Homogenization of first order equations with (u/ε)-periodic Hamiltonians. Part II: application to dislocations dynamics.

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Abstract. This paper is concerned with a result of homogenization of a non-local first order Hamilton-Jacobi equations describing the dislocations dynamics. Our model for the interaction between dislocations involve both an integro-differential operator and a (local) Hamiltonian depending periodicly on u/ε. The first two authors studied in a previous work homogenization problems involving such local Hamiltonians. Two main ideas of this previous work are used: on the one hand, we prove an ergodicity property of this equation by constructing approximate correctors which are necessarily non periodic in space in general; on the other hand, the proof of the convergence of the solution uses here a twisted perturbed test function for a higher dimensional problem. The limit equation is a nonlinear diffusion equation involving a first order Lévy operator; the nonlinearity keeps memory of the short range interaction, while the Lévy operator keeps memory of long ones. The homogenized equation is a kind of effective plastic law for densities of dislocations moving in a single slip plane.

Keywords: periodic homogenization, Hamilton-Jacobi equations, integro-differential operators, dislocations dynamics, non-periodic approximate correctors.
Mathematics Subject Classification: 35B10, 35B27, 35F20, 45K05, 47G20, 49L25

1 Introduction

Setting of the problem. In this paper, we study a non-local Hamilton-Jacobi equation describing the dislocations dynamics. We would like to say what happens when $\varepsilon \to 0$ to the solution of:

$$\begin{cases} \partial_t u^\varepsilon = \left(c\left(\frac{x}{\varepsilon}\right) + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)|\nabla u^\varepsilon| + h\left(\frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{1}$$

where M^ε is a 0 order non-local operator M^ε defined by

$$M^\varepsilon[U](x) = -U(x) + \int_{\mathbb{R}^N} dz\, J(z) U(x+\varepsilon z)$$

where $J \in C^\infty(\mathbb{R}^N)$ is an *even nonnegative* function that satisfies $\int_{\mathbb{R}^N} dz\, J(z) = 1$ and:

$$\text{there exists } R_0 > 0 \text{ and a function } g > 0 \text{ s.t. for } |z| \geq R_0 : J(z) = \frac{1}{|z|^{N+1}}\, g\left(\frac{z}{|z|}\right). \tag{2}$$

Throughout the paper, we assume:

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- $c(x)$ is Lipschitz continuous and 1-periodic;
- $h(u,p)$ is bounded Lipschitz continuous w.r.t. (u,p), is 1-periodic w.r.t. u and $h(u,p)/|p|$ is bounded;
- $u_0 \in W^{2,\infty}(\mathbb{R}^N)$.

The aim is to prove a homogenization result *i.e.* to prove that the limit u^0 of u^ε as $\varepsilon \to 0$ exists and is the (unique) solution of a homogenized equation of the form:

$$\begin{cases} \partial_t u^0 = \overline{H}^0(\ldots, \nabla u^0) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u^0(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases}$$

with $u_0 \in W^{2,\infty}(\mathbb{R}^N)$. Our aim is two-folded: to determine the so-called *effective Hamiltonian* $\overline{H}$ and to prove the convergence of u^ε towards u^0.

Main results. As usual in periodic homogenization, the limit (or effective) equation is determined by a *cell problem*, and more precisely by the long time behaviour of the solution of an evolution equation related to this cell problem. And as usual, this problem is determined by a formal computation, by an ansatz. In [28], we described the successive tries of ansatz we did to find the proper cell equation for our local homogenization problem and analogous calculi can be done here. It turns out that the proper evolution equation is, for any $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$:

$$\begin{cases} \partial_\tau w = H(L, y, p\cdot y + w, p + \nabla w, [w]) \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \text{where} \quad H(L, y, p\cdot y + w, p + \nabla w, [w]) = (c(y) + L + M[w(\tau,\cdot)])|p + \nabla w| + h(p\cdot y + w, p + \nabla w) \\ w(0,y) = 0 \quad \text{on} \quad \mathbb{R}^N \end{cases} \tag{3}$$

where $M[\,\cdot\,]$ is the non-local operator $M^\varepsilon[\,\cdot\,]$ introduced above with $\varepsilon = 1$.

Theorem 1 (Ergodicity)**.** *For any $L \in \mathbb{R}$ and $p \in \mathbb{R}^N$, there exists a unique $\lambda \in \mathbb{R}$ such that the continuous viscosity solution of* (3) *satisfies: $\frac{w(\tau,y)}{\tau}$ converges towards λ as $\tau \to +\infty$, locally uniformly in y. The real number λ is denoted by $\overline{H}^0(L,p)$.*

A superscript 0 appears in the effective Hamiltonian. The reason is that we will have to study the ergodicity of a *family* of Hamiltonians in order to prove the convergence ($\overline{H}^0(L,p) = \overline{H}(L,p,0)$ with the notations of Section 4 — see below).

We now can give the precise form of the effective equation:

$$\begin{cases} \partial_t u^0 = \overline{H}^0\left(\mathcal{I}_1[u^0(t,\cdot)], \nabla u^0\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N, \\ u^0(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases} \tag{4}$$

where $\mathcal{I}_1$ is a Lévy operator of order 1 defined for any function $U \in C_b^2(\mathbb{R}^N)$ by:

$$\begin{aligned} \mathcal{I}_1[U](x) = \quad & \int_{|z|\le r} (U(x+z) - U(x) - \nabla_x U(x)\cdot z)\, \tfrac{1}{|z|^{N+1}} g\left(\tfrac{z}{|z|}\right) dz \\ & + \int_{|z|\ge r} (U(x+z) - U(x))\, \tfrac{1}{|z|^{N+1}} g\left(\tfrac{z}{|z|}\right) dz. \end{aligned}$$

for any $r > 0$. Because J is even and satisfies (2), M can be seen like a regularized version of $\mathcal{I}_1$: the singularity of the unbounded measure $\mu(dz) = g\left(\frac{z}{|z|}\right) \frac{1}{|z|^{N+1}} dz$ is removed. We will see in the proofs that this Lévy operator $\mathcal{I}_1$ only keeps the memory of the long range behaviour of the operator M^ε. On the contrary the effective Hamiltonian $\overline{H}$ will keep the memory of the short range behaviour of M^ε. Qualitative properties of the effective Hamiltonian will be given in [27] and numerical simulation will be presented in [21].

The second main result of this paper is the following convergence result.

Theorem 2 (Convergence)**.** *The bounded continuous viscosity solution u^ε of* (1) *converges locally uniformly in (t,x) towards the bounded viscosity solution u^0 of* (4)*.*

In order to prove the convergence of u^ε towards u^0, we try to construct a so-called *corrector*, that is a bounded solution of the cell problem which, in our case, has the following form:

$$\lambda + \partial_\tau v = (c(y) + L + M[v(\tau,\cdot)])|p + \nabla v| + h(\lambda\tau + p\cdot y + v, p + \nabla v) \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^N.$$

Let us recall that, formally, $v(\tau,y) = w(\tau,y) - \lambda\tau$ where w satisfies (3). In the proof of convergence, it turns out that we need to consider regular correctors but we are not able to construct them. However, we can construct regular sub- and supercorrectors (*i.e.* Lipschitz continuous sub and supersolution of the cell problem) and it is enough to conclude. Moreover, Theorem 1 is a by-product of this construction. Let us also point out that, at the contrary of the classical case, *i.e.* Hamiltonians of the form $H(y,p)$, the (sub and super) correctors here are not periodic w.r.t. y in general. Moreover (true) correctors are necessarily time-dependent. Let us also mention that we construct sub- and supercorrectors as approximate correctors of approximate cell problems.

A specific technical difficulty of our problem is to deal with the case $\lambda = p = 0$. In order to overcome it, we consider cell problems in a higher dimensional space. Another technical point leads us to solve a family of cell problems, indexed by a real number β.

Comments. This non-local equation (1) is related to the local equation:

$$\begin{cases} \partial_t u^\varepsilon = F\left(\frac{u^\varepsilon}{\varepsilon}, \frac{x}{\varepsilon}, \nabla u^\varepsilon\right) & \text{in} \quad [0,+\infty)\times\mathbb{R}^N, \\ u^\varepsilon(0,x) = u_0(x) & \text{on} \quad \mathbb{R}^N \end{cases}$$

that the first two authors studied in [28] under the assumption that F is coercive in $p = \nabla u^\varepsilon$. As pointed out in [8], this assumption appears in all the papers dealing with homogenization of first-order Hamilton-Jacobi equations. First of all, Lions, Papanicolaou and Varadhan [34] completely solve the problem for F that are independent of u and coercive in p. After this seminal paper, this assumption appears in [29] for more general periodic situations, in [1, 25] for initial-boundary value problems, in [3] for equations with different structure, in [2] for deterministic control problems in L^∞, in [30] for almost periodic Hamiltonians and for instance in [40] in a stochastic framework. The literature about homogenization of Hamilton-Jacobi equations is highly developed and it is difficult to give an exhaustive list of references. The interested reader is refered to [17, 18, 19, 37, 10, 11, 9, 4, 5, 12, 35] and references therein for further information.

Inspired in particular by some ideas from [28], Barles [8] managed to get nice homogenization results for noncoercive Hamiltonians and gave, among other things, simpler proofs of the main results of [28] in a restricted case where it is possible to imbed the problem in dimension $N+1$ for a geometric equation.

In the present paper, the coercivity is somehow replaced with the nonlocal term. Indeed, coercivity together with a perturbation by a non-local operator (in order to get the strong maximum principle and apply the sliding method) are used in [28] in order to get estimates on space oscillations of the correctors. This is obtained here thanks to $M^\varepsilon\left[\frac{u^\varepsilon}{\varepsilon}\right]|\nabla u^\varepsilon|$ (see page 25 and Eq. (55)).

