A Gamma-convergence argument for the blow-up of a non-local semilinear parabolic equation with Neumann boundary conditions

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Abstract. In this paper we study a simple non-local semilinear parabolic equation with Neumann boundary condition. We give local existence result and prove global existence for small initial data. A natural non increasing in time energy is associated to this equation. We prove that the solution blows up at finite time T if and only if its energy is negative at some time before T. The proof of this result is based on a Gamma-convergence technique.

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1 Introduction

1.1 Setting of the problem

In this paper, we consider a bounded domain Ω in $\mathbb{R}^N$ that is uniformly regular of class C^2, and we study the solutions $u(x,t)$ of the following equation for some $p > 1$ (denoting the mean value $\frac{1}{|\Omega|}\int_\Omega f$ by $\fint_\Omega f$ for a general function f):

$$\begin{cases} u_t - \Delta u = |u|^p - \displaystyle\fint_\Omega |u|^p \quad \text{on} \quad \Omega \\ \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

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with the initial condition

$$\begin{cases} u(x,0) = u_0(x) \quad \text{on} \quad \Omega \\ \text{with} \quad \fint_\Omega u_0 = 0 \end{cases} \tag{1.2}$$

It is immediate to check that the integral (or the mean value) of u is conserved (at least once you precise the meaning of the solution).
Stationary solutions of Equation (1.1) are in fact critical points of the energy functional

$$E(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1} u|u|^p \right]$$

under the constraint that $\fint_\Omega u$ is equal to a given constant.
Without loss of generality, we can assume that $|\Omega| = 1$. Indeed, if u is a solution of (1.1)-(1.2) in Ω and $\lambda > 0$, then $v(t,x) := \lambda^{\frac{2}{p-1}}(\lambda^2 t, \lambda x)$ is a solution in $\lambda^{-1}\Omega$. Throughout the paper, we assume $|\Omega| = 1$, except when the volume $|\Omega|$ is explicitly mentioned to show the dependence of the constants.

1.2 Motivation of the problem

A lot of work has been done on scalar semilinear parabolic equations whose the most famous example is

$$u_t - \Delta u = u^p$$

and the problem of global existence or blow-up is quite well understood (see for instance [13, 4] for an energy criterion for blow-up, [16] for a study of self-similar blow-ups; see also [38, 32] and the numerous references therein). Of course, the Maximum Principle plays a fundamental role in the establishment of results in this setting. However, concerning the problem of describing the blow-up set, very few results are known. For instance, in dimension 2, the question of whether there exists a solution whose blow-up set is an ellipse is still unanswered. Recently, Zaag [39] established the first regularity results for the blow-up set, based on global estimates independent of the blow-up point obtained by Merle and Zaag [27] through the proof of a Liouville theorem.

In the case of parabolic systems or non-local scalar parabolic equations, even if some Maximum Principles may hold, it is often necessary to introduce new techniques. One of the most famous examples is the Navier-Stokes equation (see [20]), which can be written on the vorticity $\omega =$ curl u (with u the velocity and $e = \frac{1}{2}(\nabla u + {}^t\nabla u)$ the deformation velocity):

$$\omega_t - \nu\Delta\omega = -(u \cdot \nabla)\omega + e \cdot \omega$$

where the right hand side is non-local and quadratic in ω. If we consider this equation on $\Omega = \mathbb{R}^3 \backslash \mathbb{Z}^3$, the following quantity is conserved by the equation

$$\int_\Omega \omega(t,x)\ dx.$$

One of the simplest examples of non-local and quadratic equation is

$$u_t - \Delta u = u^2 - \frac{1}{|\Omega|}\int_\Omega u^2$$

with Neumann boundary condition on $\partial\Omega$ so that the quantity $\int_\Omega u(t,x)\ dx$ is conserved. This equation is also related to Navier-Stokes equations on an infinite slab for other reasons explained in [6]. Problem (1.1)-(1.2) is a natural generalization of this latter for which we provide a global existence result for small initial data as well as a new blow-up criterion based on partial Maximum Principles and on a Gamma-convergence argument.

1.3 Main results

In this subsection, we present our main results: local existence, global existence for small initial data, energetic criterion for blow-up of solutions based on an optimization result of independent interest that we prove by a Gamma-convergence technique. Furthermore we give a global existence result in the case $p = 2$, expliciting the constants as a function of the geometry of the domain.

First, let us mention that the classical semigroup theory enables us to prove, more or less directly:

- local existence and uniqueness of solutions of (1.1)-(1.2) for any initial data $u_0 \in C(\overline{\Omega})$ (see the next section),
- global existence and exponential decay of solutions of (1.1)-(1.2) for small initial data $u_0 \in C(\overline{\Omega})$. That is, there exists some (implicit) constant $\rho > 0$ depending on the geometry of Ω, so that $\|u_0\|_{L^\infty} < \rho$ implies global existence and exponential decay of u: $\|u(t)\|_{L^\infty} \leq Ce^{-\alpha t}$ for some positive constants C and α.

To prove results of this kind it suffices for instance to follow the arguments of the proof of [8, Proposition 5.3.9]. See also [5] for a 1-dimensional result in this direction.

Our main purpose is to give a natural sufficient condition for the blow-up in finite time of solutions of (1.1)-(1.2) in the case $1 < p \leq 2$. Our proof relies on the same main idea introduced by Levine [21] and Ball [4] in the

sense that the blow-up will follow from a nonlinear differential inequality that we show to be satisfied by the L^2-norm of the solution.

First, it is quite easy to see that, $\forall p > 1$, the energy

$$E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}u|u|^p \tag{1.3}$$

of a solution u of (1.1)-(1.2) is non increasing in time (see Proposition 3.1). Our main result in this direction is the following

Theorem 1.1 ***(Sufficient condition for blow-up, case $\mathbf{1 < p \leq 2}$)*** *Let us assume that $p \in (1, 2]$ and let u be a solution of (1.1)-(1.2) with $u_0 \in C(\overline{\Omega})$, $u_0 \not\equiv 0$. If the energy of u_0 is nonpositive, that is $E(u_0) \leq 0$, then the solution does not exist in $L^\infty((0,T); L^2(\Omega))$ for all $T > 0$.*

Remark 1.2 *Note that Theorem 1.1 does not imply that the L^2-norm of $u(t)$ blows-up in finite time. Indeed, the solution may simply not exist till time T.*

Recall that, for the semilinear heat equation $u_t = \Delta u + u^p$ on a bounded domain, the generic blow-up profile is given by (see [19] for details)

$$u(x,t) \to u^*(x) \qquad \text{as } t \to T,$$

with

$$u^*(x) \sim C(p)\left[\frac{|\log|x||}{|x|^2}\right]^{\frac{1}{p-1}} \qquad\qquad \text{as } |x| \to 0$$

$C(p)$ being a constant. Hence, the L^2-norm stays generically bounded whenever $p > 1 + \frac{4}{N}$, and blows-up when $p < 1 + \frac{4}{N}$.

The condition of nonpositivity of the energy in Theorem 1.1 is also necessary in the sense that, if the L^2-norm of $u(t)$ blows-up at a time $T > 0$, then the energy $E(u(t))$ needs to be negative at some time $t < T$. The situation is even worse: the energy $E(u(t))$ needs to blow-up to $-\infty$ at a time $T' \leq T$. Moreover, this property is valid for any $p \in (1, +\infty)$. Indeed, we have the following

Theorem 1.3 ***(L^2 bound on u for bounded from below energy, $\mathbf{p \in (1, +\infty)}$)*** *Let p be any real number in $(1, +\infty)$ and let u be a solution of (1.1)-(1.2) with $u_0 \in C(\overline{\Omega})$. If there exists a constant $C_0 > 0$ and a time $T_0 > 0$ such that the solution u exists on $[0, T_0)$ and satisfies*

$$E(t) \geq -C_0 \quad \text{for} \quad t \in [0, T_0),$$

then there exists a constant $C > 0$ such that

$$||u(t)||_{L^2(\Omega)} \leq C \quad \text{for} \quad t \in [0, T_0).$$

The case $p > 2$ is still not completely understood for us, however, we believe that the blow-up phenomenon of Theorem 1.1 occurs for any p in $(1, +\infty)$ and formulate the following conjecture:

Conjecture (Sufficient condition for blow-up, case p > 2)
For $p > 2$, we conjecture that if u is a solution of (1.1)-(1.2) with $E(u_0) \leq 0$ and $u_0 \not\equiv 0$, then $u(t)$ blows-up in finite time.

The proof of Theorem 1.1 is based, on one hand, on the use of maximum principles, and, on the other hand, on the following estimate of independent interest, proved by gamma-convergence:

Theorem 1.4 ***(Optimization under a L^2 constraint)***
For $p > 1$, there exist $\theta_0 > 0$ and $C > 0$ such that

$$\inf_{v \in A} \int_\Omega v|v|^p + \theta|\nabla v|^2 \geq C\sqrt{\theta} \tag{1.4}$$

for all $\theta \in [0, \theta_0]$, where

$$A := \left\{ v \in L^{p+1}(\Omega) \;\Big|\; \int_\Omega v = 0,\ \int_\Omega v^2 = 1 \text{ and } v \geq -1 \text{ on } \Omega \right\}.$$

Let us mention that the profile of blowing-up solutions for this equation seems to us an open problem in general. Besides an example given in [5] of a profile of a blowing-up solution whose the positive part concentrates at one point in the one-dimensional case, we do not know if it is possible to build blowing-up solutions with different profiles.

