On the Mushy Region Arising Between Two Fluids in a Porous Medium*

A. Bonnet[a] and R. Monneau[b]

[a] *Goldman Sachs International, UK*

[b] *Ecole Nationale des Ponts et Chaussées, CERMICS,*

6 et 8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne,

77455 Marne-la-Vallée Cedex 2, France

January 16, 2001

Abstract

We study the mushy region arising between two fluids in a porous medium. We prove that the interior of the mushy region is an epigraph in the horizontal direction. Moreover when the interior of the mushy region is empty, we give a necessary and sufficient condition to claim that the Lebesgue measure of this mushy region is zero.

AMS Classification: 35B50, 35R35.

Keywords: Free boundary problem, mushy region, blow-up.

1 Introduction

1.1 The physical problem

In this paper we study the interface between two fluids for a steady flow in a porous medium.

One fluid

Let us recall that the steady flow of one fluid in a porous medium, is characterized by its pressure $p \geq 0$ ($p = 0$ outside the fluid), and its velocity which is given by Darcy's law:

$$v = -k\nabla\phi$$

*Nonlinear Analysis TMA, submitted.

Here k is a permeability coefficient which depends on the porous medium and is assumed constant. The potential ϕ is given by $\phi = p + \rho g y$ (ρ is the volumic mass of the fluid, g the gravity, y the vertical axis upward oriented); if the soil is given by $\{y < 0\}$, then ϕ measures the difference of p to the hydrostatic pressure $p_0 = -\rho g y$. The coordinate x will denote the horizontal axis, and we will work in two dimensions $X = (x, y) \in \mathbf{R}^2$. Moreover we assume that the fluid is incompressible:

$$\text{div } v = 0$$

The research of the free surface of this monofluid can then be reduced (see [1]) to the equation

$$\Delta p = -\lambda \partial_y(\chi(p)) \tag{1.1}$$

where $\chi(p) = \begin{cases} 1 \text{ if } p > 0 \\ 0 \text{ if } p = 0 \end{cases}$, $\lambda = \rho g$. We should add some boundary conditions.

Two fluids

If now we study in a porous medium two unmiscible fluids of density ρ_1 and ρ_2, we can give a formulation of the problem using the stream function $\psi \in \mathbf{R}$ in place of the potential ϕ, and defined by $\text{curl } \psi = \nabla\phi$, where $\text{curl } \psi = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix}$. Now a stream line is given by $\{\psi = const\}$. In particular the interface Γ between these two fluids is a stream line and (up to an additive constant) we can normalize ψ such that $\Gamma = \{\psi = 0\}$. Then the problem can be reduced (see [4]) to the equation:

$$\Delta\psi = -\lambda \partial_x(\chi(\psi)) \tag{1.2}$$

where $\chi(\psi) = \begin{cases} 1 \text{ if } \psi > 0 \\ 0 \text{ if } \psi < 0 \end{cases}$, and $\lambda = (\rho_2 - \rho_1) g > 0$.

Remark 1.1 *Let us note that we expect that Γ is a curve and then $\mathcal{H}^2(\Gamma) = 0$. In this case it is not necessary to precise the value of $\chi(\psi)$ on $\Gamma = \{\psi = 0\}$. But up to our knowledge there is no general existence result of solutions with $\mathcal{H}^2(\Gamma) = 0$. The only known way to get a solution (see [4]) is to take the limit of solutions u_ϵ of equation (1.2) with a smooth function χ_ϵ in place of χ. As $\epsilon \to 0$, $\chi_\epsilon \to \chi$ and $\psi_\epsilon \to \psi$ where ψ is a weak solution to (1.2). In particular $\Gamma = \{\psi = 0\}$ could be degenerated with $\mathcal{H}^2(\Gamma) > 0$, i.e. Γ would be a mushy region. Moreover $\chi(\psi)_{|\Gamma}$ could take every value between 0 and 1. In this case the values of $\chi(\psi)_{|\Gamma}$ would be important to claim that ψ is a solution to equation (1.2).*

1.2 The mathematical formulation

From now on, let us use the notations

$$\begin{cases} u = \psi \\ \gamma = \chi(\psi) \end{cases}$$

Then the weak solutions to (1.2) (with $\lambda = 1$) over an open set $\Omega \subset \mathbf{R}^2$ are given by the following variational formulation: search $(u, \gamma) \in H^1_{loc}(\Omega) \times L^\infty(\Omega)$ such that

$$\forall v \in C_0^\infty(\Omega), \quad \int_\Omega \nabla u \nabla v + \gamma \partial_x v = 0, \text{ and } \gamma \in H(u) \tag{1.3}$$

where

$$H(u) = \begin{cases} \{1\} \text{ if } u > 0 \\ [0,1] \text{ if } u = 0 \\ \{0\} \text{ if } u < 0 \end{cases} \tag{1.4}$$

and

$$u = u_0 \text{ on } \partial\Omega \tag{1.5}$$

The existence of a solution (u, γ) to (1.3)-(1.5) is known under certain assumptions on $\partial\Omega$ and on u_0 (see [6], [13]).
Moreover let us recall:

Proposition 1.2 *Every solution (u, γ) of (1.3) belongs to $C^{0,1}_{loc}(\Omega) \times L^\infty(\Omega)$.*

Proof of proposition 1.2
See [4] page 631, [11] pages 52-53.

In this paper we are interested in getting information on the free boundary $\Gamma = \{u = 0\}$ and to know whether and when there exists a mushy region Γ with $\mathcal{H}^2(\Gamma) > 0$. The nonexistence of a mushy region is intimately related to the question of the uniqueness of the solutions (u, γ) to (1.3)-(1.5) , as it is shown in [6] in the particular case of a strip $\Omega = \mathbf{R} \times (0, 1)$. In particular we study here the properties of Γ without assuming that the function u is monotone as in [4], or satisfies a property at $+\infty$ as in [6]. See also [12], [13]. Here we study the free boundary of the solution in the general case.

1.3 Main results

Let us recall that in the region $\{u = 0\}$, γ can take any value between 0 and 1, which permits us to see γ as a coefficient of mixing of the two fluids. It justifies the terminology of mushy region (when $\mathcal{H}^2(\Gamma) > 0$), that is sometimes given to the region $\{u = 0\}$ for the Stephan problem (see [17], [18], [19], [20]).

Definition 1.3 *Let $\omega \subset \mathbf{R}^2$ an open set convex in the e_x direction, i.e. $[(x,y),(x',y)] \subset \omega$ while $(x,y),(x',y) \in \omega$. Then we say that a set $A \subset \omega$ is a epigraph on ω in the e_x direction if $[(x,y),(x',y)] \subset A$ while $(x,y) \in A$ and $[(x,y),(x',y)] \subset \omega$, $x < x'$.*

We prove the

Theorem 1.4 *If (u,γ) is a solution of (1.3) on an open set $\Omega \subset \mathbf{R}^2$, then for all open set $\omega \subset \Omega$, convex in the e_x direction, $\omega \cap \{u=0\}^0$ is an epigraph on ω in the e_x direction.*

Remark 1.5 *S. Kamin has noticed that a similar result is true for the Stephan problem: the mushy region of a one-dimensionnal Stephan problem for $(x,t) \in \mathbf{R}\times\mathbf{R}$ can diseappear in finite time. We find the analogy using the change $(x,t) \to (y,-x)$.*

Moreover we prove

Theorem 1.6 *Let (u,γ) be a solution of (1.3) on the open set Ω such that $\{u=0\}^0 = \emptyset$. Then $\mathcal{H}^2(\{u=0\}) = 0$ if and only if $(\partial\{u>0\}) \cap \Omega = (\partial\{u<0\}) \cap \Omega$.*

Open question: Is there existence of a solution (u,γ) to (1.3) on an open set Ω with $u \geq 0$, $\{u=0\}^0 = \emptyset$ and $\mathcal{H}^2(\{u=0\}) > 0$?

2 Preliminaries

The following proposition is obvious but usefull:

Proposition 2.1 *If (u,γ) is a solution to (1.3), then $(-u, 1-\gamma)$ is also a solution.*

Lemma 2.2 *(linear behaviour lemma, [8], [5]) Let $\Omega_1 \subset \mathbf{R}^n$ (respectively $\Omega_2 \subset \mathbf{R}^n$) such that there exists a ball B with*

$$B = B_r(re_n) \text{ and } B \subset \Omega_1$$

$$(\text{ respectively } B = B_r(-re_n) \text{ and } B \subset (\Omega_2)^c$$

Assume that u is a Lipschitz positive hamonic function in Ω_1 (respectively Ω_2) vanishing in $\partial\Omega_1$ (respectively $\partial\Omega_2$) and assume that $\partial\Omega_i \cap B = \{0\}$. Then near zero, u has the asymtotic development

$$u(X) = \alpha x_n + o(|X|) \text{ on } \Omega_i \text{ with } \alpha \geq 0$$

Furthermore $\alpha > 0$ in case Ω_1, because of Hopf lemma.