Possible extensions. In order to avoid unnecessary technicality, we chose to focus on a particular equation related to dislocation dynamics. With some trivial changes, it is for instance possible to treat the case of Frank-Read source considering the homogenization of the following equation

$$\partial_t u^\varepsilon = \left(c\left(\frac{x}{\varepsilon}\right) + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)|\nabla u^\varepsilon - f\left(\frac{x}{\varepsilon}\right)| + h\left(\frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right)$$

where curl $f \neq 0$ represents a source term (physically in dimension $N = 2$).
Using again the coercivity of the operator M^ε, it should be possible to deal with the homogenization of

$$\partial_t u^\varepsilon = \left(c\left(\frac{x}{\varepsilon}\right) + M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]\right)|\nabla u^\varepsilon| + h\left(\frac{u^\varepsilon}{\varepsilon}, \nabla u^\varepsilon\right) + \varepsilon F_{\mathrm{MCM}}(\nabla u^\varepsilon, D^2 u^\varepsilon)$$

where F_{MCM} is a mean curvature term.

Organization of the article. The paper is organized as follows. In Section 2.1, we give more details about the mathematical models yielding to the study of (1). In Section 3, we prove various comparison principles and existence results. Section 4 is dedicated to the proof of the convergence (Theorem 2) by assuming the existence of Lipschitz continuous sub- and supercorrectors (Proposition 5). Section 5 is the core of the paper; we construct approximate cell problems in the spirit of [28] and we construct the Lipschitz continuous supercorrectors as exact correctors of the approximate cell problems (Proposition 6). In Section 6, we prove the ergodicity of the problem (Theorem 1) and give some properties of the homogenized Hamiltonian such as its continuity (Proposition 4). Eventually, Section 2.2 is devoted to mechanical interpretation of the homogenization results obtained in this paper.

Notation. The ball of radius r centered at x is classically denoted $B_r(x)$. When x is the origin, $B_r(0)$ is simply denoted B_r. The cylinder $(t-\tau, t+\tau) \times B_r(x)$ is denoted $Q_{\tau,r}(t,x)$. The indicator function of a subset $A \subset B$ is denoted 1_A: it equals 1 on A and 0 on $B \setminus A$.

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote respectively the floor and ceil integer parts of a real number x.

It is convenient to introduce the unbounded measure on $\mathbb{R}^N$ defined on $\mathbb{R}^N \setminus \{0\}$ by:

$$\mu(dz) = \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz$$

and such that $\mu(\{0\}) = 0$. For the reader's convenience, we recall here the three integro-differential operators appearing in this work:

$$\begin{aligned} M^\varepsilon[U](x) &= \int_{\mathbb{R}^N} (U(x+\varepsilon z) - U(x)) J(z)\, dz \\ M[U](x) &= \int_{\mathbb{R}^N} (U(x+z) - U(x)) J(z)\, dz, \\ \mathcal{I}_1[U](x) &= \int_{|z|\le r} (U(x+z) - U(x) - \nabla_x U(x)\cdot z)\, \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz \\ &\quad + \int_{|z|\ge r} (U(x+z) - U(x))\, \frac{1}{|z|^{N+1}} g\left(\frac{z}{|z|}\right) dz. \end{aligned}$$

To finish with, $G, G^* > 0$ are positive numbers such that $|h(u,p)| \le G$, $|h(u,p)| \le G^*|p|$.

2 Physical application to dislocations dynamics

2.1 Motivation

Dislocations are line defects in crystals. Their typical length is of the order of $10^{-6}m$ and their thickness of the order of $10^{-9}m$. When the material is submitted to shear stress, these lines can move in the crystallographic planes and their complicated dynamics is one of the main explanation of the plastic behaviour of metals.

In the last decade the physics of dislocations have enjoyed a new boom, essentially based on new possibilities of simulations on ever more powerful computers. In the same time new tentatives have been done to build physical theories describing the collective behaviour of disocations. In particular models for the dynamics of dislocations densities have been proposed (see for instance Groma, Balogh [22], Groma, Czikor, Zaiser [23], Sethna et al. [39], Kratochvil [31], Kröner [16], *etc.*).

In the present paper we are interested in describing the effective dynamics for a collection of dislocations lines with the same Burgers's vector and all contained in a *single* slip plane $\{x_3 = 0\}$ of coordinates $x = (x_1, x_2)$, and moving in a periodic medium (see Fig. 1). At the end of this derivation, we will see that the dynamics of dislocations is described by equation (1) in dimension $N = 2$.

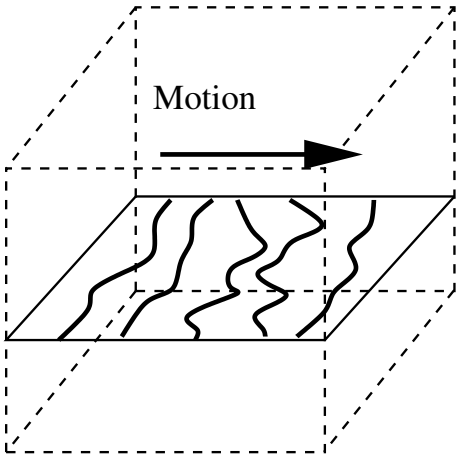

Figure 1: Dislocations in a slip plane

Several obstacles to the motion of dislocations lines can exist in real life: precipitates, inclusions, other pinned dislocations or other moving dislocations, *etc.* We will describe all these obstacles by a given and periodic field

$$c_1(x) \tag{5}$$

that we assume for simplicity not to depend on time. Another natural force exists: this is the Peach-Koehler force acting on a dislocation j. This force is the sum of the interactions with the other dislocations k for $k \neq j$, and of the self-force created by the dislocation j itself (see Fig. 2 for a schematic representation of the interactions; to be closer to reality we should draw a spring between each pair of dislocations).

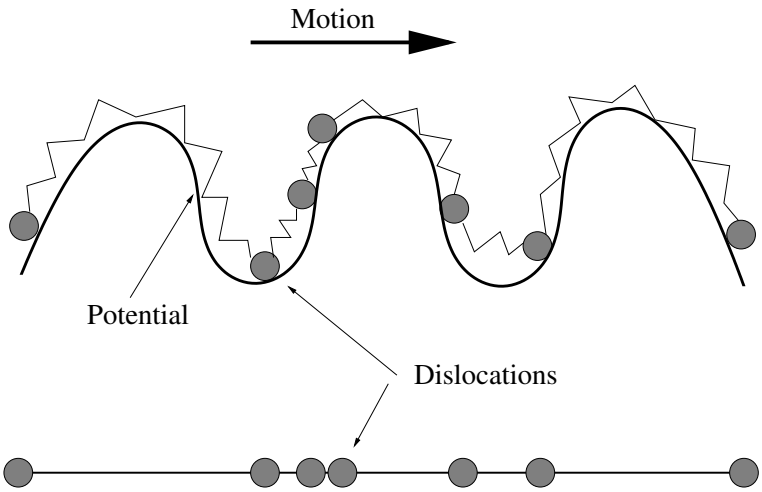

Figure 2: Schematic representation of a 1D model for dislocations related with springs

The level set approach for describing dislocation dynamics consists in considering a function v such that the dislocation $k \in \mathbb{Z}$ is basically described by the level set $\{v = k\}$. Let us first assume that v is smooth.

As explained in [7], the Peach-Koehler force at the point x created by a dislocation j is well-described by the expression

$$c_0 \star 1_{\{v>j\}}$$

where $1_{\{v>j\}}$ is the caracteristic function of the set $\{v > j\}$ which is equal to 1 or 0. In a general setting, the kernel c_0 can change sign. In the special case where the dislocations have the same Burgers vector and move in the same slip plane, a monotone formulation (see Alvarez et al. [6, 7], Da Lio et al. [15]) is physically acceptable. Indeed, the kernel can be chosen as

$$c_0 = J - \delta_0$$

where J is nonnegative. The negative part of the kernel is somehow concentrated at the origin as a Dirac mass. Moreover, we assume that J satisfies $\int_{\mathbb{R}^2} J = 1$ and the symmetry $J(-z) = J(z)$. The kernel J can be computed from physical quantities (like the elastic coefficients of the crystal, the Burgers vector of the dislocation line, the slip plane of the dislocation, the Peierls-Nabarro parameter, *etc.*). Here δ_0 is a Dirac mass where we set formally

$$\big(\delta_0 \star 1_{\{v>j\}}\big)(x) := \begin{cases} 1 & \text{if} \quad v(x) > j \\ \dfrac{1}{2} & \text{if} \quad v(x) = j \\ 0 & \text{if} \quad v(x) < j \end{cases}$$

We remark in particular that the Peach-Koehler force is discontinuous on the dislocation line in this modeling.

Let us now assume that for integers $N_1, N_2 \geq 0$, we have $-N_1 - 1/2 < v < N_2 + 1/2$. Then the Peach-Koehler force at the point x on the dislocation j (*i.e.* $v(x) = j$) created by dislocations for $k = -N_1, ..., N_2$ is given by the sum

$$\left((J - \delta_0) \star \sum_{k=-N_1}^{N_2} 1_{\{v>k\}} \right)(x) = \left(J \star \left(\lceil v - v(x) \rceil - \frac{1}{2} \right) \right)(x)$$

where $\lceil\ \rceil$ is the ceil integer part. In the general case, this Peach-Koehler force can be rewritten on the dislocations lines as

$$-v + J \star E(v) \tag{6}$$

where E is a kind of integer part (up to an additive constant 1/2) defined by

$$E(v) = k + 1/2 \quad \text{if} \quad v \in (k, k+1] \quad \text{with} \quad k \in \mathbb{Z}.$$

Defining the normal velocity to dislocations lines as the sum of the periodic field (5) and the Peach-Koehler force (6), we see that the dislocations line $\{v = j\}$ for integer j, is formally a solution of the following level set equation:

$$v_t = (c_1 - v + J \star E(v))\, |\nabla v|. \tag{7}$$

In this paper we do not study the homogenization of equation (7) whose mathematical framework is not convenient because the "integer part" E is discontinuous (let us mention that a "Slepcev" formulation of this equation could be introduced to stabilize the integer part, and such an analysis will be done in a future work).

