Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case

C. Imbert* and R. Monneau†

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Abstract

A *multi-dimensional junction* is the singular $(d+1)$-manifold obtained by gluying through their boundaries a finite number of copies of the half-space $\mathbb{R}^{d+1}_+$. We show that the general theory developed by the authors (2013) for the network setting can be adapted to this multi-dimensional case. In particular, we prove that general quasi-convex junction conditions reduce to flux-limited ones and that uniqueness holds true when flux limiters are quasi-convex and continuous. The proof of the comparison principle relies on the construction of a (multi-dimensional) vertex test function.

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1 Introduction

This paper is concerned with extending the theory developed for Hamilton-Jacobi (HJ) equations posed on junctions in [3] to the multi-dimensional setting.

A *multi-dimensional junction* is made of N copies of $\mathbb{R}^{d+1}_+$ glued through their boundaries.

$$J = \bigcup_{i=1,\dots,N} J_i \qquad \text{with} \qquad \begin{cases} J_i = \{X = (x', x_i) : x' \in \mathbb{R}^d, x_i \geq 0\} \simeq \mathbb{R}^{d+1}_+ \\ J_i \cap J_j = \Gamma \simeq \mathbb{R}^d \times \{0\} \quad \text{for} \quad i \neq j. \end{cases} \tag{1.1}$$

We emphasize that the common boundary of the half-spaces J_i is denoted by Γ and is called the *junction interface*. For points $X, Y \in J$, $d(X,Y)$ denotes $|x' - y'| + d(x,y)$ with

$$d(x,y) = \begin{cases} x + y & \text{if } X \in J_i, Y \in J_j, i \neq j \\ |x - y| & \text{if } X, Y \in J_i. \end{cases}$$

For a smooth real-valued function u defined on J, $\partial_i u(X)$ denotes the (spatial) derivative of u with respect to x_i at $X = (x', x_i) \in J_i$ and $D'u(X)$ denotes the (spatial) gradient of u with respect to x'. The "gradient" of u is defined as follows,

$$Du(X) := \begin{cases} (D'u(X), \partial_i u(X)) & \text{if} \quad X \in J_i^* := J_i \setminus \Gamma, \\ (D'u(x',0), \partial_1 u(x',0), \dots, \partial_N u(x',0)) & \text{if} \quad X = (x',0) \in \Gamma. \end{cases} \tag{1.2}$$

*CNRS, UMR 7580, Université Paris-Est Créteil, 61 avenue du Général de Gaulle, 94 010 Créteil cedex, France
†70, rue du Javelot, 75013 Paris, France

With such a notation in hand, we consider the following Hamilton-Jacobi equation posed on the multi-dimensional junction J

$$\begin{cases} u_t + H_i(Du) = 0 & t>0, X \in J_i \setminus \Gamma, \\ u_t + F(Du) = 0 & t>0, X \in \Gamma \end{cases} \tag{1.3}$$

submitted to the initial condition

$$u(0,X) = u^0(X) \quad \text{for} \quad X \in J. \tag{1.4}$$

The second equation in (1.3) is referred to as *the junction condition.*

The Hamiltonians are supposed to satisfy the following conditions:

$$\begin{cases} \textbf{(Continuity)} & H_i \in C(\mathbb{R}^{d+1}) \\ \textbf{(Quasi-convexity)} & \forall \lambda, \{H_i \le \lambda\} \text{ is convex} \\ \textbf{(Coercivity)} & \lim_{|P|\to+\infty} H_i(P) = +\infty. \end{cases} \tag{1.5}$$

We next define the A-limited flux function F_A associated with the multi-dimensional junction J. In order to do so, we first consider $\pi_i^0(p') \in \mathbb{R}$ minimal such that $p_i \mapsto H_i(p', p_i)$ reaches its minimum at $p_i = \pi_i^0(p')$ and H_i^- is defined by

$$H_i^-(p', p_i) = \begin{cases} H_i(p', p_i) & \text{if} \quad p_i \le \pi_i^0(p'), \\ H_i(p', \pi_i^0(p')) & \text{if} \quad p > \pi_i^0(p'). \end{cases}$$

In a similar way, we define

$$H_i^+(p', p_i) = \begin{cases} H_i(p', \pi_i^0(p')) & \text{if} \quad p_i < \pi_i^0(p'), \\ H_i(p', p_i) & \text{if} \quad p \ge \pi_i^0(p'). \end{cases}$$

So-called *flux-limiter functions* $A\colon \mathbb{R}^d \to \mathbb{R}$ are always assumed to be continuous and, in some important cases, to satisfy the following condition,

$$A\colon \mathbb{R}^d \to \mathbb{R} \text{ is continuous and quasi-convex.} \tag{1.6}$$

The function F_A is defined for $p = (p_1, \dots, p_N)$ and $P = (p', p)$ as

$$F_A(P) = \max\left(A(p'), \max_{i=1,\dots,N} H_i^-(p', p_i) \right). \tag{1.7}$$

We now consider the following important special case of (1.3),

$$\begin{cases} u_t + H_i(Du) = 0 & t>0, X \in J_i \setminus \Gamma, \\ u_t + F_A(Du) = 0 & t>0, X \in \Gamma. \end{cases} \tag{1.8}$$

We point out that A could be replaced with $\max(A, A_0)$ where

$$A_0(p') = \max_{i=1,\dots,N} A_i(p') \quad \text{with} \quad A_i(p') = \min_{p_i \in \mathbb{R}} H_i(p', p_i). \tag{1.9}$$

We notice (see Lemma A.1 in Appendix) that the functions A_i, $i = 0, \dots, N$ are quasi-convex, continuous and coercive.

As far as general junction conditions are concerned, we assume that the junction function $F : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}$ satisfies

$$\begin{cases} \textbf{(Continuity)} & F \in C(\mathbb{R}^d \times \mathbb{R}^N) \\ \textbf{(Monotonicity)} & \forall i,\ p_i \mapsto F(p', p_1, \dots, p_N) \text{ is non-increasing} \end{cases} \tag{1.10}$$

and, in some important cases,

$$\textbf{(Quasi-convexity)} \qquad \forall \lambda,\ \{F \le \lambda\} \text{ convex.} \tag{1.11}$$

In particular, under assumption (1.5), if A satisfies (1.6), then F_A defined in (1.7), satisfies (1.10) and (1.11).

1.1 Main results

For simplicity, we state the next theorem under a simple continuity assumption for subsolutions, but a more general result is true (see Theorem 2.13).

Theorem 1.1 (General junction conditions reduce to F_A)**.** *Let the Hamiltonians satisfy* (1.5) *and let* $F : \mathbb{R}^N \to \mathbb{R}$ *satisfy* (1.10). *There exists a unique coercive continuous function* $A_F : \mathbb{R}^d \to \mathbb{R}$, *satisfying* $A_F \geq A_0$ *with* A_0 *defined in (1.9), such that the following holds. Every relaxed viscosity super-solution (resp. sub-solution, which is moreover assumed to be continuous) of* (1.3) *is a* A_F*-flux limited super-solution (resp. sub-solution) of* (1.8). *Moreover, if* F *is quasi-convex, so is* A_F.

Remark 1.2. Let $p_i^0 \geq \pi_i^0(p')$ be minimal such that $H_i(p', p_i) = A_0$ and let p^0 denote $(p_1^0, \dots, p_N^0)$. The function A_F is defined as follows: for each $p' \in \mathbb{R}^d$, if $F(p', p^0) \leq A_0(p')$, then $A_F(p') = A_0(p')$, else $A_F(p')$ is the only $\lambda \in \mathbb{R}$ such that $\lambda \geq A_0(p') = \max_i A_i(p')$ and there exists $p_i^+ \geq p_i^0$ such that

$$H_i(p', p_i^+) = F(p', p^+) = \lambda$$

where $p^+ = (p_1^+, \dots, p_N^+)$. Notice that even if λ is unique, p^+ may be not unique.

Theorem 1.3 (Comparison principle on a multi-dimensional junction)**.** *Assume that the Hamiltonians satisfy* (1.5)*, the function* A *satisfies* (1.6) *with* $A \geq A_0$ *where* A_0 *is defined in (1.9), and that the initial datum* u_0 *is uniformly continuous. Then for all (relaxed) sub-solution* u *and (relaxed) super-solution* v *of* (1.3)-(1.4) *with* $F = F_A$ *defined in (1.7), satisfying for some* $T > 0$ *and* $C_T > 0$,

$$u(t, X) \leq C_T(1 + d(0, X)), \quad v(t, X) \geq -C_T(1 + d(0, X)), \quad \textit{for all} \quad (t, X) \in [0, T) \times J, \quad (1.12)$$

we have

$$u \leq v \quad \textit{in} \quad [0, T) \times J.$$

1.2 Comparison with known results

In the special case $N = 2$, our results are related to [1, 2] where an optimal control problem in a two-domain setting is studied. The state of the system evolves according to two different dynamics on each side of an hypersurface. Moreover, the two dynamics at the interface corresponding to the maximal and minimal Ishii's discontinuous solutions of the associated Hamilton-Jacobi equation are identified. One of the two value functions is characterized in terms of partial differential equations. We showed in [3] that, in the one-dimensional setting, both value functions can be characterized by using the notion of flux-limited solutions introduced in [3]. The result of the present paper indicates that such a connexion holds in the general two-domain setting, even if this is out of the scope of the present paper. Moreover, we can deal with quasi-convex Hamiltonians instead of convex ones.

The reader is also referred to [6, 5] for optimal control problems in multi-domains. In particular, the authors impose some transmission conditions. As we already mentioned it in [3], Definition 2.6 is strongly related to these works. See also [4] for stationary Hamilton-Jacobi problems on multi-dimensional junctions, where comparison principles are established using an optimal control approach. We finally refer the reader to the references cited in [3] and the comments there.