Proposition 2.3 *If (u,γ) is a solution of (1.3) and if*

$$u(X) = \alpha < X - X_0, \nu >^{+} - \beta < X - X_0, \nu >^{-} + o(|X - X_0|)$$

with $\nu \in \mathbf{S}^1$ and $\alpha, \beta \in \mathbf{R}$, then there exists two functions $0 \leq \gamma_\alpha^0(y), \gamma_\beta^0(y) \leq 1$ with

$$\left\{\begin{array}{lll} \gamma_\alpha^0 \equiv 1 & \textit{if} & \alpha > 0 \\ \gamma_\alpha^0 \equiv 0 & \textit{if} & \alpha < 0 \end{array}\right. \qquad \textit{and} \qquad \left\{\begin{array}{lll} \gamma_\beta^0 \equiv 0 & \textit{if} & \beta > 0 \\ \gamma_\beta^0 \equiv 1 & \textit{if} & \beta < 0 \end{array}\right.$$

such that

$$\alpha - \beta + < \nu, e_x > (\gamma_\alpha^0(y) - \gamma_\beta^0(y)) = 0 \tag{2.1}$$

Proposition 2.4 *If locally $\{u = 0\}^0 = \emptyset$, then u is locally a solution to the free boundary problem (P_G) in the Appendix with $G(\beta, \nu, X) = \beta - < e_x, \nu >$.*

Proof of proposition 2.4
From lemma 2.2 and proposition 2.3 we see (even for the particular cases $\alpha = 0$ or $\beta = 0$, because $\{u = 0\}^0 = \emptyset$) that u is a solution to problem (P_G) in the Appendix with $G(\beta, \nu, X) = \beta - < e_x, \nu >$.

The main tool that is used by L.A. Caffarelli to prove regularity theorems in [7]-[8], is the monotonicity formula:

Theorem 2.5 (lemma 5.1 [3]; lemma 18 [7]) *Let two continuous functions $u_1, u_2 \geq 0$ such that*
i) $\Delta u_i \geq 0$ (u_i subharmonic)
ii) $u_i(0) = 0$
iii) $u_1 u_2 \equiv 0$
Let

$$\phi(r) = \frac{\int_{B_r} |\nabla u_1|^2 \rho d\rho d\sigma \int_{B_r} |\nabla u_2|^2 \rho d\rho d\sigma}{r^4} \tag{2.2}$$

where (ρ, σ) are the radial and spheric coordinates in $\mathbf{R}^n$.
Then ϕ is a nondecreasing function of r. Besides ϕ is bounded near $r = 0$. In particular if the functions u_i are defined on $\mathbf{R}^2$, and if $\phi(r) = const > 0$, then there exists $\nu \in \mathbf{S}^1, \alpha_i > 0, i = 1,2$ such that $u_1(X) = \alpha_1 < X, \nu >^{+}$, $u_2(X) = \alpha_2 < X, \nu >^{-}$.

Proof of proposition 2.3
From the assumption of proposition 2.3, let us consider the blow-up:

$$\left\{\begin{array}{l} u^\epsilon(X) = \dfrac{u(X_0 + \epsilon X)}{\epsilon} \\ \gamma^\epsilon = \gamma(X_0 + \epsilon X) \end{array}\right.$$

Let us recall that $u \in C^{0,1}_{loc}(\Omega)$ and $\gamma \in L^\infty(\Omega)$, and then by Ascoli theorem up to extraction of some subsequence $(u^\epsilon, \gamma^\epsilon) \to (u^0, \gamma^0)$ on $C^{0,\alpha}(K) \times L^\infty_{weak*}(K)$ for every compact set $K \subset \mathbf{R}^2$ and every $\alpha \in (0,1)$. Then (u^0, γ^0) satisfies also (1.3) and is a solution on $\mathbf{R}^2$. We have $u^0(X) = \alpha < X, \nu >^+ - \beta < X, \nu >^-$, and for every $v \in C^\infty_0(\mathbf{R}^2)$:

$$\int_{\mathbf{R}^2} (\alpha 1_{\{<X,\nu>>0\}} + \beta 1_{\{<X,\nu><0\}})\nu \cdot \nabla v + \gamma^0 \partial_x v = 0 \tag{2.3}$$

In particular $\partial_x \gamma^0 = 0$ in $\{< X, \nu > \neq 0\}$, i.e.

$$\gamma^0(x,y) = \begin{cases} \gamma^0_\alpha(y) \text{ in } \{< X, \nu >> 0\} \\ \gamma^0_\beta(y) \text{ in } \{< X, \nu >< 0\} \end{cases}$$

Moreover by integration by part we see that

$$(\alpha - \beta)\delta_{\{<X,\nu>=0\}} + \partial_x \gamma^0 = 0$$

which gives (2.1) and proves proposition 2.3.

3 $\{u = 0\}^0$ is an epigraph in the e_x direction

Here we prove theorem 1.4. We proceed by contradiction and work in ω.

Step 1

Lemma 3.1 *Let*

$$P = (x_1, y_1) \in \{u = 0\}^0 \cap \omega$$

such that $\exists x_1'' > x_1$,

$$P'' = (x_1'', y_1) \in (\{u = 0\}^0)^c \cap \omega$$

Then there exist a small closed intervall I, *a small* $\delta_0 > 0$ *and* $x_1' > x_1$ *such that (up to change the sign of* u*)*

$$[x_1' - \delta_0, x_1'] \times I \quad \subset \quad \{u > 0\}$$

$$[x_1, x_1 + \delta_0] \times I \quad \subset \quad I_0$$

where I_0 *is the connected component of* $\{u = 0\}^0$ *which contains* P, *and*

$$\omega_0 = [x_1, x_1'] \times I \quad \subset \quad \omega$$

Proof of lemma 3.1
We know that $\exists B_r(P) \subset \{u = 0\}^0$ and by definition of P'', $\exists P', d(P'', P') < \frac{r}{2}$ with say (up to change the sign of u, see proposition 2.1) $u(P') > 0$. Then $\exists B_{r'}(P') \subset\subset \{u > 0\}$

with $r' < \frac{r}{2}$. We note $P' = (x'_1, y'_1)$ and $\omega_0 = [x_1, x'_1] \times I \subset \omega$ with $I = [y'_1 - r', y'_1 + r']$. We deduce the existence of a $\delta_0 > 0$ as in the lemma, decreasing r' if necessary. This ends the proof of lemma 3.1.

Step 2

Lemma 3.2 *Let* $\Gamma_0 = \{\Gamma_0(y) = (f_0(y), y),\ f_0(y) = \sup\{x,\ (x,y) \in I_0 \cap \omega_0\}\}$. *Then up to a change of the sign of* u *there exists a connected component* C_0 *of* $\{u > 0\} \cap \omega_0$, $\exists y_+, y_- \in I, y_+ > y_-$ *such that* $\Gamma_0(y_+), \Gamma_0(y_-) \in \partial I_0 \cap \partial C_0$.

Proof of lemma 3.2

Let $\mathcal{C}_{\omega_0}$ the set of all connected components of $(\{u > 0\} \cap \omega_0) \cup (\{u < 0\} \cap \omega_0)$. For each $C \in \mathcal{C}_{\omega_0}$, two cases appear:
i) either C is adherent to at most one point of Γ_0.
ii) or C is adherent to at least two points of Γ_0.
But $\mathcal{C}_{\omega_0}$ is a set of connected components at most countable, and Γ_0 is a set of non countable points. Then $\exists C_0 \in \mathcal{C}_{\omega_0}$ which satisfies case ii), and up to change the sign of u we can assume that $u_{|C_0} > 0$, and there exist two points $\Gamma_0(y^-), \Gamma_0(y^+) \in \Gamma_0$, with $y^- < y^+$. This ends the proof of lemma 3.2.

Step 3

Lemma 3.3 *Decreasing* ω *if necessary, we can assume (up to change the sign of* u*) that* $u \geq 0$ *on* ω.

Proof of lemma 3.3

Let g_0 be a continuous path which links together $\Gamma_0(y_-)$ to $\Gamma_0(y_+)$ in I_0. Precisely it means that there exists an injective and continuous map $\tilde{g}_0 : [-1, 1] \to \overline{I}_0$ with $\tilde{g}_0([-1, 1]) = g_0$, $\tilde{g}_0((-1, 1)) \subset I_0$, and $\tilde{g}_0(-1) = \Gamma_0(y_-)$, $\tilde{g}_0(+1) = \Gamma_0(y_+)$.
Let g_+ be a continuous path which links together $\Gamma_0(y_-)$ to $\Gamma_0(y_+)$ in C_0. Then from the maximum principle $u \geq 0$ on the bounded component of boundary $g_0 \cup g_+$. This ends the proof of lemma 3.3.

Step 4: contradiction

Let us take a ball in $\{u > 0\}$ and slide it in direction $-e_x$. Then it touches ∂I_0 at a point X_0. From the linear behaviour lemma 2.2, we get $u(X) = \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- + o(|X - X_0|)$ for some $\alpha > 0$ (because of Hopf lemma) and $\beta \leq 0$ (because $u \geq 0$) and with $< \nu, e_x > \geq 0$. Then from proposition 2.3 we get $\beta = \alpha + < \nu, e_x > (\gamma^0_\alpha(y) - \gamma^0_\beta(y)) > 0$ because $\gamma^0_\alpha(y) \equiv 1$ and $0 \leq \gamma^0_\beta(y) \leq 1$. Contradiction.
This ends the proof of theorem 1.4.

4 Examples and counter-examples

In this section we present some examples of solutions to problem (1.3).

Example: No bounds on the length of the free boundary
For every $\lambda \in \mathbf{R}$, $u(x,y) = e^{-\lambda x} \sin \lambda y, \gamma \in H(u)$ is locally a solution of problem (1.3). In particular for λ large enough $\mathcal{H}^1\left(\Gamma(u) \cap B_1(0)\right) \geq c\lambda$ where c is a constant.