First, we replace the discontinuous "integer part" E in (7), by an approximation which is a smooth function E_δ which satisfies

$$E_\delta(\cdot + 1) = E_\delta(\cdot) + 1, \quad 0 < \delta \leq E'_\delta \leq 1/\delta$$

and

$$E_\delta(j) = j \quad \text{and} \quad E_\delta(j + 1/2) = j + 1/2 \quad \text{for} \quad j \in \mathbb{Z} \tag{8}$$

and we see that $u = E_\delta(v)$ satisfies

$$u_t = (c_1 - u + J \star u)\, |\nabla u| + h(u, \nabla u) \tag{9}$$

with

$$h(u, \nabla u) = \big(u - E_\delta^{-1}(u)\big)\, |\nabla u|. \tag{10}$$

Equation (9)-(10) is still not well-posed in the framework of viscosity solutions because, like for Burgers' equation, it can create discontinuous solutions in finite time. We want to avoid this kind of mathematical difficulty which is artificial in the present modeling. We keep in mind that in order to describe well the dynamics, our model has to satisfy the following two properties:

- the function u has to be close to a half integer $k + 1/2$ on a region between two dislocations lines $\{u = k\}$ and $\{u = k + 1\}$, and to have a small transition layer around any dislocation line;
- the function h has to be close to expression (10) outside the transition layer, *i.e.* for small gradients of u, and the contribution of h to the right hand side of (9) has to be neglectible inside the transition layer, *i.e.* for large gradient of u.

This is the reason why the second thing we do in order to satisfy these assumptions and to get a good mathematical model, is to change the expression of the function h to get a bounded function with bounded derivatives. The boundedness of the derivative of h with respect to u insures mathematically that the maximum principle is applicable (see Theorem 3). We set, for instance, for $\mu > 0$ large enough:

$$h(u, p) = \mu \left(1 - e^{-|p|/\mu}\right) \left(u - E_\delta^{-1}(u)\right). \tag{11}$$

Conclusion. Finally the model that we study in this paper is Equation (9), with the function h given by (11) or even for more general h which are smooth enough, bounded with bounded derivatives and 1-periodic in u. To simplify the presentation, we will also assume that $h(u, p)/|p|$ is bounded, an assumption which is essentially technical.

Remark 1. We can remark that $u - E_\delta^{-1}(u)$ is a bistable nonlinearity on the interval $(-1/2, 1/2)$ like in [24].

Remark 2. We can also have in mind a "Phase Field" derivation of the model studied in the present paper. In a Phase Field approach, the dislocations are already represented by the transitions of a continuous function u. The associated energy is typically like

$$E(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla^{\frac{1}{2}} u|^2 + W(u) + c_1 u$$

where $\nabla^{\frac{1}{2}} u$ is a fractional derivative of u, and W is a 1-periodic potential. The dynamics of u is given by

$$u_t = -E'(u)\ |\nabla u|$$

i.e.

$$u_t = (c_1 + Lu)|\nabla u| + h(u, \nabla u)$$

with $L = \nabla^{\frac{1}{2}} \cdot \nabla^{\frac{1}{2}} u$ is a nonlocal operator and

$$h(u, \nabla u) = -W'(u)|\nabla u|.$$

In the model that we study in the present paper, the non-local term Lu is approximated by $-u + J \star u$, and to prevent some shocks like in Burgers equation, the term $h(u, \nabla u)$ is approximated by a function globally Lipschitz in $(u, \nabla u)$.

2.2 Mechanical interpretation of the homogenization

Let us briefly explain the meaning of the homogenization result. The homogenized equation (4) can be interpreted as a (generalized) plastic law, *i.e.* a relationship between the plastic strain velocity and the stress. In the homogenized equation (4),

- u_0 is a plastic strain,

- $\partial_t u^0$ is the plastic strain velocity,
- ∇u^0 is the dislocations density and
- $\mathcal{I}_1[u^0]$ is the internal stress created by the density of dislocations contained in a slip plane.

From a mechanical point of view, a shear stress is created in the three dimensional space by the density of dislocations. The trace of the shear stress on the slip plane is precisely given by $\mathcal{I}_1[u_0]$ (see [7] for similar computations for a single dislocation line). Another way to justify the fact that the operator $\mathcal{I}_1$ is a kind of half Laplacian is to remark that the physical model permits to see it as the Dirichlet-Neumann operator associated with the elasticity equation in a half space, that is more or less the Laplace equation.

Let us be more precise now. Let (e_1, e_2, e_3) denote an orthonormal basis and (x_1, x_2, x_3) the corresponding coordinates. We consider dislocations lines contained in the plane $\{x_3 = 0\}$, with a Burgers vector b that is also contained in this plane.

From a mechanical point of view (see for instance [32]), we have the following table of equivalence between our homogenized model and classical models in mechanics for elasto-visco-plasticity of crystals where Λ is the tensor of constant elastic coefficients.

	Crystal elasto-visco-plasticity	Homogenized model
resolved plastic strain	$\gamma(x_1, x_2)\delta_0(x_3)$	$u^0(x_1, x_2)\delta_0(x_3)$
Nye tensor of dislocations densities	$\alpha = b \otimes \left(\nabla^\perp \gamma\right) \delta_0(x_3)$	$\alpha = b \otimes \left(\nabla^\perp u^0\right) \delta_0(x_3)$
exterior applied stress	σ^{ext}	
microscopic resolved shear stress		$c_1 - \int_{(0,1)^2} c_1$
resolved exterior applied stress	$\sigma^{ext} : e^0$	$\int_{(0,1)^2} c_1$
displacement	$v = (v_1, v_2, v_3)$	
strain	$e := e(v) - e^0 \gamma \delta_0(x_3)$ with $e(v) := \frac{1}{2}\left(\nabla v + {}^t\nabla v\right)$ and $e^0 := \frac{1}{2}(e_3 \otimes b + b \otimes e_3)$	
total elastic energy	$E := \int_{\mathbb{R}^3} \frac{1}{2}(\Lambda : e) : e + \sigma^{ext} : e$	$E := \int_{\mathbb{R}^2} -\frac{1}{2} u^0 \mathcal{I}_1[u^0] - \left(\int_{(0,1)^2} c_1\right) u^0$
macroscopic stress	$\sigma := \Lambda : e + \sigma^{ext}$	
resolved macroscopic shear stress	$\tau := \sigma : e^0$	$\tau := \mathcal{I}_1[u^0] + \int_{(0,1)^2} c_1$
plastic law	$\frac{\partial \gamma}{\partial t} = f(\tau)$	$\frac{\partial u^0}{\partial t} = \overline{H}^0 \left(\mathcal{I}_1[u^0] + \int_{(0,1)^2} c_1, \nabla u^0\right)$
energy decay	$\frac{d}{dt} E = \int_{\mathbb{R}^2} -\tau f(\tau) \leq 0$	$\frac{d}{dt} E = \int_{\mathbb{R}^2} -\tau \overline{H}^0 \left(\tau, \nabla u^0\right) \leq 0$

where $\overline{H}^0$ has been computed for the velocity $c_1 - \int_{(0,1)^2} c_1$ with zero mean value.

The fact that $\tau \overline{H}^0 \left(\tau, \nabla u^0\right) \geq 0$ is a consequence of the monotonicity of $\overline{H}^0$ in its first argument and from (28) when $h \equiv 0$.

Identifying the function g. We recall that

$$\mathcal{I}_1[u^0](x) = \int_{\mathbb{R}^2} dz \; \frac{g(z/|z|)}{|z|^{N+1}} \; \left(u^0(x+z) - u^0(x) - z \cdot \nabla u^0(x) \; 1_{B_1}(z)\right)$$

where the function g is related to the behaviour of the function J at large scales:

$$J(z) = g(z/|z|)/|z|^{N+1} \quad \text{for} \quad |z| > R_0$$

with $N = 2$ in our case. Moreover the function $g(z) = \frac{1}{|z|} g(z/|z|)$ is given by (see Alvarez et al. [7], Da Lio et al. [15])

$$g(z) \, \frac{z}{|z|}^{\perp} \otimes \frac{z}{|z|}^{\perp} = D^2 G(z)$$

$$\text{where} \quad \begin{cases} z^{\perp} = (-z_2, z_1) \\ G(\xi_1, \xi_2) = -\frac{1}{(2\pi)^2} \int_{\mathbb{R}} B^*(\xi_1, \xi_2, \xi_3) \, d\xi_3 \quad \text{with} \\ B^*(\xi) = B(\xi/|\xi|) \quad \text{and} \quad B(\xi) = {}^t(\xi \cdot \Lambda : e^0)(\xi \cdot \Lambda \cdot \xi)^{-1}(\xi \cdot \Lambda : e^0) - e^0 : \Lambda : e^0. \end{cases}$$

In the special case of isotropic elasticity, we have

$$\Lambda_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \left(\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right)$$

and for $b = |b| e_1$, we get

$$G(z) = \frac{\mu b^2}{4\pi |z|} \left(z_2^2 + \gamma z_1^2 \right), \quad g(z) = \frac{\mu b^2}{4\pi |z|^3} \left\{ z_1^2 (2\gamma - 1) + z_2^2 (2 - \gamma) \right\}$$

with $\gamma = \frac{1}{1-\nu}$ where $\nu = \frac{\lambda}{2(\lambda+\mu)} \in (-1, 1/2)$ is the Poisson ratio.