Organization of the article. The paper is organized as follows. In Section 2, the notion of viscosity solution in the setting of multi-dimensional junction is introduced. The proof of Theorem 1.1 is done in Subsection 2.3. Section 3 is devoted to the construction of the vertex test function. The proof of Theorem 1.3 is done just after the statement of Theorem 3.1 about the vertex test function. The proof of a technical lemma is presented in an appendix.

Notation. For a function $f : D \to \mathbb{R}$, $\operatorname{epi} f$ denotes its epigraph $\{(X, r) \in D \times \mathbb{R} \colon r \geq f(X)\}$ and $\operatorname{hypo} f$ denotes its hypograph $\{(X, r) \in D \times \mathbb{R} \colon r \leq f(X)\}$. We will use the notation P to denote different objects, depending on the context.

2 Viscosity solutions on a multi-dimensional junction

2.1 Definitions

2.1.1 Class of test functions

For $T > 0$, set $J_T = (0, T) \times J$. The class of test functions on J_T is chosen as follows,

$$C^1(J_T) = \left\{\varphi \in C(J_T),\ \varphi \text{ restricted to } (0,T) \times J_i \text{ is } C^1 \text{ for } i = 1, ..., N\right\}. \tag{2.1}$$

2.1.2 Classical viscosity solutions

In order to define classical viscosity solutions, we recall the definition of upper and lower semi-continuous envelopes u^* and u_* of a (locally bounded) function u defined on $[0, T) \times J$:

$$u^*(t, X) = \limsup_{(s,Y)\to(t,X)} u(s, Y) \qquad \text{and} \qquad u_*(t, X) = \liminf_{(s,Y)\to(t,X)} u(s, Y).$$

Definition 2.1 (Classical viscosity solutions)**.** Assume the Hamiltonians satisfy (1.5) and the flux function F satisfies (1.10). Let $u : [0, T) \times J \to \mathbb{R}$ be locally bounded.

i) We say that u is a (classical viscosity) *sub-solution* (resp. *super-solution*) of (1.3) in J_T if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, X_0) \in J_T$$

with equality at (t_0, X_0) for some $t_0 > 0$, we have

$$\begin{aligned} \varphi_t + H_i(D\varphi) \leq 0 \qquad &(\text{resp.} \quad \geq 0) \quad \text{at } (t_0, X_0) \qquad \text{if } X_0 \in J_i^* = J_i \setminus \Gamma \\ \varphi_t + F(D\varphi) \leq 0 \qquad &(\text{resp.} \quad \geq 0) \quad \text{at } (t_0, X_0) \qquad \text{if } X_0 \in \Gamma. \end{aligned} \tag{2.2}$$

ii) We say that u is a (classical viscosity) *sub-solution* (resp. *super-solution*) of (1.3)-(1.4) on $[0, T) \times J$ if additionally

$$u^*(0, X) \leq u_0(X) \quad (\text{resp.} \quad u_*(0, X) \geq u_0(X)) \quad \text{for all} \quad x \in J.$$

iii) We say that u is a (classical viscosity) *solution* if u is both a sub-solution and a super-solution.

Definition 2.2 (Flux-limited solutions)**.** Consider a continuous flux-limiter function $A \colon \mathbb{R}^d \to \mathbb{R}$. Then u is a A-flux limited sub-solution (resp. super-solution, solution) of (1.8) if it is a classical sub-solution (resp. super-solution, solution) of (1.3) with $F = F_A$.

2.1.3 Relaxed viscosity solutions

We next introduce relaxed viscosity solutions.

Definition 2.3 (Relaxed viscosity solutions)**.** Assume the Hamiltonians satisfy (1.5) and the flux function F satisfies (1.10). Let $u : [0, T) \times J \to \mathbb{R}$ be locally bounded.

i) We say that u is a *relaxed sub-solution* (resp. *relaxed super-solution*) of (1.3) in J_T if for all test function $\varphi \in C^1(J_T)$ such that

$$u^* \leq \varphi \quad (\text{resp.} \quad u_* \geq \varphi) \quad \text{in a neighborhood of } (t_0, X_0) \in J_T$$

with equality at (t_0, X_0) for some $t_0 > 0$, we have

$$\varphi_t + H_i(D\varphi) \le 0 \quad (\text{resp.} \quad \ge 0) \quad \text{at } (t_0, X_0)$$

if $X_0 \in J_i^*$, and

$$\left.\begin{array}{lll} \text{either} & \varphi_t + F(D\varphi) \le 0 & (\text{resp.} \quad \ge 0) \\ \text{or} & \varphi_t + H_i(D\varphi) \le 0 & (\text{resp.} \quad \ge 0) \quad \text{for some } i \end{array}\right| \quad \text{at } (t_0, X_0)$$

if $X_0 \in \Gamma$.

ii) We say that u is a *relaxed (viscosity) solution* if u is both a sub-solution and a super-solution.

2.1.4 The "weak continuity" condition for sub-solutions

If F not only satisfies (1.10), but is also semi-coercive, that is to say if

$$F(p', p) \to +\infty \quad \text{as} \quad \max_i(\max(0, -p_i)) \to +\infty \quad \text{for each} \quad p' \in \mathbb{R}^d \tag{2.3}$$

then any F-relaxed sub-solution satisfies a "weak continuity" condition at the junction point. Precisely, the following result holds true.

Lemma 2.4 ("Weak continuity" condition on the junction interface)**.** *Assume that the Hamiltonians satisfy (1.5) and that F satisfies (1.10) and (2.3). Then any relaxed sub-solution u of (1.3) satisfies teh following "weak continuity" property*

$$u^*(t, X) = \limsup_{(s,Y)\to(t,X),\ Y\in J_i^*} u(s, Y) \quad \textit{for all} \quad i = 1, \dots, N, \quad \textit{for all} \quad (t, X) \in (0, T) \times \Gamma \tag{2.4}$$

where we recall that $J_i^* = J_i \backslash \Gamma$.

The proof of this result is a straightforward adaptation of the one of Lemma 2.3 in [3] in the case $d = 0$; so we skip the details of the proof.

As in [3], we will see that the "weak continuity" property is an important condition to avoid pathological relaxed sub-solutions (that do exist) when F is not semi-coercive. Moreover it turns ou that the notion of "weak continuity" is stable, as shows the following result.

Proposition 2.5 (Stability of the weak continuity property)**.** *Consider a familly of Hamiltonians H_i^ε satisfying (1.5). We also assume that the coercivity of the Hamiltonians is uniform in ε. Let u^ε be a family of subsolutions of*

$$u_t + H_i^\varepsilon(Du) = 0 \quad \textit{in} \quad (0, T) \times J_i^*$$

for all $i = 1, \dots, N$, and that u^ε satisfies the "weak continuity" property (2.4). If $\bar{u} = \limsup{}^* u^\varepsilon$ is everywhere finite, then $\bar{u}$ still satisfies the "weak continuity" property (2.4).

The proof of this result is also a straightforward adaptation of the one of Proposition 2.6 in [3] in the case $d = 0$; so again we skip the details of the proof.

2.1.5 A reduced set of test functions

Let $\pi_i^\pm : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be defined as follows for $\lambda \ge A_i(p') = \min H_i(p', \cdot)$

$$\pi_i^+(p', \lambda) = \inf\{p_i : H_i(p', p_i) = H_i^+(p', p_i) = \lambda\}$$
$$\pi_i^-(p', \lambda) = \sup\{p_i : H_i(p', p_i) = H_i^-(p', p_i) = \lambda\}.$$

Definition 2.6 (Reduced viscosity solutions – the flux-limited case)**.** Assume the Hamiltonians satisfy (1.5) and consider a continuous flux-limiter function $A : \mathbb{R}^d \to \mathbb{R}$ such that for all $p' \in \mathbb{R}^d$, $A(p') \geq A_0(p')$. Given $u \colon [0,T) \times J \to \mathbb{R}$ locally bounded, the function u is a *reduced sub-solution* (resp. *reduced super-solution*) of (1.3) with $F = F_A$ in J_T if and only if u is a sub-solution (resp. super-solution) outside Γ and for all test function $\varphi \in C^1(J_T)$ touching u from above at $(t_0, X_0) \in (0,+\infty) \times \Gamma$, of the following form

$$\varphi(t, x', x) = \phi(t, x') + \phi_0(x)$$

with

$$\begin{cases} \phi \in C^1((0,+\infty) \times \mathbb{R}^d) \\ D'\phi(t_0, x_0') = p_0' \end{cases} \qquad \begin{cases} \phi_0 \in C^1(\mathbb{R}) \\ \partial_i \phi_0(0) = \pi_i^+(p_0', A(p_0')) \end{cases}$$

we have

$$\varphi_t + F_A(D\varphi) \leq 0 \quad \text{(resp.} \quad \geq 0).$$

Proposition 2.7 (Equivalence of Definitions 2.2 and 2.6 under "weak continuity")**.** *Every reduced super-solution (resp. subsolution) u in the sense of Definition 2.2 is also, for Definition 2.6, a flux-limited super-solution (resp. a flux-limited subsolution if u satisfies moreover the "weak-continuity" property (2.4)).*

Proof. It is clear that flux-limited sub-solutions (resp. super-solutions) are reduced sub-solutions (resp. reduced super-solutions). To prove that the converse holds true, we proceed as in [3] by considering critical slopes in x. Precisely, it is enough to prove the following lemmas.

Lemma 2.8 (Critical slopes for super-solutions)**.** *Let u be a super-solution of* (1.8) *away from* Γ *and let φ touch u_* from below at $P_0 = (t_0, X_0)$ with $X_0 \in \Gamma$. Then the "critical slopes" defined as follows*

$$\bar{p}_i = \sup\{\bar{p} \in \mathbb{R}_+ : \exists r > 0, \varphi(t, X) + \bar{p}x \leq u_*(t, X) \text{ for } (t, X) \in B_r(P_0) \cap ((0, +\infty) \times J_i)\}$$

satisfy for all $i = 1, \dots, N$,

$$\varphi_t(P_0) + H_i(D'\varphi(P_0), \partial_i \varphi(P_0) + \bar{p}_i) \geq 0,$$

with the convention for $\bar{p}_i = +\infty$, that $H_i(p', +\infty) = +\infty$.