Example of a smooth mushy region
We recall here an example of a mushy region which is given in [6] by A. Bonnet and S. Kamin. Let $\Omega = \mathbf{R} \times (0,1)$, $u = f_0$ on $\mathbf{R} \times \{0\}$, and $u = f_1$ on $\mathbf{R} \times \{1\}$, where $f_1(x) = -f_0(x) = a \inf(1, exp(-x))$, for a constant $a > 0$ to be fixed. Let $g(y) = \frac{1}{y(1-y)}$. Let v on $\{x < g(y)\}$ be equal to the harmonic function which vanishes on $x = g(y)$ and takes the values f_0 and f_1 on $\partial\Omega$; and $v = 0$ on $\{x > g(y)\}$. Let ν the exterior unit normal to $\{x > g(y)\}$, and $\gamma_v = \chi_{[\frac{1}{2},1]}(y) + \frac{v_\nu^+(g(y),y)}{\nu \cdot e_x}$. On the free boundary one has $v_\nu^+ = O(e^{-|x|})$ and $e_x \cdot \nu = O(\frac{1}{x})$ as $x \to +\infty$. Therefore for a small enough $\gamma_v \in [0,1]$ and (v, γ_v) is a solution to (1.3), see figure 1.

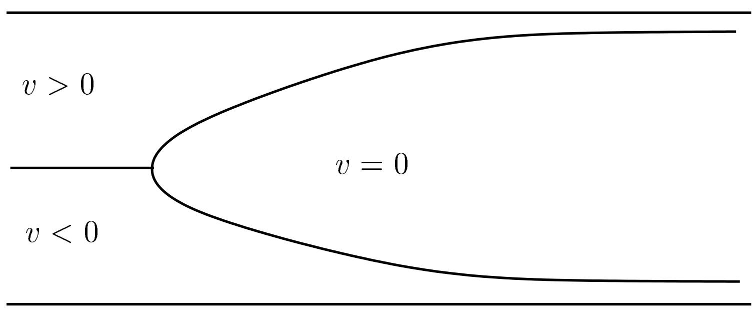

Figure 1: Example of a smooth mushy region

Let us remark that when $u \geq 0$, problem (1.3) reduces to problem (1.1) with a generalized function $\tilde{\chi}(p) = \gamma(u)$, and $u(x,y) = p(y,-x)$. Recall that if we assume that $\tilde{\chi}(0) = 0$, then it is known (see [1]) that the boundary $\partial(\{p = 0\}^0)$ (i.e. $\partial(\{u = 0\}^0)$) is an analytical graph.
In the general case we have for instance

Proposition 4.1 *If (u, γ) is a solution of (1.3), and locally*

$$\begin{cases} u \geq 0, \;\; \gamma \in C^{0,1} \\ \gamma \leq 1 - \delta < 1 \;\; on \; \{u = 0\}^0 \end{cases} \tag{4.1}$$

Then locally $\Gamma_0(u) = \partial(\{u = 0\}^0)$ *is a* $C^{1,\alpha}$ *graph in direction* e_x *and* $\gamma = \gamma(y)$ *on* $\{u = 0\}^0$.

This proposition can be proved using for the function $v(x,y) = \int_{-x}^{a} u(x',y)dx'$ the following lemma which is an adapted version of a result of Alt [1].

Lemma 4.2 *For* $B_1 \subset \mathbf{R}^2$, *if* $v \in C^1(B_1)$, $\lambda \in C^{0,1}$, $0 \in \Gamma = \partial\{v > 0\}$, *and for* $X = (x,y) \in \mathbf{R}^2$*:*

$$\begin{cases} \Delta v = \lambda(y) > 0 \ \ in \ B_1 \cap \{v > 0\} \\ \partial_x v \geq 0 \ \ in \ B_1 \\ \partial_x v > 0 \ \ in \ B_1 \cap \{v > 0\} \end{cases}$$

Then Γ *is Lipschitz in* $B_{\frac{1}{2}}$.

In proposition 4.1, the condition $\delta > 0$ is necessary. When this condition is not satisfied, we have the following counter-example:

Counter-example with a cusp where γ reaches 1

The key idea to build this counter-example is to use the fact that holomorphic transform preserves the harmonicity. Using the complex numbers, we transform the function

$$u_1(x+iy) = x^+$$

in

$$u = u_1 \circ F^{-1}$$

whete the holomorphic function F is given by

$$F(z) = -\exp(-\sqrt{-\ln(z)})$$

In particular we get

$$\Delta u = 0 \quad \text{in} \quad \{u > 0\}$$

The region $\{u > 0\}$ has a cusp because

$$F(re^{i\theta}) = Re^{i\Theta}$$

with

$$\begin{cases} R = e^{-\sqrt{-\ln r}(1+o(\frac{1}{-\ln r}))} \\ \Theta - \pi = \frac{\theta}{2\sqrt{-\ln r}}(1+o(1)) \end{cases}$$

Let us consider

$$\gamma(x,y) = \begin{cases} \gamma_0(y) & \text{if} \quad u(x,y) = 0 \\ 1 \quad \text{if} & u(x,y) > 0 \end{cases}$$

with

$$\gamma_0(y) = 1 + \frac{u_\nu^+}{< e_x, \nu >}$$

Then it is easy to check that (u, γ) satisfies (1.3) near the origin. Moreover the function u is locally Lipschitz which is a consequence of the fact that $u_\nu^+(X) \to 0$ as $X \to 0, X \in \Gamma(u)$. We also have to check that $\gamma \in [0,1]$ which is proved by an explicit computation of $\frac{u_\nu^+}{<e_x,\nu>}$ on $\Gamma(u)$.
Finally we remark that $\gamma_0 \in C^\infty$ in a small intervall which contains 0, $\Gamma(u) \backslash \{0\}$ is analytic and $\Gamma(u) \in C^{0,\alpha}$ for every $\alpha \in (0,1)$.

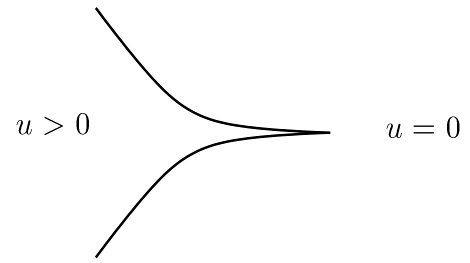

Figure 2: A mushy region $\{u = 0\}$ with a cusp

Remark 4.3 *The previous counter-example shows that the function γ can be nonmonotone in y on a connected component of $\{u = 0\}^0$.*

5 Proof of theorem 1.6

5.1 Proof of theorem 1.6

Definition 5.1 *We say that $\Gamma(u) := \{u = 0\}$ is ϵ-flat in 0 for $r \leq r_\epsilon$ if and only if $u(X) = o(|X|)$ and the following statement is true for u or $-u$:*

$$\forall r \in (0, r_\epsilon), \begin{cases} u > 0 \text{ on } \{y \geq \epsilon r\} \cap B_r(0) \\ u < 0 \text{ on } \{y \leq -\epsilon r\} \cap B_r(0) \end{cases}$$

We say that $X_0 \in \Gamma(u)$ is a flat point if $\forall \epsilon > 0, \exists r_\epsilon > 0$, such that $\Gamma(u)$ is ϵ-flat in X_0 for $r \leq r_\epsilon$.

Definition 5.2 *Let Γ_{flat} the set of flat points of Γ, and let Γ_{reg} the set of points $X_0 \in \Gamma$ such that Γ is analytic in a neighbourhood of X_0.*

We consider the following

Assumptions 5.3 *(u,γ) is a solution of (1.3) on the open set $\omega \subset \mathbf{R}^2$ simply connected with $\{u=0\}^0 \cap \omega = \emptyset$. We also assume that $\{u>0\}\cap\omega$ and $\{u<0\}\cap\omega$ are connected components such that $(\partial\{u>0\})\cap\omega = (\partial\{u<0\})\cap\omega$.*

The following two propositions will be proved later:

Proposition 5.4 *Under assumptions 5.3, if $X_0 \in \Gamma(u)\cap\omega$ is such that locally $u(X) = o(|X-X_0|)$, then $X_0 \in \Gamma_{flat}\cap\omega$.*

Proposition 5.5 *Under assumptions 5.3, if $X_0 \in \Gamma(u)\cap\omega$ is such that locally $u(X) \neq o(|X-X_0|)$, then Γ is analytic locally near X_0, i.e. $X_0 \in \Gamma_{reg}\cap\omega$.*

Proof of theorem 1.6

To prove the theorem we can work on every ball $B \subset \Omega$ and get the result summing on a countable set of balls.

Let $\mathcal{C}^+$ the set of connected components of $\{u>0\}\cap B$ and $\mathcal{C}^-$ the set of connected components of $\{u<0\}\cap B$. Then

$$\Gamma = \bigcup_{C^+\in\mathcal{C}^+,\ C^-\in\mathcal{C}^-} (\partial C^+ \cap \partial C^-)\cap B$$

Because the sets $\mathcal{C}^+$ and $\mathcal{C}^-$ are countable, we have

$$\mathcal{H}^2(\Gamma) \leq \sum_{C^+\in\mathcal{C}^+,\ C^-\in\mathcal{C}^-} \mathcal{H}^2\left((\partial C^+\cap\partial C^-)\cap B\right)$$

Let us assume that there exist $C^+\in\mathcal{C}^+, C^-\in\mathcal{C}^-$ such that

$$\mathcal{H}^2\left((\partial C^+\cap\partial C^-)\cap B\right) > 0$$

Let ω the open set in B whose boundaries are included in $(\partial C^+\cap\partial C^-)\cap B$. If $\omega\neq\emptyset$, from the maximum principle on ω, we deduce that $u=0$ on ω, which is in contradiction with the assumption that $\{u=0\}^0=\emptyset$.