Two classical plastic laws. Let us now recall two classical plastic laws usually used in mechanics. The "Orowan law" (see Sedlacek [36, p. 3739]) is

$$f(\tau) = \tau \theta$$

with θ the density of mobile dislocations. The "Norton law with threshold" (see Francois, Pineau, Zaoui [20]) is (for $C, m > 0$)

$$f(\tau) = C \text{sign}(\tau)((|\tau| - \tau_c)^+)^m$$

for τ not too large (in fact, here $m > 1$ because this kind of law already contains the Frank-read process of creation of dislocations). We can compare these laws with our homogenized law:

$$f(\tau) = \overline{H}^0(\tau, \nabla u^0)$$

where ∇u^0 is a kind of generalized dislocations density.

Conclusion. The main features of the homogenization process is that

$$\overline{H}^0(\tau, p) \simeq \tau |p| \quad \text{for} \quad \tau \quad \text{large enough}$$

but for small τ, we recover a threshold phenomenon which is due to the microstructure. The consequence is that at large scales the stress is computed as in mechanics, but the only effect of the microstructure is to introduce some nonlinearities in the plastic law with some threshold effects. Finally in this framework the dislocations density (∇u^0) is simply a derivative of the plastic strain u^0.

3 Results about viscosity solutions for non-local equations

In this paper, we have to deal with Hamilton-Jacobi equations involving integral operators. The classical notion of viscosity solution can be adapted. It is clear how to do it for (1) and as far as (4) is concerned, the reader is referred to [38] for instance. In this Section, we state (and prove if necessary) comparison principles and existence results for such equations that will be used later in the proofs.

Let us first recall the definition of relaxed lower semi-continuous (lsc for short) and upper semi-continuous (usc for short) limits of a family of functions u^ε which is locally bounded uniformly w.r.t. ε:

$$\limsup{}^* u^\varepsilon(t,x) = \limsup_{\varepsilon\to 0, s\to t, y\to x} u^\varepsilon(s,y) \quad \text{and} \quad \liminf{}_* u^\varepsilon(t,x) = \liminf_{\varepsilon\to 0, s\to t, y\to x} u^\varepsilon(s,y).$$

If the family contains only one element, we recognize the usc envelope and the lsc envelope of a locally bounded function u:

$$u^*(t,x) = \limsup_{s\to t, y\to x} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{s\to t, y\to x} u(s,y).$$

Let us recall that we define viscosity solutions of (1) similarly as in [28], where for instance a usc function u^ε is said to be a viscosity subsolution if the corresponding viscosity inequality is satisfied by the test function, keeping the function u^ε in the evaluation of the "regular" nonlocal term $M^\varepsilon\left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]$.

3.1 Viscosity solutions for (4)

We first recall the definition of a viscosity solutions for (4). The difficulty is that the measure μ is singular at the origin and the function must be at least $C^{1,1}$ with respect to x in order that $\mathcal{I}_1[u(t,\cdot)](x)$ makes sense. We refer the reader to [38] for instance. Let us introduce first the following definition of viscosity solution associated to a cut-off radius r:

Definition 1 (r-viscosity solution)**.** *Let us fix $r>0$. A usc (resp. lsc) function $u:\mathbb{R}^+\times\mathbb{R}^N\to\mathbb{R}$ is a* r-viscosity subsolution *(resp.* supersolution*) of* (4) *if for any (t,x), $t>0$, any $\tau\in(0,t)$, and any test function $\phi\in C^2(\mathbb{R}^+\times\mathbb{R}^N)$ such that $u-\phi$ attains a maximum (resp. minimum) at the point (t,x) on $Q_{\tau,r}(t,x)$, then we have*

$$\partial_t\phi(t,x) - \overline{H}^0(L_r, \nabla_x\phi(t,x)) \le 0 \quad (\textit{resp.} \ge 0)$$

where $L_r = \int_{B_r}(\phi(t,x+z)-\phi(t,x)-\nabla\phi(t,x)\cdot z)d\mu(z) + \int_{\mathbb{R}^N\setminus B_r}(u(t,x+z)-u(t,x))d\mu(z)$.

A continuous function is a r-viscosity solution of (4) *if it is a r-viscosity sub- and a supersolution of the equation.*

It is classical that the maximum can be supposed to be global (*i.e.* attained on $\mathbb{R}^+\times\mathbb{R}^N$ and not only on the cylinder) and this will be used several times. We also have the following property (see [38] for instance):

Proposition 1 (Equivalence of the definitions)**.** *Let $r>0$ and $r'>0$. A continuous (resp. usc, lsc) function u is a r-viscosity solution (resp. subsolution, supersolution) of* (4) *if and only if it is a r'-viscosity solution (resp. subsolution, supersolution) of* (4)*.*

Because of this proposition, if we do not need to emphasize the positive r we use, we simply call them viscosity solutions (resp. subsolutions, supersolutions). In order to prove a comparison result for (4), we need to be able to consider semiconcave test functions for semiconvex subsolutions.

Proposition 2. *Consider a bounded semiconvex subsolution u of* (4) *and a continuous function ϕ, semiconcave w.r.t. (t,x). Then at a local strict maximum point (t,x) of $u-\phi$, we can ensure that $\mathcal{I}_1[u(t,\cdot)](x)$ is finite when $\nabla u(t,x)\neq 0$, and then*

$$\partial_t\phi(t,x) \le \overline{H}^0(\mathcal{I}_1[u(t,\cdot)](x), \nabla u(t,x)).$$

Remark 3. Recall that a semiconvex function (in our case, the function u) is differentiable at a maximum point.

Proof. First, we notice that u and ϕ are differentiable at (t,x) because of their regularity. If $\nabla u(t,x) = \nabla\phi(t,x) = 0$, there is nothing to prove. Hence, we now assume that $\nabla u(t,x) \neq 0$.

Let us consider a cylinder $Q_{\tau,r}(t,x)$ where $u-\phi$ attains a strict maximum and let us approximate $M = \sup_{(s,y)\in Q_{\tau,r}(t,x)}\{u(s,y)-\phi(s,y)\} = u(t,x)-\phi(t,x)$ by $M^\delta = \sup_{(s,y)\in Q_{\tau,r}(t,x)}\{u(s,y)-\phi^\delta(s,y)\} = u(t_\delta,x_\delta)-\phi^\delta(t_\delta,x_\delta)$ where ϕ^δ is the sup-convolution of ϕ and $\frac{1}{2\delta}|\cdot|^2$:

$$\phi^\delta(t,x) = \sup_{y\in\mathbb{R}^N, s\in\mathbb{R}}\{\phi(s,y) - \frac{1}{2\delta}|x-y|^2 - \frac{1}{2\delta}(t-s)^2\}.$$

We know that ϕ^δ is $C^{1,1}$ w.r.t. (t,x) (see for instance [33]) and $(t_\delta,x_\delta)\to(t,x)$ as $\delta\to 0$. Moreover, we can control independently of δ the constant of semiconcavity of ϕ^δ (this is used later in the proof). Now, ϕ^δ is a proper test function for u and we therefore have:

$$\partial_t\phi^\delta(t_\delta,x_\delta) \leq \overline{H}^0(K_r^{\rho,\delta} + L_r^{\rho,\delta}, \nabla\phi^\delta(t_\delta,x_\delta)) \tag{12}$$

where

$$\begin{aligned}
K_r^{\rho,\delta} &= \int_{B_\rho}\{\phi^\delta(t_\delta,x_\delta+z) - \phi^\delta(t_\delta,x_\delta) - \nabla\phi^\delta(t_\delta,x_\delta)\cdot z\}\mu(dz),\\
L_r^{\rho,\delta} &= \int_{B_r\setminus B_\rho}\{u(t_\delta,x_\delta+z)-u(t_\delta,x_\delta)\}\mu(dz) + \int_{\mathbb{R}^N\setminus B_r}\{u(t_\delta,x_\delta+z)-u(t_\delta,x_\delta)\}\mu(dz)\\
&= \int_{B_r\setminus B_\rho}\{u(t_\delta,x_\delta+z)-u(t_\delta,x_\delta)-p^\delta\cdot z\}\mu(dz) + \int_{\mathbb{R}^N\setminus B_r}\{u(t_\delta,x_\delta+z)-u(t_\delta,x_\delta)\}\mu(dz)
\end{aligned}$$

where $p_\delta = \nabla u(t_\delta,x_\delta) = \nabla\phi^\delta(t_\delta,x_\delta)$. If (s_δ,y_δ) is a point such that: $\phi^\delta(t_\delta,x_\delta) = \phi(s_\delta,y_\delta) - \frac{1}{2\delta}|x_\delta-y_\delta|^2 - \frac{1}{2\delta}(s_\delta-t_\delta)^2$, we know that:

$$(\partial_t\phi^\delta,\nabla\phi^\delta)(t_\delta,x_\delta) = (\partial_t\phi,\nabla\phi)(s_\delta,y_\delta) \quad \text{and} \quad |(s_\delta,y_\delta)-(t_\delta,x_\delta)| = o(\sqrt{\delta}).$$

In particular, because ϕ is semiconcave and differentiable at (t,x), $\nabla\phi^\delta(t_\delta,x_\delta)\to\nabla\phi(t,x) = \nabla u(t,x)$. Using the semiconvexity of u, the definition of M^δ and the semiconcavity of ϕ^δ, we conclude that there exists $C>0$ (independent of δ and ρ) such that:

$$-C|z|^2 \leq u(t_\delta,x_\delta+z)-u(t_\delta,x_\delta)-p^\delta\cdot z \leq \phi^\delta(t_\delta,x_\delta+z)-\phi^\delta(t_\delta,x_\delta)-p^\delta\cdot z \leq C|z|^2.$$

The dominated convergence theorem asserts that $K_r^{\rho,\delta}+L_r^{\rho,\delta}$ has a limit as $\rho\to 0$ and $\delta\to 0$ successively, and this limit equals $L_r^{0,0} = \mathcal{I}_1[u(t,\cdot)](x)$. Hence (12) yields the desired result. □

3.2 Comparison principles

In this section, we state and prove comparison principles we will need later. The first theorem is a comparison principle for (1). Let us mention that, as usual in the viscosity solution theory, the fact that the derivative with respect to u of the function $h(u,p)$ is bounded uniformly in the gradient p is important.