Lemma 2.9 (Critical slopes for sub-solutions)**.** *Let u be a sub-solution of* (1.8) *away from* Γ *and let φ touch u^* from above at $P_0 = (t_0, X_0)$ with $X_0 \in \Gamma$. Then the "critical slopes" defined as follows*

$$\bar{p}_i = \inf\{\bar{p} \in \mathbb{R}_- : \exists r > 0, \varphi(t, X) + \bar{p}x \geq u^*(t, X) \text{ for } (t, X) \in B_r(P_0) \cap ((0, +\infty) \times J_i)\}$$

satisfy for all $i = 1, \dots, N$,

$$\varphi_t(P_0) + H_i(D'\varphi(P_0), \partial_i \varphi(P_0) + \bar{p}_i) \leq 0 \quad \text{if} \quad \bar{p}_i > -\infty.$$

Moreover, we have

$$\bar{p}_i > -\infty \quad \text{for each} \quad i = 1, \dots, N$$

if u satisfies the "weak continuity" property (2.4).

Remark 2.10. Even if Lemma 2.9 is not stated like that, an inspection of its proof shows that it is sufficient to have the "weak continuity" property pointwisely at (t_0, X_0) and on a single branch J_i^* to prove that $\bar{p}_i > -\infty$ for the same index i.

The proofs of these lemmas are straightforward adaptations of the corresponding ones in [3] so we skip them. The remaining of the proof is also analogous but we give some details in the sub-solution case for the reader's convenience.

Let φ touch u^* from above at $P_0 = (t_0, X_0)$ with $X_0 = (x'_0, 0) \in \Gamma$ and let λ denote $-\varphi_t(P_0)$ and $P = (p', p_1, \dots, p_N)$ denote $D\varphi(P_0)$. We want to prove

$$F_A(P) \le \lambda. \tag{2.5}$$

We know from Lemma 2.9 that for all $i = 1, \dots, N$,

$$H_i(p', p_i + \bar{p}_i) \le \lambda \tag{2.6}$$

for some $\bar{p}_i \le 0$. In particular,

$$A_0(p') \le \lambda.$$

We write next

$$\begin{aligned} F_A(P) &= \max_i (A(p'), H_i^-(p', p_i)) \\ &\le \max_i (A(p'), H_i^-(p', p_i + \bar{p}_i)) \\ &\le \max_i (A(p'), H_i(p', p_i + \bar{p}_i)) \\ &\le \max(A(p'), \lambda). \end{aligned}$$

If (2.5) does not hold true, then

$$A_0(p') \le \lambda < A(p').$$

Moreover, we have from (2.6) that

$$p_i + \bar{p}_i < \pi_i^+(p', A(p')).$$

Hence, we can consider the following test function

$$\phi(t, x', x) = \varphi(t, x', 0) + \phi_0(x)$$

with $\partial_i \phi_0(0) = \pi_i^+(p', A(p'))$ for each $i = 1, \dots, N$. From the definition of reduced sub-solutions, we thus get

$$A(p') = F_A(D\phi(P_0)) \le \lambda$$

which is the desired contradiction. □

2.2 Stability

In the following proposition, we assert that, for the special junction functions F_A, *relaxed solutions* are in fact always *classical solutions*, that is to say in the sense of Definition 2.1.

Proposition 2.11 (F_A junction conditions are always satisfied in the classical sense)**.** *Assume the Hamiltonians satisfy* (1.5) *and consider a continuous flux-limiter function A. If $F = F_A$, then relaxed viscosity solutions in the sense of Definition 2.3 coincide with viscosity solutions in the sense of Definition 2.1.*

Remark 2.12. Because relaxed solutions are always stable (see [3]), we also deduce from Proposition 2.11 that for the special case $F = F_A$, classical solutions are also stable.

Proof. We treat successively the super-solution case and the sub-solution case.

Case 1: the super-solution case. Let u be a relaxed super-solution and let us assume by contradiction that there exists a test function φ touching u_* from below at $P_0 = (t_0, X_0)$ for some $t_0 \in (0,T)$ and $X_0 \in \Gamma$, such that

$$\varphi_t + F_A(D\varphi) < 0 \quad \text{at} \quad P_0. \tag{2.7}$$

Consider next the test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \leq \varphi$ in a neighborhood of P_0, with equality at P_0 such that

$$\begin{array}{rl} \tilde{\varphi}_t(P_0) & = \varphi_t(P_0) \\ D'\tilde{\varphi}(P_0) & = D'\varphi(P_0) \end{array} \quad \text{and} \quad \partial_i \tilde{\varphi}(P_0) = \min(\pi_i^0(D'\varphi(P_0)), \partial_i\varphi(P_0)) \quad \text{for} \quad i = 1, ..., N.$$

Using the fact that $F_A(D\varphi) = F_A(D\tilde{\varphi}) \geq H_i^-(D'\tilde{\varphi}, \partial_i\tilde{\varphi}) = H_i(D'\tilde{\varphi}, \partial_i\tilde{\varphi})$ at P_0 for all i, we deduce a contradiction with (2.7) using the viscosity inequality satisfied by $\tilde{\varphi}$ for some $i \in \{1, \ldots, N\}$.

Case 2: the sub-solution case. Let now u be a relaxed sub-solution and let us assume by contradiction that there exists a test function φ touching u^* from above at $P_0 = (t_0, X_0)$ for some $t_0 \in (0,T)$ and $X_0 \in \Gamma$, such that

$$\varphi_t + F_A(D\varphi) > 0 \quad \text{at} \quad P_0. \tag{2.8}$$

Let us define

$$I = \left\{ i \in \{1, ..., N\}, \quad H_i^-(D'\varphi, \partial_i\varphi) < F_A(D\varphi) \quad \text{at} \quad P_0 \right\}$$

and for $i \in I$, let $q_i \geq \pi_i^0(D'\varphi(P_0))$ be such that

$$H_i(D'\varphi(P_0), q_i) = F_A(D\varphi(P_0))$$

where we have used the fact that $H_i(D'\varphi(P_0), +\infty) = +\infty$. Then we can construct a test function $\tilde{\varphi}$ satisfying $\tilde{\varphi} \geq \varphi$ in a neighborhood of P_0, with equality at P_0, such that

$$\begin{array}{rl} \tilde{\varphi}_t(P_0) & = \varphi_t(P_0) \\ D'\tilde{\varphi}(P_0) & = D'\varphi(P_0) \end{array} \quad \text{and} \quad \partial_i\tilde{\varphi}(P_0) = \begin{cases} \max(q_i, \partial_i\varphi(P_0)) & \text{if} \quad i \in I, \\ \partial_i\varphi(P_0) & \text{if} \quad i \notin I. \end{cases}$$

Using the fact that $F_A(D\varphi) = F_A(D\tilde{\varphi}) \leq H_i(D'\tilde{\varphi}, \partial_i\tilde{\varphi})$ at P_0 for all i, we deduce a contradiction with (2.8) using the viscosity inequality for $\tilde{\varphi}$ for some $i \in \{1, \ldots, N\}$. □

2.3 General junction conditions reduce to flux-limited ones

We have the following result which implies immediately Theorem 1.1.

Theorem 2.13 (General junction conditions reduce to F_A)**.** *Let the Hamiltonians satisfy* (1.5) *and let $F : \mathbb{R}^N \to \mathbb{R}$ satisfy* (1.10). *There exists a unique coercive continuous function $A_F : \mathbb{R}^d \to \mathbb{R}$, satisfying $A_F \geq A_0$ with A_0 defined in (1.9), such that the following holds.*

i) Every relaxed viscosity super-solution (resp. sub-solution satisfying moreover the "weak continuity" property (2.4)) of (1.3) *is a A_F-flux limited super-solution (resp. sub-solution) of* (1.8).

ii) Reciprocically, every A_F-flux limited super-solution (resp. sub-solution) of (1.8)*, is a F-relaxed viscosity super-solution (resp. sub-solution) of* (1.3).

iii) Moreover, if F is quasi-convex, so is A_F.

We prove Theorem 2.13.

Proof of Theorem 2.13. With the notation of Remark 1.2 in hand, we first recall that if $F(p', p^0) \geq A_0(p')$, then there exists only one $\lambda \geq A_0(p')$ such that there exists $p^+ = (p_1^+, \dots, p_N^+)$ with $p_i^+ \geq p_i^0$ such that

$$H_i(p', p_i^+) = F(p', p^+) = \lambda.$$

The coercivity of A_F is a direct consequence of the fact that $A_F \geq A_0$. We thus prove next that A_F is continuous. Consider a sequence $(p'_n)_n$ converging towards p'. Then we have two cases.

Case 1n

There exists $p_n^+ = (p_{1,n}^+, \dots, p_{N,n}^+)$ with $p_{i,n}^+ \geq p_i^0 = p_i^0(p'_n)$ such that

$$H_i(p'_n, p_{i,n}^+) = F(p'_n, p_n^+) = A_n = A_F(p'_n) \geq A_0(p'_n) \quad \text{if} \quad F(p'_n, p^0(p'_n)) \geq A_0(p'_n). \tag{2.9}$$

Case 2n

$$A_n = A_0(p'_n) = A_F(p'_n) \quad \text{if} \quad F(p'_n, p^0(p'_n)) \leq A_0(p'_n).$$

We first claim that $(p_{i,n}^+)_n$ is bounded. Indeed, if not, then $A_n \to +\infty$ and, for n large enough,

$$F(p'_n, p^0(p'_n)) \geq A_n$$

which is impossible. The claim also implies that $(A_n)_n$ is also bounded. Consider now to converging subsequences, still denoted by $(p_n^+)_n$ and $(A_n)_n$, and let p^+ and A be their limits.

Case 1∞

We can pass to the limit in (2.9) and get

$$H_i(p', p_i^+) = F(p', p^+) = A \geq A_0(p')$$

with $p_i^+ \geq p_i^0(p')$ and then $A = A_F(p')$.

