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Équations aux dérivées partielles/*Partial differential equations*

Estimations de convexité pour des équations elliptiques non-linéaires et application à des problèmes de frontière libre

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Résumé. On établit la convexité de l'ensemble délimité par la frontière libre correspondant à une équation quasi-linéaire dans un domaine convexe en dimension 2. La méthode repose sur l'étude de la courbure des lignes de niveau aux points qui réalisent le maximum de la dérivée normale à niveau donné, pour des solutions analytiques d'équations elliptiques complètement non-linéaires. La méthode fournit aussi une estimation du gradient en fonction du minimum de la courbure (signée) du bord du domaine, qui n'est pas nécessairement supposé convexe. © Académie des Sciences/Elsevier, Paris

Convexity estimates for nonlinear elliptic equations and application to free boundary problems

Abstract. *We prove the convexity of the set which is delimited by the free boundary corresponding to a quasi-linear elliptic equation in a 2-dimensional convex domain. The method relies on the study of the curvature of the level lines at the points which realize the maximum of the normal derivative at a given level, for analytic solutions of fully nonlinear elliptic equations. The method also provides an estimate of the gradient in terms of the minimum of the (signed) curvature of the boundary of the domain, which is not necessarily assumed to be convex.* © Académie des Sciences/Elsevier, Paris

Version Française abrégée

Considérons dans un domaine $\Omega \subset I\!R^2$ le problème de frontière libre défini par

$$\mathrm{div}\big(a(|\nabla u|^2)\nabla u\big) = f(u) \quad dans \quad \Omega \setminus \Lambda \tag{1}$$

$$0 = u_{|\partial\Lambda} < u < u_{|\partial\Omega} = u_0 \ , \tag{2}$$

$$\partial_n u = 0 \quad sur \quad \partial\Lambda \ . \tag{3}$$

où u_0 est une constante positive donnée, $\Lambda \subset \Omega$ un ensemble fermé et $\partial_n u$ la dérivée de u normale au bord de Λ, sortante.

Note présentée par Pierre-Louis Lions.

THÉORÈME 1 – *Supposons $a(0)$, $f(0)>0$. Si $q\mapsto a(q)$, $u\mapsto f(u)$ sont des fonctions croissantes de classe C^1 et C^0 respectivement, si Ω est convexe et si u est solution de (1)-(2)-(3), alors Λ est aussi convexe.*

Ce résultat généralise des travaux antérieurs correspondant à des choix particuliers pour f et a: voir [6], [5], et [2] lorsque la condition (3) est remplacée par $\partial_n u = Const > 0$. Nous allons démontrer le Théorème 1 en nous plaçant dans un cadre beaucoup plus général, mais en supposant les solutions analytiques. Nous supposerons que

$$\partial_n u = \lambda(K) \geq 0 \quad sur \quad \partial\Lambda\ , \tag{4}$$

où λ est une fonction de la courbure K de $\partial\Lambda$. Soit n et τ les vecteurs unitaires respectivement normal et tangent à la courbe de niveau, tels que (τ, n) est une base directe de $I\!R^2$. Nous allons considérer les solutions analytiques de l'équation complètement non-linéaire

$$\mathcal{G}(D_{nn}u, D_{\tau\tau}u, D_{n\tau}u, |\nabla u|, u) = 0\ , \tag{5}$$

qui n'a de sens que sous la

(A0) Condition de compatibilité: *il existe une fonction $\overline{\mathcal{G}}$ telle que*

$$\mathcal{G}(a,b,c,0,u) = \overline{\mathcal{G}}(a+b, ab-c^2, u)\ .$$

Définissons $\alpha = (\mathcal{G})'_{D_{nn}u}$, $\beta = (\mathcal{G})'_{D_{\tau\tau}u}$ et $\gamma = (\mathcal{G})'_{D_{n\tau}u}$. Nous introduisons maintenant les hypothèses suivantes:

(A1) Condition de gradient non nul:

$$\forall t \in (0,u_0), \quad \exists\, x \in \Omega\setminus\Lambda, \quad u(x)=t, \quad \nabla u(x) \neq 0\ .$$

(A2) Conditions d'ellipticité:

$$\inf_X \alpha > 0, \quad \inf_X \beta > 0, \quad \inf_X (4\alpha\beta - \gamma^2) \geq 0\ ,$$

X étant l'ensemble des points qui réalisent le maximum du gradient sur leur ligne de niveau:

$$X = \left\{ x\in\Omega\setminus\Lambda :\ |\nabla u(x)| = \max_{\substack{y\in\Omega\setminus\Lambda\\ u(y)=u(x)}} |\nabla u(y)| \right\}\ .$$

(A3) Condition sur la frontière libre:

$\partial\Lambda$ est analytique et l'application $K\mapsto\lambda(K)$ est analytique, décroissante. De plus si $\lambda(K)\equiv 0$ sur $\partial\Lambda$, alors on demande que $\mathcal{G}(0,0,0,0,0)<0$ et que le champ de vecteur $n=\frac{\nabla u}{|\nabla u|}$ soit analytique jusqu'au bord $\partial\Lambda$.