Theorem 3. *Consider a bounded usc subsolution u_1 and a bounded lsc supersolution u_2 of* (1). *If $u_1(0,x)\leq u_0(x)\leq u_2(0,x)$ with $u_0\in W^{1,\infty}$, then $u_1\leq u_2$ on $(0,+\infty)\times\mathbb{R}^N$.*

Remark 4. We will use later a comparison principle for a higher dimensional problem, precisely for (20), in the class of functions $f(t,x,x_{N+1})$ such that for any $(t,x,x_{N+1})\in(0,T)\times\mathbb{R}^N\times\mathbb{R}$:

$$|f(t,x,x_{N+1})| \leq C(T)(1+|x_{N+1}|).$$

Such a comparison principle can also be proved.

Remark 5. We will use later comparison principles for two other equations very similar to (1) (see (50) and (65)). The reader can check that these results can be easily adapted from classical techniques.

Remark 6. Let us finally point out that when passing from (50) to (51), the nonlocal operator is modified and the solution U is a function that is the sum of a linear function and of a bounded one (and the linear part is *a priori* known). In such a context and with such a modification of the nonlocal operator, a comparison principle also holds true.

The third comparison principle is stated on a bounded domain Q. Because we deal with a nonlocal equation, we need to be able to compare the sub- and the supersolution everywhere outside Q.

Theorem 4 (Comparison principle on bounded domains for (1))**.** *Consider a bounded usc function $u : Q \to \mathbb{R}$ and a bounded lsc function $v : Q \to \mathbb{R}$ that are respectively a sub- and supersolution of* (1) *on a bounded open domain $Q \subset \mathbb{R}^+ \times \mathbb{R}^N$. If $u \le v$ outside Q, then $u \le v$ on Q.*

The main result of this section is a comparison result for (4). We will use Proposition 2 to prove it.

Theorem 5. *Consider a bounded usc subsolution u and a bounded lsc supersolution v of* (4)*. If $u(0,x) \le u_0(x) \le v(0,x)$ with $u_0 \in W^{1,\infty}$, then $u \le v$ on $(0,+\infty) \times \mathbb{R}^N$.*

Remark 7. Remark 4 is also valid when passing from (4) to (32).

Proof. Consider $M = \sup_{t\in[0,T[,x\in\mathbb{R}^N}\{u(t,x) - v(t,x)\}$, suppose that $M > 0$ and let us exhibit a contradiction. First, we approximate the previous supremum by $M^\varepsilon = \sup_{t\in[0,T[,x\in\mathbb{R}^N}\{u^\varepsilon(t,x) - v_\varepsilon(t,x)\}$ where u^ε (resp. v_ε) is the sup-convolution (resp. inf-convolution) of u (resp. v) in the space variable x. It is well-known that u^ε (resp. v_ε) is a semi-convex subsolution (resp. semi-concave supersolution) of (4); in particular, this means that $u^\varepsilon(t,\cdot) + \frac{1}{2\varepsilon}|\cdot|^2$ is convex (resp. $v_\varepsilon(t,\cdot) - \frac{1}{2\varepsilon}|\cdot|^2$ is concave) and we have, at a point (t,x) of differentiability of u_ε and v_ε,

$$u^\varepsilon(t,x+z) - u^\varepsilon(t,x) - \nabla u^\varepsilon(t,x)\cdot z \ge -\frac{1}{2\varepsilon}|z|^2 \text{ for any } z \in \mathbb{R}^N, \tag{13}$$

$$v_\varepsilon(t,x+z) - v_\varepsilon(t,x) - \nabla v_\varepsilon(t,x)\cdot z \le \frac{1}{2\varepsilon}|z|^2 \text{ for any } z \in \mathbb{R}^N. \tag{14}$$

Now consider $M^\varepsilon_{\alpha,\eta,\nu} = \sup_{t,s\in[0,T[,x\in\mathbb{R}^N}\{u^\varepsilon(t,x) - v_\varepsilon(s,x) - \alpha\varphi(x) - \frac{(t-s)^2}{2\nu} - \frac{\eta}{T-t}\}$ where φ is a C^2 function with bounded first and second derivatives, such that there exists C such that $|\varphi(x)| \le C(1+\sqrt{|x|})$ and $\varphi(x) \to +\infty$ as $|x| \to \infty$. The supremum is attained at $(\bar t, \bar s, \bar x)$. Choosing the parameters small enough, we can ensure that $M^\varepsilon_{\alpha,\eta,\nu} \ge M/2 > 0$. Classical results about penalization (see [14] for instance) show that $\frac{(\bar t - \bar s)^2}{\nu} \to 0$ as $\nu \to 0$.

Suppose that there are sequences $\nu_n \to 0$, $\alpha_n \to 0$, $\eta_n \to 0$, $\varepsilon_m \to 0$ such that $\bar t = 0$ or $\bar s = 0$. Then by letting first n tend to $+\infty$, we obtain:

$$0 < M \le M^{\varepsilon_m} \le \sup_{x\in\mathbb{R}^N}\{u^{\varepsilon_m}(0,x) - v_{\varepsilon_m}(0,x)\} \le \sup_{x\in\mathbb{R}^N}\{(u_0)^{\varepsilon_m}(x) - (u_0)_{\varepsilon_m}(x)\} \to 0 \text{ as } m \to +\infty$$

which is a contradiction. Therefore, we can assume that $\bar t, \bar s > 0$, and with $B = B_1(0)$ we have

$$\begin{aligned}
&\left(\frac{\bar t - \bar s}{\nu} + \frac{\eta}{(T-\bar t)^2}, \nabla u^\varepsilon(\bar t,\bar x)\right) \in D^{1,+}u^\varepsilon(\bar t,\bar x),\\
&\left(\frac{\bar t - \bar s}{\nu}, \nabla v_\varepsilon(\bar s,\bar x)\right) \in D^{1,-}v_\varepsilon(\bar t,\bar x), \qquad \nabla u^\varepsilon(\bar t,\bar x) - \alpha\nabla\varphi(\bar x) = \nabla v_\varepsilon(\bar s,\bar x) =: p_\varepsilon,\\
&u^\varepsilon(\bar t,\bar x+z) - u^\varepsilon(\bar t,\bar x) - \nabla u^\varepsilon(\bar t,\bar x)\cdot z\mathbf{1}_B(z)\\
&\le v_\varepsilon(\bar s,\bar x+z) - v_\varepsilon(\bar s,\bar x) - p_\varepsilon\cdot z\mathbf{1}_B(z) - \alpha[\phi(\bar x+z) - \phi(\bar x) - \nabla\phi(\bar x)\cdot z\mathbf{1}_B(z)].
\end{aligned} \tag{15}$$

Apply Proposition 2 and get the following viscosity inequalities:

$$\begin{aligned}
\frac{\bar{t}-\bar{s}}{\nu}+\frac{\eta}{(T-\bar{t})^2} &\leq \overline{H}^0\left(\int[u^\varepsilon(\bar{t},\overline{x}+z)-u^\varepsilon(\bar{t},\overline{x})-\nabla u^\varepsilon(\bar{t},\overline{x})\cdot z\mathbf{1}(z)]\mu(dz), \quad p_\varepsilon+\alpha\nabla\varphi(\overline{x})\right)\\
&\leq \overline{H}^0\left(\int[v_\varepsilon(\bar{s},\overline{x}+z)-v_\varepsilon(\bar{s},\overline{x})-p_\varepsilon\cdot z\mathbf{1}(z)]\mu(dz)-\alpha\mathcal{I}_1[\varphi](\overline{x}), \quad p_\varepsilon+\alpha\nabla\varphi(\overline{x})\right),\\
\frac{\bar{t}-\bar{s}}{\nu} &\geq \overline{H}^0\left(\int[v_\varepsilon(\bar{s},\overline{x}+z)-v_\varepsilon(\bar{s},\overline{x})-\nabla v_\varepsilon(\bar{s},\overline{x})\cdot z\mathbf{1}(z)]\mu(dz), \quad p_\varepsilon\right).
\end{aligned}$$

Substracting these two inequalities yield

$$\frac{\eta}{T^2}\leq \overline{H}^0(I_\varepsilon-\alpha\mathcal{I}_1[\varphi](\overline{x}),p_\varepsilon+\alpha\nabla\varphi(\overline{x}))-\overline{H}^0(I_\varepsilon,p_\varepsilon) \tag{16}$$

with $I_\varepsilon := \int[v_\varepsilon(\bar{s},\overline{x}+z)-v_\varepsilon(\bar{s},\overline{x})-\nabla v_\varepsilon(\bar{s},\overline{x})\cdot z\mathbf{1}(z)]\mu(dz)=I_\varepsilon^1+I_\varepsilon^2$ and

$$\begin{aligned}
I_\varepsilon^1 &= \int_{|z|\leq 1}[v_\varepsilon(\bar{s},\overline{x}+z)-v_\varepsilon(\bar{s},\overline{x})-\nabla v_\varepsilon(\bar{s},\overline{x})\cdot z]\mu(dz)\\
I_\varepsilon^2 &= \int_{|z|\geq 1}[v_\varepsilon(\bar{s},\overline{x}+z)-v_\varepsilon(\bar{s},\overline{x})]\mu(dz).
\end{aligned}$$

Let us estimate these two integrals by using (13), (14) and (15):

$$\begin{aligned}
-C\left(\frac{1}{\varepsilon}+\alpha\right)\leq -\frac{1}{2\varepsilon}\int_{B_1}\frac{dz}{|z|^{N-1}}-\alpha\int_{B_1}[\varphi(\overline{x}+z)d-\varphi(\overline{x}-\nabla\varphi(\overline{x})\cdot z]\mu(dz)\leq I_\varepsilon^1\\
I_\varepsilon^1\leq\frac{1}{2\varepsilon}\int_{B_1}\frac{dz}{|z|^{N-1}}\leq\frac{C}{\varepsilon}\\
-2\|v\|_\infty\int_{\mathbb{R}^N\setminus B_1}\mu(dz)\leq I_2^\varepsilon\leq 2\|v\|_\infty\int_{\mathbb{R}^N\setminus B_1}\mu(dz).
\end{aligned}$$

We now use the classical estimate about inf-convolutions: $|p_\varepsilon|\leq\frac{\sqrt{2\|v\|_\infty}}{\sqrt{\varepsilon}}$. We therefore see that the right hand side of (16) tends to zero as $\alpha\to 0$ for fixed ε. We then get the desired contradiction: $\frac{\eta}{T^2}\leq 0$. □

3.3 Existence results

Theorem 6. *Consider $u_0\in W^{2,\infty}(\mathbb{R}^N)$ and $c\in W^{1,\infty}(\mathbb{R}^N)$. For $\varepsilon>0$, there exists a (unique) bounded continous viscosity solution u^ε of (1) satisfying $u^\varepsilon(0,x)=u_0(x)$. Moreover, u^ε is locally bounded in (t,x), uniformly in ε and:* $\limsup^* u^\varepsilon(0,x)=\liminf_* u^\varepsilon(0,x)=u_0(x)$.