Now we have for $\Gamma_0 = (\partial C^+\cap\partial C^-)\cap B$:

$$\underline{\theta}(X) = \liminf_{r\to 0}\frac{|B_r(X)\cap(K\cap\Gamma_0)|}{|B_r(X)|} = 1 \quad \mathcal{H}^2\text{-a.e.} \quad X\in\Gamma_0$$

In particular, there exists $X_0\in\Gamma_0$ such that $\underline{\theta}(X_0)=1$. Then $X_0\notin\Gamma_{reg}$, and from proposition 5.4 we have $X_0\in\Gamma_{flat}$. By definition 5.1, this implies that $\underline{\theta}(X_0)<\epsilon$ for all $\epsilon>0$. Contradiction. Then

$$\mathcal{H}^2\left((\partial C^+\cap\partial C^-)\cap B\right) = 0$$

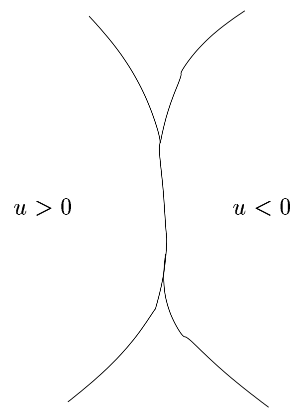

Figure 3: The contact between two connected components of $\{u > 0\}$ and $\{u < 0\}$.

and $\mathcal{H}^2(\Gamma) = 0$.
Conversely if $\mathcal{H}^2(\Gamma) = \emptyset$, then $\Delta u = -\partial_x \gamma = 0$ on $\{u \geq 0\}^0 \cup \{u \leq 0\}^0$, and from the strong maximum principle we deduce that

$$(\partial \{u > 0\}) \cap \Omega = (\partial \{u < 0\}) \cap \Omega$$

This ends the proof of the theorem.

Remark 5.6 *Following the previous proof, we can get*

$$\mathcal{H}^{1+\epsilon}(\Gamma) = 0$$

for every $\epsilon > 0$, if $\{u = 0\}^0 = \emptyset$ and $(\partial \{u > 0\}) \cap \Omega = (\partial \{u < 0\}) \cap \Omega$.

5.2 Flat points: proof of proposition 5.4

In this subsection we are interested in the case where $u(X) = o(|X|)$ near 0.

Lemma 5.7 *For all $\eta_0 > 0$, there exists $\epsilon > 0$ such that if $|u| < \epsilon$ on $[-1, 1] \times [y_1, y_2]$ with $y_2 - y_1 = 2\eta_0$, then there exists $x_0 \in (-1, 1)$ such that if there exists a continuous path $g_0 \subset \{u > 0\} \cap [x_0, 1] \times [y_1, y_2]$ with $\{P_1 = (x_1, y_1)\} = g_0 \cap [x_0, 1] \times \{y_1\}$, $\{P_2 = (x_2, y_2)\} = g_0 \cap [x_0, 1] \times \{y_2\}$, then for $y_0 = \frac{1}{2}(y_1 + y_2)$:*

$$\{y = y_0\} \cap R^{g_0} \subset \{u > 0\}$$

where $P_1' = (-1, y_1), P_2' = (-1, y_2)$ and R^{g_0} is the bounded connected component of boundary $[P_1 P_1'] \cup [P_1' P_2'] \cup [P_2' P_2] \cup g_0$, i.e. the component of $[-1, 1] \times [y_1, y_2]$ at the left of g_0.

Proof of lemma 5.7

Step 1: Construction of a subsolution

Let $\alpha_0 > 0$ be a small number to fix later. Let $\mathcal{C}(\alpha_0) = \{\rho(\cos\phi, \sin\phi), \rho > 0,\ \phi \in [-\alpha_0, \alpha_0]\}$, and for $\delta > 0$, $\mathcal{C}_\delta(\alpha_0) = \{X,\ d(X, \mathcal{C}(\alpha_0)) < \delta\}$. For $L > 0$, let $\mathcal{C}^L_\delta(\alpha_0) = \{(x,y) \in \mathcal{C}_\delta(\alpha_0),\ x < L\}$. We want to construct a subsolution to problem (1.3) on $\mathcal{C}^L_{2\delta}(\alpha_0)$. To this end, we introduce the function v_1 defined on $\overline{\mathcal{C}^L_{2\delta}(\alpha_0)\backslash\mathcal{C}^L_\delta(\alpha_0)}$ by

$$\begin{cases} \Delta v_1 = 0 \text{ on } \mathcal{C}^L_{2\delta}(\alpha_0)\backslash\overline{\mathcal{C}^L_\delta(\alpha_0)} \\ v_1 = 0 \text{ on } (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\} \\ v_1 = 1 \text{ on } (\partial\mathcal{C}_{2\delta}(\alpha_0)) \cap \{x < L\} \\ v_1 = \dfrac{d(x, \mathcal{C}_\delta(\alpha_0))}{\delta} \text{ on } \{x = L\} \cap \overline{\mathcal{C}_{2\delta}(\alpha_0)\backslash\mathcal{C}_\delta(\alpha_0)} \end{cases} \tag{5.1}$$

And on $\mathcal{C}^L_\delta(\alpha_0)$ we define v^1 by

$$\begin{cases} \Delta v^1 = 0 \text{ on } \mathcal{C}^L_\delta(\alpha_0) \\ v^1 = 0 \text{ on } (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\} \\ v^1 = \cos(\frac{y}{y_L}) \text{ on } \mathcal{C}_\delta(\alpha_0) \cap \{x = L\} \\ \text{where } 2y_L = \text{ length of } \mathcal{C}_\delta(\alpha_0) \cap \{x = L\} \end{cases} \tag{5.2}$$

We are now in position to define our subsolution:

$$v^\eta_\epsilon = \begin{cases} -\epsilon v_1 \text{ on } \overline{\mathcal{C}^L_{2\delta}(\alpha_0)\backslash\mathcal{C}^L_\delta(\alpha_0)} \\ \eta v^1 \text{ on } \overline{\mathcal{C}^L_\delta(\alpha_0)} \end{cases} \tag{5.3}$$

On the free boundary, $\forall X_0 \in \Gamma(v^\eta_\epsilon) = (\partial\mathcal{C}_\delta(\alpha_0)) \cap \{x < L\}$, we have

$$u(X) = \alpha < X - X_0, \nu_0 >^+ -\beta < X - X_0, \nu_0 >^- + o(|X - X_0|)$$

where ν_0 is the normal to $\Gamma(v^\eta_\epsilon)$ in X_0. Here $\alpha = \eta\alpha_1$, $\beta = \epsilon\beta_1$, $< e_x, \nu > \geq \sin\alpha_0 > 0$, and we want to satisfy the condition of subsolution on the boundary $\Gamma(v^\eta_\epsilon)$:

$$\alpha > \beta - < e_x, \nu > \tag{5.4}$$

Then we see that $\exists\epsilon = \epsilon(\alpha_0, \delta, L) > 0$ (and $\epsilon \to 0$ as $\alpha_0, \delta_0 \to 0$) such that we obtain for all $\eta \geq 0$ a strict subsolution v^η_ϵ of problem (P_G) in the Appendix with $G(\beta, x, \nu) = \beta - < e_x, \nu >$. Let us chose δ and α_0 such that $\eta_0 \geq 2\delta + L\sin\alpha_0$ and $L = 4$.

Step 2

Let $v^t(x,y) = v^\eta_\epsilon(x + t, y - y_0)$. For $t < -2$ we have $R^{g_0} \cap supp(v^t) = \emptyset$. Then we apply a sliding method, increasing t continuously. By construction v^t is a subsolution on $supp(v^t) \cap R^{g_0}$ (see figure 4), and v^t can not touch u on $\partial(supp(v^t) \cap R^{g_0}) \subset supp(v^t) \cup g_0$, because

i) $v^t = -\epsilon$ and $u > -\epsilon$ on $\partial(supp(v^t))$.
ii) $u > 0$ on g_0 and then we can chose $\eta > 0$ such that $u > \eta \geq v^t$ on g_0.
Then v^t can only touch u on $\Gamma(v^t) = \partial\{v^t > 0\}$. But v^t is a strict subsolution on $\Gamma(v^t)$, consequently it is impossible (see lemma 7 in [7]). Then if we have chosen δ and α_0 small enough we can increase t until $\Gamma(v^t)$ touches $\{x = -2 + \delta\}$, which proves lemma 5.7.

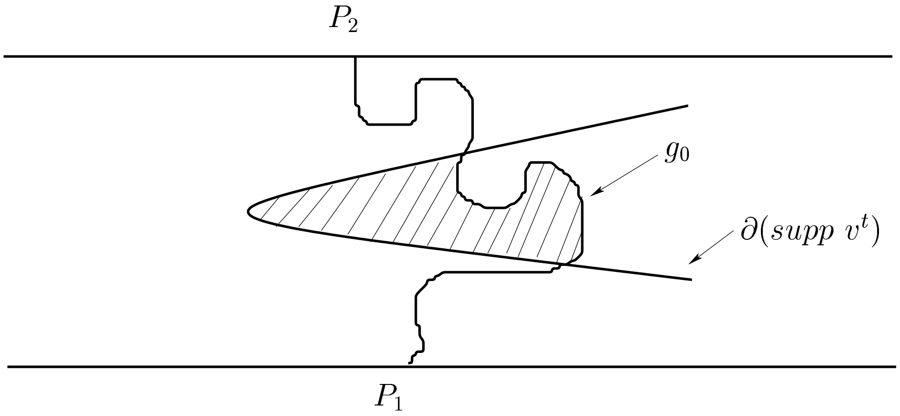

Figure 4: Sliding of the subsolution v^t below the solution u

We note $R_\mu = (-\mu, \mu)^2$ of center 0 for $\mu > 0$.