Dans ce cadre et pour des solutions analytiques uniquement, on montre les résultats suivants.

PROPOSITION 2 – *Sous les hypothèses (A0)-(A1)-(A2)-(A3), si u est une solution analytique du problème de frontière libre (5)-(2)-(4) (y compris dans le cas $\Lambda=\emptyset$), alors on a les propriétés suivantes:*

i) Le gradient de u est borné par une constante qui ne dépend que de $\mathcal{G}$, de $|\lambda(K)|_{L^\infty(\Lambda)}$, de u_0 et du minimum de la courbure signée de $\partial\Omega$.

ii) Le minimum de la courbure signée de $\partial\Lambda$ est plus grand que le minimum de la courbure signée de $\partial\Omega$. En particulier, si Ω est convexe, alors Λ est aussi convexe.

Ce résultat repose sur des estimations données par un système de deux équations différentielles ordinaires, que l'on définit de la manière suivante. Pour une solution analytique de (5) sur $\Omega \setminus \Lambda$ (y compris dans le cas $\Lambda = \emptyset$), on définit

$$\Gamma^t = \{x \in \Omega \ : \ u(x) = t\} \ , \quad m(t) = \max_{y\in\Gamma^t} |\nabla u(y)| \ ,$$
$$et \quad X^t = \{x \in \Gamma^t \ : \ |\nabla u(x)| = m(t)\} \ ,$$

pour tout $t \in (0, u_0)$. Soit $K(t) := \max_{y\in X^t} \frac{D_{\tau\tau}u}{|\nabla u|}(y)$.

THÉORÈME 3 – *Sous les hypothèses (A0)-(A1)-(A2), soit u une solution analytique de (5). Avec les notations précédentes, m et K sont solutions sur $(0, u_0)$ de*

$$\mathcal{G}(m\frac{dm}{dt}, mK, 0, m, t) = 0 \ , \tag{6}$$

$$\frac{dK}{dt} \le -\frac{K^2}{m} \ . \tag{7}$$

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1. Main results

Consider a solution of the following free boundary problem

$$\mathrm{div}\big(a(|\nabla u|^2)\nabla u\big) = f(u) \quad in \quad \Omega \setminus \Lambda \tag{1}$$

$$0 = u_{|\partial\Lambda} < u < u_{|\partial\Omega} = u_0 \ , \tag{2}$$

$$\partial_n u = 0 \quad on \quad \partial\Lambda \ . \tag{3}$$

where u_0 is a given nonnegative constant, Λ is a closed subset of a bounded domain Ω in $\mathbb{R}^2$ and $\partial_n u$ is the normal (to $\partial\Lambda$) outgoing derivative of u. This problem arises for instance from an obstacle problem.

THEOREM 1 – *Assume that $a(0)$, $f(0) > 0$ and that $q \mapsto a(q)$, $u \mapsto f(u)$ are increasing functions of class C^1 and C^0 respectively. If Ω is convex and if u is a solution of (1)-(2)-(3), then Λ is also convex.*

This theorem has been proved in the special case where $a \equiv a_0$ and $f \equiv f_0$ are constants by A. Friedman and D. Phillips [6] in two dimensions, and then extended to any dimension by B. Kawohl [5]. Similar results were also proved (in any dimensions) for $a \equiv a_0$, $f \equiv 0$ and $\partial_n u = const > 0$ in place of $\partial_n u = 0$ by L.A. Caffarelli and J. Spruck [2]. We will prove Theorem 1 in a much more general framework, except that we will deal only with analytic solutions for reasons that will be made clear later (see Lemma 5). We will assume

$$\partial_n u = \lambda(K) \ge 0 \quad on \quad \partial\Lambda \ , \tag{4}$$

where λ is a function of the curvature K of $\partial\Lambda$. Here we denote by n and τ normal and tangent unit vectors to a level set, so that (τ, n) is a direct orthonormal basis in $\mathbb{R}^2$, and $n = \frac{\nabla u}{|\nabla u|}$. When

$\nabla u \neq 0$, the curvature is defined by $K = \frac{D_{\tau\tau}u}{|\nabla u|}$. We shall consider the analytic solutions of the fully nonlinear elliptic equation

$$\mathcal{G}(D_{nn}u, D_{\tau\tau}u, D_{n\tau}u, |\nabla u|, u) = 0 \quad in \quad \Omega \setminus \Lambda \,, \tag{5}$$

where $\mathcal{G}$ is an analytic function. The vectors n and τ are well defined if $\nabla u \neq 0$. For the equation to make sense in case of a patch of zero gradient, we therefore require the following