Our next purpose is to focus on global existence results in the case $p = 2$. As mentioned previously, in usual global existence results, the constant ρ that determines the smallness of the initial data should depend on the geometry of the domain Ω. It is interesting to understand this dependence. That is precisely the aim of our Theorem 1.5. In particular, we relax here the assumption $|\Omega| = 1$ to show the dependence on the volume $|\Omega|$. We need first to introduce the following two invariants:

- the first positive eigenvalue $\lambda_1(\Omega)$ of the Laplacian in Ω under Neumann boundary conditions. Recall that we have the following isoperimetric-type inequality due to Szegö [34] and Weinberger [37]:

  $$\lambda_1(\Omega)|\Omega|^{2/N} \leq \lambda_1^*(N), \tag{1.5}$$

  where $\lambda_1^*(N)$ is the first positive Neumann eigenvalue of the N-dimensional Euclidean ball of volume 1.

- the constant $H(\Omega)$ defined as the supremum over $(0,+\infty)\times\Omega$ of the function $t^{N/2}[K(t,x,x)-\frac{1}{|\Omega|}]$ where $K(t,x,y)$ is the heat kernel associated to the Laplacian in Ω with Neumann boundary conditions (see [10] and section 2.2 for the precise definition of K and the existence of $H(\Omega)$). Notice that one has (see remark 5.2)

$$H(\Omega)\geq(4\pi)^{-\frac{N}{2}}.$$

It is also well known that the constant $H(\Omega)$ is closely related to the so-called Neumann Sobolev constant $C(\Omega)$ defined as the best constant in the inequality : $\forall f\in H^1(\Omega)$ such that $\int_\Omega f=0$, we have $\|f\|^2_{\frac{2N}{N-2}}\leq C(\Omega)\|\nabla f\|_2^2$ (see, for instance, [35, section 3] for results about this relationship).

Let us first remark that we have the following property for $p=2$

$$\text{if}\quad ||u_0||_{L^\infty(\Omega)}\leq\frac{3}{2}\lambda_1(\Omega),\quad\text{then}\quad E(u_0)\geq 0,$$

which may indicate (from Theorem 1.1) that the corresponding solutions may not necessary blow-up in finite time. This shows in particular that it is natural to compare $||u_0||_{L^\infty(\Omega)}$ with the first eigenvalue $\lambda_1(\Omega)$ as it can also be seen from the scaling of the equation for $p=2$.

The following theorem gives a global existence result under an explicit smallness condition on the initial data, depending on $\lambda_1(\Omega)$ on the one hand and on N, and $H(\Omega)$ on the other hand. For simplicity, we state it only for $p=2$ although a general version is possible.

Theorem 1.5 ***(Global existence for small initial data with explicit constants, case*** $p=2$***).*** *Let $\rho(\Omega)$ be the constant given by*

$$\frac{\rho(\Omega)}{\lambda_1(\Omega)}=\frac{1}{2N}\cdot\exp\Big(-\gamma_N\,\lambda_1^*(N)\,(H(\Omega))^{\frac{2}{N}}\Big),$$

where

$$\gamma_N=\frac{2^7}{N}$$

For every $u_0\in C(\overline{\Omega})$ satisfying $\int_\Omega u_0=0$ and

$$\|u_0\|_{L^\infty(\Omega)}\leq\rho(\Omega),\tag{1.6}$$

the (unique) solution of the problem (1.1)-(1.2) is defined for all t and tends to zero as $t\to\infty$.

Remark 1.6 *We also provide an exponential decay (see Theorem 5.1).*

Note that $H(\Omega)$ is invariant by dilation of the domain Ω, and then that $\rho(\lambda\Omega) = \lambda^{-2}\rho(\Omega) < \rho(\Omega)$ for $\lambda > 1$, but there is no reason in general to get $\rho(\Omega) \leq \rho(\Omega')$ if $\Omega' \subset \Omega$.

Remark 1.7 *Actually, the volume of Ω being fixed, one could reasonably expect that the constant $\rho(\Omega)$ is maximal when Ω is a ball.*

1.4 Brief review of the literature

Parabolic problems involving non local terms have been recently studied extensively in the literature (see for instance [12, 14, 29]). For local existence and continuation results for general semilinear equations under the Neumann boundary condition setting, one can see for example [1] and [33]. In [30, Appendix A] Souplet gives very general local existence results for a large class of non local problems in time and in space but in the Dirichlet boundary condition setting. The problem treated in the present work has been first considered by Budd, Dold and Stuart ([5]) for $p = 2$ and in the one dimensional case. They obtained a theorem like our Theorem 1.5 as well as a blow-up type result for solutions whose Fourier coefficients of the initial data satisfy an infinite number of conditions.

Hu and Yin ([18]) considered slightly different problems. They showed in particular blow-up result (see [18, Theorem 2.1]), based on energy criteria, considering $u|u|^{p-1}$ instead of $|u|^p$. They showed also (see [18, Theorem 3.1 and 3.2]) global existence for positive solutions and p not too large. A radial blowing-up solution for p large is also given.

Wang and Wang ([36]) considered a more general problem of the form

$$u_t - \Delta u = ku^p - \int u^q \tag{1.7}$$

with Neumann or Dirichlet boundary conditions and positive initial data. They showed global existence and exponential decay in the case where $p = q$, $|\Omega| \leq k$ and Neumann boundary condition. They also obtain a blow-up result under the assumption that the initial data is bigger than some "gaussian function" in the case where $|\Omega| > k$.

Finally, in [31, Theorem 2.2], Souplet determines exact behavior of the blow-up rate for equations of the form (1.7) with $k = 1$ and $p \neq q$.

1.5 Organization of the article

Our paper is organized as follows. In the second section we first set the space under which problem (1.1)-(1.2) admits a unique local solution. Section 3 is devoted to the proof of Theorems 1.1 and 1.3. In section 4 we give the proof of the optimization result (Theorem 1.4) which is based on a result of Gamma-convergence of Modica [25]. For the convenience of the

reader we provide in the appendix (section 6) a self-contained proof of the corresponding Gamma-convergence-like result. In section 5, we give the proof of Theorem 1.5.

2 Local existence result

We recall that Ω is bounded. The basic space to be considered in this paper is the space $C(\overline{\Omega})$ of continuous functions. Following the notations [1] of Stewart [33] denote for $q > N$ by

$$D^q := \left\{ u \in C(\overline{\Omega});\ u \in W^{2,q}(\Omega),\ \Delta u \in C(\overline{\Omega}), \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

Set

$$D := \bigcup_{N<q<+\infty} D^q.$$

Then we have as a direct application of [33, Theorem 2]:

Theorem 2.1 *The operator $-\Delta$ with domain D generates an analytic semigroup in the space $C(\overline{\Omega})$ with the supremum norm.*

See Lunardi [23] for the definition of analytic semigroups.

Then we have

Theorem 2.2 ***(Local existence result,*** $1 < p < +\infty$***)***
For every $u_0 \in C(\overline{\Omega})$ there is a $0 < t_{max} \leq \infty$ such that the problem (1.1)-(1.2) has a unique mild solution, i.e. a unique solution

$$u \in C\left([0, t_{max}); C(\overline{\Omega})\right) \cap C^1\left((0, t_{max}); C(\overline{\Omega})\right) \cap C\left((0, t_{max}); D\right)$$

of the following integral equation

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} f(u(s))\, ds$$

on $[0, t_{max})$, with

$$f(u(s)) = |u(s)|^p - \fint_\Omega |u(s)|^p.$$

Moreover, we have

$$\int_\Omega u(t) = 0 \quad \textit{for all} \quad t \in [0, t_{max}) \tag{2.8}$$

and if $t_{max} < \infty$ then

$$\lim_{t \uparrow t_{max}} \|u(t)\|_{L^\infty(\Omega)} = \infty.$$

[1] Since Ω is bounded and $\partial\Omega \in C^2$, we can easily check that D^q is dense in $C(\overline{\Omega})$

Proof of Theorem 2.2
First let us remark that the non linearity in (1.1): $u \in C(\overline{\Omega}) \longmapsto f(u) = |u|^p - \fint_\Omega |u|^p \in C(\overline{\Omega})$ satisfies the hypothesis of [28, Theorem 6.1.4], namely, f is locally Lipschitz in u, uniformly in t on bounded intervals of time.
Then a standard semigroups result ([28, Theorem 6.1.4]) gives the local existence result. Reminder to show (2.8). This comes from the simple computation for $t \in (0, t_{max})$:

$$\frac{d}{dt}\left(\int_\Omega u\right) = \int_\Omega u_t = \int_\Omega \Delta u + \int_\Omega f(u) = 0$$

because of the definition of f, the integration by part on Δu, and the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$. □

From this result, we see that the solution is a classical solution of equation (1.1) on $(0, t_{max}) \times \Omega$ with initial condition (1.2), and then from the standard parabolic estimates (see Lieberman [22]) and classical bootstrap arguments, we get

$$u \in C^\infty((0, t_{max}) \times \Omega).$$

3 Blow-up: proofs of Theorems 1.1 and 1.3

As mentioned in the introduction, we follow the energetic method introduced by Levine [21] and Ball [4]. The main idea is to show that the L^2-norm of the solution satisfies some super-linear differential inequality which implies the finite time blow-up.

All along this section, we denote by u a solution of (1.1)-(1.2) whose initial data u_0 satisfies $\int_\Omega u_0 = 0$. Also, we will assume, without lack of generality, that $|\Omega| = 1$ so that we have, in particular, $\fint_\Omega u^2 = \int_\Omega u^2$.

Let us recall the expression of the energy of the problem

$$E(u) := \int \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1} u|u|^p.$$

Proposition 3.1 ***(Energy decay)***
The energy $E(t) := E(u(t))$ is a non increasing function of t in $(0, \infty)$.