Lemma 5.8 *We are under assumptions 5.3. Let us assume that* $|u| < \epsilon$ *on* R_2*. Then we can not find three points* $P_i = (x_i, y_i) \in \partial\{u < 0\} \cap R_1, i = 1, 2, 3$ *such that* $-1 \leq y_1 < y_1 + 5\eta_0 \leq y_2 < y_2 + 5\eta_0 \leq y_3 \leq 1$.

Proof of lemma 5.8
Let us assume that there exists three such points. Then there exists $P_{12} = (x_{12}, y_{12}) \in (-1, 1) \times (y_1 + 2\eta_0, y_2 - 2\eta_2)$ such that $u(P_{12}) \neq 0$. By symmetry let us assume that $u(P_{12}) > 0$. Then there exists a continuous path $g_{12} \subset \{u < 0\}$ which links together P_1 to P_2. It is possible that g_{12} goes outside R_2, but in every cases $u < \epsilon$ on the bounded component of boundary $[P_1P_2] \cup g_{12}$, because of the maximum principle.
case 1): g_{12} goes on the right of P_{12}
Because g_{12} goes on the right of P_{12}, we can apply the proof of lemma 5.7: let $P_1' = (-1, y_1), P_2' = (-1, y_2)$ and $R^{g_{12}}$ the bounded component of boundary $[P_1P_1'] \cup [P_1'P_2'] \cup [P_2'P_2] \cup g_{12}$. Then $P_{12} \in R^{g_{12}}_{\eta_0} = R^{g_{12}} \cap \{y_1 + \eta_0 \leq y \leq y_2 - \eta_0\} \subset \{u < 0\}$. Contradiction.
case 2): g_{12} goes on the left of P_{12}
Let $g_{23} \subset \{u > 0\}$ a continuous path which links together P_{12} to P_3.
subcase 2)a): g_{23} goes on the left of g_{12} (see figure 5)
Let $R^{g_{23}}$ the bounded component of boundary $g_{23} \cup [P_{12}P_3]$. Then as previously we get

$P_1 \in R^{g_{23}}_{\eta_0} = R^{g_{23}} \cap \{y \leq y_{12} - \eta_0\} \subset \{u > 0\}$. Contradiction.
subcase 2)b): g_{23} goes on the right of g_{12} (see figure 5)
Then $P_2 \in R^{g_{23}}_{\eta_0} = R^{g_{23}} \cap \{y_{12} + \eta_0 \leq y \leq y_3 - \eta_0\} \cap R_1 \subset \{u > 0\}$. Contradiction.
In every case we get a contradiction. Then it proves lemma 5.8.

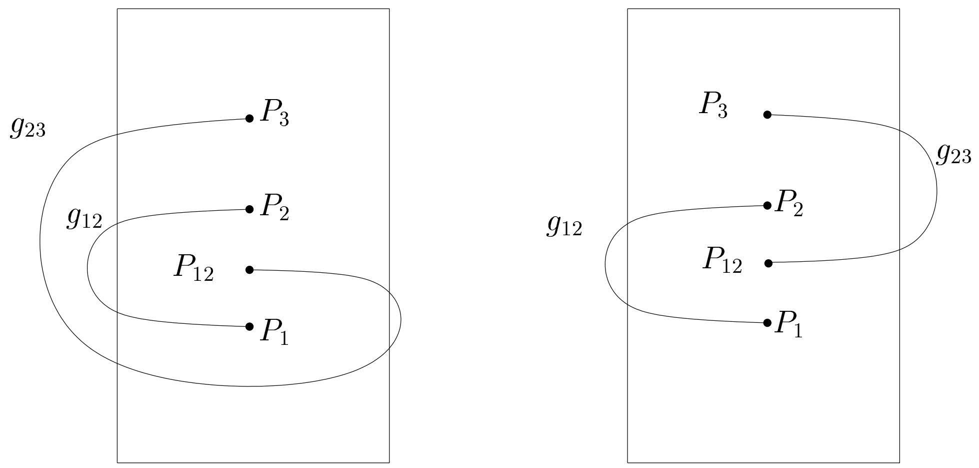

Figure 5: Subcases 2)a) and 2)b)

The following proposition is a straightforward corollary of lemma 5.8.

Proposition 5.9 *We assume 5.3.* $\forall \eta_0 > 0, \exists \epsilon > 0$, *such that if* $|u| < \epsilon$ *on* $R_1 \subset \Omega$, *then for*

$$y^+ = \sup\{y, (x,y) \in \partial\{u < 0\} \cap R_1\}$$
$$y^- = \inf\{y, (x,y) \in \partial\{u < 0\} \cap R_1\}$$

we have

$$\partial\{u < 0\} \cap R_1 \quad \subset \quad (-1,1) \times ([y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+])$$

Proof of proposition 5.4
Up to consider $u^{\epsilon_1}(X) = \frac{u(\epsilon_1 X)}{\epsilon_1}, \gamma^{\epsilon_1}(X) = \gamma(\epsilon_1 X)$ with $0 < \epsilon_1 < 1$ in place of (u, γ), we deduce from proposition 5.9 that we are in one of the following cases:
case C3
We have three parts: $|u| > 0$ on $(-1,1) \times (-1, y^-), (-1,1) \times (y^- + 5\eta_0, y^+ - 5\eta_0), (-1,1) \times (y^+, 1)$, where $-1 < y^- < y^- + 10\eta_0 < y^+ < 1$ and $0 \in [y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+]$.
case C2
We have two parts: $|u| > 0$ on $(-1,1) \times (-1, -10\eta_0), (-1,1) \times (10\eta_0, 1)$.
And each case has subcases: we look at the sign of u on each part from the top to the

bottom. For instance we note $C3++-$ a situation in case C3 where $u>0$ on the first two parts on the top and $u<0$ on the last part below. We will note more generally $C3aab$ the case $C3++-$ or the case $C3--+$.

subcase C3aaa

Let us consider for instance the case $C3---$. Then the method of proof of lemma 5.8 applies and gives a contradiction (see figure 6).

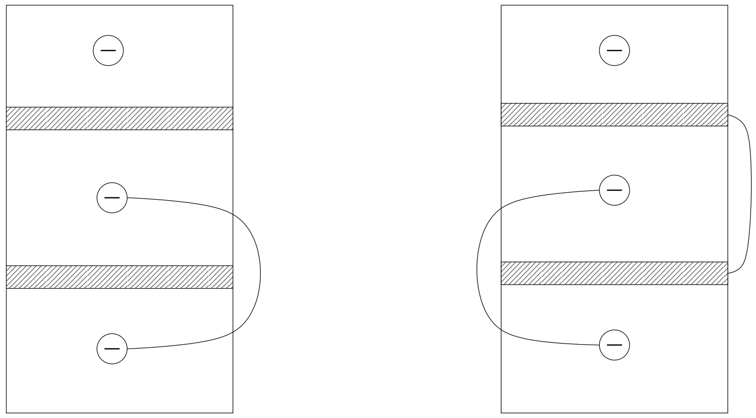

Figure 6: Subcase C3 - - -

subcase C3aab or C3abb

Let us consider for instance the case $C3--+$. Then the method of proof of lemma 5.8 applies and gives a contradiction (see figure 7).

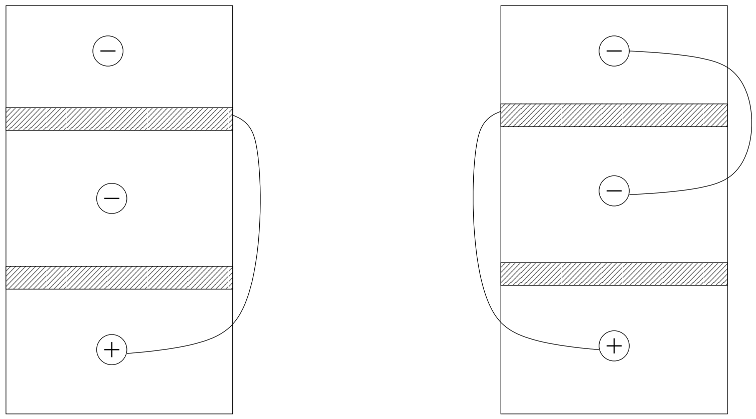

Figure 7: Subcase C3 - - +

subcase C3aba

Let us consider for instance the case $C3+-+$. Then using a zoom with some $0<\epsilon_1<1$ we get the case C2ab for $(u^{\epsilon_1},\gamma^{\epsilon_1})$ (see figure 8).

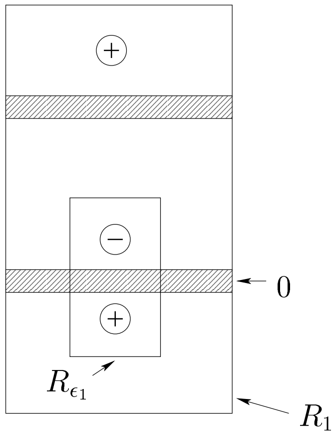

Figure 8: Subcase C3 + - +

subcase C2ab

Let us consider for instance the case $C2+-$. Then with a continuous zoom with $0<\epsilon_1<1$ we get the case C2ab for $(u^\lambda,\gamma^\lambda)$ (see figure 9), because the only other cases C3aab or C3abb are impossible. Then the configuration C2ab is "stable" by zoom.