Case 2∞

We get

$$A = A_0(p')$$

Subcase 2∞.1: $F(p', p^0(p')) \leq A_0(p')$

Then $A_F(p') = A_0(p') = A$.

Subcase 2∞.2: $F(p', p^0(p')) > A_0(p')$

Then we have to enter in more details in the results of the limit process. We get

$$F(p', \bar{p}^0) \leq A_0(p') \quad \text{and} \quad A = A_0(p') = H_i(p', \bar{p}_i^0) \quad \text{where} \quad \bar{p}_i^0 \geq \pi_i^0(p')$$

with

$$\bar{p}^0 = \lim\ p^0(p'_n) \quad \text{for a subsequence}$$

which implies $\bar{p}_i^0 \geq p_i^0(p')$. Then we can choose some $p_i^+ \in [p_i^0(p'), \bar{p}_i^0]$ such that

$$H_i(p', p_i^+) = F(p', p^+) = A_0(p') = A$$

which shows again that $A_F(p') = A$.

This ends the proof that A_F is contiuous.

Proof of i)

We only do the proof for sub-solutions since the proof for super-solutions follows along the same lines. Let φ be a test function touching u^* from above at $P_0 = (t_0, X_0)$. We only need to consider the case where $X_0 \in \Gamma$. From Proposition 2.7, we can also assume that

$$\varphi(t, X) = \phi(t, x') + \phi_0(x)$$

with

$$D'\phi(t_0, x'_0) = p'_0 \quad \text{and} \quad \partial_i \phi_0(0) = \pi_i^+(p'_0, A_F(p'_0)).$$

We have

$$\varphi_t(P_0) + \min(F(D\varphi(P_0)), \min_i H_i(D'\varphi(P_0), \partial_i\varphi(P_0)) \leq 0$$

which yields

$$\varphi_t(P_0) + \max(F(p_0', \pi^+(p_0', A_F(p_0'))), A_F(p_0')) \leq 0.$$

In view of the definition of A_F, we get

$$\varphi_t(P_0) + A_F(p_0') \leq 0.$$

Now compute

$$F_{A_F}(D\varphi(P_0)) = \max(A_F(p_0'), \max_i H_i^-(p_0', \pi_i^+(p_0', A_F(p_0'))) = A_F(p_0').$$

This ends the proof of i).

Proof of ii)
We only do the proof for super-solutions since the proof for sub-solutions follows along the same lines. Let φ be a test function touching u_* from below at $P_0 = (t_0, X_0)$. We want to show that it is a F-relaxed viscosity supersolution, i.e.

$$\max(F(D\varphi(P_0)), \max_i H_i(D'\varphi(P_0), \partial_i\varphi(P_0)) \geq \lambda := -\varphi_t(P_0) \tag{2.10}$$

We set

$$D\varphi(P_0) = (p_0', p) \quad \text{with} \quad p = (p_1, \ldots, p_N)$$

We know that u is a F_A-reduced viscosity solution with $A = A_F$, i.e.

$$\max(A_F(p_0'), \max_i H_i^-(p_0', p_i)) = F_{A_F}(D\varphi(P_0)) \geq \lambda \tag{2.11}$$

Moreover, we have

$$F(p_0', \pi^+(p_0', A_F(p_0'))) = A_F(p_0') > A_0(p_0') \tag{2.12}$$

or

$$A_F(p_0') = A_0(p_0') \tag{2.13}$$

We now distinguish two cases.

Case 1
Assume first that there exists an index i_0 such that $H_{i_0}(p_0', p_{i_0}) \geq \max(A_F(p_0'), \max_i H_i(p_0', p_i))$. Then (2.11) implies the result (2.10).

Case 2
Assume that for all i, we have $H_i(p_0', p_i) < A_F(p_0')$. Then $p_i < \pi_i^+(p_0', A_F(p_0'))$ and $F(p_0', p_i) \geq F(p_0', \pi^+(p_0', A_F(p_0')) = A_F(p_0') \geq \lambda$ in case of (2.12).
In the case of (2.13), we have $A_F(p_0') = A_0(p_0')$ and the inequality for all i

$$H_i(p_0', p_i) < A_F(p_0') = A_0(p_0') = \max_j \left(\min_{q_j} H_j(p_0', q_j) \right)$$

leads to a contradiction.
The proof of ii) is now complete.

Proof of iii)
It follows from Proposition 2.14 below.
The proof is now complete. □

We now turn to the following useful proposition.

Proposition 2.14 (Quasi-convex Hamiltonians and flux functions generate quasi-convex flux limiters)**.** *If the Hamiltonians H_i satisfy* (1.5) *and the flux function F satisfies* (1.10)-(1.11)*, then A_F is continuous, quasi-convex and coercive.*

The proof of this proposition is postponed and can be found in Appendix.

2.4 Existence

Theorem 2.15 (Existence)**.** *Let $T > 0$. Assume that Hamiltonians satisfy* (1.5)*, that the junction function F satisfies* (1.10) *and that the initial datum u^0 is uniformly continuous. Then there exists a relaxed viscosity solution u of* (1.3)*-*(1.4) *in $[0,T) \times J$ and a constant $C_T > 0$ such that*

$$|u(t,X) - u^0(X)| \le C_T \quad \textit{for all} \quad (t,X) \in [0,T) \times J.$$

Moreover u is continuous.

Sketch of the proof of Theorem 2.15. Using Perron's method as in [3], we easily get existence of relaxed viscosity solutions for general junction functions F satisfying (1.10). The only new result which needs some comment is the continuity of the solution u. To this end, we first construct u (by Perron's method) as a F_A-relaxed solution with $A = A_F$ given by Theorem 1.1 and Remark 1.2. For this problem we can apply the compariton principle (Theorem 1.3) which implies both the uniqueness and the continuity of u. Using Theorem 2.13 ii), we conclude that u is also a F-relaxed viscosity solution. □

3 Vertex test function

This section is devoted to the construction of the vertex test function to be used in the proof of the comparison principle.

We will use below the following shorthand notation

$$H(X,p',p) = \begin{cases} H_i(p',p) & \text{for} \quad p = p_i & \text{if} \quad X \in J_i \setminus \Gamma, \\ F_A(p',p) & \text{for} \quad p = (p_1,...,p_N) & \text{if} \quad X \in \Gamma. \end{cases} \tag{3.1}$$

We also introduce a modulus of continuity ω_R (with obviously $\omega_R(0) = 0$), such that

$$|H(X,P) - H(X,\hat{P})| \le \omega_R(|P - \hat{P}|) \quad \text{for all} \quad |P|, |\hat{P}| \le R \tag{3.2}$$

In particular, keeping in mind the definition of Du (see (1.2)), Problem (1.8) on the junction can be rewritten as follows

$$u_t + H(X,Du) = 0 \quad \text{for all} \quad (t,X) \in (0,+\infty) \times J.$$

In the spirit of the definition of test function in (2.1), we set

$$C^1(J) = \left\{\phi \in C(J), \quad \phi \text{ restricted to } J_i \text{ is } C^1 \text{ for } i = 1,\dots,N\right\}$$

Then our key result is the following one.

Theorem 3.1 (The vertex test function)**.** *Let A satisfying* (1.6) *with $A \ge A_0$ and let $\gamma \in (0,1]$. Assume the Hamiltonians satisfy* (1.5)*. Then there exists a function $G : J^2 \to \mathbb{R}$ enjoying the following properties.*

i) (Regularity)

$$G \in C(J^2) \quad \textit{and} \quad \begin{cases} G(X,\cdot) \in C^1(J) & \textit{for all} \quad X \in J, \\ G(\cdot,Y) \in C^1(J) & \textit{for all} \quad Y \in J. \end{cases}$$

ii) (Bound from below) $G \ge 0 = G(0,0)$.

iii) (Compatibility condition on the diagonal) *For all $X \in J$,*

$$0 \le G(X,X) - G(0,0) \le \gamma. \tag{3.3}$$

iv) (Compatibility condition on the gradients) *For all* $(X,Y) \in J^2$ *with* $d(X,Y) \le K$,

$$H(Y, -D_Y G(X,Y)) - H(X, D_X G(X,Y)) \le \omega_R(\gamma C_K) \tag{3.4}$$

with $R = C_K$ *given in* (3.6) *where notation introduced in* (1.2), (3.1) *and* (3.2) *are used.*

v) (Superlinearity) *There exists* $g : [0,+\infty) \to \mathbb{R}$ *nondecreasing and s.t. for* $(X,Y) \in J^2$

$$g(d(X,Y)) \le G(X,Y) \quad \text{and} \quad \lim_{a \to +\infty} \frac{g(a)}{a} = +\infty. \tag{3.5}$$

vi) (Gradient bounds) *For all* $K > 0$, *there exists* $C_K > 0$ *such that for all* $(X,Y) \in J^2$,

$$d(X,Y) \le K \quad \Longrightarrow \quad |G_X(X,Y)| + |G_Y(X,Y)| \le C_K. \tag{3.6}$$

We now assert that Theorem 1.3 is a direct consequence of Theorem 3.1.

Proof of Theorem 1.3. Use Theorem 3.1 and proceed as in [3] (indeed the modification of estimate (3.4) with respect to the corresponding one in [3], does not affect the arguments of the proof). □

3.1 The case of smooth convex Hamiltonians

Assume that the Hamiltonians H_i satisfy the following assumptions for $i = 1, ..., N$,

$$\begin{cases} H_i \in C^2(\mathbb{R}^{d+1}) \quad \text{with} \quad D^2 H_i > 0 \quad \text{in} \quad \mathbb{R}^{d+1}, \\ \lim_{|P| \to +\infty} \frac{H_i(P)}{|P|} = +\infty \end{cases} \tag{3.7}$$

and the flux limiter

$$A_0 \le A \in C^2(\mathbb{R}^d) \quad \text{and } D^2 A > 0 \quad \text{in} \quad \mathbb{R}^{d+1}. \tag{3.8}$$

It is useful to associate with each H_i satisfying (3.7) its partial inverse functions $\pi_i^\pm$:

$$\text{for } \lambda \ge A_i(p'), \quad H_i(p', \pi_i^\pm(p',\lambda)) = \lambda \quad \text{such that} \quad \pi_i^-(p',\lambda) \le \pi_i^0(p') \le \pi_i^+(p',\lambda) \tag{3.9}$$

where we recall that $A_i(p') = \min_{p_i} H_i(p', p_i)$. is convex in p' (see Lemma A.1).