Proof of proposition 3.1
A direct computation using (1.1)-(1.2) and the fact that $\int_\Omega u_t = 0$ yields

$$\frac{d}{dt}E(u(t)) = \int_\Omega \left(-\Delta u - |u|^p\right) u_t = -\int_\Omega (u_t)^2 \leq 0.$$

□

Lemma 3.2 ***(L^2 bound from below)***
Let us define

$$F(t) = \int_\Omega u^2(t)$$

Then we have, $\forall p > 1$,

$$\frac{1}{2}F'(t) = -(p+1)E(t) + \frac{p-1}{2}\int_\Omega |\nabla u|^2.$$

In particular, we get

$$\frac{1}{2}F'(t) \geq -(p+1)E(0) + \frac{p-1}{2}\lambda_1(\Omega)F(t).$$

Consequently, if $E(0) \leq 0$, then

$$F(t) \geq F(0)e^{(p-1)\lambda_1(\Omega)t}.$$

Proof of Lemma 3.2
The lemma is a consequence of the following computation

$$\begin{aligned}
\tfrac{1}{2}F'(t) &= \int_\Omega u u_t \\
&= \int_\Omega u\left(\Delta u + |u|^p\right) \\
&= \int_\Omega -|\nabla u|^2 + u|u|^p \\
&= -(p+1)E(t) + \frac{p-1}{2}\int_\Omega |\nabla u|^2
\end{aligned}$$

□

Lemma 3.3 ***(L^2 bound from below when $\inf_{\mathbf{x}} \mathbf{u(t)} \geq -||\mathbf{u}||_{\mathbf{L^2(\Omega)}}$)***
Let $0 < t_1 < t_2 < \infty$, $p > 1$ and assume that

$$\inf_x u(t) \geq -||u||_{L^2(\Omega)} \quad \textit{for all} \quad t \in (t_1, t_2).$$

Then for all $\beta \in (2, p+1)$, there exist two constants $C_\beta > 0$ and $\lambda_\beta > 0$ such that if

$$||u(t)||_{L^2(\Omega)} \geq \lambda_\beta, \qquad \textit{for all} \qquad t \in (t_1, t_2)$$

then

$$\frac{1}{2}F'(t) \geq -\beta E(t) + C_\beta F(t)^{\frac{p+3}{4}} \qquad \textit{for all} \qquad t \in (t_1, t_2).$$

Proof of Lemma 3.3
Let us consider a parameter $\beta \in (2, p+1)$. We have

$$\begin{aligned}
\frac{1}{2}F'(t) &= \int_\Omega -|\nabla u|^2 + u|u|^p \\
&= -\beta\left(\int_\Omega \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}u|u|^p\right) + \frac{\beta-2}{2}\int_\Omega |\nabla u|^2 + \left(1 - \frac{\beta}{p+1}\right)\int_\Omega u|u|^p \\
&= -\beta E(t) + \frac{p+1-\beta}{p+1}\left(\int_\Omega u|u|^p + \gamma|\nabla u|^2\right)
\end{aligned}$$

with

$$\gamma = \frac{\beta-2}{2} \times \frac{p+1}{p+1-\beta}$$

Here we will use Theorem 1.4. To this end, we define

$$\lambda = ||u||_{L^2}, \quad v = \frac{u}{\lambda}.$$

Then Theorem 1.4 claims that

$$\int_\Omega v|v|^p + \theta|\nabla v|^2 \geq C\sqrt{\theta}, \quad \text{if} \quad 0 \leq \theta \leq \theta_0.$$

Then we get

$$\begin{aligned}
\frac{1}{2}F'(t) &= -\beta E(t) + \frac{p+1-\beta}{p+1}\lambda^{p+1}\left[\int_\Omega v|v|^p + \gamma\lambda^{1-p}|\nabla v|^2\right] \\
&\geq -\beta E(t) + \frac{p+1-\beta}{p+1}C\gamma^{\frac{1}{2}}\lambda^{\frac{p+3}{2}}
\end{aligned}$$

if $\gamma\lambda^{1-p} \leq \theta_0$. □

Lemma 3.4 ***($\mathbf{L^\infty}$ bound from below for $\mathbf{1 < p \leq 2}$)***
Let $p \in (1, 2]$, and let u be a solution of the problem (1.1)-(1.2) with the initial data $u(t=0) = u_0$ satisfying the condition

$$u_0 \geq -||u_0||_{L^2(\Omega)},$$

and

$$E(u_0) \leq 0 \quad \textit{and} \quad u_0 \not\equiv 0$$

Then for all $t > 0$ (where the solution exists), we have

$$u(t) > -||u(t)||_{L^2(\Omega)}.$$

Proof of Lemma 3.4
Let us define the set for every $T > 0$

$$\Sigma_T = \left\{(x,t) \in \Omega \times (0,T), \quad u(x,t) < -||u(t)||_{L^2(\Omega)}\right\},$$

and the function

$$v(x,t) = -||u(t)||_{L^2(\Omega)}.$$

If $\Sigma_T \neq \emptyset$, then the functions u and v satisfy (using the condition $p \leq 2$)

$$\left\{\begin{array}{l} \Delta u - u_t = ||u||^p_{L^p(\Omega)} - |u|^p < ||u||^p_{L^p(\Omega)} - ||u||^p_{L^2(\Omega)} \leq 0 \\ \\ \Delta v - v_t = \frac{F'(t)}{2\sqrt{F(t)}} \geq 0 \end{array}\right| \quad \text{on} \quad \Sigma_T$$

where we have used the fact that $F' \geq 0$ if $E(u_0) \leq 0$ (see Lemma 3.2). Consequently we have for $w = u - v$:

$$\left\{\begin{array}{l} \Delta w - w_t < 0 \quad \text{on} \quad \Sigma_T \\ \\ w = 0 \quad \text{on} \quad (\partial\Sigma_T) \setminus \{t = T\}. \end{array}\right.$$

The maximum principle implies $w \geq 0$ on Σ_T which gives a contradiction with the definition of Σ_T. Therefore $\Sigma_T = \emptyset$ for every $T > 0$, and then w satisfies

$$w \geq 0 \quad \text{on} \quad \Omega \times (0, +\infty).$$

From Lemma 3.2, if $E(0) \leq 0$ and $u_0 \not\equiv 0$, then

$$\frac{F'(t)}{2\sqrt{F(t)}} \geq \sqrt{F(0)}\frac{(p-1)\lambda_1(\Omega)}{2} e^{\frac{(p-1)\lambda_1(\Omega)}{2}t} > 0.$$

Then, if there is a point $P = (x_0, t_0) \in \overline{\Omega} \times (0, +\infty)$ such that $w(P) = 0$, we have $\Delta w(P) - w_t(P) = -\frac{F'(t_0)}{2\sqrt{F(t_0)}} < 0$, and then there is a connected open neighborhood σ_P of P in $\Omega \times (0, +\infty)$ such that

$$\left\{\begin{array}{l} \Delta w - w_t < 0 \quad \text{on} \quad \sigma_P \\ \\ w \geq 0 \quad \text{on} \quad \sigma_P \\ \\ w(P) = 0. \end{array}\right. \tag{3.9}$$

As a consequence of the strong maximum principle, we get that $w = 0$ on σ_P which does not satisfies the parabolic equation (3.9). Contradiction. We conclude that

$$w > 0 \quad \text{on} \quad \Omega \times (0, +\infty)$$

□

Lemma 3.5 ***(Monotonicity of the infimum of* u *for* p $\in (1, +\infty)$*)***
Let us consider $p \in (1, +\infty)$. We assume that there exists $0 \le t_1 < t_2$, such that

$$\inf_x u(t) < -||u(t)||_{L^p(\Omega)} \quad \textit{for all} \quad t \in (t_1, t_2) \tag{3.10}$$

and $u \in L^\infty_{loc}((0, t_2); L^p(\Omega))$. Then the infimum

$$m(t) = \inf_x u(t)$$

is nondecreasing on (t_1, t_2).

Proof of Lemma 3.5
For every $t_0 \in (t_1, t_2)$, let us consider the solution $g^{t_0} = g$ of the following ODE:

$$\begin{cases} g'(t) = |g|^p - ||u(t)||^p_{L^p(\Omega)} \quad \text{on} \quad (t_0, t_2) \\ \\ g(t_0) = m(t_0), \end{cases}$$

and the set

$$\Sigma = \left\{(x, t) \in \Omega \times (t_0, t_2), \quad u(x, t) < g^{t_0}(t)\right\}$$

If $\Sigma \neq \emptyset$, then u satisfies

$$\begin{cases} \Delta u - u_t \le ||u(t)||^p_{L^p} - |g^{t_0}|^p = \Delta g^{t_0} - \left(g^{t_0}\right)_t \quad \text{on} \quad \Sigma \\ \\ u = g^{t_0} \quad \text{on} \quad \partial\Sigma. \end{cases}$$

Therefore the maximum principle implies that $u \ge g^{t_0}$ on Σ, which gives a contradiction with the definition of Σ. Thus $\Sigma \neq \emptyset$ and $u \ge g^{t_0}$ on $\Omega \times (t_0, t_2)$, which implies

$$m(t) \ge g^{t_0}(t) \quad \text{for all} \quad t \in (t_0, t_2).$$

Now using (3.10), we get

$$(g^{t_0})'(t) = |g^{t_0}(t)|^p - ||u(t)||^p_{L^p(\Omega)} \ge |m(t)|^p - ||u(t)||^p_{L^p(\Omega)} \ge 0 \tag{3.11}$$

and then for t_0' satisfying $t_1 < t_0 < t_0' < t_2$, we get

$$m(t_0') \ge g^{t_0}(t_0') \ge g^{t_0}(t_0) = m(t_0) \tag{3.12}$$

□

Proof of Theorem 1.3
We assume that $E(t) \ge -C_0$ on $(0, T_0)$. Then we compute

$$\begin{aligned} \frac{1}{2}F'(t) &= \int_\Omega u u_t \\ &\le \frac{1}{2}\left(\int_\Omega u^2 + u_t^2\right) \\ &= \frac{1}{2}\left(F(t) - E'(t)\right) \end{aligned}$$

We deduce

$$\begin{aligned}(F+E+C_0)'(t) &\leq F(t)\\ &\leq (F+E)(t)+C_0\end{aligned}$$

Consequently for $t \in (0, T_0)$ we get

$$F(t) \leq (F+E)(t)+C_0 \leq (F(0)+E(0)+C_0)e^t$$

which proves that $F(t)$ is bounded on $[0, T_0)$. In particular F can not blow up at time T_0. □

Proof of Theorem 1.1

We assume that $E(u_0) \leq 0$ and $u_0 \not\equiv 0$.

