Dynamics of dislocation densities in a bounded channel. Part I: smooth solutions to a singular coupled parabolic system

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Abstract

We study a coupled system of two parabolic equations in one space dimension. This system is singular because of the presence of one term with the inverse of the gradient of the solution. Our system describes an approximate model of the dynamics of dislocation densities in a bounded channel submitted to an exterior applied stress. The system of equations is written on a bounded interval with Dirichlet conditions and requires a special attention to the boundary. The proof of existence and uniqueness is done under the use of a certain comparison principle on the gradient of the solution.

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Key words: Boundary value problems, parabolic systems, comparison principle.

1 Introduction

1.1 Setting of the problem

In this paper, we are concerned in the study of the following singular parabolic system:

$$\begin{cases} \kappa_t = \varepsilon\kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau\rho_x & \text{on} \quad I \times (0,\infty) \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau\kappa_x & \text{on} \quad I \times (0,\infty), \end{cases} \tag{1.1}$$

with the initial conditions:

$$\kappa(x,0) = \kappa^0(x) \quad \text{and} \quad \rho(x,0) = \rho^0(x), \tag{1.2}$$

and the boundary conditions:

$$\begin{cases} \kappa(0,.) = \kappa^0(0) \quad \text{and} \quad \kappa(1,.) = \kappa^0(1), \\ \rho(0,.) = \rho(1,.) = 0, \end{cases} \tag{1.3}$$

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where

$$\varepsilon > 0, \quad \tau \in \mathbb{R},$$

are fixed constants, and

$$I := (0,1)$$

is the open and bounded interval of $\mathbb{R}$.

The goal is to show the long-time existence and uniqueness of a smooth solution of (1.1), (1.2) and (1.3). Our motivation comes from a problem of studying the dynamics of dislocation densities in a constrained channel submitted to an exterior applied stress. In fact, system (1.1) can be seen as an approximate model of the one described in [7]. This approximate model (presented in [7] for $\varepsilon = 0$) reads:

$$\begin{cases} \theta_t^+ = \varepsilon\theta_{xx}^+ + \left[\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau\right)\theta^+\right]_x & \text{on} \quad I \times (0,\infty), \\ \theta_t^- = \varepsilon\theta_{xx}^- - \left[\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau\right)\theta^-\right]_x & \text{on} \quad I \times (0,\infty), \end{cases} \tag{1.4}$$

with τ representing the exterior stress field. System (1.4) can be deduced from (1.1), by spatially differentiating (1.1), and by considering

$$\rho_x^{\pm} = \theta^{\pm}, \quad \rho = \rho^+ - \rho^-, \quad \kappa = \rho^+ + \rho^-, \tag{1.5}$$

which explains the presence of the factor $(1+\varepsilon)$ in the second equation of (1.1). Here θ^+ and θ^- represent the densities of the positive and negative dislocations respectively (see [18, 9] for a physical study of dislocations).

The part II of this work will be presented in [12]. There, we will show some kind of convergence of the solution $(\rho^\varepsilon, \kappa^\varepsilon)$ as $\varepsilon \to 0$.

1.2 Statement of the main result

The main result of this paper is:

Theorem 1.1 ***(Existence and uniqueness of a solution).*** *Let ρ^0, κ^0 satisfying:*

$$\rho^0, \kappa^0 \in C^\infty(\bar{I}), \quad \rho^0(0) = \rho^0(1) = \kappa^0(0) = 0, \quad \kappa^0(1) = 1,$$

$$\begin{cases} (1+\varepsilon)\rho_{xx}^0 = \tau\kappa_x^0 & \textit{on} \quad \partial I \\ (1+\varepsilon)\kappa_{xx}^0 = \tau\rho_x^0 & \textit{on} \quad \partial I, \end{cases} \tag{1.6}$$

and

$$\kappa_x^0 > |\rho_x^0| \quad \textit{on} \quad \bar{I}.$$

Then there exists a unique global solution (ρ, κ) of system (1.1), (1.2) and (1.3) satisfying

$$(\rho, \kappa) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{I} \times [0,\infty)) \cap C^\infty(\bar{I} \times (0,\infty)), \quad \forall \alpha \in (0,1). \tag{1.7}$$

Moreover, this solution also satisfies :

$$\kappa_x > |\rho_x| \quad \textit{on} \quad \bar{I} \times [0,\infty). \tag{1.8}$$

Remark 1.2 *Conditions (1.6) are natural here. Indeed, the regularity (1.7) of the solution of (1.1) with the boundary conditions (1.2) and (1.3) imply in particular (1.6).*

Remark 1.3 *Remark that the choice $\kappa^0(0) = 0$ and $\kappa^0(1) = 1$ does not reduce the generality of the problem, because equation (1.1) does not see the constants and has the following invariance: if (ρ, κ) is a solution, then $(\lambda\rho, \lambda\kappa)$ is also a solution for any $\lambda \in \mathbb{R}$.*

1.3 Brief review of the literature

To our knowledge, systems of equations involving the singularity in $1/\kappa_x$ as in (1.1) has not been directly handled elsewhere in the literature. However, parabolic problems involving singular terms have been widely studied in various aspects. Fast diffusion equations:

$$u_t - \Delta u^m = 0, \quad 0 < m < 1,$$

are examined, for instance, in [2, 4, 5]. These equations are singular at points where $u = 0$. In dimension 1, setting $u = v_x$ we get, up to a constant of integration:

$$v_t - m v_x^{m-1} v_{xx} = 0$$

which makes appear a singularity like $1/v_x$. Other class of singular parabolic equations are for instance of the form:

$$u_t = u_{xx} + \frac{b}{x} u_x, \tag{1.9}$$

where b is a certain constant. Such an equation is related to axially symmetric problems and also occurs in probability theory (see [3, 16]). An important type of equations that can be indirectly related to our system are semilinear parabolic equations:

$$u_t = \Delta u + |u|^{p-1} u, \quad p > 1. \tag{1.10}$$

Many authors have studied the blow-up phenomena for solutions of the above equation (see for instance [17, 8]). Equation (1.10) can be somehow related to the first equation of (1.1), but with a singularity of the form $1/\kappa$. This can be formally seen if we first suppose that $u \geq 0$, and then we apply the following change of variables $u = 1/v$. In this case, equation (1.10) becomes:

$$v_t = \Delta v - \frac{2|\nabla v|^2}{v} - v^{2-p},$$

and hence if $p = 3$, we obtain:

$$v_t = \Delta v - \frac{1}{v}(1 + 2|\nabla v|^2). \tag{1.11}$$

Since the solution u of (1.10) may blow-up at a finite time $t = T$, then v may vanishes at $t = T$, and therefore equation (1.11) faces similar singularity to that of the first equation of (1.1), but in terms of the solution v instead of v_x.

1.4 Strategy of the proof

The existence and uniqueness is made by using a fixed point argument after a slight artificial modification in the denominator κ_x of the first equation of (1.1) in order to avoid dividing by zero. We will first show the short time existence, proving in particular that

$$M(x,t) = \kappa_x(x,t) - \sqrt{\gamma^2(t) + \rho_x^2(x,t)} \geq 0,$$

for some well chosen initial data and a suitable function $\gamma(t) = ce^{-ct}$, $c > 0$. This follows from the PDE satisfied by M. Let us mention that one of the key points here is that $\left|\frac{\rho_x}{\kappa_x}\right| \leq 1$ which somehow linearize the first equation of (1.1). After that, due to some *a priori* estimates, we can prove the global time existence.

