal to 1) such that

$$|F(P)-F(Q)|\le L|P-Q|$$

we get that with the choice $y_0=x_I$

$$|A|\le L|D^2\psi|_{L^\infty}\Delta x$$

which ends the proof of the Lemma.

10 Appendix: proof of some results for the FMM

In this section, we prove some results related to the FMM, namely a comparison principle for the scheme (Proposition 3.4) and the monotonicity of the sequence of times $(t_n)_n$ (Proposition 4.3).

10.1 Proof of the comparison principle for the scheme used in the FMM

In this subsection we prove Proposition 3.4 which claims a certain comparison principle for the scheme satisfying (H1),(H2),(H3),(H4),(H5).

Proof of Proposition 3.4
Step 1: Preliminaries
Let us consider two functions $u = (u_I)_I$ and $v = (v_I)_I$ as in the statement of Proposition 3.4. Let us proceed by contradiction, and assume that

$$M = \sup_I (u_I - v_I) > 0. \tag{10.1}$$

We define the set

$$\mathcal{K} = \left\{ I \in \mathbb{Z}^2, \quad M + v_I = u_I \right\}$$

Because $u_I \le v_I$ for $x_I \in \mathbb{R}^2 \backslash \Omega$, we deduce that

$$\mathcal{K} \subset \left\{ I \in \mathbb{Z}^2, \quad x_I \in \Omega \right\}$$

and then $\mathcal{K}$ is bounded. Let us choose an index $I_0 \in \mathcal{K}$ such that

$$v_{I_0} = \inf_{J \in \mathcal{K}} v_J$$

Because $x_{I_0} \in \Omega$, we deduce from (3.14) that

$$S_{I_0}[v] \ge 0 \tag{10.2}$$

Step 2: Proof of $v_{I_0} > \inf_{J \in V(I_0)\backslash\{I_0\}} v_J$
Assume by contradiction that

$$v_{I_0} \le \inf_{J \in V(I_0)\backslash\{I_0\}} v_J \tag{10.3}$$

Let us define

$$\hat{v}_J = \begin{cases} v_J & \text{if} \quad v_J < v_{I_0} \quad \text{or} \quad J = I_0 \\ \\ +\infty & \text{otherwise} \end{cases}$$

In particular, we have

$$\hat{v}_J = \begin{cases} v_{I_0} & \text{if} \quad J = I_0 \\ \\ +\infty & \text{if} \quad J \in V(I_0) \backslash \{I_0\} \end{cases}$$

Then by the causality asumption (H3), we have

$$S_{I_0}[v] = S_{I_0}[\hat{v}] = S_{I_0}(\{\hat{v}_J\}_{J \in V(I_0)})$$

where we have also used assumption (H4) in the last equality. Defining now the constant function

$$\tilde{v}_J = v_{I_0} \quad \text{for all} \quad J$$

the same raisoning also gives that

$$S_{I_0}[\tilde{v}] = S_{I_0}(\{\hat{v}_J\}_{J \in V(I_0)})$$

Therefore

$$S_{I_0}[v] = S_{I_0}[\tilde{v}] = S_{I_0}[0] < 0$$

where we have used (H2) for the last equality and (H5) for the last inequality. This gives a contradiction with (10.2) and shows that (10.3) is not true, i.e. we have

$$v_{I_0} > \inf_{J \in V(I_0) \setminus \{I_0\}} v_J$$

Step 3: Proof that $M + v_J > u_J$ for $J \in \mathcal{J}_0$
Let us define the set

$$\mathcal{J}_0 = \{J \in V(I_0) \setminus \{I_0\}, \quad v_J < v_{I_0}\}$$

and consider a point $J \in \mathcal{J}_0$. On the one hand, from (10.1), we have

$$u_J \leq M + v_J \tag{10.4}$$

On the other hand, if

$$u_J = M + v_J \tag{10.5}$$

this shows that $J \in \mathcal{K}$. Therefore by definition of I_0, we have

$$v_{I_0} \leq v_J$$

This is in contradiction with the defintion of the set $\mathcal{J}_0$. Therefore (10.5) is impossible and we deduce from (10.4) that

$$u_J < M + v_J \quad \text{for all} \quad J \in \mathcal{J}_0 \tag{10.6}$$

Step 4: Getting a contradiction
Let us define the functions

$$\bar{v}_J = \begin{cases} v_{I_0} & \text{if} \quad J = I_0 \\ v_J & \text{if} \quad J \in \mathcal{J}_0 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\check{v}_J = \begin{cases} v_{I_0} & \text{if} \quad J = I_0 \\ v_J - \varepsilon & \text{if} \quad J \in \mathcal{J}_0 \\ +\infty & \text{otherwise} \end{cases}$$

for some $\varepsilon > 0$ small enough such that we can strengthen inequality (10.6) as

$$u_J < M + v_J - \varepsilon \quad \text{for all} \quad J \in \mathcal{J}_0$$

Then we have

$$\begin{cases} u_J \le M + \check{v}_J & \text{for all} \quad J \\ u_{I_0} = M + \check{v}_{I_0} \end{cases}$$