(A0) Compatibility condition: *we assume the existence of a function $\overline{\mathcal{G}}$ such that*

$$\mathcal{G}(a, b, c, 0, u) = \overline{\mathcal{G}}(a + b, ab - c^2, u) \quad \forall\, a, b, c, u \;.$$

We define the coefficients $\alpha = (\mathcal{G})'_{D_{nn}u}$, $\beta = (\mathcal{G})'_{D_{\tau\tau}u}$ and $\gamma = (\mathcal{G})'_{D_{n\tau}u}$ and introduce the following assumptions:

(A1) Non zero gradient condition:

$$\forall t \in (0, u_0), \quad \exists\, x \in \Omega \setminus \Lambda, \quad u(x) = t, \quad \nabla u(x) \neq 0 \;.$$

(A2) Ellipticity conditions:

$$\inf_X \alpha > 0, \quad \inf_X \beta > 0, \quad \inf_X (4\alpha\beta - \gamma^2) \geq 0 \;,$$

where X is the set of points which realize the maximum of the gradient on their level line:

$$X = \left\{ x \in \Omega \setminus \Lambda \;:\; |\nabla u(x)| = \max_{\substack{y \in \Omega \setminus \Lambda \\ u(y) = u(x)}} |\nabla u(y)| \right\} \,.$$

(A3) Condition on the free boundary:

We assume that $\partial\Lambda$ is analytic and that the map $K \mapsto \lambda(K)$ is analytic nonincreasing. Moreover if $\lambda(K) \equiv 0$ on $\partial\Lambda$, then we assume that $\mathcal{G}(0,0,0,0,0) < 0$ and that the vector field $n = \frac{\nabla u}{|\nabla u|}$ is analytic up to $\partial\Lambda$.

These assumptions cover the case of quasilinear elliptic equations as well as the Monge-Ampère equations

$$\det(D^2 u) = f_0(u, |\nabla u|) > 0 \;.$$

In this framework and for analytic solutions only, we prove the following result.

PROPOSITION 2 – *Under assumptions (A0)-(A1)-(A2)-(A3), if u is an analytic solution to the free boundary problem (5)-(2)-(4) (including the case $\Lambda = \emptyset$), then we have the following properties:*

i) There exists a constant M which only depends on $\mathcal{G}$, $|\lambda(K)|_{L^\infty(\Lambda)}$, u_0 and the minimum of the signed curvature of $\partial\Omega$ such that

$$||\nabla u||_{L^\infty(\Omega\setminus\Lambda)} \leq M \;.$$

ii) The minimum of the signed curvature of $\partial\Lambda$ is bigger than the minimum of the signed curvature of $\partial\Omega$:

$$\inf_{\partial\Lambda} K \geq \inf_{\partial\Omega} K \;.$$

As a consequence, if Ω is convex, Λ is also convex.

In this note we shall only sketch the proofs and refer to [3] for more details. The key point of the method is Theorem 4 (see below).

2. Curvature and maximum of the gradient

Throughout this section, we assume that u is an analytic solution of Equation (5) on $\Omega \setminus \Lambda$ (with eventually $\Lambda = \emptyset$).

Notations. We shall note $\partial_\sigma u = \sigma \cdot \nabla u$ the derivative of u along the unit vector σ and $D_{\sigma\sigma} u := (\sigma, D^2 u\ \sigma)$. Since the tangent and normal unit vectors τ and ν depend on x, $\partial_\tau(\partial_\tau u) \neq D_{\tau\tau} u$ in general, and one has to use the Fréchet formula

$$\partial_\tau n = K\tau\ , \quad \partial_\tau \tau = -Kn\ ,$$

$$\partial_n n = \rho\tau\ , \quad \partial_n \tau = -\rho n\ ,$$

where $K = \frac{1}{|\nabla u|} D_{\tau\tau} u$ is the curvature of the level line and $\rho = \frac{1}{|\nabla u|} D_{n\tau} u$.

For $t > 0$, let

$$\Gamma^t = \{x \in \Omega\ :\ u(x) = t\}\ , \quad m(t) = \max_{y \in \Gamma^t} |\nabla u(y)|\ ,$$

$$\textit{and} \quad X^t = \{x \in \Gamma^t\ :\ |\nabla u(x)| = m(t)\}\ .$$

With a straightforward abuse of notations, we define $K(t) := \inf_{y \in X^t} \frac{D_{\tau\tau} u}{|\nabla u|}(y)$. The following result is the core of our method.

THEOREM 3 – *Under assumptions (A0)-(A1)-(A2), consider an analytic solution u of Equation (5). With the above notations, m is continuous and derivable outside an enumerable closed set in $(0, u_0)$ such that*

$$\mathcal{G}(m\frac{dm}{dt}, mK, 0, m, t) = 0\ \textit{for a.e.}\ t \in (0, u_0)\ , \tag{6}$$

$$\frac{dK}{dt} \leq -\frac{K^2}{m}\ , \tag{7}$$

where the inequality has to be understood in the sense of distributions.