Remark 1.4 *In a previous version of the present paper (see the PhD thesis of H. Ibrahim [10] and the preprint [11]), our main arguments of the proof involved some estimates on higher derivatives of the solution, which required the use of a parabolic logarithmic Sobolev inequality of the Kozono-Taniuchi type (see [13]). We are grateful to an anonymous referee whose suggestions have simplified the presentation of this paper.*

1.5 Organization of the paper

This paper is organized as follows: in Section 2, we present the tools needed throughout this work, this includes a brief recall on the L^p and C^α theory for parabolic equations. In Section 3, we show a comparison principle associated to (1.1) that will play a crucial rule in the long time existence of the solution as well as the positivity of κ_x. In Section 4, we present a result of short time existence, uniqueness and regularity of a solution (ρ, κ) of (1.1). Section 5 is devoted to give some exponential bounds on the spatial derivatives up to order 2 of ρ and κ. In Section 6, we prove our main result: Theorem 1.1. Finally, we show in an Appendix, the proofs of some technical results.

2 Tools: theory of parabolic equations

We start with some basic notations and terminology:

Abridged notation.

• I_T is the cylinder $I \times (0,T)$; $\bar{I}$ is the closure of I; $\overline{I_T}$ is the closure of I_T; ∂I is the boundary of I.

• $\|.\|_{L^p(\Omega)} = \|.\|_{p,\Omega}$, Ω is an open set, $p \geq 1$.

• S_T is the lateral boundary of I_T, or more precisely, $S_T = \partial I \times (0,T)$.

• $\partial^p I_T$ is the parabolic boundary of I_T, i.e. $\partial^p I_T = \overline{S_T} \cup (I \times \{t = 0\})$.

- $D_y^s u = \frac{\partial^s u}{\partial y^s}$, u is a function depending on the parameter y, $s \in \mathbb{N}$.

- $[l]$ is the floor part of $l \in \mathbb{R}$.

- $|\Omega|$ is the n-dimensional Lebesgue measure of the open set $\Omega \subset \mathbb{R}^n$.

2.1 L^p and C^α theory of parabolic equations

A major part of this work deals with the following typical problem in parabolic theory:

$$\begin{cases} u_t = \varepsilon u_{xx} + f & \text{on} \quad I_T \\ u(x,0) = \phi & \text{on} \quad I \\ u = \Phi & \text{on} \quad \partial I \times (0,T), \end{cases} \tag{2.1}$$

where $T > 0$ and $\varepsilon > 0$. A wide literature on the existence and uniqueness of solutions of (2.1) in different function spaces could be found for instance in [14], [6] and [15]. We will deal mainly with two types of spaces:

The Sobolev space $W_p^{2,1}(I_T)$, $1 < p < \infty$ which is the Banach space consisting of the elements in $L^p(I_T)$ having generalized derivatives of the form $D_t^r D_x^s u$, with r and s two non-negative integers satisfying the inequality $2r + s \leq 2$, also in $L^p(I_T)$. The norm in this space is defined as $\|u\|_{W_p^{2,1}(I_T)} = \sum_{i=0}^{2} \sum_{2r+s=i} \|D_t^r D_x^s u\|_{p,I_T}$.

The Hölder spaces $C^\ell(\bar{I})$ and $C^{\ell,\ell/2}(\overline{I_T})$, $\ell > 0$ a nonintegral positive number. We do not recall the definition of the space $C^\ell(\bar{I})$ which is very standard. The Hölder space $C^{\ell,\ell/2}(\overline{I_T})$ is the Banach space of functions $v(x,t)$ that are continuous in $\overline{I_T}$, together with all derivatives of the form $D_t^r D_x^s v$ for $2r + s < \ell$, and have a finite norm $|v|_{I_T}^{(\ell)} = \langle v \rangle_{I_T}^{(\ell)} + \sum_{j=0}^{[\ell]} \langle v \rangle_{I_T}^{(j)}$, where

$$\langle v \rangle_{I_T}^{(0)} = |v|_{I_T}^{(0)} = \|v\|_{\infty,I_T}, \quad \langle v \rangle_{I_T}^{(j)} = \sum_{2r+s=j} |D_t^r D_x^s v|_{I_T}^{(0)}, \quad \langle v \rangle_{I_T}^{(\ell)} = \langle v \rangle_{x,I_T}^{(\ell)} + \langle v \rangle_{t,I_T}^{(\ell/2)},$$

and

$$\langle v \rangle_{x,I_T}^{(\ell)} = \sum_{2r+s=[\ell]} \langle D_t^r D_x^s v \rangle_{x,I_T}^{(\ell-[\ell])}, \quad \langle v \rangle_{t,I_T}^{(\ell/2)} = \sum_{0<\ell-2r-s<2} \langle D_t^r D_x^s v \rangle_{t,I_T}^{\left(\frac{\ell-2r-s}{2}\right)},$$

with

$$\langle v \rangle_{x,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t) - v(x',t)| \leq c|x - x'|^\alpha,\ (x,t),(x',t) \in \overline{I_T}\}, \quad 0 < \alpha < 1,$$

$$\langle v \rangle_{t,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t) - v(x,t')| \leq c|t - t'|^\alpha,\ (x,t),(x,t') \in \overline{I_T}\}, \quad 0 < \alpha < 1.$$

The above definitions could be found in details in [14, Section 1]. Now, we write down the compatibility conditions of order 0 and 1. These compatibility conditions concern the given data ϕ, Φ and f of problem (2.1).

Compatibility condition of order 0. Let $\phi \in C(\bar{I})$ and $\Phi \in C(\overline{S_T})$. We say that the compatibility condition of order 0 is satisfied if

$$\phi\big|_{\partial I} = \Phi\big|_{t=0}. \tag{2.2}$$

Compatibility condition of order 1. Let $\phi \in C^2(\bar{I})$, $\Phi \in C^1(\overline{S_T})$ and $f \in C(\overline{I_T})$. We say that the compatibility condition of order 1 is satisfied if (2.2) is satisfied and in addition we have:

$$(\varepsilon\phi_{xx} + f)\big|_{\partial I} = \frac{\partial \Phi}{\partial t}\Big|_{t=0}. \tag{2.3}$$

We state two results of existence and uniqueness adapted to our special problem. We begin by presenting the solvability of parabolic equations in Hölder spaces.

Theorem 2.1 ***(Solvability in Hölder spaces, [14, Theorem 5.2]).*** *Suppose $0 < \alpha < 2$, a non-integral number. Then for any*

$$\phi \in C^{2+\alpha}(\bar{I}), \quad \Phi \in C^{1+\alpha/2}(\overline{S_T}) \quad \text{and} \quad f \in C^{\alpha,\alpha/2}(\overline{I_T})$$

satisfying the compatibility condition of order 1 *(see (2.2) and (2.3)), problem (2.1) has a unique solution $u \in C^{2+\alpha,1+\alpha/2}(\overline{I_T})$ satisfying the following inequality:*

$$|u|_{I_T}^{(2+\alpha)} \leq ce^{cT}\left(|f|_{I_T}^{(\alpha)} + |\phi|_I^{(2+\alpha)} + |\Phi|_{S_T}^{(1+\alpha/2)}\right), \tag{2.4}$$

for some $c = c(\varepsilon, \alpha) > 0$.