First case: $\mathbf{u_0 \geq -||u_0||_{L^2(\Omega)}}$

Then by lemma 3.4, we get $u(t) \geq -||u(t)||_{L^2(\Omega)}$ for all $t > 0$. For some $\beta \in (2,3)$, let us consider the real λ_β given by lemma 3.3. Then by lemma 3.2, there exists a time $t_\beta \geq 0$, such that $||u(t)||_{L^2(\Omega)} \geq \lambda_\beta > 0$ for every $t \geq t_\beta$. Using lemma 3.3 (and the monotonicity of the energy E given by proposition 3.1), we get for $t \geq t_\beta$

$$F'(t) \geq 2C_\beta F^{\frac{p+3}{4}}(t)$$

which blows up in finite time $T > 0$.

Second case: $\mathbf{\inf_x u_0 < -||u_0||_{L^2(\Omega)}}$

We know by lemma 3.5 that $m(t) = \inf_x u(t)$ is nondecreasing as long as

$$\inf_x u(t) < -||u(t)||_{L^2(\Omega)} \leq -||u(t)||_{L^p(\Omega)}$$

because we have

$$p \leq 2$$

Then

$$m(0) \leq m(t) \leq -||u(t)||^2_{L^2(\Omega)}$$

Lemma 3.2 proves that there is necessarily one time t_0 such that $\inf_x u(t_0) = -||u(t_0)||^2_{L^2(\Omega)}$. We can then apply the first case with initial time t_0. □

Let us conclude this section with a partial result in the case $p > 2$:

Proposition 3.6 ***($\mathbf{L^\infty}$ bound from below for $\mathbf{p \in (1, +\infty)}$)***

Let $p \in (1, +\infty)$, and let u be a solution of the problem (1.1)-(1.2) with the initial data $u(t=0) = u_0$ satisfying

$$u_0 \geq -||u_0||_{L^p(\Omega)},$$

and

$$E(u_0) \leq 0 \quad \textit{and} \quad u_0 \not\equiv 0.$$

Then for all $t > 0$ (where the solution exists), we have

$$u(t) > -\sup_{s\in[0,t]} ||u(s)||_{L^p(\Omega)}.$$

Proof of Proposition 3.6
The proof is similar to the proof of proposition 3.4, where we use the function $v(x,t) = -\sup_{s\in[0,t]} ||u(s)||_{L^p(\Omega)}$, which satisfies

$$\Delta v - v_t \geq 0 \quad \text{on} \quad \Omega \times (0, +\infty)$$

In the last part of the proof, we remark that $w = 0$ on σ_P, and by connexity of $\Omega \times (0, +\infty)$, we get $w = 0$ on $\Omega \times (0, +\infty)$. The equation on u implies $u(x,t) = constant$ on $\Omega \times (0, +\infty)$ which is in contradiction with the fact that $\int_\Omega u(t) = 0$ and $u_0 \not\equiv 0$. □

4 Optimization by a gamma-convergence technique: proof of Theorem 1.4

In this whole section we assume that Ω is a bounded domain.

To do the proof of Theorem 1.4, we first need to rewrite an integral as follows:

Lemma 4.1 ***(Rewrite $\int_\Omega \mathbf{v}|\mathbf{v}|^{\mathbf{p}}$ as the integral of nonnegative function, for $\mathbf{p} > \mathbf{1}$)***
Let us denote

$$A := \left\{ v \in L^{p+1}(\Omega); \quad \int_\Omega v = 0; \quad \int_\Omega v^2 = 1; \quad v \geq -1 \text{ on } \Omega \right\}.$$

Then there exists a function $f \in C^2([-1, +\infty))$ which satisfies

$$f > 0 \quad \text{on} \quad (-1,1) \cup (1, +\infty), \quad \text{and} \quad f(-1) = f(1) = 0$$

such that for every $v \in A$ we have

$$\int_\Omega v|v|^p = \int_\Omega f(v) \geq 0.$$

Proof of Lemma 4.1
Here we use the function

$$f(v) = v|v|^p - v + \frac{p}{2}(1 - v^2)$$

and use the fact that $\int_\Omega v = \int_\Omega (1 - v^2) = 0$. The properties of this function can be easily checked, computing

$$f'(v) = (p+1)|v|^p - pv - 1, \quad f''(v) = p(p+1)v|v|^{p-2} - p.$$

□

Proof of Theorem 1.4
To prove Theorem 1.4, we simply observe that for $\theta = \varepsilon^2$, and $v \in A$, we can write

$$\int_\Omega v|v|^p + \theta|\nabla v|^2 = \varepsilon J^\varepsilon(v)$$

with

$$J^\varepsilon(v) = \begin{cases} \displaystyle\int_\Omega \varepsilon|\nabla v|^2 + \frac{1}{\varepsilon} f(v) & \text{if} \quad v \in A \cap H^1(\Omega) \\ \\ +\infty & \text{if} \quad v \notin A \cap H^1(\Omega). \end{cases}$$

As ε goes to zero, the minimizers of J^ε will concentrate on the minima of the function f, namely on the values $v = -1$ or $v = 1$. We see formally that at the limit, we will get discontinuous functions. To perform rigorously the analysis, we need to introduce the space $BV(\Omega)$ of functions of bounded variations on Ω.
For a function $v \in L^1(\Omega)$, we define the total variation of v as

$$|\nabla v|(\Omega) = \sup\left\{\int_\Omega -\sum_{i=1}^n v\frac{\partial \phi_i}{\partial x_i}, \quad \phi = (\phi_1, ..., \phi_n) \in (C^1(\overline{\Omega}))^n, \quad \sum_{i=1}^n \phi_i^2 \leq 1 \quad \text{on} \quad \Omega\right\}.$$

Then the norm in $BV(\Omega)$ is defined by

$$\|v\|_{BV(\Omega)} := \int_\Omega |v|\, dx + |\nabla v|(\Omega)$$

and the space $BV(\Omega)$ is naturally defined by

$$BV(\Omega) = \left\{v \in L^1(\Omega), \quad \|v\|_{BV(\Omega)} < +\infty\right\}$$

It is known that $BV(\Omega)$ is a Banach space.

Then Theorem 1.4 is a consequence of the following result:

Proposition 4.2 ***(Limit inf of the energy)***
Assume that Ω is a bounded domain and $\varepsilon > 0$. Then the energy

$$I^\varepsilon = \inf_{u \in A} J^\varepsilon(u)$$

satisfies

$$\liminf_{\varepsilon \to 0} I^\varepsilon \geq I^0 > 0$$

where I^0 is a constant.

We give the sketch of the proof of proposition 4.2 below, based on a Gamma-convergence technique, but for the convenience of the reader, we provide in the appendix a self-contained proof (see Proposition 6.1 and its proof).

Proof of Proposition 4.2
We remark that

$$\inf_{u\in A} J^\varepsilon(u) \geq \inf_{u\in A_0} J^\varepsilon(u)$$

where

$$A \subset A_0 := \left\{ v \in L^1(\Omega); \quad \int_\Omega v = 0; \quad v \geq -2 \text{ on } \Omega \right\}$$

with the function f extended on $[-2,-1]$ by $f(v) = |v+1|$. Now we apply the result of Modica [25, Theorem I page 132, Proposition 3 page 138], with $W(t) = f(t-2)$, $\alpha = 1, \beta = 3, m = 2, |\Omega| = 1$, $k = p+1$. It is easy to see that there exists a constant $I_0 > 0$ as stated in Proposition 4.2. □

See also the overview of Alberti [2], where the full Gamma-convergence result is stated. The concept of Gamma-convergence has been introduced by De Giorgi [11], and one of the first illustration of this concept was the work of Modica, Mortola [26]. For an introduction to Gamma-convergence and many references, we refer the reader to the book of Dal Maso [9].

5 Explicit global existence for $p = 2$ and proof of Theorem 1.5

In this section, in order to make clear the dependence on the volume $|\Omega|$, we do not assume $|\Omega| = 1$.

Theorem 1.5 is actually a special case of the following more general result:

Theorem 5.1 ***(Global existence for small initial data with explicit constants, case*** $p = 2$***).*** *Let r be any real number satisfying $r > \frac{N}{2}$ and $r \geq 2$. Let $\rho_r(\Omega)$ be the constant given by*

$$\rho_r(\Omega) = \frac{\lambda_1(\Omega)}{4r} \cdot \exp\left(- \left(\gamma_{N,r} \, (\lambda_1(\Omega))^{\frac{N}{2}} \, |\Omega| \, H(\Omega) \right)^{\frac{2}{N}} \right),$$

where

$$\gamma_{N,r} = \left(1 + \frac{e^{-1}}{2(1-\frac{N}{2r})} \right)^r 2^{r-1} \, r^{-\frac{N}{2}}$$

For every $u_0 \in C(\overline{\Omega})$ satisfying $\int_\Omega u_0 = 0$ and

$$\|u_0\|_{L^\infty(\Omega)} \leq \rho_r(\Omega), \tag{5.13}$$

the (unique) solution of the problem (1.1)-(1.2) is defined for all t. Moreover for all $t \geq 1$ the solution satisfies:

$$\|u(t)\|_{L^\infty(\Omega)} \leq C(r) \|u_0\|_{L^\infty} \, e^{-\frac{\lambda_1(\Omega)}{r} t},$$

where

$$C(r) := 2^{\frac{r-1}{r}} |\Omega|^{\frac{1}{r}} H(\Omega)^{\frac{1}{r}} \left[1 + \frac{2\|u_0\|_{L^\infty}}{1 - \frac{N}{2r}} \right].$$

To deduce Theorem 1.5, we simply apply Theorem 5.1 with $r = 2N$, and use the fact that $(\gamma_{N,2N})^{\frac{2}{N}} \leq \gamma_N$ and the inequality (1.5).