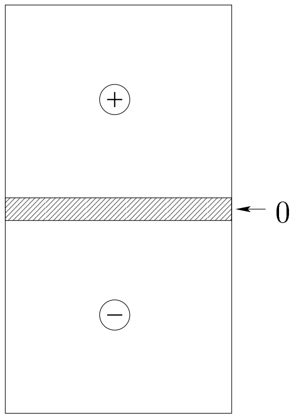

Figure 9: Subcase C2 + -

subcase C2aa (see figure 10)

Let us consider for instance the case $C2--$. Then using a zoom with $0<\epsilon_1<1$ we can only get cases:

i) C3aaa: impossible
ii) C3aba, and then C2ab
iii) C2aa
Let us assume that we always get case C2aa for every $0 < \epsilon_1 < 1$. Then for the cones $C^{\pm} = \{X, \pm\frac{<X,e_y>}{|X|} > \frac{1}{2}\}$ we have

$$(C^+ \cup C^-) \cap B_r(0) \subset \{u < 0\} \tag{5.5}$$

if η_0 is small enough. But $0 \in \partial\{u > 0\}$, and if $C_l = \{u > 0\} \cap \{x < 0\} \cap B_r(0) \neq \emptyset$, $C_d = \{u > 0\} \cap \{x > 0\} \cap B_r(0) \neq \emptyset$ for every $r > 0$ small enough, then there exists a continuous path $g \subset \{u > 0\}$ which connects C_l to C_d. This path can not go near 0 because of (5.5), then g goes around one of the components $(-1,1)\times(-1,-10\eta_0)$, $(-1,1)\times(10\eta_0,1)$ where $u < 0$. Contradiction. Then we have $C_l = \emptyset$ or $C_d = \emptyset$. For instance on $\{x < 0\}$ we have locally either $u < 0$ or $u > 0$. Finally Hopf lemma gives a contradiction to the fact that $u(X) = o(|X|)$.

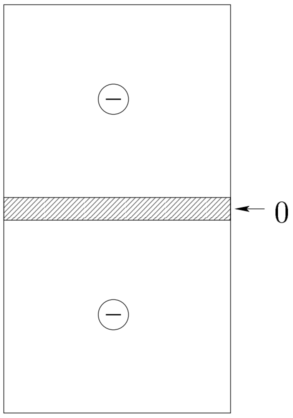

Figure 10: Subcase C2 - -

Consequently in every case for $0 < \epsilon_1 < 1$ small enough, $(u^{\epsilon_1}, \gamma^{\epsilon_1})$ is in the case C2ab, and it proves that 0 is a flat point which ends the proof of proposition 5.4.

5.3 Proof of proposition 5.5

Let us consider a point $X_0 \in \Gamma \cap \omega$ such that $u(X) \neq o(|X - X_0|)$. Let us set $X_0 = 0$, $u^\epsilon(X) = \frac{u(\epsilon X)}{\epsilon}$. Then there exists a subsequence such that $u^\epsilon \to u^0$ and $u^0 \not\equiv 0$. In particular from theorem 2.5 if we set $u_1 = u^+$, $u_2 = u^-$, we get that $\phi_u(r) := \phi(r)$ is nondecreasing. Moreover because $\phi_{u^\epsilon}(r) = \phi_u(\epsilon r)$, we get $\phi_{u^0}(r) = \phi_u(0)$.
case 1: $\phi_u(0) > 0$
If $\phi_u(0) > 0$ we conclude that $u^0(X) = \alpha < X, \nu >^+ - \beta < X, \nu >^-$, with $\alpha, \beta > 0, \nu \in$

$\mathbf{S}^1$. From proposition 2.4 and from theorem 6.3 i) in the Appendix we conclude that Γ is locally $C^{1,\alpha}$ near X_0, and from the result of Kinderlehrer-Mirenberg [16], Γ is locally analytic.
case 2: $\phi_u(0) = 0$
In this case $u^0 \geq 0$ or $u^0 \leq 0$. Up to change the sign of u (see proposition 2.1) we can always assume that $u^0 \geq 0$ and $u^0 \not\equiv 0$. In particular from theorem 1.4, $\{u^0 = 0\}^0$ is an epigraph in the e_x direction. Here we prove:

Lemma 5.10 *If $u^0 \not\equiv 0$, $u^0 \geq 0$ and (u^0, γ^0) is a solution of (1.3), then $\{u^0 = 0\}$ is an epigraph in the e_x direction.*

Proof of lemma 5.10
Let us assume that $u > 0$ on $B = B_r(x_0, y_0)$ and $\exists x_1 < x_0,\ u(x_1, y_0) = 0$. We have $u^0 \geq 0$ and $\Delta u^0 = 0$ in $\{u^0 > 0\}$ and u^0 is (Lipschitz-) continuous. Then u^0 is a subsolution and $\partial_x \gamma^0 = -\frac{1}{\lambda}\Delta u^0 \leq 0$. We know that $\gamma^0 \equiv 1$ in B and then $\Delta u^0 = 0$ on the left of B. From the strong maximum principle we deduce that $u^0 > 0$ on the left of B, because $u^0 > 0$ on B. Consequently $u^0(x_0, y_0) > 0$. Contradiction. This proves lemma 5.10.

Lemma 5.11 *If $u^0 \not\equiv 0$, $u^0 \geq 0$ and (u^0, γ^0) is a solution of (1.3), then $\forall y \in \mathbf{R}, \exists x \in \mathbf{R}, u^0(x, y) > 0$*

Proof of lemma 5.11
Let us assume that $\exists y_0 \in \mathbf{R}, \forall x \in \mathbf{R}, u^0(x, y_0) = 0$. Then up to a translation we can assume that $y_0 = 0$. Because $u^0 \not\equiv 0$, $\exists P \in \{y > 0\} \cup \{y < 0\}$, $u^0(P) > 0$.
Case i): $\exists P^+ \in \{y > 0\}, \exists P^- \in \{y < 0\}, u^0(P^+) > 0, u^0(P^-) > 0$
Then let us consider the blow-in with $\mu \to +\infty$:

$$\left\{ \begin{array}{l} u^{0,\mu}(X) = \dfrac{u^0(\mu X)}{\mu} \\ \gamma^{0,\mu}(X) = \gamma^0(\mu X) \end{array} \right.$$

If we set $u_1 = u^0 1_{\{y>0\}}, u_2 = u^0 1_{\{y<0\}}$ we know from theorem 2.5 that $\phi_{u^0}(r) := \phi(r)$ is nondecreasing in r. In particular $\phi_{u^{0,\mu}}(r) = \phi_{u^0}(\mu r) > 0$ for $\mu > 0$ large enough. Then up to extraction of some subsequence $(u^{0,\mu}, \gamma^{0,\mu}) \to (u^{0,\infty}, \gamma^{0,\infty})$ which is a solution of (1.3) on $\mathbf{R}^2$, and $\phi_{u^{0,\infty}}(r) = \phi_{u^0}(+\infty) > 0$. Then from theorem 2.5, $u^{0,\infty}(X) = \alpha y^+ - \beta y^- \geq 0$ with $\alpha > 0, \beta < 0$ This is imposible from proposition 2.3.
Case ii): $u^0 = 0$ **on** $\{y > 0\}$ **or** $\{y < 0\}$
Let us assume that $u^0 = 0$ on $\{y > 0\}$. Let for some $\epsilon_0 > 0$:

$$\left\{ \begin{array}{l} \tilde{u}^0(x,y) = \left\{ \begin{array}{l} u^0(x, y + \epsilon_0) \text{ if } y \leq 0 \\ u^0(x, -y + \epsilon_0) \text{ if } y \geq 0 \end{array} \right. \\ \tilde{\gamma}^0(x,y) = \left\{ \begin{array}{l} \gamma^0(x, y + \epsilon_0) \text{ if } y \leq 0 \\ \gamma^0(x, -y + \epsilon_0) \text{ if } y \geq 0 \end{array} \right. \end{array} \right.$$

Then $(\tilde{u}^0, \tilde{\gamma}^0)$ is a solution of (1.3) on $\mathbf{R}^2$ and we get a contradiction as in case i).
This ends the proof of lemma 5.11.

End of the proof of proposition 5.5
To finish the proof, we have to discard the case 2 of a solution (u^0, γ^0) of (1.3) on $\mathbf{R}^2$ with $u^0 \geq 0$ and $u^0(0) = 0$, $u^0(X) \neq o(|X|)$.
We have $0 \in \partial\{u^0 = 0\}$, then from lemma 5.10, $u^0(x, 0) = 0$ for $x \geq 0$. Let for $\epsilon \geq 0$, $\lambda \geq 1$, $t \geq 0$:

$$\begin{cases} u^\epsilon_{t,\lambda}(x,y) = \frac{u^\epsilon(\lambda x + t, \lambda y)}{\lambda} \\ \gamma^\epsilon_{t,\lambda}(x,y) = \gamma^\epsilon(\lambda x + t, \lambda y) \end{cases}$$

In particular $u^0_{t,1} \to u^0_{\infty,1}$ uniformly on every compact sets (up to extraction of some subsequence), and $\gamma^0_{t,1} \to \gamma^0_{\infty,1}$ in L^∞_{weak*}. Then $(u^0_{\infty,1}, \gamma^0_{\infty,1})$ is a solution of (1.3), $u^0_{\infty,1} \geq 0$, and $\forall x \in \mathbf{R}, u^0_{\infty,1}(x,0) = 0$, therefore from lemma 5.11 we have $u^0_{\infty,1} \equiv 0$ on $\mathbf{R}^2$. Now we deduce for each $\lambda \geq 1$ that

$$|u^{\epsilon_1}_{t,\lambda} - u^0_{\infty,\lambda}| \leq \epsilon \quad \text{on} \quad R_2$$

if ϵ_1 is small enough and t is large enough. Then from proposition 5.9 we deduce that there exist $y^\pm = (y^{\epsilon_1}_{t,\lambda})^\pm$ such that

$$\begin{cases} |u^{\epsilon_1}_{t,\lambda}| > 0 \quad \text{on} \quad R_1 \backslash [-1,1] \times ([y^-, y^- + 5\eta_0] \cup [y^+ - 5\eta_0, y^+]) \\ \gamma^{\epsilon_1}_{t,\lambda} = 0 \quad \text{where} \quad u^{\epsilon_1}_{t,\lambda} < 0 \\ \gamma^{\epsilon_1}_{t,\lambda} = 1 \quad \text{where} \quad u^{\epsilon_1}_{t,\lambda} > 0 \end{cases}$$