We now present the solvability in Sobolev spaces. Recall the norm of fractional Sobolev spaces. If $f \in W_p^s(a,b)$, $s > 0$ and $1 < p < \infty$, then

$$\|f\|_{W_p^s(a,b)} = \|f\|_{W_p^{[s]}(a,b)} + \left(\int_a^b \int_a^b \frac{|f^{([s])}(x) - f^{([s])}(y)|^p}{|x-y|^{1+(s-[s])p}}\right)^{1/p}. \tag{2.5}$$

Theorem 2.2 ***(Solvability in Sobolev spaces, [14, Theorem 9.1]).*** *Let $p > 1$, $\varepsilon > 0$ and $T > 0$. For any $f \in L^p(I_T)$, $\phi \in W_p^{2-2/p}(I)$ and $\Phi \in W_p^{1-1/2p}(S_T)$, with $p \neq 3/2$ ($p = 3/2$ is called the* ***singular*** *index) satisfying in the case $p > 3/2$ the compatibility condition of order zero (see (2.2)), there exists a unique solution $u \in W_p^{2,1}(I_T)$ of (2.1) satisfying the following estimate:*

$$\|u\|_{W_p^{2,1}(I_T)} \leq c\left(\|f\|_{p,I_T} + \|\phi\|_{W_p^{2-2/p}(I)} + \|\Phi\|_{W_p^{1-1/2p}(S_T)}\right), \tag{2.6}$$

for some $c = c(\varepsilon, p, T) > 0$.

For a better understanding of the spaces stated in the above two theorems, especially fractional Sobolev spaces, we send the reader to [1] or [14]. The dependence of the constant c of Theorem 2.2 on the variable T will be of notable importance and this what is emphasized by the next lemma.

Lemma 2.3 ***(The constant*** c ***given by (2.6): case*** $\phi = 0$ ***and*** $\Phi = 0$***).*** *Under the same hypothesis of Theorem 2.2, with* $\phi = 0$ *and* $\Phi = 0$*, the estimate (2.6) can be written as:*

$$\frac{\|u\|_{p,I_T}}{T} + \frac{\|u_x\|_{p,I_T}}{\sqrt{T}} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \leq c\|f\|_{p,I_T}, \tag{2.7}$$

where $c = c(\varepsilon, p) > 0$ *is a positive constant depending only on* p *and* ε*.*

The proof of this lemma will be done in Appendix A. Moreover, We will frequently make use of the following two lemmas also depicted from [14].

Lemma 2.4 ***(Sobolev embedding in Hölder spaces, [14, Lemma 3.3]).***
(i) (Case $p > 3$*). For any function* $u \in W_p^{2,1}(I_T)$*, if* $\alpha = 1 - 3/p > 0$*, i.e.* $p > 3$*, then* $u \in C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{I_T})$ *with* $|u|_{I_T}^{(1+\alpha)} \leq c\|u\|_{W_p^{2,1}(I_T)}$*,* $c = c(p, T) > 0$*. However, in terms of* u_x*, we have that* $u_x \in C^{\alpha, \alpha/2}(\overline{I_T})$ *satisfies the following estimates:*

$$\|u_x\|_{\infty,I_T} \leq c\left\{\delta^{\alpha}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{\alpha-2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0, \tag{2.8}$$

$$\langle u_x\rangle_{I_T}^{(\alpha)} \leq c\left\{\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \frac{1}{\delta^2}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0.$$

(ii) (Case $p > 3/2$*). If* $u \in W_p^{2,1}(I_T)$ *with* $p > 3/2$*, then* $u \in C(\overline{I_T})$*, and we have the following estimate:*

$$\|u\|_{\infty,I_T} \leq c\left\{\delta^{2-3/p}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + \delta^{-3/p}\|u\|_{p,I_T}\right\}, \quad c = c(p) > 0. \tag{2.9}$$

In the above two cases $\delta = \min\{1/2, \sqrt{T}\}$*.*

Lemma 2.5 ***(Trace of functions in*** $W_p^{2,1}(I_T)$***, [14, Lemma 3.4]).*** *If* $u \in W_p^{2,1}(I_T)$*,* $p > 1$*, then for* $2r + s < 2 - 2/p$*, we have* $D_t^r D_x^s u\big|_{t=0} \in W_p^{2-2r-s-2/p}(I)$ *with*

$$\|u\|_{W_p^{2-2r-s-2/p}(I)} \leq c(T)\|u\|_{W_p^{2,1}(I_T)}.$$

In addition, for $2r + s < 2 - 1/p$*, we have* $D_t^r D_x^s u\big|_{\overline{S_T}} \in W_p^{1-r-s/2-1/2p}(\overline{S_T})$ *with*

$$\|u\|_{W_p^{1-r-s/2-1/2p}(\overline{S_T})} \leq c(T)\|u\|_{W_p^{2,1}(I_T)}.$$

A useful technical lemma will now be presented. The proof of this lemma will be done in Appendix A.

Lemma 2.6 ***(***L^∞ ***control of the spatial derivative).*** *Let* $p > 3$ *and let* $0 < T \leq 1/4$ *(this condition is taken for simplification). Then for every* $u \in W_p^{2,1}(I_T)$ *with* $u = 0$ *on* $\partial^p(I_T)$ *in the trace sense (see Lemma (2.5)), there exists a constant* $c(T, p) > 0$ *such that*

$$\|u_x\|_{\infty,I_T} \leq c(T,p)\|u\|_{W_p^{2,1}(I_T)}, \quad \text{with} \quad c(T,p) = c(p)T^{\frac{p-3}{2p}} \to 0 \text{ as } T \to 0.$$

3 A comparison principle

Proposition 3.1 ***(A comparison principle for system (1.1)).*** *Let*

$$(\rho,\kappa)\in\left(C^{3+\alpha,\frac{3+\alpha}{2}}\left(\overline{I_T}\right)\right)^2 \quad \textit{for some} \quad 0<\alpha<1,$$

be a solution of (1.1), (1.2) and (1.3) with $\kappa_x>0$*, and the initial conditions* ρ^0, κ^0 *satisfying:*

$$\kappa_x^0\geq\sqrt{\gamma_0^2+(\rho_x^0)^2} \quad \textit{on} \quad I, \quad \gamma_0\in(0,1). \tag{3.1}$$

Choose

$$\gamma(t)=\gamma_0 e^{-\tau^2 t/4\varepsilon}, \tag{3.2}$$

then we have

$$\kappa_x(x,t)\geq\sqrt{\gamma^2(t)+\rho_x^2(x,t)} \quad \textit{on} \quad I_T. \tag{3.3}$$

Proof. Throughout the proof, we will extensively use the following notation:

$$G_a(y)=\sqrt{a^2+y^2}, \quad a,y\in\mathbb{R}.$$

Without loss of generality (up to a change of variables in (x,t) and a re-definition of τ), assume in the proof that

$$I=(-1,1).$$

Define the quantity M by:

$$M(x,t)=\kappa_x(x,t)-G_{\gamma(t)}(\rho_x(x,t)), \quad (x,t)\in\overline{I_T},$$

$\gamma(t)>0$ is a function to be determined. The proof could be divided into five steps.