Lemma 3.2 (Properties of $\pi_i^\pm$)**.** *Assume* (3.7). *Then* $\pi_i^\pm(p', \cdot) \in C^2(A_i(p'), +\infty)$ *and* $\pi_i^\pm \in C(\operatorname{epi} A_i)$. *Moreover,* $\pi_i^\pm$ *is concave w.r.t.* (p',λ) *in* $\operatorname{epi} A_i$ *and* $\pm\pi_i^\pm$ *is non-decreasing w.r.t.* λ.

Proof. The regularity of $\pi^\pm$ can be derived thanks to the inverse function theorem. As far as the concavity of π_i^+ is concerned, we can drop the subscript i and we do so for clarity. let $(p',\lambda), (q',\mu) \in \operatorname{epi} A$ and $t \in (0,1)$. Then

$$\begin{aligned} t\lambda + (1-t)\mu &= tH(p', \pi^+(p',\lambda)) + (1-t)H(q', \pi^+(q',\mu)) \\ &\ge H(tp' + (1-t)q', t\pi^+(p',\lambda) + (1-t)\pi^+(q',\mu)). \end{aligned}$$

Hence

$$\pi^+(tp' + (1-t)q', t\lambda + (1-t)\mu) \ge t\pi^+(p',\lambda) + (1-t)\pi^+(q',\mu)$$

which is the desired result. The monotonicity of π^+ is easy to derive from the convexity of H. The proof of the lemma is now complete. □

We next define the function G^0 for $X \in J_i, Y \in J_j$, $i,j = 1, ..., N$, as follows,

$$G^0(X,Y) = \sup_{(P,\lambda) \in \mathcal{G}_A^{ij}} (p' \cdot (x' - y') + p_i x - p_j y - \lambda) \tag{3.10}$$

where

$$\mathcal{G}_A^{ij} = \begin{cases} \{(P,\lambda) \in \mathbb{R}^{d+3} \times \mathbb{R} : P = (p', p_i, p_j), \lambda = H_i(p', p_i) = H_j(p', p_j) \ge A(p')\} & \text{if } i \ne j \\ \{(P,\lambda) \in \mathbb{R}^{d+2} \times \mathbb{R} : P = (p', p_i), \lambda = H_i(p', p_i) \ge A(p')\} & \text{if } i = j \end{cases} \tag{3.11}$$

with $A \ge A_0$.

Proposition 3.3 (The vertex test function – the smooth convex case)**.** *Let $A \geq A_0$ with A_0 given by* (1.9) *and assume that the Hamiltonians satisfy* (3.7) *and the limiter A satisfies (3.8). Then G^0 satisfies*

i) (Regularity)

$$G^0 \in C(J^2) \quad and \quad \begin{cases} G^0 \in C^1(\{(X,Y) \in J \times J, \quad x \neq y\}), \\ G^0(0,\cdot) \in C^1(J) \quad and \quad G^0(\cdot,0) \in C^1(J); \end{cases}$$

ii) (Bound from below) $G^0 \geq G^0(0,0)$;

iii) (Compatibility conditions) (3.3) *holds with $\gamma = 0$; and* (3.4) *holds with $\gamma = 0$ for $X = (x', x)$, $Y = (y', y)$ with $x \neq y$ or $x = y = 0$;*

iv) (Superlinearity) (3.5) *holds for some $g = g^0$;*

v) (Gradient bounds) (3.6) *holds only for $(X,Y) \in J^2$ such that $x \neq y$ or $(x,y) = (0,0)$;*

The proof of this proposition is postponed until Subsection 3.4. With such a result in hand, we can now prove Theorem 3.1 in the case of smooth convex Hamiltonians.

Lemma 3.4 (The case of smooth convex Hamiltonians)**.** *Assume that the Hamiltonians satisfy* (3.7) *and the limiter A satisfies (3.8) with $A \geq A_0$. Then the conclusion of Theorem 3.1 holds true.*

Proof. Recall that

$$G^0_{ii}(X,Y) = \mathfrak{G}_{ii}(Z) \quad \text{with} \quad Z = X - Y.$$

Up to substract $G^0(0,0)$ to G^0, we can assume that $G^0(0,0) = 0$. It is enough (and it is our goal) to regularize G^0_{ii} in a neighborhood of $\{x_i = y_i\} \setminus \{x_i = y_i = 0\}$. Let $\varepsilon_0 \in (0,1]$ small to fix later, and consider a smooth nondecreasing function $\zeta : \mathbb{R} \to [0,1]$ satisfying $\zeta = 0$ on $(-\infty, 0]$, $\zeta > 0$ on $(0,+\infty)$, and $\zeta = 1$ on $[B,+\infty)$, with $B \geq 1$ large. We also consider a smooth nonincreasing function $\xi : [0,+\infty) \to (0,+\infty)$ with $\xi(+\infty) = 0$, which satisfies in particular for $Z = (z', z_i)$ and a real $\bar{z}_i$

$$|\mathfrak{G}_{ii}(z', z_i) - \mathfrak{G}_{ii}(z', \bar{z}_i)| \leq \frac{|z_i - \bar{z}_i|}{\xi(|z'|)} \quad \text{if} \quad |z_i|, |\bar{z}_i| \leq 2\xi(|z'|)$$

We will regularize G^0_{ii} in a neighborhood of the diagonal of half thickness $\varepsilon_0 \theta$ with

$$\theta(z', x_i + y_i) := \xi(|z'|)\zeta(x_i + y_i)$$

To this end, we consider a smooth cut-off function $\Psi : \mathbb{R} \to [0,1]$ such that $\operatorname{supp}\Psi \subset [-1,1]$ with $\Psi = 1$ on $[-1/2, 1/2]$. We will also use a one-dimensional non-negative mollifier

$$\rho_\eta(z_i) = \frac{1}{\eta}\rho(\frac{z_i}{\eta})$$

with $\operatorname{supp}\rho \subset [-1,1]$ to regularize by convolution the function $\mathfrak{G}_{ii}(Z)$ in the direction of z_i only, because $\mathfrak{G}_{ii}(Z)$ is already C^1 in the other directions z'. Finally we define with $Z = (z', z_i)$ and $z' = x' - y'$, $z_i = x_i - y_i$, the function

$$G_{ii}(X,Y) = \left(1 - \Psi\left(\frac{z_i}{\varepsilon_0 \theta(z', x_i + y_i)}\right)\right)\mathfrak{G}_{ii}(z', z_i) + \Psi\left(\frac{z_i}{\varepsilon_0 \theta(z', x_i + y_i)}\right)\int_{a \in \mathbb{R}} \rho_{\varepsilon_0 \theta(z', x_i + y_i)}(a)\mathfrak{G}_{ii}(z', z_i - a).$$

This regularization procedure preserves the desired properties like estimates (3.5) (with a possible different function g but independent on any $\varepsilon_0 \in (0,1]$) and (3.6) with a possible different constant C_K. Moreover, for $\varepsilon_0 > 0$ small enough, this regularization procedure introduces a small error γ in (3.3) and another small error γ in (3.4). This ends the proof of the lemma. □

3.2 The vertex test function in $J_i \times J_j$ with $i \neq j$

In order to prove Proposition 3.3, we first need to study G^0 for $X \in J_i$ and $Y \in J_j$ with $i \neq j$. Then, one can write

$$G^0_{ij}(X,Y) = \mathfrak{G}_{ij}(x' - y', x_i, -y_j)$$

with

$$\mathfrak{G}_{ij}(Z) = \sup_{(P,\lambda)\in\mathcal{G}^{ij}_A} (P \cdot Z - \lambda)$$

where $\mathcal{G}^{ij}_A$ is defined in (3.11). Remark that for $X \in J_i$ and $Y \in J_j$, we have $Z = X - Y \in \mathcal{Q}$ where

$$\mathcal{Q} = \mathbb{R}^d \times [0, +\infty[\times] - \infty; 0].$$

We also consider the simplex

$$\mathcal{T} = \{(\alpha_i, \alpha_j, \alpha_0) \in [0,1]^3 : \alpha_i + \alpha_j + \alpha_0 = 1\}.$$

Lemma 3.5 (Necessary conditions for the maximiser : ij-version)**.** *Given $Z \in \mathcal{Q}$, the supremum defining $\mathfrak{G}_{ij}(Z)$ is reached for some $(P,\lambda) \in \mathcal{G}^{ij}_A$ and there exists $(\alpha_i, \alpha_j, \alpha_0) \in \mathcal{T}$ such that*

$$Z = D(\alpha \cdot H)(P)$$

with $H = (H_i, H_j, A)$.

Proof. $\mathfrak{G}_{ij}(Z)$ is defined by maximizing a linear function under a equality constraint and an inequality constraint. Constraints are qualified if

$$D(H_i - H_j) \text{ is not colinear with } D(H_i - A).$$

When constraints are qualified, Karush-Kuhn-Tucker theorem asserts (computing $D_P(P \cdot Z - \lambda)$) that there exists $\alpha_j \in \mathbb{R}$ and $\alpha_0 \geq 0$ such that

$$Z = \nabla_P H_i + \alpha_j(\nabla_P H_j - \nabla_P H_i) + \alpha_0 \nabla_P (A - H_i)$$

with

$$\alpha_0 = 0 \quad \text{ if } A(p') < H_i(p', p_i).$$

If one sets $\alpha_i = 1 - \alpha_0 - \alpha_j$, Equivalently, we have

$$\begin{cases} z_i = \alpha_i \partial_i H_i(p', p_i) \geq 0 \\ z_j = \alpha_j \partial_j H_j(p', p_i) \leq 0 \\ z' = \alpha_i \nabla_{p'} H_i + \alpha_j \nabla_{p'} H_j + \alpha_0 \nabla_{p'} A \end{cases}$$

The constraints are qualified in particular if

$$\partial_i H_i(p', p_i) > 0 \text{ and } \partial_j H_j(p', p_j) < 0. \tag{3.12}$$

In this case we deduce that $(\alpha_i, \alpha_j, \alpha_0) \in \mathcal{T}$. Hence, the result is proved in case (3.12).