REMARKS: *In higher dimensions, we can formally get a similar result for the mean curvature of the level sets.*

In the case of a radially symmetric solution (when Ω is a ball), Inequality (7) becomes an equality. The result of Theorem 3 can therefore be compared with comparison results based on rearrangement techniques, like the ones obtained by G. Talenti in another context [9].

In the nonradial case, we prove a refined version of (7): $\frac{dK}{dt} \leq -\frac{K^2}{m} - \frac{1}{m}\Big(2\sqrt{\frac{\alpha}{\beta}} - \gamma\Big) \cdot \min |\partial_\tau(\frac{D_{\tau\tau} u}{|\nabla u|})|$, where the minimum has to be taken over the set $\{y \in \Omega \setminus \Lambda\ :\ u(y) = t\}$.

We will not give a complete proof of this result, but simply sketch the method in a simple case: assume that $t \mapsto x^t$ is a curve (on an open set in t) such that $|\nabla u(x^t)| = \partial_n u(x^t) = m(t)$ and $u(x^t) = t$. It is straightforward to establish that $n \cdot \frac{dx^t}{dt} = 1/m(t)$, and then to prove that $m\frac{dm}{dt} = \frac{d}{dt}(\frac{1}{2}|\nabla u(x^t)|^2) = D_{nn} u(x^t)$, which using the definition of K and

$$0 = \partial_\tau(\frac{1}{2}|\nabla u|^2)_{|x=x^t} \quad \Longrightarrow \quad D_{n\tau} u(x^t) = 0\ ,$$

gives (6).

Inequality (7) is slightly more difficult to establish. It is essentially based on

$$0 \geq G(t) := \partial_\tau^2(\frac{1}{2}|\nabla u|^2)_{|x=x^t}\ ,$$

which allows to prove that either $\partial_\tau K = 0$ and $\frac{dK}{dt} = -\frac{K^2}{m} + \frac{G}{m^3}$, or $\partial_\tau K \neq 0$, $G < 0$ and $\frac{dK}{dt} = -\frac{K^2}{m} + \frac{G}{m^3} + \frac{\alpha}{\beta} m (\partial_\tau K)^2 \frac{1}{G} + \frac{\gamma}{\beta} m \partial_\tau K$, where $\partial \tau K(t)$ stands for $\partial_\tau (\frac{D_{\tau\tau} u}{|\nabla u|})_{|x=x^t}$. In that case, the conclusion follows by an optimization on G. $\square$

2. Sketch of the proofs

As a conclusion, we shall justify the assumptions made in the proof of Theorem 4 and briefly sketch the main steps of the proofs of Proposition 2, and Theorem 1.

LEMMA 4 – *Under assumption (A1), if u is an analytic solution of (5)-(2)-(4), then either $X = \cup_{t\in(0,u_0)} X^t$ is locally included in a finite union of analytic curves, or $X \equiv \Omega \setminus \Lambda$.*

According to this lemma, let us call $\mathcal{C}$ the collection of all such curves:

$$m(t) = \sup_{\gamma_0 \in \mathcal{C}} |\nabla u(\gamma_0(t))| \, .$$

Given two curves $\gamma_1, \gamma_2 \in \mathcal{C}$ the curvatures $K(\gamma_1(t))$ and $K(\gamma_2(t))$ can be different and in general this implies that

$$K(t) = \inf_{\gamma_0 \in \mathcal{C}} K(\gamma_0(t))$$

may have some jumps. A detailed analysis of $t \mapsto K(\gamma_0(t))$ for t corresponding to an endpoint of such a curve shows that the jumps of $t \mapsto K(t)$ are always nonpositive. This allows to complete the proof of Theorem 4 (in the generic case).

Any accumulation point of $\cup_{t\in(0,\epsilon)} X^t$ as $\epsilon \to 0$ realizes the minimum of the curvature of $\partial\Lambda$. This is based on a delicate study of the asymptotic behaviour of u near $\partial\Lambda$ and one has to distinguish between the cases $\lambda > 0$ and $\lambda = 0$. Proposition 2 is then easy to prove and is a consequence of Theorem 4. Details are left to the reader.

Theorem 1 is deduced from Proposition 2 and from considerations on uniqueness and regularity. The free boundary corresponding to an analytic solution of (1)-(2)-(3) is indeed analytic. The proof uses a control of the regularity of the free boundary by a perturbation method based on the papers by L.A. Caffarelli [1] on the obstacle problem and by D. Kinderlehrer and L. Nirenberg [7] on the analyticity of the free boundary. Theorem 1 for non necessarily analytic solutions then follows by a density argument.

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