Step 1. (Partial differential inequality satisfied by M)

We first do the following computations on I_T:

$$M_t=\kappa_{xt}-G_\gamma'(\rho_x)\rho_{xt}-\frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}, \tag{3.4}$$

$$M_x=\kappa_{xx}-G_\gamma'(\rho_x)\rho_{xx}, \quad M_{xx}=\kappa_{xxx}-G_\gamma''(\rho_x)\rho_{xx}^2-G_\gamma'(\rho_x)\rho_{xxx}. \tag{3.5}$$

Deriving (1.1) with respect to x, we deduce that

$$\begin{cases}\kappa_{xt}=\varepsilon\kappa_{xxx}+\dfrac{\rho_{xx}^2}{\kappa_x}+\dfrac{\rho_x\rho_{xxx}}{\kappa_x}-\dfrac{\rho_x\rho_{xx}\kappa_{xx}}{\kappa_x^2}-\tau\rho_{xx},\\ \rho_{xt}=(1+\varepsilon)\rho_{xxx}-\tau\kappa_{xx}.\end{cases} \tag{3.6}$$

We set

$$\Gamma=\frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}, \quad F_\gamma(y)=y-\gamma\arctan(y/\gamma).$$

Doing again some direct computations, and using (3.4), (3.5) and (3.6), we obtain

$$\begin{aligned} M_t = \varepsilon M_{xx} + \left(\tau G_\gamma'(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) M_x + \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx}G_\gamma'(\rho_x)}{\kappa_x}\right) M \\ + \varepsilon G_\gamma''(\rho_x)\rho_{xx}^2 + \frac{\rho_{xx}^2}{\kappa_x^2}[G_\gamma(\rho_x) - G_\gamma'(\rho_x)\rho_x] - \tau(1 - F_\gamma'(\rho_x))\rho_{xx} - \Gamma. \end{aligned} \tag{3.7}$$

Using Young's inequality $2ab \leq a^2 + b^2$, we have:

$$\frac{\tau\gamma^2|\rho_{xx}|}{\gamma^2 + \rho_x^2} \leq \frac{\varepsilon\gamma^2\rho_{xx}^2}{(\gamma^2+\rho_x^2)^{3/2}} + \frac{\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}}. \tag{3.8}$$

Plugging (3.8) into (3.7), and using some properties of G_γ and F_γ, we get:

$$M_t \geq \varepsilon M_{xx} + \left(\tau G_\gamma'(\rho_x) - \frac{\rho_x\rho_{xx}}{\kappa_x^2}\right) M_x + \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx}G_\gamma'(\rho_x)}{\kappa_x}\right) M + A,$$

with

$$A = -\frac{\gamma^2\tau^2}{4\varepsilon\sqrt{\gamma^2+\rho_x^2}} - \frac{\gamma\gamma'}{\sqrt{\gamma^2+\rho_x^2}}.$$

Remark that choosing γ as in (3.2) gives $A = 0$.

Step 2. (The boundary conditions for M)

The boundary conditions (1.3), and the PDEs of system (1.1) imply the following equalities on the boundary (using the smoothness of the solution up to the boundary),

$$\begin{cases} \varepsilon\kappa_{xx} + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x = 0 & \text{on} \quad \partial I \times [0,T] \\ (1+\varepsilon)\rho_{xx} - \tau\kappa_x = 0 & \text{on} \quad \partial I \times [0,T]. \end{cases} \tag{3.9}$$

In particular (3.9) implies

$$M_x = -\frac{\tau}{1+\varepsilon}G_\gamma'(\rho_x)M \quad \text{on} \quad \partial I \times [0,T]. \tag{3.10}$$

To deal with the boundary condition (3.10), we now introduce the following change of unknown function:

$$\overline{M}(x,t) = \cosh(\beta x)M(x,t), \quad (x,t) \in \overline{I_T}.$$

We calculate $\overline{M}$ on the boundary of I to get:

$$\overline{M}_x = \left(\beta\tanh(\beta x) - \frac{\tau}{1+\varepsilon}G_\gamma'(\rho_x)\right)\overline{M} \quad \text{on} \quad \partial I \times [0,T]. \tag{3.11}$$

We claim that, for any fixed time t, it is impossible for $\overline{M}$ to have a positive minimum at the boundary of I. Indeed we have the following two cases:

$$\overline{M} \text{ has a positive minimum at } x = 1 \quad \Rightarrow \quad \overline{M}_x \leq 0;$$

$$\overline{M} \text{ has a positive minimum at } x = -1 \quad \Rightarrow \quad \overline{M}_x \geq 0.$$

Both cases violate the equation (3.11) in the case of the choice of $\beta = \beta(\varepsilon, \tau)$ large enough, and hence the minimum of $\overline{M}$ is attained inside the interval I. Direct computations give:

$$\begin{aligned} \overline{M}_t \geq \varepsilon \overline{M}_{xx} &+ \left[\tau G'_\gamma(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2} - 2\beta\varepsilon \tanh(\beta x) \right] \overline{M}_x \\ &+ \left[\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G'_\gamma(\rho_x)}{\kappa_x} - \beta \tanh(\beta x) \left(\tau G'_\gamma(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2} \right) + \varepsilon \beta^2 (2 \tanh^2(\beta x) - 1) \right] \overline{M}. \end{aligned} \tag{3.12}$$

Step 3. (The inequality satisfied by the minimum of $\overline{M}$)

Let

$$\overline{m}(t) = \min_{x \in I} \overline{M}(x,t).$$

Since the minimum is attained inside I, and since $\overline{M}$ is regular, there exists $x_0(t) \in I$ such that $\overline{m}(t) = \overline{M}(x_0(t), t)$. We remark that we have:

$$\overline{M}_x(x_0(t), t) = 0, \quad \text{and} \quad \overline{M}_{xx}(x_0(t), t) \geq 0,$$

and hence, using (3.12), we can write down the equation satisfied by $\overline{m}$, we get (indeed in the viscosity sense at $x = x_0(t)$):

$$\overline{m}_t \geq \left(\frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} G'_\gamma(\rho_x)}{\kappa_x} - \beta \tanh(\beta x) \left(\tau G'_\gamma(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2} \right) + \varepsilon \beta^2 (2 \tanh^2(\beta x) - 1) \right) \overline{m}$$

therefore we deduce that $\overline{m}(0) \geq 0$ directly implies $\overline{m}(t) \geq 0$, $\forall t \in (0, T)$. □

4 Short time existence, uniqueness, and regularity

In this section, we will prove a result of short time existence, uniqueness and regularity of a solution of problem (1.1), (1.2) and (1.3).