Now assume that $\partial_i H_i(p', p_i) \leq 0$. We remark that in all cases, $\partial_i H_i(p', p_i) \geq 0$ since $z_i \geq 0$. Hence, $\partial_i H_i(p', p_i) = 0$ or, in other words, $H_i(p', p_i) = A_i(p')$. But the constraint $H_i(p', p_i) \geq A(p')$, the assumption $A(p') \geq A_0(p')$ and the simple fact that $A_i(p') \leq A_0(p')$ imply in particular that $A(p') = A_0(p')$. We arrive at the same conclusion if $\partial_j H_j(p', p_j) \geq 0$. In other words,

$$\text{Condition (3.12) holds true as soon as} \quad \forall p', \; A(p') > A_0(p'). \tag{3.13}$$

In particular, the result of the lemma holds true under this latter condition: $A(p') > A_0(p')$ for all $p' \in \mathbb{R}^d$. If now there are some p' such that $A(p') = A_0(p')$, we remark that

$$\mathfrak{G}_{ij}(Z) = \lim_{\varepsilon \to 0} \mathfrak{G}^\varepsilon_{ij}(Z)$$

where $\mathfrak{G}^\varepsilon_{ij}(Z)$ is associated with $A^\varepsilon(p') = \varepsilon + A(p')$. From the previous case, we know that there exists P_ε and λ_ε such that

$$\mathfrak{G}^\varepsilon_{ij}(Z) = P_\varepsilon \cdot Z - \lambda_\varepsilon$$

and $\alpha^\varepsilon = (\alpha^\varepsilon_i, \alpha^\varepsilon_j, \alpha^\varepsilon_0) \in \mathcal{T}$ such that

$$Z = D(\alpha \cdot H)(P_\varepsilon).$$

We can extract a subsequence such that $\alpha^\varepsilon \to \alpha$. Moreover, $P_\varepsilon \cdot Z - \lambda_\varepsilon$ is bounded from above and

$$\lambda_\varepsilon = H_i(p'^\varepsilon, p^\varepsilon_i) = H_j(p'^\varepsilon, p^\varepsilon_j).$$

Since H_i and H_j are assumed to be superlinear, we conclude that we can also extract a converging subsequence from P_ε. This achieves the proof of the lemma. □

Lemma 3.6 (Uniqueness of P : ij-version)**.** *Let $Z = (z', z_i, z_j) \in \mathcal{Q}$. If there exists α, P, λ and β, Q, μ such that $\alpha, \beta \in \mathcal{T}$ and*

$$\begin{cases} \mathfrak{G}_{ij}(Z) = P \cdot Z - \lambda = Q \cdot Z - \mu, \\ Z = D(\alpha \cdot H)(P) = D(\beta \cdot H)(Q). \end{cases}$$

Then $\lambda = \mu$, $p' = q'$ and

$$p_i = q_i = \pi^+_i(p', \lambda) \tag{3.14}$$

except in the case

$$\alpha_i = \beta_i = 0 = z_i \tag{3.15}$$

and

$$p_j = q_j = \pi^-_j(p', \lambda) \tag{3.16}$$

except in the case

$$\alpha_j = \beta_j = 0 = z_j \tag{3.17}$$

Moreover under the previous assulptions, and in all cases, we can define

$$\hat{P} = (p', \pi^+_i(p', \lambda), \pi^-_j(p', \lambda))$$

and then we have

$$\mathfrak{G}_{ij}(Z) = \hat{P} \cdot Z - \lambda \quad \textit{and} \quad Z = D(\alpha \cdot H)(\hat{P})$$

Proof. We consider the function $\Psi : \mathbb{R}^{d+2} \times \mathcal{T} \to \mathbb{R}$ defined as follows

$$\Psi(P, \alpha) = D(\alpha \cdot H)(P).$$

By assumption, we have

$$0 = D(\alpha \cdot H)(P) - D(\beta \cdot H)(Q).$$

If $\bar{P}$ denotes $Q - P$ and $\bar{\alpha}$ denotes $\beta - \alpha$, then

$$\begin{aligned} 0 &= \int_0^1 \begin{pmatrix} \bar{P} \\ \bar{\alpha} \end{pmatrix} \cdot D\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha}) d\theta \\ &= \int_0^1 D_P\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha})\bar{P} d\theta + \int_0^1 D_\alpha\Psi(P + \theta\bar{P}, \alpha + \theta\bar{\alpha})\bar{\alpha} d\theta. \end{aligned}$$

Taking the scalar product with $\bar{P}$ yields

$$\begin{aligned} 0 &= \int_0^1 D^2_{PP}((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P} d\theta + \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P} d\theta \\ &= T_1 + T_2 \end{aligned}$$

with $T_i \geq 0$, $i = 1, 2$ and

$$T_1 = \int_0^1 D^2_{PP}((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P} d\theta \geq 0$$
$$T_2 = \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P} d\theta \geq 0.$$

Indeed, keeping in mind that

$$\begin{cases} H_i(P) = H_j(P) \\ H_i(Q) = H_j(Q) \end{cases} \quad \text{and} \quad \begin{cases} \alpha_0(A(P) - H_i(P)) = 0 \\ \beta_0(A(Q) - H_i(Q)) = 0 \end{cases}$$

we remark that

$$\begin{aligned} \int_0^1 D_P H(P + \theta\bar{P})\bar{\alpha} \cdot \bar{P} d\theta &= \bar{\alpha} \cdot (H(Q) - H(P)) \\ &= \bar{\alpha}_i(H_i(Q) - H_i(P)) + \bar{\alpha}_j(H_j(Q) - H_j(P)) + \bar{\alpha}_0(A(Q) - A(P)) \\ &= (\beta_0 - \alpha_0)(A(Q) - H_i(Q) - A(P) + H_i(P)) \\ &= \beta_0(H_i(P) - A(P)) + \alpha_0(H_i(Q) - A(Q)) \geq 0. \end{aligned}$$

Hence, we get

$$\begin{aligned} 0 &= \int_0^1 D^2_{PP}((\alpha + \theta\bar{\alpha}) \cdot H)(P + \theta\bar{P})\bar{P} \cdot \bar{P} d\theta \\ 0 &= \beta_0(H_i(P) - A(P)) \\ 0 &= \alpha_0(H_i(Q) - A(Q)). \end{aligned}$$

We distinguish three cases. We will use several times the fact that $H_i(p', p_i) = \lambda$ and $\partial_i H_i(p', p_i) \geq 0$ implies that $p_i = \pi_i^+(p', \lambda)$. We will also use the corresponding property for p_j: $p_j = \pi_j^-(p', p_j)$.

- *Case 1.* If there exists $\theta \in (0, 1)$ such that $\alpha + \theta\bar{\alpha} \in \operatorname{int} \mathcal{T}$, then $P = Q$ and

$$\lambda = P \cdot Z - \mathfrak{G}_{ij}(Z) = \mu.$$

- *Case 2.* If $\alpha = \beta$ is a vertex of $\mathcal{T}$, then either $\alpha = (1, 0, 0)$ or $\alpha = (0, 1, 0)$ or $\alpha = (0, 0, 1)$.
  - In the first subcase, $\alpha_i = 1$, we get $p' = q'$ and $p_i = q_i$ and $Z = \nabla_P H_i(P)$ and

  $$0 = (p_j - q_j)z_j = (P - Q) \cdot Z = \lambda - \mu.$$

  We conclude by remarking that we can choose $p_j = \pi_j^-(p', \lambda) = q_j$ when $\alpha_j = \beta_j = 0 = z_j$. The second subcase is similar.
  - If now $\alpha = (0, 0, 1)$, then $p' = q'$ and $Z = \nabla_P A(P)$ and

  $$0 = (p_i - q_i)z_i + (p_j - q_j)z_j = P \cdot Z = \lambda - \mu$$

  and we conclude as in the two previous subcases.

- *Case 3.* Assume finally that there exists $\theta \in (0, 1)$ such that $\alpha + \theta\bar{\alpha} \in \partial\mathcal{T}$ but is not a vertex. In this third case, this implies that two components of $a = \alpha + \theta\bar{\alpha} = (a_i, a_j, a_0)$ are not 0.
  - If $a_0 = 0$ then $p' = q'$ and $p_i = q_i$ and $p_j = q_j$, i.e. $P = Q$.
  - If $a_i = 0$ then $p' = q'$ and $p_j = q_j$ and $z_i = 0$ and $\lambda = \mu$ and we can choose $p_i = \pi^+(p', \lambda) = q_i$ when $\alpha_i = \beta_i = 0 = z_i$. The third subcase $a_j = 0$ is similar to the second one.

The proof of the lemma is now complete. □

The two previous lemmas imply the following one.

Lemma 3.7 (Gradients of G^0_{ij})**.** *The function G^0_{ij} is C^1 in $J_i \times J_j$, up to the boundary, and*

$$DG^0_{ij}(X,Y) = (p', p_i, -p', -p_j), \quad p_i = \pi_i^+(p',\lambda), \quad p_j = \pi_j^-(p',\lambda) \quad \text{and} \quad P = (p', p_i, p_j)$$

where $(p',\lambda) = (\mathfrak{P}(X,Y), \mathfrak{L}(X,Y))$ are uniquely determined by the relation for some $\alpha \in \mathcal{T}$

$$\begin{cases} G^0_{ij}(X,Y) = p' \cdot (x' - y') + p_i x_i - p_j y_j - \lambda, \\ Z = D(\alpha \cdot H)(P) \quad \text{with} \quad Z = (x' - y', x_i, -y_j) \end{cases}$$

In particular, the maps $\mathfrak{P}$ and $\mathfrak{L}$ are continuous in $J_i \times J_j$.