Proof. From the classical theory of viscosity solutions, we know that it suffices to construct barriers in order to apply Perron's method. To do so, we prove that there exists a constant $C>0$ (independent of ε) such that $u^+(t,x)=u_0(x)+Ct$ (resp. $u_-(t,x)=u_0(x)-Ct$) is a supersolution (resp. subsolution) of (1). By the comparison principle for (1), this implies that for any $\varepsilon>0$, $u_-\leq u^\varepsilon\leq u^+$. It is therefore possible to construct by Perron's method a bounded viscosity solution of (1). It suffices to adapt [26, Theorem 3]. It also implies that $\limsup^* u^\varepsilon(0,x)=\liminf_* u^\varepsilon(0,x)=u_0(x)$.

Let us now determine C. First, we prove that for any $C>0$,

$$\left|M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right]\right|\leq C_1$$

where C_1 depends only on $R_0,N,\|u_0\|_{2,\infty}$ (in particular it does not depend on C or ε). To see this, we

write,

$$M^\varepsilon\left[\frac{u_0(\cdot)+Ct}{\varepsilon}\right] = \int_{B_{R_0}} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz \tag{17}$$

$$+\int_{B_{1/\varepsilon}\setminus B_{R_0}} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz \tag{18}$$

$$+\int_{\mathbb{R}^N\setminus B_{1/\varepsilon}} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz \tag{19}$$

and we estimate each term. The first one, (17), is treated as follows:

$$\left|\int_{B_{R_0}} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz\right| \le \|\nabla u_0\|_\infty \int_{B_{R_0}} J(z)|z|dz \le R_0\|\nabla u_0\|_\infty.$$

Next we estimate (18):

$$\left|\int_{R_0\le|z|\le 1/\varepsilon} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz\right| = \left|\int_{\varepsilon R_0\le|y|\le 1} (u_0(x+y)-u_0(x))\,\mu(dy)\right|$$
$$= \left|\int_{\varepsilon R_0\le|y|\le 1} [u_0(x+y)-u_0(x)-\nabla u_0(x)\cdot y]\,\mu(dy)\right| \le \|D^2u_0\|_\infty \int_{B_1}\frac{|y|^2}{2}\mu(dy).$$

Eventually, we estimate (19):

$$\left|\int_{|z|\ge 1/\varepsilon} J(z)\left(\frac{u_0(x+\varepsilon z)}{\varepsilon} - \frac{u_0(x)}{\varepsilon}\right)dz\right| = \left|\int_{|y|\ge 1} (u_0(x+y)-u_0(x))\,\mu(dy)\right|$$
$$\le 2\|u_0\|_\infty \int_{\mathbb{R}^N\setminus B_1}\mu(dy).$$

Then define $C_1 = R_0\|\nabla u_0\|_\infty + \|D^2u_0\|_\infty\int_{B_1}\frac{|y|^2}{2}\mu(dy) + 2\|u_0\|_\infty\int_{\mathbb{R}^N\setminus B_1}\mu(dy)$. Plugging u_0+Ct into (1), we see that it is a supersolution if $C \ge (\|c\|_\infty + C_1)\|\nabla u_0\|_\infty + G$. It also ensures that $u_0 - Ct$ is a subsolution of (1). □

Proposition 3. *The homogenized equation* (4) *has a unique bounded continuous viscosity solution* u^0.

Proof. Adapting the argument of [26] (already adapted from the classical argument), we can construct a solution by Perron's method if we construct "good" barriers. But since u_0 is $W^{2,\infty}$, the two functions $u_\pm(t,x) = u_0(x) \pm Ct$ are respectively a super and a subsolution for

$$C \ge \sup\{\overline{H}^0(L,p) : |L| \le D_N\|u_0\|_{2,\infty}, |p| \le \|\nabla u_0\|_\infty\}$$

with D_N only depending on the dimension. Moreover we have $(u^+)_*(0,x) = (u^-)^*(0,x) = u_0(x)$. □

4 The proof of convergence

This section is devoted to the proof of Theorem 2. As announced in Introduction, the proof is essentially the same as the one for local equations [28]: imbed the problem in a higher dimensional one, argue by contradiction, consider a twisted perturbed test function and combine the relaxed semi-limit technique with the existence of Lipschitz continuous approximate correctors to exhibit a contradiction. The additional difficulty in comparison with [28] lies in the way to handle the non-local term.

As announced above, let us consider a higher dimensional problem:

$$\begin{cases} \partial_t U^\varepsilon = \left[c\left(\frac{x}{\varepsilon}\right) + M^\varepsilon\left[\frac{U^\varepsilon(t,\cdot,x_{N+1})}{\varepsilon}\right]\right]|\nabla_x U^\varepsilon| + h\left(\frac{U^\varepsilon}{\varepsilon}, \nabla_x U^\varepsilon\right) & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ U^\varepsilon(0,X) = u_0(x) + x_{N+1} & \text{on} \quad \mathbb{R}^{N+1} \end{cases} \tag{20}$$

where $X = (x, x_{N+1})$. Let us exhibit the link between the problems in $\mathbb{R}^N$ and in $\mathbb{R}^{N+1}$.

Lemma 1. *If u^ε and U^ε denote respectively the solutions of* (1) *and* (20)*, then we have*

$$\left|U^\varepsilon(t,x,x_{N+1}) - u^\varepsilon(t,x) - \varepsilon\left\lfloor\frac{x_{N+1}}{\varepsilon}\right\rfloor\right| \leq \varepsilon, \tag{21}$$

$$U^\varepsilon\left(t,x,x_{N+1} + \varepsilon\left\lfloor\frac{a}{\varepsilon}\right\rfloor\right) = U^\varepsilon(t,x,x_{N+1}) + \varepsilon\left\lfloor\frac{a}{\varepsilon}\right\rfloor. \tag{22}$$

This lemma is a straightforward consequence of a comparison principle for (20) and of invariance by ε-integer translations w.r.t. x_{N+1}.

We previously explained that the existence of regular sub- and supercorrectors of a family of cell problems is necessary. Precisely, we consider for $P = (p,1) \in \mathbb{R}^{N+1}$:

$$\begin{cases} \lambda + \partial_\tau V = H(L, y, \lambda\tau + P\cdot Y + V, P + \nabla V, [V]) + \beta \quad \text{in } (0,+\infty)\times\mathbb{R}^{N+1}, \\ \text{where} \quad H(...) = (c(y) + L + M[V(\tau,\cdot,y_{N+1})])|p + \nabla_y V| + h(\lambda\tau + P\cdot Y + V, p + \nabla_y V) \\ V(0,Y) = 0 \quad \text{on} \quad \mathbb{R}^{N+1} \end{cases} \tag{23}$$

where $Y = (y, y_{N+1})$.

We also need to make more precise how the real number λ given by Theorem 1 for the Hamiltonian $H + \beta$ depends on its variables. The two following properties are by-products of the construction of approximate correctors we will do in the next section.

Proposition 4 (Properties of the effective Hamiltonian)**.** *Let $p \in \mathbb{R}^N$ and $L \in \mathbb{R}$. For any $\beta \in \mathbb{R}$, let $\overline{H}(L,p,\beta)$ be the constant defined by Theorem 1 for the Hamiltonian $H + \beta$. Then $\overline{H}$ is continuous w.r.t. (L,p,β) and nondecreasing w.r.t. L and β and*

$$\overline{H}(L,p,\beta) \to \pm\infty \quad \text{as } L \to \pm\infty \text{ if } p \neq 0, \tag{24}$$

$$\overline{H}(L,p,\beta) \to \pm\infty \quad \text{as } \beta \to \pm\infty. \tag{25}$$

In particular, if $\lambda(\beta)$ denotes $\overline{H}(L,p,\beta)$, then

$$\begin{cases} \lambda(\beta) \text{ is nondecreasing and continuous w.r.t. } \beta, \\ \forall\lambda_0 \in \mathbb{R}, \exists\beta_0 \in \mathbb{R}, \text{ such that } \lambda(\beta_0) = \lambda_0. \end{cases} \tag{26}$$