Proof of Theorem 5.1

Let us denote by $K(t, x, y)$ the heat kernel associated to the Laplacian in Ω with Neumann boundary conditions[2]. That is

$$\begin{cases} \frac{\partial}{\partial t} K(t,x,y) - \Delta_x K(t,x,y) = 0, & \forall x, y \in \Omega,\ t > 0 \\ \frac{\partial}{\partial n_x} K(t,x,y) = 0, & \forall x \in \partial\Omega,\ y \in \Omega,\ t > 0 \\ K(t,x,y) \longrightarrow \delta_y(x) \text{ in } \mathcal{D}'(\Omega) \text{ as } t \to 0^+, & \forall y \in \Omega. \end{cases}$$

This function is related to the eigenvalues and eigenfunctions of the Neumann Laplacian $-\Delta$ in Ω by the following identity:

$$K(t,x,y) = \sum_{k \geq 0} e^{-\lambda_k(\Omega)t} f_k(x) f_k(y) = \frac{1}{|\Omega|} + \sum_{k \geq 1} e^{-\lambda_k(\Omega)t} f_k(x) f_k(y), \quad (5.14)$$

where $\{\lambda_k(\Omega)\ ;\ k \geq 0\}$ are the eigenvalues of $-\Delta$ and $\{f_k\ ;\ k \geq 0\}$ is an L^2-orthonormal family of corresponding eigenfunctions (recall that $\lambda_0(\Omega) = 0$ and $f_0 = \frac{1}{|\Omega|^{\frac{1}{2}}}$). Let us set $K_0(t,x,x) := K(t,x,x) - \frac{1}{|\Omega|}$. From the classical results on heat kernels (see for instance [10, Theorem 2.4.4]) we know that the function $t^{N/2} K(t,x,x)$ is bounded on $(0,1] \times \Omega$, and then the same is true for $t^{N/2} K_0(t,x,x)$. On the other hand, it follows immediately from (5.14) that $e^{\lambda_1(\Omega)(t-1)} K_0(t,x,x)$ is decreasing on $[1, +\infty)$ and then, for any $t \geq 1$,

$$K_0(t,x,x) \leq e^{-\lambda_1(\Omega)(t-1)} K_0(1,x,x) \leq C_1 e^{-\lambda_1(\Omega)(t-1)} \leq C_2 t^{-N/2}$$

for some constants C_1 and C_2. Hence, $t^{N/2} K_0(t,x,x)$ is bounded on $(0, +\infty) \times \Omega$ and we denote by $H(\Omega)$ its supremum. Since for all $t > 0$, $K(t,x,y)$ achieves its supremum on the diagonal of $\Omega \times \Omega$, the constant $H(\Omega)$ is actually the best constant in the following inequality

$$K(t,x,y) \leq \frac{1}{|\Omega|} + H(\Omega) t^{-N/2}$$

valid in $(0, +\infty) \times \Omega \times \Omega$. Notice that, in contrast to the Dirichlet boundary condition case, there is no universal upper bound to $H(\Omega)$ (even for domains of fixed volume). Indeed, it is rather easy to see that, in the Dirichlet case, the heat kernel is bounded above by $(4\pi t)^{-N/2}$ whatever the domain is.

[2] In fact for this section it suffices to consider a domain Ω satisfying the extension property (see [10, Section 1.7]).

Remark 5.2 *Notice that*

$$H(\Omega) \geq \frac{t^{\frac{N}{2}}}{|\Omega|}\left[\left(\int_\Omega K(t,x,x)\right)-1\right] = \frac{t^{\frac{N}{2}}}{|\Omega|}\left[tr\, e^{t\Delta}-1\right]$$

with

$$tr\, e^{t\Delta} = (4\pi t)^{\frac{-N}{2}}\left[|\Omega| + \frac{\sqrt{\pi}}{2}\left(\mathcal{H}^{N-1}(\partial\Omega)\right)t^{\frac{1}{2}} + o(t)\right] \quad as \quad t\to 0^+,$$

see [15], see also [3], [7] and [24] for the dependance of $\lambda_1(\Omega)$ on the geometry of Ω. This implies

$$H(\Omega) \geq (4\pi)^{-N/2}.$$

The following property seems to be a standard one, we give it for completeness.

Lemma 5.3 ***($\mathbf{L^p - L^q}$-estimate for the linear heat equation)***
Let $v_0 \in C(\overline{\Omega})$ be such that $\int_\Omega v_0 = 0$ and let

$$v(t,x) = \left(e^{t\Delta}v_0\right)(x) = \int_\Omega K(t,x,y)v_0(y)\,dy$$

be the solution of the heat equation with Neumann boundary condition and v_0 as initial data. For any positive t and all $1 < p \leq q \leq +\infty$, we have

$$\|v(t)\|_{L^q} \leq \left[2^{p-1}H(\Omega)t^{-N/2}\right]^{\frac{1}{p}-\frac{1}{q}}\|v_0\|_{L^p}. \tag{5.15}$$

Proof. Since $\int_\Omega v_0 = 0$ we have, for any $p > 1$ and any $(t,x) \in (0,+\infty)\times\Omega$,

$$|v(t,x)| = \left|\int_\Omega K_0(t,x,y)\,v_0(y)\,dy\right| \leq \|v_0\|_{L^p}\|K_0(t,x,\cdot)\|_{L^{p'}}$$

with $\frac{1}{p}+\frac{1}{p'} = 1$. Now,

$$\begin{aligned}\|K_0(t,x,\cdot)\|_{L^{p'}}^{p'} &= \int_\Omega |K_0(t,x,y)|^{p'}\,dy \\ &\leq \left[H(\Omega)t^{-N/2}\right]^{p'-1}\|K_0(t,x,\cdot)\|_{L^1}\end{aligned}$$

with $\|K_0(t,x,\cdot)\|_{L^1} = \int_\Omega |K(t,x,y) - \frac{1}{|\Omega|}|\,dy \leq \int_\Omega (K(t,x,y)+\frac{1}{|\Omega|})\,dy = 2$, since $K(t,x,y) \geq 0$ and $\int_\Omega K(t,x,y)\,dy = 1$. Hence, $\|K_0(t,x,\cdot)\|_{L^{p'}}^{p'} \leq 2\left[H(\Omega)t^{-N/2}\right]^{p'-1}$ and then,

$$\|v(t)\|_{L^\infty} \leq 2^{\frac{1}{p'}}\|v_0\|_{L^p}\left[H(\Omega)t^{-N/2}\right]^{\frac{p'-1}{p'}} = \|v_0\|_{L^p}\left[2^{p-1}H(\Omega)t^{-N/2}\right]^{\frac{1}{p}}.$$

Now let us remark that

$$\begin{aligned} ||v(t)||_{L^q}^q &= \int_\Omega |v(t)|^{q-p}|v(t)|^p \\ &\leq ||v(t)||_{L^\infty}^{q-p}||v(t)||_{L^p}^p \\ &\leq \|v_0\|_{L^p}^{q-p}\left[2^{p-1}H(\Omega)t^{-N/2}\right]^{\frac{q-p}{p}}||v(t)||_{L^p}^p. \end{aligned}$$

We get finally the $L^p - L^q$ estimate of the lemma, using the contraction property of the Heat equation with Neumann boundary condition (see [23, Section 3.1.1]): $||v(t)||_{L^p} \leq ||v_0||_{L^p}$. □

The following elementary property will also be useful

Lemma 5.4 *Let $\alpha, \beta \geq 1$. For all $f \in L^{\alpha+\beta}(\Omega)$, we have*

$$\left(\int_\Omega |f|^\alpha\right) \cdot \left(\int_\Omega |f|^\beta\right) \leq |\Omega| \int_\Omega |f|^{\alpha+\beta}.$$

Proof. Using Hölder inequality, the following holds

$$\begin{aligned} \int_\Omega |f|^\alpha &\leq \left[\int_\Omega |f|^{\alpha+\beta}\right]^{\frac{\alpha}{\alpha+\beta}} \times |\Omega|^{\frac{\beta}{\alpha+\beta}} \\ \int_\Omega |f|^\beta &\leq \left[\int_\Omega |f|^{\alpha+\beta}\right]^{\frac{\beta}{\alpha+\beta}} \times |\Omega|^{\frac{\alpha}{\alpha+\beta}} \end{aligned}$$

Thus

$$\int_\Omega |f|^\alpha \times \int_\Omega |f|^\beta \leq |\Omega| \int_\Omega |f|^{\alpha+\beta}$$

□

From now on, we denote by u a solution of (1.1)-(1.2) for $p = 2$ whose initial data u_0 satisfies $\int_\Omega u_0 = 0$. The proof of Theorem 1.5 is based on the following a priori estimate:

Lemma 5.5 (A priori estimate)
Let $r > \frac{N}{2}$ satisfying $r \geq 2$, $0 < K < \frac{\lambda_1}{2r}$, and assume that there exists a positive T such that $\|u(t)\|_{L^\infty} \leq K$ for all $t \in [0, T]$. Then, for any $\beta \in (0, 1)$, we have

$$\begin{aligned} \|u(T)\|_{L^\infty} &\leq \|u_0\|_\infty 2^{1-\frac{1}{r}} H(\Omega)^{\frac{1}{r}} |\Omega|^{\frac{1}{r}} [\beta T]^{-\frac{N}{2r}} \times \\ &\left[e^{-2[\frac{\lambda_1}{r}-K](1-\beta)T} + \frac{\beta e^{-1}\|u_0\|_{L^\infty}}{(1-\beta)[\lambda_1/r - 2K]\left(1-\frac{N}{2r}\right)}\right]. \end{aligned} \quad (5.16)$$

Proof of the lemma. For simplicity we will write λ_1 and H for $\lambda_1(\Omega)$ and $H(\Omega)$. Let $q \geq 2$ be an even integer. For all $t \in [0,T]$, we have $|u|^{q+1} \leq Ku^q$ and then, setting $F_q(t) := \frac{1}{q}\int u^q(t)\,dx \geq 0$, $\int |u|^{q+1} \leq qKF_q(t)$. Using lemma 5.4 we also have

$$\left|\left(\int u^{q-1}\right)\left(\fint u^2\right)\right| \leq \left(\int |u|^{q-1}\right)\left(\fint u^2\right) \leq \int |u|^{q+1} \leq qKF_q(t).$$

Therefore,

$$\left|\int u^{q+1} - \left(\int u^{q-1}\right)\left(\fint u^2\right)\right| \leq 2qKF_q(t).$$

Multiplying Equation (1.1) (with $p = 2$) by u^{q-1} and integrating over Ω we get, after integrating by parts,

$$\int u^{q-1}u_t + 4\frac{q-1}{q^2}\int |\nabla u^{q/2}|^2 \leq 2qKF_q(t). \tag{5.17}$$

This implies that $F_q'(t) - 2qKF_q(t) \leq 0$. Hence, for all $t \in [0,T]$, we have $F_q(t) \leq F_q(0)e^{2qKt}$, or equivalently

$$\|u(t)\|_{L^q} \leq \|u(0)\|_{L^q}e^{2Kt}.$$

Making $q \to \infty$, one can deduce that

$$\|u(t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}\, e^{2Kt}. \tag{5.18}$$

On the other hand, we clearly have $\int u(t) = 0$ for all t. Poincaré's inequality gives, for all $t \in (0,T]$, $\int |\nabla u|^2 \geq \lambda_1 \int u^2$. Taking $q = 2$ in (5.17) we then get, for all $t \in [0,T]$,

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}\, e^{-(\lambda_1 - 2K)t} \leq \|u_0\|_{L^\infty}|\Omega|^{\frac{1}{2}}e^{-(\lambda_1-2K)t}. \tag{5.19}$$

By $L^2 - L^\infty$ interpolation, we obtain, since $2 \leq r < +\infty$

$$\|u(t)\|_{L^r} \leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty}e^{-2[\frac{\lambda_1}{r}-K]t}. \tag{5.20}$$

Now, Equation (1.1) leads to the following integral equation

$$u(t+t_0) = e^{t_0\Delta}u(t) + \int_0^{t_0} e^{(t_0-s)\Delta}f(u(t+s))\,ds,$$

with $f(u) := u^2 - \fint u^2$. Taking $t_0 := \beta T$ and $t = t_1 := (1-\beta)T$ and using the $L^r - L^\infty$ estimates (5.15) we get

$$\begin{aligned}\|u(T)\|_{L^\infty} &= \|u(t_1+t_0)\|_{L^\infty} \leq \|e^{t_0\Delta}u(t_1)\|_{L^\infty} + \int_0^{t_0}\|e^{(t_0-s)\Delta}f(u(t_1+s))\|_{L^\infty}\,ds \\ &\leq 2^{1-\frac{1}{r}}H^{\frac{1}{r}}t_0^{-\frac{N}{2r}}\|u(t_1)\|_{L^r} + \int_0^{t_0} 2^{1-\frac{1}{r}}H^{\frac{1}{r}}(t_0-s)^{-\frac{N}{2r}}\|f(u(t_1+s))\|_{L^r}\,ds.\end{aligned}$$

Using Hölder inequality for the first line and (5.20) for the second, we get

$$\begin{aligned} \|f(u(t))\|_{L^r} &\leq 2\|u(t)\|^2_{L^{2r}} \\ &\leq 2|\Omega|^{\frac{1}{r}}\|u_0\|^2_{L^\infty} e^{-2[\frac{\lambda_1}{r}-2K]t}. \end{aligned} \tag{5.21}$$

Setting $\alpha_1 := 2[\frac{\lambda_1}{r} - K] > 0$ and $\alpha_2 := 2[\frac{\lambda_1}{r} - 2K] > 0$ we obtain using (5.20) and (5.21):

$$\begin{aligned} \frac{\|u(T)\|_{L^\infty}}{\left(2^{1-\frac{1}{r}}H^{\frac{1}{r}}\right)} &\leq t_0^{-\frac{N}{2r}}\|u(t_1)\|_{L^r} + \int_0^{t_0}(t_0-s)^{-\frac{N}{2r}}\|f(u(t_1+s))\|_{L^r}\,ds \\ &\leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty} t_0^{-\frac{N}{2r}} e^{-\alpha_1 t_1} \\ &\quad +2|\Omega|^{\frac{1}{r}}\|u_0\|^2_{L^\infty}\int_0^{t_0}(t_0-s)^{-\frac{N}{2r}}e^{-\alpha_2(t_1+s)}\,ds \\ &\leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty} t_0^{-\frac{N}{2r}} e^{-\alpha_1 t_1} \\ &\quad +2|\Omega|^{\frac{1}{r}}\|u_0\|^2_{L^\infty}e^{-\alpha_2 t_1}\int_0^{t_0}(t_0-s)^{-\frac{N}{2r}}\,ds \\ &= |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty} t_0^{-\frac{N}{2r}}\left[e^{-\alpha_1 t_1} + 2\|u_0\|_{L^\infty}\frac{t_0 e^{-\alpha_2 t_1}}{1-\frac{N}{2r}}\right] \\ &\leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty} t_0^{-\frac{N}{2r}}\left[e^{-\alpha_1(1-\beta)T} + \frac{2\|u_0\|_{L^\infty}\beta T e^{-\alpha_2(1-\beta)T}}{1-\frac{N}{2r}}\right] \\ &\leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty} t_0^{-\frac{N}{2r}}\left[e^{-\alpha_1(1-\beta)T} + \frac{2\beta\|u_0\|_{L^\infty}}{1-\frac{N}{2r}}\sup_{\overline{T}>0}\left(\overline{T}e^{-\alpha_2(1-\beta)\overline{T}}\right)\right] \\ &\leq |\Omega|^{\frac{1}{r}}\|u_0\|_{L^\infty}(\beta T)^{-\frac{N}{2r}}\left[e^{-\alpha_1(1-\beta)T} + \frac{2\beta e^{-1}\|u_0\|_{L^\infty}}{1-\frac{N}{2r}}\cdot\frac{1}{\alpha_2(1-\beta)}\right]. \end{aligned}$$

□

End of the proof of Theorem 5.1

First step: Global existence.

Let u be a solution of (1.1)-(1.2) whose initial data u_0 satisfies (5.13). Let us suppose, for a contradiction, that the maximal time of existence T_{max} of u is finite. Put $K = \frac{\lambda_1}{4r}$ in the last lemma. Let T be the maximal time such that $\|u(t)\|_{L^\infty} \leq K$ in $[0,T]$. Hence (see (5.18))

$$K = \|u(T)\|_{L^\infty} \leq \|u_0\|_{L^\infty} e^{2KT}. \tag{5.22}$$

From the assumption (5.13), we have $\|u_0\|_{L^\infty} \leq \alpha\frac{\lambda_1}{4r} = \alpha K$, with

$$\alpha = \exp\left(-\left(\gamma_{N,r}\,H|\,\Omega|\,\lambda_1^{\frac{N}{2}}\right)^{\frac{2}{N}}\right) \quad < \quad 1$$

After replacing into (5.22), this gives

$$T \geq -\frac{\ln \alpha}{2K}. \tag{5.23}$$

Applying Lemma 5.5 with $\beta = \frac{1}{2}$, we get:

$$\|u(T)\|_{L^\infty} \leq 2^{\frac{r-1}{r}} H^{\frac{1}{r}} |\Omega|^{\frac{1}{r}} (T/2)^{-\frac{N}{2r}} \|u_0\|_{L^\infty} \left[e^{-\frac{3\lambda_1}{4r}T} + \frac{2e^{-1}\|u_0\|_{L^\infty}}{\frac{\lambda_1}{r}\left(1-\frac{N}{2r}\right)} \right].$$

Since $\|u_0\|_{L^\infty} \leq \alpha \frac{\lambda_1}{4r} = \alpha K$, it follows that

$$\begin{aligned}
\frac{\|u(T)\|_{L^\infty}}{K} &\leq 2^{\frac{r-1}{r}} H^{\frac{1}{r}} |\Omega|^{\frac{1}{r}} (T/2)^{-\frac{N}{2r}} \alpha \left[e^{-\frac{3\lambda_1}{4r}T} + \frac{e^{-1}\alpha}{2\left(1-\frac{N}{2r}\right)} \right] \\
&< 2^{\frac{r-1}{r}} H^{\frac{1}{r}} |\Omega|^{\frac{1}{r}} (T/2)^{-\frac{N}{2r}} \left[1 + \frac{e^{-1}}{2\left(1-\frac{N}{2r}\right)} \right] \\
&\leq 2^{\frac{r-1}{r}} H^{\frac{1}{r}} |\Omega|^{\frac{1}{r}} \left(\frac{(-\ln \alpha)}{\frac{\lambda_1}{r}} \right)^{-\frac{N}{2r}} \left[1 + \frac{e^{-1}}{2\left(1-\frac{N}{2r}\right)} \right] \\
&= \left(\gamma_{N,r} H |\Omega| \lambda_1^{\frac{N}{2}} (-\ln \alpha)^{-\frac{N}{2}} \right)^{\frac{1}{r}} \\
&= 1
\end{aligned}$$

which shows that

$$\|u(T)\|_{L^\infty} < K.$$

This contradicts the definition of T.