The following lemma is elementary but it will be used below.

Lemma 3.8 (G^0_{ij} at the boundary)**.** *The restriction of $\mathfrak{G}_{ij}$ to $\{z_i = 0\}$ and $\{z_j = 0\}$ equals respectively $(H_j \vee A)^\star$ and $(H_i \vee A)^\star$, where the star exponent denotes here the Legendre-Fenchel transform.*

3.3 The vertex test function in $J_i \times J_i$

In view of the definition of G^0, see (3.10), we have the following Legendre-Fenchel transform equality

$$G^0_{ii}(X,Y) = (H_i \vee A)^\star (X - Y).$$

In particular, we derive from Lemma 3.8 the following one.

Lemma 3.9 (Continuity of G^0)**.** *The function G^0 is continuous in $J \times J$.*

We now state (without proof, because the proofs are even easier) the following two analogues of Lemmas 3.5 and 3.6.

Lemma 3.10 (Necessary conditions for the maximiser : ii-version)**.** *Let $\mathcal{T}_i = \{(\alpha_i, \alpha_0) \in [0,1]^2, \quad \alpha_i + \alpha_0 = 1\}$, and $\alpha \cdot H = \alpha_i H_i + \alpha_0 A$, and $Z = (z', z_i)$. If the suppremum defining $\mathfrak{G}_{ii}(Z)$ is reached at some $(P,\lambda) \in \mathcal{G}^{ii}_A$, then there exists $\alpha \in \mathcal{T}_i$ such that*

$$Z = D(\alpha \cdot H)(P)$$

Lemma 3.11 (Uniqueness of P: ii-version)**.** *Let $Z = (z', z_i) \in \mathbb{R}^{d+1}$. If there exists α, P, λ and β, Q, μ such that $\alpha, \beta \in \mathcal{T}_i$ and*

$$\begin{cases} \mathfrak{G}_{ii}(Z) = P \cdot Z - \lambda = Q \cdot Z - \mu, \\ Z = D(\alpha \cdot H)(P) = D(\beta \cdot H)(Q). \end{cases}$$

Then $\lambda = \mu$, $p' = q'$ and

$$p_i = q_i = \pi_i^+(p',\lambda) \quad \text{if} \quad z_i > 0 \tag{3.18}$$

and

$$p_i = q_i = \pi_i^-(p',\lambda) \quad \text{if} \quad z_i < 0 \tag{3.19}$$

Moreover under the previous assumptions, and in all cases, we can define either

$$\hat{P} = (p', \pi_i^+(p',\lambda)) \quad \text{if} \quad z_i \geq 0$$

or

$$\hat{P} = (p', \pi_i^-(p',\lambda)) \quad \text{if} \quad z_i \leq 0$$

and then we always have

$$\mathfrak{G}_{ij}(Z) = \hat{P} \cdot Z - \lambda \quad \text{and} \quad Z = D(\alpha \cdot H)(\hat{P})$$

We now turn to the regularity of G^0_{ii}.

Lemma 3.12 (Gradients of G^0_{ii})**.** *G^0_{ii} is C^1 in $J_i \times J_i \setminus \{x_i = y_i > 0\}$. For $(X,Y) \in J_i \times J_i$ such that $x_i \neq y_i$, we have*

$$DG^0_{ii}(X,Y) = (p', p_i, -p', -p_i) \quad \textit{and} \quad P = (p', p_i)$$

with $p_i = \pi_i^\pm(p', \lambda)$ if $\pm(x_i - y_i) > 0$. Here $(p', \lambda) = (\mathfrak{P}(X,Y), \mathfrak{L}(X,Y))$ is uniquely determined by

$$\begin{cases} G^0_{ii}(X,Y) = p' \cdot (x' - y') + p_i(x_i - y_i) - \lambda \\ Z = \alpha_i DH_i(P) + (1-\alpha_i) DA(P) \quad \textit{with} \quad Z = (x' - y', x_i - y_i) \end{cases}$$

which holds true for some $\alpha_i \in [0,1]$. In particular, the maps $\mathfrak{P}$ and $\mathfrak{L}$ are continuous in $J_i \times J_i$. Moreover the restrictions of G^0_{ii} to $(J_i \times J_i) \cap \{\pm(x_i - y_i) \geq 0\}$ are C^1 and

$$G^0_{ii}(x', 0, y', 0) = p' \cdot (x' - y') - \lambda$$

with

$$DG^0_{ii}(x', 0, y', 0) = (p', \pi_i^+(p', \lambda), -p', -\pi_i^-(p', \lambda))$$

3.4 Proof of Proposition 3.3

We now turn to the proof of Proposition 3.3.

Proof of Proposition 3.3. The proof proceeds in several steps.

Step 1: Regularity. We already noticed in Lemma 3.9 that $G^0 \in C(J^2)$ and Lemmas 3.7 and 3.12 imply that $G^0 \in C^1(\mathcal{R})$ for each region $\mathcal{R}$ given by

$$\mathcal{R} = \begin{cases} J_i \times J_j & \text{if} \quad i \neq j, \\ T_i^\pm = \{(X,Y) \in J_i \times J_i, \quad \pm(x_i - y_i) \geq 0\} & \text{if} \quad i = j. \end{cases} \tag{3.20}$$

Step 2: Computation of the gradients. For each $\mathcal{R}$ given by (3.20) and for all $(X,Y) \in \mathcal{R} \subset J_i \times J_j$, Lemmas 3.7 and 3.12 imply that

$$G^0(X,Y) = p' \cdot (x' - y') + p_i x_i - p_j y_j - \lambda$$

and

$$(D', \partial_i) G^0_{|\mathcal{R}}(X,Y) = (p', p_i) \quad \text{and} \quad -(D', \partial_j) G^0_{|\mathcal{R}}(X,Y) = (p', p_j)$$

with $\lambda = \mathfrak{L}(X,Y)$ and $p' = \mathfrak{P}(X,Y)$ with

$$(p_i, p_j) = \begin{cases} (\pi_i^+(p', \lambda), \pi_j^-(p', \lambda)) & \text{if} \quad \mathcal{R} = J_i \times J_j & \text{with} \quad i \neq j, \\ (\pi_i^\pm(p', \lambda), \pi_i^\pm(p', \lambda)) & \text{if} \quad \mathcal{R} = T_i^\pm & \text{with} \quad i = j. \end{cases} \tag{3.21}$$

Notice in particular that $\mathfrak{P}$ and $\mathfrak{L}$ are continuous in $J \times J$. We also easily deduce that $G^0(X,Y) \geq G^0(X,X) = G^0(0,0)$.

Step 3: Checking the compatibility condition on the gradients. Let us consider $(X,Y) \in J^2$, $X = (x', x)$, $Y = (y', y)$ with $x = y = 0$ or $x \neq y$. We have

$$\begin{aligned} D_X(G^0(\cdot, Y))(X) &\in \{(p', \pi_i^\pm(\lambda))\} \\ -(D_Y G^0(X, \cdot))(Y) &\in \{(p', \pi_j^\pm(\lambda))\} \end{aligned}$$

with $\lambda \geq A(p')$. We claim that

$$H(X, D_X G^0(X,Y)) = \lambda \quad \text{for} \quad N \geq 1 \tag{3.22}$$

and

$$H(Y, -D_Y G^0(X,Y)) \le \lambda \quad \text{for} \quad N \ge 1 \tag{3.23}$$

with equality for $N \ge 2$ (we use here once again the short hand notation (3.1).

Equality (3.22) is clear except if $x = 0$. In this case, if $y \neq 0$, say $Y \in J_j$, the desired equality is rewritten as

$$\max(A(p'), \max_i H_i^-(p', p_i)) = \lambda$$

with $p_i = \pi_i^+(p', \lambda)$ if $i \neq j$ and $p_j = \pi_j^-(p', \lambda)$. Since $\lambda \ge A(p')$ and $H_j^-(p', p_j) = \lambda$, we get the result for $N \ge 2$. For $N = 1$, we have $x - y < 0$ and then $p_i = \pi_i^-(p', \lambda)$ which gives again the result. If now $(x, y) = (0, 0)$, then $p_i = \pi_i^+(p', \lambda)$ for all index i and $\lambda = A(p') \ge A_0(p')$. Hence, we get (3.22) in this case too.

One can derive (3.23) in the same way, even with equality for $N \ge 2$. For $N = 1$, where $y = 0$, $X = (x', x_i) \in J_i^*$, i.e. $x_i - y_i > 0$, this gives $p_i = \pi_i^+(p', \lambda)$, and we only get

$$H(Y, -D_Y G^0(X,Y)) = \max(A(p'), \min H_i(p', \cdot)) \le \lambda$$

with a strict inequality (for $\lambda > A(p')$). On the other hand, we recover equality for $y \neq 0$.