4.1 Short-time existence and uniqueness of a truncated system

We denote

$$I_{a,b} := I \times (a, a+b), \quad a, b \geq 0.$$

Fix $T_0 \geq 0$. Consider the following system defined on $I_{T_0,T}$ by:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_x \rho_{xx}}{\kappa_x} - \tau \rho_x & \text{on} \quad I_{T_0,T} \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \kappa_x & \text{on} \quad I_{T_0,T}, \end{cases} \tag{4.1}$$

with the initial conditions:

$$\kappa(x, T_0) = \kappa^{T_0}(x) \quad \text{and} \quad \rho(x, T_0) = \rho^{T_0}(x), \tag{4.2}$$

and the boundary conditions:

$$\begin{cases} \kappa(0,.) = 0 \quad \text{and} \quad \kappa(1,.) = 1 & \text{for} \quad T_0 < t < T_0 + T \\ \rho(0,.) = \rho(1,.) = 0, & \text{for} \quad T_0 < t < T_0 + T. \end{cases} \tag{4.3}$$

Remark 4.1 ***(The terms*** p ***and*** α***).*** *In all what follows, the terms* p *and* $\alpha \in (0,1)$ *are two fixed positive real numbers such that*

$$p > 3 \quad \text{and} \quad \alpha = 1 - 3/p.$$

Concerning system (4.1), (4.2) and (4.3), we have the following existence and uniqueness result.

Proposition 4.2 ***(Short time existence and uniqueness).*** *Let* $p > 3$*, and* $T_0 \geq 0$*. Let*

$$\rho^{T_0}, \kappa^{T_0} \in C^\infty(\bar{I} \times \{T_0\})$$

be two given functions such that $\rho^{T_0}(0) = \rho^{T_0}(1) = \kappa^{T_0}(0) = 0$*, and* $\kappa^{T_0}(1) = 1$*. Suppose furthermore that*

$$\kappa_x^{T_0} \geq \gamma_0 \quad on \quad I \times \{t = T_0\},$$

and

$$\|(D_x^s \rho^{T_0}, D_x^s \kappa^{T_0})\|_{\infty,I} \leq M_0 \quad on \quad I \times \{t = T_0\}, \quad s = 1, 2,$$

where $\gamma_0 > 0$ *and* $M_0 > 0$ *are two given positive real numbers. Then there exists*

$$T = T(M_0, \gamma_0, \varepsilon, \tau, p) > 0, \tag{4.4}$$

such that the system (4.1), (4.2) and (4.3) admits a unique solution

$$(\rho, \kappa) \in (W_p^{2,1}(I_{T_0,T}))^2.$$

Moreover, this solution satisfies

$$\kappa_x \geq \gamma_0/2 \quad on \quad \overline{I_{T_0,T}}, \tag{4.5}$$

and

$$|\rho_x| \leq 2M_0 \quad on \quad \overline{I_{T_0,T}}. \tag{4.6}$$

Proof. The short time existence is done by using a fixed point argument. Since we are looking for solutions satisfying (4.5) and (4.6), we artificially modify (4.1), and look for a solution of

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_{xx} T_{2M_0}(\rho_x)}{(\gamma_0/2) + (\kappa_x - \gamma_0/2)^+} - \tau \rho_x & \text{in} \quad I_{T_0,T} \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \kappa_x & \text{in} \quad I_{T_0,T}, \end{cases} \tag{4.7}$$

with the truncation function $T_\zeta(x) = x 1\!\!1_{(-\zeta,\zeta)} + \zeta 1\!\!1_{\{x \geq \zeta\}} - \zeta 1\!\!1_{\{x \leq -\zeta\}}$, $\zeta > 0$, and satisfying the same initial and boundary data (4.2), (4.3). Denote

$$Y = W_p^{2,1}(I_{T_0,T}).$$

For any constant $\lambda > 0$, let us define D^{ρ}_{λ} and D^{κ}_{λ} as the two closed subsets of Y given by:

$$D^{\rho}_{\lambda} = \{u \in Y;\ \|u_x\|_{p,I_{T_0,T}} \leq \lambda,\ \ u = \rho^{T_0} \text{ on } \partial^p I_{T_0,T}\}$$

and

$$D^{\kappa}_{\lambda} = \{v \in Y;\ \|v_x\|_{p,I_{T_0,T}} \leq \lambda,\ \ v = \kappa^{T_0} \text{ on } \partial^p I_{T_0,T}\}.$$

We choose λ large enough such that these sets are nonempty. Define the application Ψ by:

$$\begin{aligned} \Psi : D^{\rho}_{\lambda} \times D^{\kappa}_{\lambda} &\longmapsto \quad D^{\rho}_{\lambda} \times D^{\kappa}_{\lambda} \\ (\hat{\rho}, \hat{\kappa}) &\longmapsto \quad \Psi(\hat{\rho}, \hat{\kappa}) = (\rho, \kappa), \end{aligned}$$

where (ρ, κ) is a solution of the following system:

$$\begin{cases} \kappa_t = \varepsilon \kappa_{xx} + \dfrac{\rho_{xx} T_{2M_0}(\hat{\rho}_x)}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+} - \tau \hat{\rho}_x & \text{in} \quad I_{T_0,T}, \\ \rho_t = (1+\varepsilon)\rho_{xx} - \tau \hat{\kappa}_x & \text{in} \quad I_{T_0,T}, \end{cases} \tag{4.8}$$

with the same initial and boundary conditions given by (4.2) and (4.3) respectively. The existence of the solution of (4.8), (4.2) and (4.3) is a direct consequence of Theorem 2.2. Taking $\bar{\rho}(x,t) = \rho(x,t) - \rho^{T_0}(x)$ and $\bar{\kappa}(x,t) = \kappa(x,t) - \kappa^{T_0}(x)$, we can easily check that $(\bar{\rho}, \bar{\kappa})$ satisfies a parabolic system similar to (4.8) with $(\bar{\rho}, \bar{\kappa}) = 0$ on $\partial^p I_{T_0,T}$. Using Sobolev estimates for parabolic equations to the system satisfied by $(\bar{\rho}, \bar{\kappa})$, particularly (2.7), we deduce that for sufficiently small $T > 0$, we have $\|\rho_x\|_{p,I_{T_0,T}} \leq \lambda$, $\|\kappa_x\|_{p,I_{T_0,T}} \leq \lambda$, and hence the application Ψ is well defined.