<u>Second step</u>: Exponential decay.

Note that, since $K = \frac{\lambda_1}{4r}$, estimate (5.20) becomes

$$\|u(t)\|_{L^r} \leq |\Omega|^{\frac{1}{r}} \|u_0\|_{L^\infty} e^{-\frac{3\lambda_1}{2r}t}$$

for all $t \in [0, +\infty[$. Using again the integral equation, we have

$$u(t+1) = e^{\Delta} u(t) + \int_0^1 e^{(1-s)\Delta} f(u(t+s))\, ds.$$

Using the same computation as in the proof of Lemma 5.5, we obtain for all $t > 0$

$$\begin{aligned}
\left(2^{\frac{r-1}{r}} H^{\frac{1}{r}}\right)^{-1} \|u(t+1)\|_{L^\infty} &\leq \|u(t)\|_{L^r} + \int_0^1 (1-s)^{-\frac{N}{2r}} \|f(u(t+s))\|_{L^r}\, ds \\
&\leq |\Omega|^{\frac{1}{r}} \|u_0\|_{L^\infty} e^{-\frac{\lambda_1}{r}t} \left[1 + 2\|u_0\|_{L^\infty} \int_0^1 (1-s)^{-\frac{N}{2r}} \right] ds \\
&\leq \left(2^{\frac{r-1}{r}} H^{\frac{1}{r}}\right)^{-1} C(r) \|u_0\|_{L^\infty} e^{-\frac{\lambda_1}{r}t}.
\end{aligned}$$

□

6 Appendix : proof of a Gamma-convergence-like result

We give here the following result which is more precise than proposition 4.2, and propose a self-contained proof.

Proposition 6.1 ***(Limits of the energy of the minimizers)***
Assume that Ω is a bounded domain. Then for every $\varepsilon > 0$, there exists at least one minimizer u^ε of the following problem

$$I^\varepsilon = \inf_{u \in A} J^\varepsilon(u)$$

and

$$\liminf_{\varepsilon \to 0} I^\varepsilon \geq J^0(u^0) \geq I^0 > 0$$

More precisely, there exists a subsequence $(u^{\varepsilon'})_{\varepsilon'}$ such that

$$u^{\varepsilon'} \longrightarrow u^0 \quad in \quad L^1(\Omega)$$

and $u^0 \in B$, where

$$B := \left\{ u \in BV(\Omega); \quad u = \pm 1 \ a.e. \ in \ \Omega; \quad \int_\Omega u = 0 \right\}$$

and

$$I^0 = \inf_{u \in B} J^0(u)$$

where

$$J^0(u) = c|\nabla u|(\Omega) \quad with \quad c = \int_{-1}^{1} \sqrt{f(s)}\, ds.$$

To prove proposition 6.1, we will use the following classical compactness result in $BV(\Omega)$.

Proposition 6.2 ***(Compactness in $BV(\Omega)$, [17])***
Let Ω be a bounded domain. For every sequence $(v^n)_n$, bounded in $BV(\Omega)$, there exists a subsequence $(v^{n'})_{n'}$ and $v^\infty \in BV(\Omega)$ such that

$$v^{n'} \longrightarrow v^\infty \quad in \quad L^1(\Omega)$$

and

$$\liminf_{n' \to +\infty} |\nabla v^{n'}|(\Omega) \quad \geq \quad |\nabla v^\infty|(\Omega).$$

Proof of Proposition 6.1
The proof of this proposition is done in the following steps.

Step 1: there exists a minimizer u^ε.
It is easy to see that there exists a constant $C > 0$ such that

$$|v|^{p+1} + C \geq f(v) \geq |v|^{p+1} - C \quad \text{for} \quad v \geq -1. \tag{6.24}$$

Therefore, for $\varepsilon > 0$ fixed, every minimizing sequence of J^ε in A is bounded in $H^1(\Omega) \cap L^{p+1}(\Omega)$. From the compactness of the injection $H^1(\Omega) \longrightarrow L^2(\Omega)$, it is classical to get the existence of a minimizer $u^\varepsilon \in A$ of J^ε.

Step 2: there exists $C_0 > 0$ such that $J^\varepsilon(u^\varepsilon) \leq C_0$ for ε small enough.
Here for ε small enough we will build a function $w^\varepsilon \in A$ such that $J^\varepsilon(w^\varepsilon) \leq C_0$.
Let us consider the direction x_1 and assume that the hyperplane $\{x_1 = 0\}$ separates Ω in two equal volumes:

$$|\Omega \cap \{x_1 < 0\}| = |\Omega \cap \{x_1 > 0\}|\,.$$

For $\delta > 0$, we define the function

$$v^\delta(x_1) = \begin{cases} -1 & \text{if} \quad x_1 < -\varepsilon \\ \dfrac{x_1}{\varepsilon} & \text{if} \quad -\varepsilon \leq x_1 \leq \varepsilon(1+\delta) \\ 1+\delta & \text{if} \quad x_1 > \varepsilon(1+\delta). \end{cases}$$

Next we define the translation of v^δ, for a small parameter $a \in \mathbb{R}$:

$$v_a^\delta(x_1) = v^\delta(x_1 - a).$$

Then for a close enough to zero and fixed, there is a unique $\delta = \delta(a) > 0$ such that

$$\int_\Omega |v_a^{\delta(a)}|^2 = 1.$$

In particular, this implies that

$$\int_{\Omega \cap \{x_1 > a+\varepsilon\}} \left(|v_a^{\delta(a)}|^2 - 1\right) = \int_{\Omega \cap \{a-\varepsilon < x_1 < a+\varepsilon\}} \left(1 - |v_a^{\delta(a)}|^2\right)$$

and then

$$\frac{1}{2}\,|\Omega|\,(1 - o(|a| + \varepsilon(1+\delta)))\left((1+\delta)^2 - 1\right) \leq 2\varepsilon(\mathrm{diam}(\Omega))^{n-1}$$

i.e. for a close enough to zero, there exists a constant $C > 0$ such that

$$\delta(a) \leq C\varepsilon \leq 1 \quad \text{for} \quad \varepsilon \quad \text{small enough.}$$

Then we consider the map

$$a \longmapsto \Phi(a) = \int_\Omega v_a^{\delta(a)}.$$

We have $\Phi(-2\varepsilon) > 0$. On the other hand, because the open set Ω is connected, there exists a constant $C_2 > 0$ such that $|\Omega \cap \{0 < x_1 < \eta\}| \geq C_2\eta$ for $\eta > 0$ small enough. Therefore $\Phi(a) \leq |\Omega|/2\delta(a) - C_2(a - \varepsilon)$ and then $\Phi(\varepsilon(1 + C|\Omega|/C_2)) < 0$. Using the continuity of the map Φ, we deduce that

$$\exists a \in (-2\varepsilon, (1 + C|\Omega|/C_2)\varepsilon) \quad \int_\Omega v_a^{\delta(a)} = 0.$$

We set $w^\varepsilon = v_a^{\delta(a)} \in A$ and estimate $J^\varepsilon(w^\varepsilon)$ as follows

$$\int_\Omega \varepsilon|\nabla w^\varepsilon|^2 \leq (2 + \delta(a))(\mathrm{diam}(\Omega))^{n-1}$$

$$\int_\Omega \frac{1}{\varepsilon} f(w^\varepsilon) \leq \left(\sup_{[-1,1]} f\right) 2(\mathrm{diam}(\Omega))^{n-1} + |\Omega|\frac{1}{\varepsilon} \sup_{[1,1+\delta(a)]} f.$$

We then remark that

$$\sup_{[1,1+\delta]} f \leq \frac{1}{2} f''(1)(\delta)^2 + o(\delta^2)$$

because $f(1) = f'(1) = 0$. We deduce that

$$\frac{1}{\varepsilon} \sup_{[1,1+\delta(a)]} f \leq \frac{1}{2} f''(1) C^2 \varepsilon + o(\varepsilon)$$

Putting all together we get the existence of a constant $C_0 > 0$ such that for ε small enough we get

$$J^\varepsilon(w^\varepsilon) \leq C_0.$$

Because $w^\varepsilon \in A$, and u^ε is a minimizer of J^ε on A, we deduce that

$$J^\varepsilon(u^\varepsilon) \leq J^\varepsilon(w^\varepsilon) \leq C_0.$$

This ends the proof of Step 2.

Step 3: there exists $C_1 > 0$ such that $||\mathbf{v}^\varepsilon||_{\mathbf{BV}(\mathbf{\Omega})} \leq \mathbf{C_1}$ for ε small enough.