Step 4: Superlinearity. In view of the definition of G^0, we deduce from (3.21) that for all $R > 0$ and $\lambda > A(R(x' - y')/|x' - y'|)$,

$$G^0(X,Y) \ge R|x' - y'| + \begin{cases} x\pi_i^+(R\widehat{x' - y'}, \lambda) - y\pi_j^-(R\widehat{x' - y'}, \lambda) - \lambda & \text{if} \quad i \neq j, \\ (x - y)\pi_i^\pm(R\widehat{x' - y'}, \lambda) - \lambda & \text{if} \quad i = j, \pm(x - y) \ge 0 \end{cases}$$

where $\hat{z} = z/|z|$. For $R > 0$, we define

$$\pi^0(R, \lambda) := \min_{\pm,\ i=1,\dots,N, |p'| \le R} \pm\pi_i^\pm(p', \lambda) \ge 0.$$

Hence we get

$$G^0(X,Y) \ge R|x' - y'| + \pi^0(R, \lambda) d(x, y) - \lambda$$

where

$$d(x,y) = \begin{cases} |x_i - y_i| & \text{if } X, Y \in J_i \\ x_i + y_j & \text{if } X \in J_i, Y \in J_j, i \neq j. \end{cases}$$

From the definition (3.9) of $\pi_i^\pm$ and the assumption (3.7) on the Hamiltonians, we deduce that

$$\pi^0(R, \lambda) \to +\infty \quad \text{as} \quad \lambda \to +\infty$$

and fix some $\lambda(R) \ge \sup_{|p'| \le R} A(p')$ such that $\pi^0(R, \lambda(R)) \ge R$. This gives

$$G^0(X,Y) \ge R d(X,Y) - \lambda(R).$$

Therefore we get (3.5) with

$$g^0(a) = \sup_{R \ge 0} (Ra - \lambda(R)).$$

Step 5: Gradient bounds. Because each component of the gradients of G^0 are equal to one of the $\left\{(p', \pi_k^\pm(p', \lambda))\right\}_{\pm, k=1,\dots,N}$ with $\lambda = \mathfrak{L}(X,Y)$ and $p' = \mathfrak{P}(X,Y)$, we deduce (3.6) from the continuity of $\mathfrak{L}$, $\mathfrak{P}$ and $\pi_k^\pm$. We use in particular the fact that $\mathfrak{L}$ and $\mathfrak{P}$ only depend on $x' - y'$ and $x_i - y_i$ if $X, Y \in J_i$; and $x' - y'$ and $(x_i, -y_j)$ if $X \in J_i$, $Y \in J_j$ with $i \neq j$. $\square$

3.5 The general case

Let us consider a slightly stronger assumption than (1.5), namely

$$\begin{cases} H_i \in C^2(\mathbb{R}^{d+1}) \quad \text{with} \quad \min H_i = H_i(P_i^0) \quad \text{and} \quad D^2 H_i(P_i^0) > 0, \\ D^2 H > 0 \quad \text{on} \quad (DH_i)^\perp, \quad \text{and} \quad DH_i(P) \neq 0 \quad \text{for} \quad P \neq P_i^0 \\ \lim_{|P| \to +\infty} H_i(P) = +\infty. \end{cases} \tag{3.24}$$

Notice that the second line basically says that the sub-level sets are strictly convex. The following technical result will allow us to reduce a large class of quasi-convex Hamiltonians to convex ones.

Lemma 3.13 (From quasi-convex to convex Hamiltonians)**.** *Given Hamiltonians H_i satisfying* (3.24)*, there exists a function $\beta : \mathbb{R} \to \mathbb{R}$ such that the functions $\beta \circ H_i$ satisfy* (3.7) *for $i = 1, ..., N$. Moreover, we can choose β such that*

$$\beta \quad \textit{is convex}, \quad \beta \in C^2(\mathbb{R}) \quad \textit{and} \quad \beta' \geq \delta > 0. \tag{3.25}$$

Proof. In view of (3.24), it is easy to check that $D^2(\beta \circ H_i) > 0$ if and only if we have

$$0 < \{(\ln \beta')'(\lambda)\} \left(\widehat{DH_i} \times \widehat{DH_i}\right) \circ \pi_i^\pm(p', \lambda) + \frac{D^2 H_i}{|DH_i|^2} \circ \pi_i^\pm(p', \lambda) \quad \text{for} \quad \lambda > H_i(P_i^0), \quad p' \in \mathbb{R}^d. \tag{3.26}$$

Because $D^2 H_i(p_i^0) > 0$, we see that the right hand side is positive for λ close enough to $H_i(P_i^0)$. Then it is easy to choose a function β satisfying (3.26) and (3.25) (looking at each level set $\{H_i = \lambda\}$). Finally, compositing β with another convex increasing function which is superlinear at $+\infty$ if necessary, we can ensure that $\beta \circ H_i$ superlinear. □

Lemma 3.14 (The case of smooth Hamiltonians)**.** *Theorem 3.1 holds true if the Hamiltonians satisfy* (3.24)*.*

Proof. We assume that the Hamiltonians H_i satisfy (3.24). Let β be the function given by Lemma 3.13. If u solves (1.8) on J_T, then u is also a viscosity solution of

$$\begin{cases} \bar{\beta}(u_t) + \hat{H}_i(Du) = 0 \quad \text{for} \quad t \in (0,T) \qquad \text{and} \quad X \in J_i^*, \\ \bar{\beta}(u_t) + \hat{F}_{\hat{A}}(Du) = 0 \quad \text{for} \quad t \in (0,T) \qquad \text{and} \quad X \in \Gamma \end{cases} \tag{3.27}$$

with $\hat{F}_{\hat{A}}$ constructed as F_A where H_i and A are replaced with $\hat{H}_i$ and $\hat{A}$ defined as follows

$$\hat{H}_i = \beta \circ H_i, \quad \hat{A} = \beta(A)$$

and $\bar{\beta}(\lambda) = -\beta(-\lambda)$. We can then apply Theorem 3.1 in the case of smooth convex Hamiltonians to construct a vertex test function $\hat{G}$ associated to problem (3.27) for every $\hat{\gamma} > 0$. This means that we have with $\hat{H}(X,P) = \beta(H(X,P))$,

$$\hat{H}(Y, -D_Y G) \leq \hat{H}(X, D_X G) + \hat{\gamma}.$$

This implies

$$H(Y, -D_Y G) \leq \beta^{-1}(\beta(H(X, D_X G)) + \hat{\gamma}) \leq H(X, D_X G) + \hat{\gamma} |(\beta^{-1})'|_{L^\infty(\mathbb{R})}.$$

Because of the lower bound on β' given by Lemma 3.13, we get $|(\beta^{-1})'|_{L^\infty(\mathbb{R})} \leq 1/\delta$ which yields the compatibility condition (3.4) with $\gamma = \hat{\gamma}/\delta$ arbitrarily small. □

We are now in position to prove Theorem 3.1 in the general case.

Proof of Theorem 3.1. Let us now assume that the Hamiltonians only satisfy (1.5). In this case, we simply approximate the Hamiltonians H_i by other Hamiltonians $\tilde{H}_i$ satisfying (3.24) such that

$$|H_i - \tilde{H}_i| \leq \gamma.$$

We then apply Theorem 3.1 to the Hamiltonians $\tilde{H}_i$ and construct an associated vertex test function $\tilde{G}$ also for the parameter γ. We deduce that

$$H(Y, -\tilde{G}_Y) \leq H(X, \tilde{G}_X) + 3\gamma$$

with $\gamma > 0$ arbitrarily small, which shows again the compatibility condition on the Hamiltonians (3.4) for the Hamiltonians H_i's. The proof is now complete in the general case. □

A Proof of Proposition 2.14

Before proving Proposition 2.14, we state and prove the following elementary lemma.

Lemma A.1 (Quasi-convexity of the functions A_i)**.** *If the Hamiltonians H_i are quasi-convex (resp. convex), continuous and coercive, so are the functions A_i defined in (1.9). In particular, $A_0 = \max_i A_i$ is quasi-convex (resp. convex), continuous and coercive.*

Proof. We only address the question of the quasi-convexity of the functions A_i since their continuity and coercivity are simpler.

Consider p' and q' such that $A_i(p') \leq \lambda$ and $A_i(q') \leq \lambda$ for some $\lambda \in \mathbb{R}$. There exists $p_i, q_i \in \mathbb{R}$ such that

$$A_i(p') = H_i(p', p_i) \quad A_i(q') = H_i(q', q_i).$$

Then $(p', p_i), (q', q_i) \in \{H_i \leq \lambda\}$ and we conclude from the convexity of $\{H_i \leq \lambda\}$ that for $t, s \geq 0$ with $t + s = 1$,

$$A_i(tp' + sq') \leq H_i(tp' + sq', tp_i + sq_i) \leq \lambda.$$

This achieves the proof of the lemma. □

Proof of Proposition 2.14. We assume that the Hamiltonians H_i are convex, $p_i \mapsto H_i(p', p_i)$ is increasing in $[\pi_i^0(p'), +\infty)$ and decreasing in $(-\infty, \pi_i^0(p')]$ and F is convex in all variables and $p \mapsto F(p', p)$ is decreasing in each variable for every p' fixed. In particular, the functions $\pm\pi_i^{\pm}$ are concave. The general case follows by an approximation argument and by remarking that it is enough to find β increasing such that $\beta \circ F$ and $\beta \circ H_i$ satisfy the previous assumptions (see Lemma 3.13).

We now prove that

$$G(p', \lambda) = F(p', \pi^+(p', \lambda))$$

is convex w.r.t. $(p', \lambda) \in \operatorname{epi} A_0$. For $(p', \lambda), (q', \mu) \in \operatorname{epi} A_0$ and $t, s \geq 0$ with $t + s = 1$, we can use the monotonicity of F together with the concavity of π_i^+ (see Lemma 3.2) to get

$$\begin{aligned} tG(p', \lambda) + sG(q', \mu) &\geq F(tp' + sq', t\pi^+(p', \lambda) + s\pi^+(q', \mu)) \\ &\geq F(tp' + sq', \pi^+(tp' + sq', t\lambda + s\mu)) \\ &= G(tp' + sq', t\lambda + s\mu). \end{aligned}$$

Similarly, we can see that G is non-increasing with respect to λ.

We next remark that

$$A_F(p') = G(p', A_F(p'))$$

and for $p', q' \in \mathbb{R}^d$ and $t, s \geq 0$ with $t + s = 1$, we can write

$$\begin{aligned} tA_F(p') + sA_F(q') &= tG(p', A_F(p')) + sG(q', A_F(q')) \\ &\geq G(tp' + sq', tA_F(p') + sA_F(q')) \end{aligned}$$

and

$$A_F(tp' + sq') = G(tp' + sq', A_F(tp' + sq')).$$

We thus deduce from the monotonicity of G in λ that

$$A_F(tp' + sq') \leq tA_F(p') + sA_F(q').$$

The proof is now complete. □

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