Therefore we deduce that

$$\begin{aligned} S_{I_0}[u] &\ge S_{I_0}[M + \check{v}] \\ &= S_{I_0}[\varepsilon + \check{v}] \\ &\ge \delta\varepsilon + S_{I_0}[\bar{v}] \\ &= \delta\varepsilon + S_{I_0}[v] > S_{I_0}[v] \end{aligned}$$

where we have used (H1) for the first line, (H2) for the second line, (H5) for the third line (with $\varepsilon + \check{v}_J = \bar{v}_J$ for $J \neq I_0$), and finally (H3) to get the last line. This gives a constradiction with (3.14). Therefore $u \le v$ and this ends the proof. □

10.2 Proof of the monotonicity of the times $(t_n)_n$ for the FMM

We recall that Proposition 4.3 claims that the sequence of times constructed by the FMM satisfies

$$t_n < t_{n+1}$$

In this subsection, we give the proof of this fact.

Proof of Proposition 4.3
If $n = 0$, the fact that $t_1 > 0 = t_0$ follows from the second line of (H5). Therefore we can assume that $n \ge 1$. Let us assume by contradiction that $t_{n+1} \le t_n$, i.e.

$$\text{there exists a point} \quad I \in NA^{n+1} \subset F^n \quad \text{with} \quad \tilde{u}_I^n = t_{n+1} \le t_n. \tag{10.7}$$

We recall that $\tilde{u}_I^n$ solves

$$S_I(\tilde{u}_I^n, \{u_J^n\}_{J \in V^*(I)}) = 0 \quad \text{with} \quad V^*(I) = V(I) \setminus \{I\} \tag{10.8}$$

Let us define

$$B^n := V^*(I) \cap NA^n$$

We have

$$\begin{cases} \tilde{u}_I^n \le t_n = u_J^n < +\infty = u_J^{n-1} & \text{for every} \quad J \in B^n \\ u_J^n = u_J^{n-1} & \text{for every} \quad J \in V^*(I) \setminus B^n \end{cases}$$

Combining assumptions (H3) and (H4), we see that the scheme has the following property:

$$S_I(\tilde{u}_I^n, \{u_J^n\}_{J \in V^*(I)}) = S_I(\tilde{u}_I^n, \{\hat{u}_J^n\}_{J \in V^*(I)}) \quad \text{with} \quad \hat{u}_J^n = \begin{cases} u_J^n & \text{if} \quad u_J^n < \tilde{u}_I^n \\ +\infty & \text{if} \quad u_J^n \ge \tilde{u}_I^n \end{cases} \tag{10.9}$$

Therefore we deduce that

$$S_I(\tilde{u}_I^n, \{u_J^{n-1}\}_{J\in V^*(I)}) = S_I(\tilde{u}_I^n, \{u_J^n\}_{J\in V^*(I)}) = 0 \tag{10.10}$$

Case 1: $I \in F^{n-1}$
Recall that the uniqueness of the solution to equation (10.10) follows from Proposition 3.3. Therefore we get

$$\tilde{u}_I^{n-1} = \tilde{u}_I^n$$

Then (10.7) implies in particular that

$$\tilde{u}_I^{n-1} \le t_n \tag{10.11}$$

On the other hand, because $I \in NA^{n+1}$ and $NA^{n+1} \cap NA^n = \emptyset$, this implies that $I \in F^{n-1}$ does not belong to NA^n, and then the only possibility is that

$$\tilde{u}_I^{n-1} > t_n$$

This gives a contradiction with (10.11).

Case 2: $I \in F^n \setminus F^{n-1}$
By assumption $I \in F^n$ and then $I \in \mathbb{Z}^2 \backslash A^n$. We have moreover $A^{n-1} \subset A^n$, which implies $I \in \mathbb{Z}^2 \backslash A^{n-1}$. If there exists a point $K \in V^*(I) \cap A^{n-1}$, this shows that $I \in F^{n-1}$, which is impossible by assumption. Therefore

$$V^*(I) \cap A^{n-1} = \emptyset$$

which implies

$$V^*(I) \cap A^n = V^*(I) \cap NA^n$$

Because $I \in F^n$, we deduce that there exists $J \in V^*(I)$ such that $J \in NA^n$ and $u_J^n = t_n$ and then

$$V^*(I) \cap NA^n \neq \emptyset$$

We finally deduce from (10.8) and Proposition 3.3 that

$$\tilde{u}_I^n > \min_{J\in V^*(I)} u_J^n = \min_{J\in V^*(I)\cap A^n} u_J^n = \min_{J\in V^*(I)\cap NA^n} u_J^n = t_n$$

This gives a contradiction with (10.7). $\square$

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