We define

$$G(s) = \begin{cases} \displaystyle\int_{-1}^{s} \sqrt{f}(t)\ dt & \text{if} \quad s \geq -1 \\ 0 & \text{if} \quad s < -1 \end{cases}$$

and

$$v^\varepsilon = G(u^\varepsilon).$$

From (6.24), we get for $t \geq -1$

$$f(t) \leq |t|^{p+1} + C$$

and then there exists some constants $C, C' > 0$ such that (using $p > 1$)

$$G(s) \leq C\left(|s|^{\frac{p+3}{2}} + |s| + 1\right) \leq C'\left(|s|^{p+1} + |s| + 1\right) \tag{6.25}$$

To estimate $||v^\varepsilon||_{L^1(\Omega)}$, we will first estimate $||u^\varepsilon||_{L^1(\Omega)}$ and $||u^\varepsilon||_{L^{p+1}(\Omega)}$. Because $\int_\Omega u^\varepsilon = 0$, we remark that $\int_\Omega (u^\varepsilon)^+ = \int_\Omega (u^\varepsilon)^- \leq |\Omega|$ and then

$$||u^\varepsilon||_{L^1(\Omega)} \leq 2|\Omega|.$$

From Step 2, we have $J^\varepsilon(u^\varepsilon) \leq C_0$, and then because of (6.24), we get

$$\int_\Omega |u^\varepsilon|^{p+1} - C|\Omega| \leq C_0\varepsilon$$

which gives

$$||u^\varepsilon||_{L^{p+1}(\Omega)} \leq (C|\Omega| + C_0\varepsilon)^{\frac{1}{p+1}}.$$

Putting all together we deduce from (6.25):

$$||v^\varepsilon||_{L^1(\Omega)} = ||G(u^\varepsilon)||_{L^1(\Omega)} \leq C'\left(||u^\varepsilon||^{p+1}_{L^{p+1}(\Omega)} + ||u^\varepsilon||_{L^1(\Omega)} + |\Omega|\right).$$

To estimate the whole norm in $BV(\Omega)$ we only need to estimate $|\nabla v^\varepsilon|(\Omega)$. This is done, using the following classical trick of Modica [25] for every $u \in A \cap H^1(\Omega)$ (and $a^2 + b^2 \geq 2ab$)

$$J^\varepsilon(u) = \int_\Omega \varepsilon|\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon}f(u) \geq \int_\Omega 2|\nabla u|\sqrt{f(u)} = \int_\Omega 2|\nabla G(u)|. \tag{6.26}$$

Applied to v^ε, we get

$$\int_\Omega |\nabla v^\varepsilon| = \int_\Omega |\nabla G(u^\varepsilon)| \leq \frac{1}{2}J^\varepsilon(u^\varepsilon) \leq \frac{1}{2}C_0$$

This proves the expected inequality and ends Step 3.

Step 4: $II^0 \geq 2|\nabla v^0|(\Omega)$**.**
Let us define

$$II^0 = \liminf_{\varepsilon \to 0} J^\varepsilon(u^\varepsilon).$$

Then we extract a subsequence $(u^{\varepsilon'})_{\varepsilon'}$ such that

$$J^{\varepsilon'}(u^{\varepsilon'}) \longrightarrow II^0.$$

From (6.26), we have

$$J^{\varepsilon'}(u^{\varepsilon'}) \geq \int_\Omega 2|\nabla G(u^{\varepsilon'})| = \int_\Omega 2|\nabla v^{\varepsilon'}| = 2|\nabla v^{\varepsilon'}|(\Omega).$$

From Step 3 and the compactness result in BV (proposition 6.2), up to extract a new subsequence, we can assume that there exists $v^0 \in BV(\Omega)$ such that

$$v^{\varepsilon'} \longrightarrow v^0 \quad \text{in} \quad L^1(\Omega) \tag{6.27}$$

and

$$\liminf_{\varepsilon' \to 0} |\nabla v^{\varepsilon'}|(\Omega) \quad \geq \quad |\nabla v^0|(\Omega)$$

so that

$$II^0 \geq 2|\nabla v^0|(\Omega).$$

Moreover from (6.27), and the converse Lebesgue theorem, up to extraction of a subsequence, we can assume that

$$v^{\varepsilon'}(x) \longrightarrow v^0(x) \quad \text{for a.e.} \quad x \in \Omega. \tag{6.28}$$

Step 5. $\mathbf{u^{\varepsilon''} \longrightarrow u^0}$ in $\mathbf{L^{p+1}(\Omega)}$ and $\mathbf{u^0 = \pm 1}$ a.e. in $\mathbf{\Omega}$.
From Step 2, we have

$$\int_\Omega f(u^\varepsilon) \leq \varepsilon J^\varepsilon(u^\varepsilon) \leq C\varepsilon$$

and then

$$f(u^\varepsilon) \longrightarrow 0 \quad \text{in} \quad L^1(\Omega).$$

Then from the converse Lebesgue theorem, there exists a function $h \in L^1(\Omega)$, that we can always choose satisfying $h \geq 1$, such that there exists a subsequence $(u^{\varepsilon''})_{\varepsilon''}$ with

$$f(u^{\varepsilon''}(x)) \leq h(x) \quad \text{for a.e.} \quad x \in \Omega$$

and

$$f(u^{\varepsilon''}(x)) \longrightarrow 0 \quad \text{for a.e.} \quad x \in \Omega. \tag{6.29}$$

Our goal is now to prove that there exists a subsequence of $(u^{\varepsilon''})_{\varepsilon''}$ which is convergent to some u^0 in $L^{p+1}(\Omega)$, and $u^0 = \pm 1$ a.e. in Ω.
We remark that from (6.24), we have for $v \geq 1$, $f^{-1}(|v|^{p+1} - C) \leq v$, and then setting $h = |v|^{p+1} - C$, we get

$$f^{-1}(h) \leq |h + C|^{\frac{1}{p+1}}.$$

This proves that $f^{-1}(h) \in L^{p+1}(\Omega)$, and

$$|u^{\varepsilon''}| \leq 1 + f^{-1}(h) \in L^{p+1}(\Omega). \tag{6.30}$$

Now from (6.28) and the continuity of G^{-1} on $(0,+\infty)$, we have

$$u^{\varepsilon''} = G^{-1}(v^{\varepsilon''}) \longrightarrow G^{-1}(v^0) =: u^0 \quad \text{for a.e.} \quad x \in \Omega.$$

Moreover from (6.30), we deduce that $u^0 \in L^{p+1}(\Omega)$ and

$$u^{\varepsilon''} \longrightarrow u^0 \quad \text{in} \quad L^{p+1}(\Omega). \tag{6.31}$$

Consequently from (6.29), we deduce

$$f(u^{\varepsilon''}(x)) \longrightarrow 0 = f(u^0(x)) \quad \text{for a.e.} \quad x \in \Omega$$

and then

$$u^0(x) = \pm 1 \quad \text{for a.e.} \quad x \in \Omega.$$

This ends the proof of Step 5.

Step 6: $\int_\Omega \mathbf{u^0} = \mathbf{0}$, $\mathbf{u^0} \in \mathbf{BV}(\mathbf{\Omega})$ and $|\nabla \mathbf{v^0}|(\mathbf{\Omega}) = \frac{\mathbf{G(1)}}{\mathbf{2}}|\nabla \mathbf{u^0}|(\mathbf{\Omega})$.
(i) From (6.31), we get in particular that $u^{\varepsilon''} \longrightarrow u^0$ in $L^1(\Omega)$, and then $0 = \int_\Omega u^\varepsilon \longrightarrow \int_\Omega u^0$ which proves that

$$\int_\Omega u^0 = 0.$$

(ii) We have $v^0 = G(u^0) \in BV(\Omega)$ and $u^0 = \pm 1$ a.e. in Ω. Therefore $v^0 = G(\pm 1)$ a.e. in Ω. Moreover, because each of these two functions only takes two values, we can express u^0 as a function of v^0, i.e. (using $G(-1) = 0$)

$$u^0 = \frac{2}{G(1)} v^0 - 1.$$

Thus $u^0 \in BV(\Omega)$ and $|\nabla u^0|(\Omega) = \frac{2}{G(1)}|\nabla v^0|(\Omega)$. This ends the proof of Step 6. Consequently we get

$$II^0 \geq J^0(u^0) \quad \text{with} \quad u^0 \in B.$$

Step 7: $\inf_{w \in B} J^0(w) > 0$.
First notice that $I^0 = \inf_{w \in B} J^0(w) < +\infty$, because $J^0(u^0) \leq II^0 \leq C_0 < +\infty$.
Let us assume that $I^0 = 0$. Then we can consider a minimizing sequence $w^k \in B$ such that $J^0(w^k) \longrightarrow 0$. By definition of the BV-norm, of B and of J^0, we see that the sequence $(w^k)_k$ is bounded in $BV(\Omega)$:

$$\|w^k\|_{BV(\Omega)} \leq \|w^k\|_{L^1(\Omega)} + \|\nabla w^k\|_{L^1(\Omega)} = |\Omega| + \frac{1}{c} J^0(w^k) \leq |\Omega| + \frac{C_0}{c}.$$

From the compactness result for BV (proposition 6.2), up to extract a subsequence, we get the existence of a function $w^\infty \in BV(\Omega)$, such that

$$w^k \longrightarrow w^\infty \quad \text{in} \quad L^1(\Omega) \tag{6.32}$$

and

$$0 = \liminf_{k \to +\infty} |\nabla w^k|(\Omega) \geq |\nabla w^\infty|(\Omega)$$

Therefore w^∞ is constant on Ω, and because of (6.32) and $\int_\Omega w^k = 0$, we have

$$\int_\Omega w^\infty = 0$$

and then $w^\infty \equiv 0$ on Ω.
On the other hand, because of (6.32), up to extract a subsequence, we have

$$w^k(x) \longrightarrow w^\infty(x) \quad \text{a.e. in} \quad \Omega$$

and then $w^\infty(x) = \pm 1$ a.e. in Ω. Contradiction.
Then $\inf_{w \in B} J^0(w) > 0$. This ends the proof of proposition 6.1. $\square$

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