For $\beta = 0$, we have $\overline{H}^0(L,p) = \overline{H}(L,p,0)$, and

$$\overline{H}^0(L,0) = 0 \text{ for any } L. \tag{27}$$

Moreover if $\int_{[0,1)^N} c = 0$ and $h \equiv 0$, then

$$\overline{H}^0(0,p) = 0 \text{ for any } p. \tag{28}$$

Proposition 5 (Existence of approximate correctors)**.** *For any fixed $p \in \mathbb{R}^N$, $\beta \in \mathbb{R}$ and $K > 0$ large enough, there exist real numbers $\lambda_K^+(\beta)$, $\lambda_K^-(\beta)$, a constant $C > 0$ (independent on K, β and p) and bounded super and subcorrectors V_K^+, V_K^- (depending on β) i.e. a super and a subsolution of* (23) *respectively with $\lambda = \lambda_K^\pm(\beta)$, such that*

$$\lambda(\beta) := \lim_{K\to+\infty} \lambda_K^+(\beta) = \lim_{K\to+\infty} \lambda_K^-(\beta) \tag{29}$$

with $\lambda_K^+(\beta)$ and $\lambda_K^-(\beta)$ satisfying (26) and for any $(\tau, Y) \in \mathbb{R}^+ \times \mathbb{R}^{N+1}$:

$$|V_K^\pm(\tau, Y)| \leq C. \tag{30}$$

For any $\lambda_0 \in \mathbb{R}$, there exists $\beta_0^\pm, \beta_K^\pm$ such that

$$\begin{cases} \lambda_K^\pm(\beta_K^\pm) = \lambda(\beta_0^\pm) = \lambda_0, \\ \left|\nabla_Y V_K^\pm\right| \leq D_K, \\ \beta_K^\pm \longrightarrow \beta_0^\pm \quad as \quad K \to +\infty \end{cases} \tag{31}$$

for the supercorrector V_K^+ and the subcorrector V_K^- respectively associated to β_K^+ and β_K^-, and for some constant $D_K = D(K, p) > 0$.

Proof of Theorem 2. Classically, we prove that $U^+ = \limsup^* U^\varepsilon$ is a subsolution of

$$\begin{cases} \partial_t W = \overline{H}^0\left(\mathcal{I}_1[W(t, \cdot, x_{N+1})], \nabla_x W\right) & \text{in} \quad (0, +\infty) \times \mathbb{R}^{N+1}, \\ W(0, X) = u_0(x) + x_{N+1} & \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{32}$$

We will need the following lemma.

Lemma 2. *If u^0 and U^0 denote respectively the solutions of (4) and (32), then we have:*

$$U^0(t, x, x_{N+1}) = u^0(t, x) + x_{N+1}, \tag{33}$$

$$U^0(t, x, x_{N+1} + a) = U^0(t, x, x_{N+1}) + a. \tag{34}$$

Like Lemma 1, it is a consequence of invariance by integer translations and of a comparison principle for the higher dimensional problem (see Remark 7).

Using barriers analogous to the ones of Theorem 6, we know that the family of functions $\{U^\varepsilon\}_{\varepsilon>0}$ is locally bounded so that U^+ is everywhere finite. Analogously, we can prove that $U^- = \liminf_* U^\varepsilon$ is a supersolution of (32). Using a second time the barriers, that are uniform with respect to ε, we deduce that $U^+(0, X) = U^-(0, X) = u_0(x) + x_{N+1}$. The comparison principle for (32) thus implies that $U^+ \leq U^-$. Since $U^- \leq U^+$ always holds true, we conclude that the two functions coincide with U^0, the unique continuous viscosity solution of (32). This last fact is equivalent to the local convergence of U^ε towards U^0. By Lemmata 1 and 2, this proves in particular the local convergence of u^ε towards u^0.

Arguing by contradiction, we prove that U^+ is a 2-viscosity subsolution of (32). We consider a test function ϕ such that $U^+ - \phi$ attains a strict global zero maximum at (t_0, X_0) with $X_0 = (x_0, x_{N+1}^0)$ and $t_0 > 0$ and we suppose that there exists $\theta > 0$ such that:

$$\partial_t \phi(t_0, X_0) = \overline{H}(L_0, \nabla_x \phi(t_0, X_0)) + \theta$$

where

$$\begin{aligned} L_0 \quad = \quad & \int_{|x| \leq 2} \left(\phi(t_0, x_0 + x, x_{N+1}^0) - \phi(t_0, x_0, x_{N+1}^0) - \nabla_x \phi(t_0, x_0, x_{N+1}^0) \cdot x\right) \mu(dx) \\ & + \int_{|x| \geq 2} \left(U^+(t_0, x_0 + x, x_{N+1}^0) - U^+(t_0, x_0, x_{N+1}^0)\right) \mu(dx). \end{aligned} \tag{35}$$

First, let us consider $\delta > 0$ such that:

$$\overline{H}(L_0, \nabla_x \phi(t_0, X_0)) + \theta = \overline{H}(L_0 + \delta, \nabla_x \phi(t_0, x_0)) + \frac{\theta}{2}.$$

Secondly, let us consider $\beta_0^+ > 0$ and β_K^+ such that $\lambda(\beta_0^+) = \lambda_K^+(\beta_K^+) = \partial_t \phi(t_0, X_0)$ and choose K large enough so that $\beta_K^+ \geq \beta_0^+/2 > 0$. Then let V_K^+ be the supersolution of (23) given by Proposition 5 for $\beta = \beta_K^+$ and for $L = L_0 + \delta$. In the following, $\lambda = \lambda_K^+(\beta_K^+)$, ¡$p = \nabla_x \phi(t_0, X_0)$ and $V = V_K^+$.

For $\varepsilon \le r < \frac{1}{2}$, we consider the function $F(t,X) = \phi(t,X) - p\cdot x - \lambda t$ and we define a "x_{N+1}-twisted perturbed test function" ϕ^ε by:

$$\phi^\varepsilon(t,X) = \begin{cases} \phi(t,X) + \varepsilon V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{F(t,X)}{\varepsilon}\right) + k_\varepsilon \varepsilon & \text{on } (t_0/2, 2t_0)\times B_1(X_0), \\ U^\varepsilon(t,X) & \text{outside} \end{cases} \tag{36}$$

where $k_\varepsilon \in \mathbb{Z}$ will be chosen later. We are going to prove that ϕ^ε is a supersolution of (20) on $\mathcal{V}_r = Q_{r,r}(t_0,X_0)$ for some $r = r_K < 1/2$ properly chosen, satisfying in particular $\mathcal{V}_r \subset (t_0/2, 2t_0)\times B_1(X_0)$.

Let us first focus on the boundary conditions. For ε small enough (*i.e.* $0 < \varepsilon \le \varepsilon_0(r) < r$), since $U^+ - \phi$ attains a strict maximum at (t_0, X_0), we can ensure that:

$$U^\varepsilon(t,X) \le \phi(t,X) + \varepsilon V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{F(t,X)}{\varepsilon}\right) - \eta_r \text{ on } (t_0/3, 3t_0)\times B_2(X_0)\setminus \mathcal{V}_r \tag{37}$$

for some $\eta_r = o_r(1) > 0$. Hence, choosing $k_\varepsilon = \lceil \frac{-\eta_r}{\varepsilon}\rceil$, we ensure that $U^\varepsilon \le \phi^\varepsilon$ outside $\mathcal{V}_r$.

Let us next study the equation. From (22), we deduce that $U^+(t,x,x_{N+1}+a) = U^+(t,x,x_{N+1}) + a$. Hence, we first derive that:

$$\frac{\partial \phi}{\partial x_{N+1}}(t_0,X_0) = 1.$$

Consider a test function ψ such that $\phi^\varepsilon - \psi$ attains a global zero minimum at $(\bar t, \overline{X}) \in \mathcal{V}_r$. Then for any $(t,x) \in \mathbb{R}^+\times\mathbb{R}^{N+1}$:

$$V\left(\frac{\bar t}{\varepsilon}, \frac{\overline{x}}{\varepsilon}, \frac{F(\bar t,\overline{X})}{\varepsilon}\right) - \frac{1}{\varepsilon}(\psi-\phi)(\bar t,\overline{X}) \le V\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \frac{F(t,X)}{\varepsilon}\right) - \frac{1}{\varepsilon}(\psi-\phi)(t,X).$$

We have $\frac{\partial F}{\partial x_{N+1}}(t_0,X_0) = \frac{\partial \phi}{\partial x_{N+1}}(t_0,X_0) = 1$. Consequently, there exists $r_0 > 0$ such that the map

$$\begin{array}{rlcl} Id\times F: & \mathcal{V}_{r_0} & \longrightarrow & \mathcal{U}_{r_0} \subset \mathbb{R}\times\mathbb{R}^N\times\mathbb{R} \\ & (t,x,x_{N+1}) & \longmapsto & (t,x,F(t,x,x_{N+1})) \end{array}$$

is a C^1-diffeomorphism from $\mathcal{V}_{r_0}$ onto its range $\mathcal{U}_{r_0}$. Let $G:\mathcal{U}_{r_0} \longrightarrow \mathbb{R}$ be the map such that

$$\begin{array}{rlcl} Id\times G: & \mathcal{U}_{r_0} & \longrightarrow & \mathcal{V}_{r_0} \\ & (t,x,\xi_{N+1}) & \longmapsto & (t,x,G(t,x,\xi_{N+1})) \end{array}$$

is the inverse of $Id\times F$. Let us consider the variables $\tau = t/\varepsilon$, $Y = (y, y_{N+1})$ with $y = x/\varepsilon$ and $y_{N+1} = F(t,X)/\varepsilon$ and define

$$\Gamma^\varepsilon(\tau,Y) = \frac{1}{\varepsilon}\left(\psi\left(\varepsilon\tau,\varepsilon y, G(\varepsilon\tau,\varepsilon y,\varepsilon y_{N+1})\right) - \phi\left(\varepsilon\tau,\varepsilon y, G(\varepsilon\tau,\varepsilon y,\varepsilon y_{N+1})\right)\right).$$