If we pass to the limit firstly on $\epsilon_1 \to 0$, we deduce that $\gamma^0_{t,\lambda} = u^0_{t,\lambda} = 0$ on certain regions of the form $[-1,1] \times [a,b]$. But $\gamma^0_{t,\lambda}$ is nonincreasing in x, therefore $\forall t' > t, \gamma^0_{t',\lambda} = u^0_{t',\lambda} = 0$ on $[-1,1] \times [a,b]$.
Now if we take λ large enough we deduce that there exist $y^+, y^- \in \mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ such that at $x = +\infty$, γ^0 is equal to 0 or 1 on each interval $(-\infty, y^-), (y^-, y^+), (y^+, +\infty)$.
Case A: $\gamma^0_{\infty,1} = 1$ on $(a,0)$ or $(0,a)$
It implies by construction, and because of lemma 5.11 that $u^0 > 0$ on $\mathbf{R} \times (0,a)$. Then we get a contradiction as in the proof of lemma 5.11 because $u^0(x,0) = 0$ for $x \geq 0$.
Case B: $\gamma^0_{\infty,1} = 0$ on $(a,b) \ni 0$
Then there exists a ball $B_r(\Lambda e_x) \subset \{u^0 = 0\}^0$ for some $r, \Lambda > 0$, where $\gamma^0 = 0$. We define $v^0(x,y) = \int_{-x}^{\Lambda} u^0(s,y)ds$ which satisfies

$$\begin{cases} v^0 \geq 0 \\ \Delta v^0 = \lambda 1_{\{v^0 > 0\}} \\ |D^2 v^0|_{L^\infty} \leq \max(Lip(u), \lambda) < +\infty \end{cases} \tag{5.6}$$

Moreover $(v^0)'_x = u^0 \geq 0$ and $(v^0)'_x > 0$ in $\{v^0 > 0\}$. From a result of Alt [1] (see lemma 4.2), $\partial\{v^0 > 0\} = \partial\{u^0 > 0\}$ is locally Lipschitz. We make a new blow-up

$u^{0,\epsilon}(X) = \frac{u^0(\epsilon X)}{\epsilon} \to u^{00}(X)$, $v^{0,\epsilon}(X) = \frac{v^0(\epsilon X)}{\epsilon^2} \to v^{00}(X)$. From L.A. Caffarelli results on the obstacle problem [10] we get $v^{00}(x,y) = \frac{(<X,\nu>^+)^2}{2}$ or $v^{00}(x,y) = \frac{<X,\nu>^2}{2}$ where $< e_x, \nu > \neq 0$ because Γ is Lipschitz. But $\partial_x v^{00} = u^{00} \geq 0$, then $u^{00}(X) =< X, \nu >^+$ for $< e_x, \nu >< 0$. Once more, from [10] we get that $\partial\{u^0 > 0\}$ is analytic near 0. We conclude (see lemma 6.5 and the proof of theorem 6.3 iii) in the Appendix) that $\Gamma(u)$ is analytic near X_0. Then Hopf lemma applies and gives locally $u(X) = \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- +o(|X|)$ with $\alpha, \beta > 0$. We deduce that $\phi_u(0) > 0$, which proves that case 2 is impossible (for the two phases problem). This ends the proof of proposition 5.5.

6 Appendix: extension of L.A. Caffarelli results for free boundaries with a general function $G(u_\nu^+, \nu, X)$

Definition 6.1 *A function u is a solution of problem (P_G) on the open set $\Omega \subset \mathbf{R}^n$ if and only if:*
i) $u \in C^{0,1}_{loc}(\Omega)$
ii) $\Delta u = 0$ in $\Omega^+(u) := \{u > 0\}, \Omega^-(u) := \{u \leq 0\}^0$
iii) On $\Gamma(u) := (\partial\Omega^+(u)) \cap \Omega$, we have $u_\nu^+ = G(u_\nu^-, \nu, X_0)$ in the following weak sense. For every ball $B = B_r(Y_0)$ with $X_0 \in \partial B \cap \Gamma(u)$ and $r = |X_0 - Y_0|$:
a) If $B \subset \Omega^+(u)$, let $\nu = \frac{Y_0 - X_0}{r} \in \mathbf{S}^{n-1}$. Then

$$\exists \alpha > 0, \beta \geq 0, \quad u(X) \leq \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- +o(|X - X_0|)$$

b)If $B \subset \Omega^-(u)$, let $\nu = -\frac{Y_0 - X_0}{r} \in \mathbf{S}^{n-1}$. Then

$$\exists \alpha \in \mathbf{R}, \beta \geq 0, \quad u(X) \geq \alpha < X - X_0, \nu >^+ -\beta < X - X_0, \nu >^- +o(|X - X_0|)$$

where in each case $\alpha = G(\beta, \nu, X_0)$.

We assume that G satisfies the assumptions:

Assumptions 6.2
i) $G(\beta, x, \nu) \in \mathbf{R}$
ii) G is strictly increasing in β.
iii) $\forall M > 0$, G is a lipschitz continuous function in $(\beta, \nu, X) \in [-M, M] \times \mathbf{S}^{n-1} \times \overline{\Omega}$

Then we have the following two local results:

Theorem 6.3 (Regularity of the free boundary)
i)If locally

$$u(X) = \alpha_0 < X - X_0, \nu_0 >^+ -\beta_0 < X - X_0, \nu_0 >^- +o(|X - X_0|) \quad \text{and} \quad \alpha_0, \beta_0 > 0$$

then

$$\text{locally} \quad \Gamma(u) \in C^{1,\alpha}$$

and for every $X_1 \in \Gamma(u)$ near X_0, there exist $\nu_1 \in \mathbf{S}^{n-1}$ and $\alpha, \beta \geq const(\alpha_0, \beta_0) > 0$ such that we have locally $u(X) = \alpha < X - X_1, \nu_1 >^+ - \beta < X - X_1, \nu_1 >^- + o(|X - X_1|)$.
ii)Let us assume that

$$\forall \epsilon > 0, \exists r_\epsilon > 0, \forall r < r_\epsilon, \quad \Omega^-(u) \supset B_r(X_0) \cap \{< X - X_0, \nu_0 > \leq -\epsilon r\}$$

If $u \geq 0$ locally near X_0 with

$$u(X) = \alpha_0 < X - X_0, \nu_0 >^+ + o(|X - X_0|) \quad \text{and} \quad \alpha_0 > 0$$

then

$$\text{locally} \quad \Gamma(u) \in C^{1,\alpha}$$

and for $X_1 \in \Gamma(u)$ near X_0, there exists $\nu_1 \in \mathbf{S}^{n-1}$ and $\alpha \geq const(\alpha_0) > 0, \beta \geq 0$ such that we have locally $u(X) = \alpha < X - X_1, \nu_1 >^+ - \beta < X - X_1, \nu_1 >^- + o(|X - X_1|)$.
iii)The conclusion of ii) is also true if we change the condition $u \geq 0$ in a neighbourhood of X_0 by the condition $G(0, \nu_0, X_0) > 0$.

Remark 6.4 *Here L.A. Caffarelli's approach [7]-[8] does not apply to improve the regularity of Γ near a flat point (see definition 5.1) because the solution is degenerate near a flat point.*

Lemma 6.5 *If u is a solution of problem (P_G), with the condition i) of theorem 6.3 (resp. ii) or iii)), then locally, $\forall \theta_0 \in (0, \frac{\pi}{2})$, $\exists C_{\theta_0} > 0$, $\exists \epsilon > 0$, $\forall \tau \in C^+(\theta_0, \nu_0) \cap \mathbf{S}^{n-1}$ where $C^+(\theta_0, \nu_0) = \{\tau \in \mathbf{R}^n \backslash \{0\},\ \text{angle}\ (\tau, \nu_0) \leq \theta_0\}$, we have locally*

$$u(X + \epsilon\tau) - u(X) \geq C_{\theta_0}\epsilon$$

$$(\text{resp.} \quad u(X + \epsilon\tau) - u(X) \geq C_{\theta_0}\epsilon \quad \text{in} \quad \{u > 0\})$$

Proof of lemma 6.5
It is an easy consequence of lemma 1, lemma 5, lemma 4 of [7], and of an adaptation of the proof of theorem 2' of [8].

Proof of the theorem 6.3
Cases i) and ii)
Under conditions i) or ii) , the difficulty, is to discard the values of $G(u_\nu^-, \nu, X) \leq 0$, with the help of a control on the normal ν of Γ. So we adapt the proof of L.A. Caffarelli [8],

using the fact that initially for $\mathcal{C}_M = B_1^{n-1} \times [-M, M] \subset \mathbf{R}^n$ where $e_n = \nu_0$, $\theta_0 \sim \frac{\pi}{2}$, $\alpha_1 > 0$, there exists $\epsilon_0 > 0, \epsilon_0 << 1$, such that

$$\forall \epsilon > \epsilon_0, \quad \begin{cases} v = \sup_{|Y|<\sin\theta_0} u(X - \epsilon(e_n + Y)) \leq u(X) - \epsilon\alpha_0 \cos\theta_0 \text{ in } \{v > 0\} \cap \mathcal{C}_M \\ v = \sup_{|Y|<\sin\theta_0} u(X - \epsilon(e_n + Y)) \leq u(X) \text{ in } \mathcal{C}_M \end{cases} \tag{6.1}$$

What is important in the proof, is the condition on the boundary $\partial\{\overline{v_t} > 0\}$. We recall that $\overline{v_t} = v_t + \eta w$ is a subsolution for the free boundary problem, where $v_t(X) = \sup_{Y \in B_{\sigma\phi_t(X)}} u(Y)$, $\eta = C\epsilon^{\frac{1}{4}}$, and $w \geq 0$ is a corrector function to permit to satisfy the boundary condition of subsolution on the free boundary of $\overline{v_t}$:

$$\overline{v_t}(X) \geq \alpha < X - \tilde{X}_1, \tilde{\nu} >^+ - \beta < X - \tilde{X}_1, \tilde{\nu} >^- + o(| < X - \tilde{X}_1, \tilde{\nu} > |) \tag{6.2}$$

with $\alpha = G(\beta, \tilde{\nu}, \tilde{X}_1)$.