The application Ψ is a contraction map. Let $\Psi(\hat{\rho}, \hat{\kappa}) = (\rho, \kappa)$ and $\Psi(\hat{\rho}', \hat{\kappa}') = (\rho', \kappa')$. Direct computations, using in particular (2.7), give:

$$\|\rho - \rho'\|_Y \leq c\sqrt{T}\|\hat{\kappa} - \hat{\kappa}'\|_Y, \tag{4.9}$$

and

$$\|\kappa - \kappa'\|_Y \leq c\|F\|_{p,I_{T_0,T}}, \tag{4.10}$$

with the function F satisfying:

$$\begin{aligned} F + \tau(\hat{\rho} - \hat{\rho}')_x = {} & \overbrace{\frac{T_{2M_0}(\hat{\rho}_x)}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+}(\rho_{xx} - \rho'_{xx})}^{A_1} + \overbrace{\frac{\rho'_{xx}(T_{2M_0}(\hat{\rho}_x) - T_{2M_0}(\hat{\rho}'_x))}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+}}^{A_2} \\ & + \overbrace{\rho'_{xx} T_{2M_0}(\hat{\rho}'_x)\left(\frac{1}{(\gamma_0/2) + (\hat{\kappa}_x - \gamma_0/2)^+} - \frac{1}{(\gamma_0/2) + (\hat{\kappa}'_x - \gamma_0/2)^+}\right)}^{A_3}. \end{aligned} \tag{4.11}$$

In order to prove the contraction for some small $T > 0$, we need to estimate all the terms appearing in (4.11). The term A_1 can be easily handled. However, for the term A_2, we proceed as follows. We apply the L^{∞} control of the spatial derivative (see Lemma 2.6) to the function $\hat{\rho} - \hat{\rho}'$, we get:

$$\|(\hat{\rho} - \hat{\rho}')_x\|_{\infty,I_{T_0,T}} \leq cT^{\frac{p-3}{2p}}\|\hat{\rho} - \hat{\rho}'\|_Y. \tag{4.12}$$

For the term ρ'_{xx}, we apply (2.7), and hence we deduce that

$$\|\rho'_{xx}\|_{p,I_{T_0,T}} \leq c(M_0+\lambda). \tag{4.13}$$

From (4.12) and (4.13), we deduce that

$$\|A_2\|_{p,I_{T_0,T}} \leq c\frac{(M_0+\lambda)}{\gamma_0}T^{\frac{p-3}{2p}}\|\hat{\rho}-\hat{\rho}'\|_Y.$$

The term A_3 could be treated in a similar way as the term A_2. The above arguments, particularly (4.9) and (4.10), give the contraction of Ψ in the short time interval (T_0, T_0+T) with $T = T(M_0, \gamma_0, \varepsilon, \tau, p) > 0$. Finally, inequalities (4.5) and (4.6) directly follow using the Sobolev embedding in Hölder spaces (Lemma 2.4). □

4.2 Regularity of the solution

This subsection is devoted to show that the solution of (4.1), (4.2) and (4.3) enjoys more regularity than the one indicated in Proposition 4.2. This will be done using a special bootstrap argument, together with the Hölder regularity of solutions of parabolic equations.

Proposition 4.3 ***(Regularity of the solution: bootstrap argument).*** *Under the same hypothesis of Proposition 4.2, let ρ^{T_0} and κ^{T_0} satisfy:*

$$\begin{cases} (1+\varepsilon)\rho^{T_0}_{xx} = \tau\kappa^{T_0}_x & at \quad \partial I, \\ (1+\varepsilon)\kappa^{T_0}_{xx} = \tau\rho^{T_0}_x & at \quad \partial I. \end{cases} \tag{4.14}$$

Then the unique solution (ρ,κ) given by Proposition 4.2 is in fact more regular. Precisely, it satisfies for $\alpha = 1-3/p$:

$$\rho,\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_{T_0,T}}) \cap C^\infty(\bar{I}\times(T_0,T_0+T)), \tag{4.15}$$

where T is the time given by Proposition 4.2.

Proof. For the sake of simplicity, let us suppose that $T_0 = 0$.

The Hölder regularity. Since $\kappa \in W^{2,1}_p(I_T)$, we use Lemma 2.4 to deduce that $\kappa_x \in C^{\alpha,\alpha/2}(\overline{I_T})$. We apply the Hölder theory for parabolic equations Theorem 2.1, to the second equation of (4.1) (using in particular the regularity of the initial data ρ^0), we deduce that:

$$\rho \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}). \tag{4.16}$$

Here the compatibility condition is satisfied by (4.14). Using (4.16) and (4.5), we deduce that $\frac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ and similar arguments as above give that:

$$\kappa \in C^{2+\alpha,1+\alpha/2}(\overline{I_T}). \tag{4.17}$$

Repeating the above arguments, using this time (see (4.17)) that $\kappa_x \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T})$, and hence

$$\rho \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}), \tag{4.18}$$

where (4.18) directly implies that $\frac{\rho_x \rho_{xx}}{\kappa_x} - \tau\rho_x \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I_T})$, and therefore

$$\kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I_T}). \tag{4.19}$$

The compatibility condition of order 1 which is needed to apply Theorem 2.1 is always satisfied by (4.14). The Hölder regularity of (ρ, κ) directly follows from (4.18) and (4.19).

The C^∞ regularity. In order to get the C^∞ regularity, we argue as in the case of the Hölder regularity (bootstrap argument). In this case the compatibility condition is replaced by multiplying by a test function that vanishes near $t = 0$. □.

5 Exponential bounds

In this section, we will give some exponential bounds of the solution given by Proposition 4.2, and having the regularity shown by Proposition 4.3. It is very important, throughout all this section, to precise our notation concerning the constants that may certainly vary from line to line. Let us mention that a constant depending on time will be denoted by $c(T)$. Those which do not depend on T will be simply denoted by c. In all other cases, we will follow the changing of the constants in a precise manner.

Proposition 5.1 ***(Exponential bound in time for ρ_x and κ_x).*** *Let*

$$\rho, \kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I} \times [0,\infty)) \cap C^\infty(\bar{I} \times (0,\infty)),$$

be a solution of (1.1), (1.2) and (1.3), with $\rho^0(0) = \rho^0(1) = 0$, $\kappa^0(0) = 0$ and $\kappa^0(1) = 1$. Suppose furthermore that the function

$$B = \frac{\rho_x}{\kappa_x} \quad \textit{satisfies} \quad \|B\|_{L^\infty(I\times(0,\infty))} \le 1.$$

Then, for small $T^* = T^*(\varepsilon, \tau, p) > 0$, and $A = 1 + \|\rho^0\|_{W_p^{2-2/p}(I)} + \|\kappa^0\|_{W_p^{2-2/p}(I)}$, we have for all $t \ge 0$:

$$|\rho_x|^{(\alpha)}_{I_{t,T^*}}, |\kappa_x|^{(\alpha)}_{I_{t,T^*}} \le cAe^{ct}, \tag{5.1}$$

and c is a fixed constant independent of the initial data.

Proof. We use the special coupling of the system (1.1) to find our *a priori* estimate. Roughly speaking, the fact that κ_x appears as a source term in the second equation of system (1.1) permits, by the L^p theory for parabolic equations, to have L^p bounds, in terms of $\|\kappa_x\|_{p,I_T}$, on ρ_x and ρ_{xx} which in their turn appear in the source terms of the first equation of (1.1) satisfied by κ. All this permit to deduce our estimates. To be more precise, let $T > 0$ an arbitrarily fixed time, the proof is divided into four steps:

Step 1. (estimating κ_x in the L^p norm)

Let κ' be the solution of the following equation:

$$\begin{cases} \kappa'_t = \kappa'_{xx} & \text{on} \quad I_T \\ \kappa' = \kappa & \text{on} \quad \partial^p I_T. \end{cases} \tag{5.2}$$

As a solution of a parabolic equation, we use the L^p parabolic estimate (2.6) to the function $\kappa^{'}$ to deduce that:

$$\|\kappa^{'}\|_{W_p^{2,1}(I_T)} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)}+1\right), \tag{5.3}$$

where the term 1 comes from the value of $\kappa' = \kappa$ on S_T. Take

$$\bar{\kappa} = \kappa - \kappa^{'}, \tag{5.4}$$

then the system satisfied by $\bar{\kappa}$ reads:

$$\begin{cases} \bar{\kappa}_t = \bar{\kappa}_{xx} - (\kappa_t^{'} - \varepsilon\kappa_{xx}^{'}) + \dfrac{\rho_x\rho_{xx}}{\kappa_x} - \tau\rho_x \quad \text{on} \quad I_T \\ \bar{\kappa} = 0 \quad \text{on} \quad \partial^p I_T. \end{cases}$$

Using the special version (2.7) of the parabolic L^p estimate to the function $\bar{\kappa}$, we obtain:

$$\|\bar{\kappa}_x\|_{p,I_T} \leq c\sqrt{T}\left(\|\kappa_t^{'}\|_{p,I_T} + \|\kappa_{xx}^{'}\|_{p,I_T} + \|\rho_{xx}\|_{p,I_T} + \|\rho_x\|_{p,I_T}\right), \tag{5.5}$$

where we have plugged into the constant c the terms ε, τ, p and $\|B\|_\infty$. Combining (5.3), (5.4) and (5.5), we get:

$$\|\kappa_x\|_{p,I_T} \leq c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)}+1\right) + c\sqrt{T}\|\rho\|_{W_p^{2,1}(I_T)}. \tag{5.6}$$

The term $\|\rho\|_{W_p^{2,1}(I_T)}$ appearing in the previous inequality is going to be estimated in the next step.

Step 2. (estimating ρ in the $W_p^{2,1}$ norm)

As in Step 1, let $\rho^{'}$, $\bar{\rho}$ be the two functions defined similarly as $\kappa^{'}$, $\bar{\kappa}$ respectively (see (5.2) and (5.4)). The function $\rho^{'}$ satisfies an inequality similar to (5.3) that reads:

$$\|\rho^{'}\|_{W_p^{2,1}(I_T)} \leq c(T)\|\rho^0\|_{W_p^{2-2/p}(I)}. \tag{5.7}$$

The term 1 disappeared here because $\rho^{'} = \rho = 0$ on $\overline{S_T}$. We write the system satisfied by $\bar{\rho}$, we obtain:

$$\begin{cases} \bar{\rho}_t = (1+\varepsilon)\bar{\rho}_{xx} + ((1+\varepsilon)\rho_{xx}^{'} - \rho_t^{'}) - \tau\kappa_x \quad \text{on} \quad I_T \\ \bar{\rho}(x,0) = 0 \quad \text{on} \quad \partial^p I_T, \end{cases}$$

hence the following estimate on $\bar{\rho}$, due to the special L^p interior estimate (2.7), holds:

$$\|\bar{\rho}\|_{W_p^{2,1}(I_T)} \leq c\left(\|\rho_t^{'}\|_{p,I_T} + \|\rho_{xx}^{'}\|_{p,I_T} + \|\kappa_x\|_{p,I_T}\right). \tag{5.8}$$

Again, we have plugged ε, τ and p into the constant c, and we have assumed that $T \leq 1$. Combining (5.7) and (5.8), we get in terms of ρ:

$$\|\rho\|_{W_p^{2,1}(I_T)} \leq c(T)\|\rho^0\|_{W_p^{2-2/p}(I)} + c\|\kappa_x\|_{p,I_T}. \tag{5.9}$$

We will use this estimate in order to have a control on $\|\kappa_x\|_{p,I_T}$ for sufficiently small time.

Step 3. (Estimate on a small time interval)

From (5.6) and (5.9), we deduce that:

$$\|\kappa_x\|_{p,I_T} \le c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right) + c\sqrt{T}\|\kappa_x\|_{p,I_T}. \tag{5.10}$$

Let us remind the reader that all constants c and $c(T)$ have been changing from line to line. In fact, the important thing is whether they depend on T or not. Let

$$T^* = \frac{1}{2c^2}, \quad c \text{ is the constant appearing in (5.10)},$$

we deduce, from (5.10), that

$$\|\kappa_x\|_{p,I_{T^*}} \le c_3\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right),$$

where $c_3 = c_3(T^*) > 0$ is a positive constant which depends on T^*. Recall the special coupling of system (1.1), together with the above estimate, we can deduce that:

$$\|(\rho,\kappa)\|_{W_p^{2,1}(I_{T^*})} \le c_4\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right), \tag{5.11}$$

with $c_4 = c_4(T^*) > 0$ is also a positive constant depending on T^* but independent of the initial data.

Step 4. (The exponential estimate by iteration)

Now we move to show the exponential bound. Set

$$f(t) = \|(\rho,\kappa)\|_{W_p^{2,1}(I\times(t,t+T^*))}, \quad \text{and} \quad g(t) = \|\kappa(\cdot,t)\|_{W_p^{2-2/p}(I)} + \|\rho(\cdot,t)\|_{W_p^{2-2/p}(I)}.$$

Using estimate (5.11) of Lemma 2.5, together with estimate (5.11) of Step 3, we get

$$g(T^*) \le c_5 f(0) \le c_5 c_4 (g(0)+1), \quad c_5 = c_5(T^*).$$

In this case, the Sobolev embedding in Hölder spaces (see Lemma 2.5), and the time iteration give immediately the result. □

Proposition 5.2 ***(Exponential bound in time for*** ρ_{xx}***).*** *Under the same hypothesis of Proposition 5.1, and for some* $T^* = T^*(\varepsilon,\tau,p) > 0$*, we have:*

$$|\rho|_{I_{t,T^*}}^{(2+\alpha)} \le ce^{ct}, \quad t \ge 0, \tag{5.12}$$

where $c > 0$ *is a positive constant depending only on the initial data.*

Proof. The proof is very similar to the proof of Proposition 5.1. It uses in particular the Hölder estimate for parabolic equations (namely (2.4)), the Hölder embedding in Sobolev spaces (Lemma 2.4), and finally the iteration in time. □

Proposition 5.3 ***(Exponential bound in time for κ_{xx}).*** *Under the same hypothesis of Proposition 5.1, and for some $T^* = T^*(\varepsilon, \tau, p) > 0$, we have:*

$$|\kappa|^{(2+\alpha)}_{I_{t,T^*}} \leq ce^{ct}, \quad t \geq 0, \tag{5.13}$$

where $c > 0$ is a positive constant depending only on the initial data.