1)Firstly, from lemma 2 [8], v_t is monotone in a cone $\mathcal{C}(\overline{\theta}_0)$, where $\overline{\theta}_0$ is very close to $\frac{\pi}{2}$, if θ_0 is close enough to $\frac{\pi}{2}$ initially. Then the normal $\tilde{\nu}$ to $\partial\{v_t > 0\}$ is very close to e_n. This fact permits us to control the free boundary in a neighbourhood of X_0. In particular the normal ν in the proof of lemma 4 in [8] is close to e_n because $\tilde{\nu} = \frac{\nu + \sigma\nabla\phi_t}{|\nu + \sigma\nabla\phi_t|}$ and $|\sigma\nabla\phi_t| \leq C\epsilon^{\frac{1}{2}}$.

2) Secondly, we must satisfy the boundary condition (6.2). But here we have changed the condition

$$\exists C > 0, \ \beta^{-C} G(\beta, x, \nu) \text{ is a decreasing function in } \beta \tag{6.3}$$

which was required in the proof of L.A. Caffarelli, by the condition iii) of assumption 6.2. The boundary condition is satisfied, because the function w constructed in [8] is now nondegenerate because of (6.1). More precisely we can find a ball $B \subset \Omega^+(v_t)$ with B tangent to $\partial\Omega^+(v_t)$ and the diameter of B is of order ϵ. Now $w \geq v$ and at a distance $\frac{CM\epsilon}{2}$ where $v \sim u \geq \epsilon \frac{CM}{2} \alpha_0 \cos\theta_0$ and by Harmack inequality on B, we can construct a barrier subsolution which proves that $\exists C = C(\alpha_1 \cos\theta_0) > 0, \partial_{\tilde{\nu}} w \geq C > 0$.

Remark 6.6 *Let us remark that this modification of the proof of L.A. Caffarelli is suffisant to apply his proof without other knowledge on the sign of G in a neighboorhood of X_0 (a priori G could be negative in some points).*

Remark 6.7 *In particular it proves that the results of L.A. Caffarelli in [7]-[8] with $\inf_{\nu,X} G(0, \nu, X) > 0$ are true without the conditions (6.3), but only assuming that u is Lipschitz and assumption 6.2 iii).*

Then independently on ϵ, $\exists\lambda \in (0, 1)$ such that we obtain (6.1) with $\epsilon, \theta_0, \alpha_1, \mathcal{C}_M$ respectively changed by $\lambda\epsilon, \theta_0 - \epsilon^{\frac{1}{4}}, \alpha_1 - \epsilon^{\frac{1}{16}}, \mathcal{C}_{M(1 - C\epsilon^{\frac{1}{8}})}$. Then the proof of L.A. Caffarelli applies and proves that Γ is Lipschitz. Moreover the proof of [7] applies with the same modification.

Remark 6.8 *We have not used the fact that $G(\beta_0, \nu_0, X_0) > 0$. In fact it is a consequence of (6.1) which implies at the limit $\epsilon \to 0^+$,*

$$\exists \theta_0^*, \alpha_1^*, \ \forall \tau \in C^+(\theta_0^*), \ (u^+)'_\tau \geq C\alpha_1^* \cos\theta_0^* > 0$$

This proves that $\alpha \geq const(\alpha_0) > 0$. In particular under condition i) of theorem 6.3 we have $\beta_0 > 0$, then the same reasoning applies in $\{u < 0\}$ which proves that $\beta \geq const(\beta_0) > 0$.

Case iii)
In this case we would like to apply the proof of theorem 2 in [8]. But here is a new difficulty: we do not know a priori if u^+ in nondegenerate, i.e. in a neighbourhood of X_0, $u^+_\nu \geq C > 0$. To prove the iii) we must modify the proof of lemma 6 in [8] as follows. Let us take the new criteria (for some $1 >> \delta_1 > 0$ fixed): $u^-(-\frac{1}{2}e_n) \geq C\epsilon^{1-\delta_1}$ for alternative a), and $u^-(-\frac{1}{2}e_n) < C\epsilon^{1-\delta_1}$ for alternative b).
alternative a)
Then in $C_{1-C\epsilon^{\tau_1}}$ for some $\tau_1 > 0$ small enough and for $C_1\epsilon^{\tau_1} < |X_1 - X_2| < C_2\epsilon^{\tau_1}$ we get (see p 72 in [8] for the function v defined page 70) $v(X_2) - v(X_1) \geq \frac{C}{\delta_0} u^-(-\frac{1}{2}e_n)\epsilon^{(\alpha+1)\tau_1} \geq C\epsilon^{1-(\delta_1+(\alpha+1)\tau_1)}$, and because u is Lipschitz and $d(X_3, \Gamma) \leq \epsilon$ for every $X_3 \in A$, we get $u^- \leq C\epsilon$ on A and then $v \leq u^- \leq v + C\epsilon$ (as in p 70 in [8]). We conclude that $u(X_2) \geq u(X_1)$ for $X_1 - X_2 \in \Gamma(\theta_1, e_n)$ and u is $C\epsilon^{\tau_1}$-monotone.
alternative b)
We have $u^+_\nu \geq G(u^-_\nu, \nu, X) \sim G(0, \nu_0, X_0) > 0$ in the points of interest of $\Gamma(u)$ (point of comparison for $\overline{v}_t$). In particular for these points we get (from the monotonicity formula) $0 \leq u^-_\nu \leq C\epsilon^{\frac{1-\delta_1}{2}-\mu}$ and then we conclude similarly as in [8] with $\eta \geq C\epsilon^{\frac{1-\delta_1}{2}-\mu}$.

Aknowledgement: The authors are very grateful to S. Kamin and C.J. van Duijn for stimulating discussions.

References

[1] H.W. Alt, *The fluid flow through porous media. Regularity of the free surface*, Manuscripta Math. 21 1977, 255-272.

[2] H.W. Alt, L.A. Caffarelli, A. Friedman, *The dam problem with two fluids*, Comm. in Pure and Appl. Math. 37 1984, 601-645.

[3] H.W. Alt, L.A. Caffarelli, A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. 282(2) 1984, 431-461.

[4] H.W. Alt, C.J. van Duijn, *A stationary flow of fresh and salt groundwater in a coastal aquifer*, Nonlinear Anal. TMA 14(8) 1990, 625-656.

[5] H. Berestycki, A. Bonnet, C.J. van Duijn, *The Regularity of the Free Boundary Between Two Fluids in a Porous Medium*, to appear.

[6] A. Bonnet, S. Kamin, *A stationary flow in a strip*, Nonlinear Anal. TMA 35A(6) 1999, 765-780.

[7] L.A. Caffarelli, *A Harnack Inequality Approach to the Regularity of Free Boundaries. Part I: Lipschitz Free Boundaries are $C^{1,\alpha}$*, Rev. Math. Iberoamericana 3(2) 1987, 139-162.

[8] L.A. Caffarelli, *A Harnack Inequality Approach to the Regularity of Free Boundaries. Part II: Flat Free boundaries are Lipschitz*, Comm. Pure Appl. Math. 15 1989, 55-78.

[9] L.A. Caffarelli, *A Harnack Inequality Approach to the Regularity of Free Boundaries. Part III: Existence Theory, Compactness, and Dependence on X*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci (4), 15(4) 1988/1989, 583-602.

[10] L.A. Caffarelli, *Compactness method in free boundary problems*, Comm. Partial Diff. Equ. 5(4) 1980, 427-448.

[11] L.A. Caffarelli, *Free Boundary Problems, a Survey*, 1986.

[12] J. Carrillo, *On the uniqueness of the solution of the evolution dam problem*, Nonlinear Anal. TMA, 22 (5) 1994, 573-607.

[13] J. Carrillo, *Entropy solutions for nonlinear degenerate problems*, to appear.

[14] M. Chipot, *Variational inequalities and flow in porous media*, Springer-Verlag Ed. 52, 1984.

[15] A. Friedman, *Variational Principles and Free Boundary Problems*, Interscience, New York, 1982.

[16] D. Kinderlehrer, L. Nirenberg, *Regularity in free boundary problems*, Ann. Sculoa Norm. Sup. Pisa, 4 1977, 373-391.

[17] J.-F. Rodrigues, F. Yi, *On a two-phase continuous casting Stefan problem with nonlinear flux*, European J. Appl. Math. 1 1990, 259-278.

[18] J.-F. Rodrigues, B. Zaltzmann, *On classical solutions of the two-phase steady-state Stefan problem in strips*, Nonlinear Anal. Theory Methods Appl. 19(3) 1992, 207-208.

[19] J.-F. RODRIGUES, B. ZALTZMANN, *Regular solutions of a Stefan problem in strips*, preprint, Universidade de Lisboa, 1994.

[20] B. ZALTZMANN, *Multidimensional two-phase quasistationary Stefan problem*, Manuscripta Math. 78 1993, 287-301.