Proof. Let $T > 0$. Using both equations of (1.1), estimates (5.1) and (5.12), together with the following elementary identities:

$$\left\langle \frac{f}{g} \right\rangle^{(\alpha/2)}_{t,\mathcal{D}} \leq \left\| \frac{f}{g} \right\|_{\infty,\mathcal{D}} \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle g \rangle^{(\alpha/2)}_{t,\mathcal{D}} + \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle f \rangle^{(\alpha/2)}_{t,\mathcal{D}},$$

and

$$\left\langle \frac{f}{g} \right\rangle^{(\alpha)}_{x,\mathcal{D}} \leq \left\| \frac{f}{g} \right\|_{\infty,\mathcal{D}} \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle g \rangle^{(\alpha)}_{x,\mathcal{D}} + \left\| \frac{1}{g} \right\|_{\infty,\mathcal{D}} \langle f \rangle^{(\alpha)}_{x,\mathcal{D}},$$

applied for $f = \rho_x \rho_{xx}$ and $g = \kappa_x$, we deduce, from Hölder estimate for parabolic equations (see inequality (2.4) of Theorem 2.1), that

$$|\kappa|^{(2+\alpha)}_{I_{0,t}} \leq \frac{ce^{ct}}{\gamma(t)} \left(1 + |\kappa_x|^{(\alpha)}_{I_{0,t}} \right), \tag{5.14}$$

where $\gamma(t)$ is given by (3.2), and $c > 0$ is a positive constant depending only on the initial data. Inequality (5.14) directly implies (5.13) by iteration. □

Summarizing the above results we obtain the following corollary:

Corollary 5.4 *Under the same hypothesis of Proposition 3.1, we have $\forall t \in (0, T)$:*

$$\|\kappa_x(., t)\|_{\infty,I} \geq ce^{-ct} \tag{5.15}$$

$$\|D^s_x \rho(., t)\|_{\infty,I}, \|D^s_x \kappa(., t)\|_{\infty,I} \leq ce^{ct}, \quad s = 1, 2, \tag{5.16}$$

where $c > 0$ is a positive constant only depending on the initial data.

Proof. Directly follows from (3.3), (5.1), (5.12) and (5.13). □

6 Long time existence and uniqueness

Now we are ready to show the main result of this paper, namely Theorem 1.1.

Proof of Theorem 1.1. Define the set $\mathcal{B}$ by:

$$\mathcal{B} = \left\{ \begin{array}{l} T > 0;\ \exists !\ \text{ solution } (\rho, \kappa) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I_T}) \text{ of} \\ (1.1), (1.2) \text{ and } (1.3), \text{ satisfying } (1.8) \end{array} \right\}.$$

This set is non empty by the short time existence result (Theorem 4.2). Set

$$T_\infty = \sup \mathcal{B}.$$

We claim that $T_\infty = \infty$. Assume, by contradiction that $T_\infty < \infty$. In this case, let $\delta > 0$ be an arbitrary small positive constant, and apply the short time existence result (Theorem 4.2) with $T_0 = T_\infty - \delta$. Indeed, by the exponential bounds (5.15) and (5.16), we deduce that the time of existence T given by (4.4) is in fact independent of δ. Hence, choosing δ small enough, we obtain $T_0 + T \in \mathcal{B}$ with $T_0 + T > T_\infty$ and hence a contradiction. □

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Appendix. Miscellaneous parabolic estimates

Proof of Lemma 2.3 (L^p estimate for parabolic equations). As a first step, we will prove the result in the case where $\varepsilon = 1$, and in a second step, we will move to the case $\varepsilon > 0$. It is worth noticing that the term c may take several values only depending on p.

Step 1. (The estimate: case $\varepsilon = 1$)

Suppose $\varepsilon = 1$. We have $u = 0$ on $\partial I \times [0, T]$. Take $\tilde{u} = u^{asym}$ where we define u^{asym} over $\mathbb{R} \times (0, T)$, first by considering the antisymmetry of u with respect to the line $x = 0$ over the interval $(-1, 0)$, and then by spatial periodicity. We also take $\tilde{f} = f^{asym}$. Define $\bar{u}$ by

$$\bar{u} = \tilde{u}\phi^n,$$

with

$$\begin{cases} \phi^n(x) = 1 & \text{if } x \in (0, 2n) \\ \phi^n(x) = 0 & \text{if } x \geq 2n + 1 \text{ or } x \leq -1. \end{cases}$$

This function satisfies

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{f}, & \text{on} \quad \mathbb{R} \times (0, T) \\ \bar{u}(x, 0) = 0, & \text{on} \quad \mathbb{R}, \end{cases}$$

with

$$\bar{f} = \tilde{f}\phi^n - \tilde{u}\phi^n_{xx} - 2\tilde{u}_x\phi^n_x.$$

The proof that

$$\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T} \leq c\|f\|_{p,I_T} \tag{6.1}$$

can be easily deduced by applying the Calderon-Zygmund estimates to the function $\bar{u}$ satisfying the above equation, and passing to the limit $n \to \infty$. Now, since $u \in W^{2,1}_p(I_T)$ with $u|_{t=0} = 0$, we use [14, Lemma 4.5, page 305] to get

$$\|u\|_{p,I_T} \leq cT(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) \tag{6.2}$$

and

$$\|u_x\|_{p,I_T} \leq c\sqrt{T}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}). \tag{6.3}$$

Combining (6.1), (6.2) and (6.3), we deduce that

$$\frac{1}{T}\|u\|_{p,I_T} + \frac{1}{\sqrt{T}}\|u_x\|_{p,I_T} + \|u_{xx}\|_{p,I_T} + \|u_t\|_{p,I_T} \leq c\|f\|_{p,I_T}.$$

Step 2. (The estimate: general case $\varepsilon > 0$)

To get the general inequality, we consider the following rescaling of the function u:

$$\hat{u}(x,t) = u(x, t/\varepsilon), \quad (x,t) \in I_{\varepsilon T},$$

which allows to get the desired result. □

Proof of Lemma 2.6 (L^∞ control of the spatial derivative). Since $u \in W_p^{2,1}(I_T)$ for $p > 3$, we know from Lemma 2.4 that $u_x \in C^{\alpha,\alpha/2}(\overline{I_T})$ for $\alpha = 1 - \frac{3}{p}$. In this case, we use the estimate (2.8) with $\delta = \sqrt{T}$, we obtain

$$\|u_x\|_{\infty,I_T} \leq c(p)\{T^{\frac{\alpha}{2}}(\|u_t\|_{p,I_T} + \|u_{xx}\|_{p,I_T}) + T^{\frac{\alpha}{2}-1}\|u\|_{p,I_T}\}. \tag{6.4}$$

Remark that the fact that $u = 0$ on the parabolic boundary $\partial^p I_T$, and that it obviously satisfies the equation:

$$\begin{cases} u_t = u_{xx} + f, & \text{with} \quad f = u_t - u_{xx} \\ u = 0 \quad \text{on} & \partial^p I_T, \end{cases}$$

then we can apply estimate (2.7) to bound the term $\|u\|_{p,I_T}$. Hence (6.4) becomes (with a different constant $c(p)$):

$$\begin{aligned} \|u_x\|_{\infty,I_T} &\leq c(p)\{T^{\frac{\alpha}{2}}\|u_t - u_{xx}\|_{p,I_T} + T^{\frac{\alpha}{2}-1}T\|u_t - u_{xx}\|_{p,I_T}\} \\ &\leq c(p)T^{\frac{\alpha}{2}}\|u\|_{W_p^{2,1}(I_T)} \\ &\leq c(p)T^{\frac{p-3}{2p}}\|u\|_{W_p^{2,1}(I_T)}, \end{aligned}$$

and the result follows. □

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