Let $\overline{\tau} = \frac{\bar t}{\varepsilon}$, $\overline{y} = \frac{\overline{x}}{\varepsilon}$, $\overline{y}_{N+1} = \frac{F(\bar t,\overline{X})}{\varepsilon}$, $\overline{Y} = (\overline{y},\overline{y}_{N+1})$. Then:

$$V(\overline{\tau},\overline{Y}) - \Gamma^\varepsilon(\overline{\tau},\overline{Y}) \quad \le \quad V(\tau,Y) - \Gamma^\varepsilon(\tau,Y) \quad \text{for all} \quad (\varepsilon\tau,\varepsilon Y) \in \mathcal{U}_{r_0}.$$

From Proposition 5, we know that V is D_K-Lipschitz continuous with respect to Y. This implies that:

$$|\nabla_Y \Gamma^\varepsilon(\overline{\tau},\overline{Y})| \le D_K. \tag{38}$$

Simple computations yield with $P = (p,1) \in \mathbb{R}^{N+1}$:

$$\begin{cases} \lambda + \partial_\tau \Gamma^\varepsilon(\overline{\tau},\overline{Y}) = \partial_t\psi(\bar t,\overline{X}) + (1+\partial_{y_{N+1}}\Gamma^\varepsilon(\overline{\tau},\overline{Y}))(\partial_t\phi(t_0,X_0) - \partial_t\phi(\bar t,\overline{X})), \\ p + \nabla_y\Gamma^\varepsilon(\overline{\tau},\overline{Y}) = \nabla_x\psi(\bar t,\overline{X}) + (1+\partial_{y_{N+1}}\Gamma^\varepsilon(\overline{\tau},\overline{Y}))(\nabla_x\phi(t_0,X_0) - \nabla_x\phi(\bar t,\overline{X})), \\ \lambda\overline{\tau} + P\cdot\overline{Y} + V(\overline{\tau},\overline{Y}) = \frac{\phi^\varepsilon(\bar t,\overline{X})}{\varepsilon} - k_\varepsilon. \end{cases} \tag{39}$$

Using (39) and (38), Equation (23) yields:

$$\partial_t\psi(\bar{t},\overline{X})+o_r(1)\geq\left[c\left(\frac{\overline{x}}{\varepsilon}\right)+T_{nl}\right]|\nabla_x\psi(\bar{t},\overline{X})+o_r(1)|+h\left(\frac{\phi^\varepsilon(\bar{t},\overline{X})}{\varepsilon},\nabla_x\psi(\bar{t},\overline{X})+o_r(1)\right)+\beta_K^+ \quad (40)$$

with the nonlocal term

$$T_{nl} \quad = \quad L_0+\delta+M[V(\overline{\tau},\cdot,\overline{y}_{N+1})](\overline{y}).$$

We then have the following result whose proof is postponed:

Lemma 3. *The quantity* $M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x})$ *is bounded independently of* ε *and we have*

$$M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x})\leq L_0+\delta+M[V(\overline{\tau},\cdot,\overline{y}_{N+1})](\overline{y})=T_{nl}. \quad (41)$$

Hence, from (40), we obtain with $\beta_K^+\geq\beta_0^+/2>0$

$$\begin{aligned}\partial_t\psi(\bar{t},\overline{X})+o_r(1) \quad \geq \quad & \left[c\left(\frac{\overline{x}}{\varepsilon}\right)+M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x})\right]|\nabla_x\psi(\bar{t},\overline{X})+o_r(1)| \\ & +h\left(\frac{\phi^\varepsilon(\bar{t},\overline{X})}{\varepsilon},\nabla_x\psi(\bar{t},\overline{X})+o_r(1)\right)+\beta_K^+.\end{aligned}$$

Therefore, there exists $r_K>0$ such that for any (ε,r) such that $0<\varepsilon\leq\varepsilon_0(r)<r\leq r_K$: ϕ^ε is a supersolution of (20) on $\mathcal{V}_r$. By Theorem 4, we conclude that $U^\varepsilon\leq\phi^\varepsilon$ on $\mathcal{V}_r$, *i.e.* we get: $U^\varepsilon(t,X)\leq\phi(t,X)+\varepsilon V(\dots)+\varepsilon k_\varepsilon$ on $\mathcal{V}_r$ and we obtain the desired contradiction by passing to the upper limit at (t_0,X_0) using the fact that $U^+(t_0,X_0)=\phi(t_0,X_0)$: $0\leq-\eta_r$. □

Proof of Lemma 3. We proceed in two steps.
Step 1: bound on M^ε uniformly in ε. We write:

$$\begin{aligned}& M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x}) \\ =\int_{E_1(\overline{X})}J(z)&\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1})+\varepsilon V(\overline{\tau},\overline{y}+z,F\left(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}\right)/\varepsilon)-\phi(\bar{t},\overline{X})-\varepsilon V(\overline{\tau},\overline{Y})}{\varepsilon}\right) \\ &+\int_{E_2(\overline{X})}J(z)\left(\frac{U^\varepsilon(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1})-\phi(\bar{t},\overline{X})-\varepsilon V(\overline{\tau},\overline{Y})}{\varepsilon}\right) \\ &=M_1+M_2\end{aligned}$$

with

$$\begin{aligned}E_1(\overline{X})&=\{z\in\mathbb{R}^N:(\overline{x}+\varepsilon z,\overline{x}_{N+1})\in B_1(X_0)\}\subset B_{\frac{2}{\varepsilon}}(0), \\ E_2(\overline{X})&=\{z\in\mathbb{R}^N:(\overline{x}+\varepsilon z,\overline{x}_{N+1})\notin B_1(X_0)\}\subset\mathbb{R}^N\setminus B_{\frac{1}{2\varepsilon}}(0)\subset\mathbb{R}^N\setminus B_{R_0}(0).\end{aligned}$$

because $r<1/2$ and $(\bar{t},\overline{X})\in\mathcal{V}_r$. Hence,

$$\begin{aligned}|M_2|&\leq\int_{|x|\geq1/2}|U^\varepsilon(\bar{t},\overline{x}+x,\overline{x}_{N+1})-\phi(\bar{t},\overline{X})-\varepsilon V(\overline{\tau},\overline{Y})|\mu(dx) \\ &\leq(\|U^\varepsilon\|_{L^\infty(|x_{N+1}-x^0_{N+1}|\leq1)}+\|\phi\|_\infty+\varepsilon||V||_\infty)\int_{|x|\geq1/2}\mu(dx).\end{aligned}$$

Moreover using the fact that for $\rho = \sqrt{1-(\overline{x}_{N+1}-x^0_{N+1})^2}$, $a = \overline{x} - x_0$, we have $E_1(\overline{X}) = B_{\rho/\varepsilon}(a/\varepsilon)$ and we remark that it contains $B_{\tilde{\rho}/\varepsilon}$ with $\tilde{\rho}^2 = \rho^2 - |x_0 - \overline{x}|^2 \geq 0$. Now we write:

$$\begin{aligned} M_1 &= \int_{E_1(\overline{X})} J(z)\left(V(\overline{\tau},\overline{y}+z, F\left(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}\right)/\varepsilon) - V(\overline{\tau},\overline{Y})\right) dz \\ &\quad + \int_{B_{\rho/\varepsilon}(a/\varepsilon)\setminus B_{\tilde{\rho}/\varepsilon}} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi(\bar{t},\overline{X})}{\varepsilon}\right) dz \\ &\quad + \int_{B_{\tilde{\rho}/\varepsilon}} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi(\bar{t},\overline{X}) - \nabla_x\phi(\bar{t},\overline{x},\overline{x}_{N+1})\cdot\varepsilon z}{\varepsilon}\right) dz \\ &= M_{11} + M_{12} + M_{13}. \end{aligned}$$

We have $|M_{11}| \leq 2||V||_\infty$, $|M_{13}| \leq ||D^2_{xx}\phi||_\infty\ \varepsilon \int_{|z|\leq\tilde{\rho}/\varepsilon} \frac{|z|^2}{2} J(z)dz \leq C||D^2_{xx}\phi||_\infty$ and

$$|M_{12}| \leq 2||\phi||_\infty\ \varepsilon^{-1} \int_{B_{\rho/\varepsilon}(a/\varepsilon)\setminus B_{\tilde{\rho}/\varepsilon}} \mu(dz) \leq C.$$

Collecting all the estimates, we get the desired bound on $M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x})$.

Step 2: More precise estimate from above. Let us now estimate more precisely from above the quantity:

$$M^\varepsilon\left[\frac{\phi^\varepsilon(\bar{t},\cdot,\overline{x}_{N+1})}{\varepsilon}\right](\overline{x}) = \int J(z)\left(\frac{\phi^\varepsilon(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi^\varepsilon(\bar{t},\overline{X})}{\varepsilon}\right) dz.$$

Choose ε such that $R_0 \leq 2/\varepsilon$ where R_0 is defined in (2), use (37), the definition (36) of ϕ^ε, and the fact that J is even to get:

$$\begin{aligned} M^\varepsilon\left[\frac{\phi^\varepsilon}{\varepsilon}\right] &\leq \int_{|z|\leq 2/\varepsilon} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) + \varepsilon V(\dots) - \phi(\bar{t},\overline{X}) - \varepsilon V(\bar{\tau},\bar{Y})}{\varepsilon}\right) dz + \int_{|z|>2/\varepsilon}\{\dots\} \\ &\leq \int_{|z|\leq R_0} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi(\bar{t},\overline{x},\overline{x}_{N+1}) - \nabla_x\phi(\bar{t},\overline{x})\cdot\varepsilon z}{\varepsilon}\right) \\ &\quad + \int_{R_0\leq|z|\leq 2/\varepsilon} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi(\bar{t},\overline{x},\overline{x}_{N+1}) - \nabla_x\phi(\bar{t},\overline{x},\overline{x}_{N+1})\cdot\varepsilon z}{\varepsilon}\right) dz \\ &\quad + \int_{|z|\leq 2/\varepsilon} J(z)\left(V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1})}{\varepsilon}\right) - V\left(\overline{\tau},\overline{y},\frac{F(\bar{t},\overline{x},\overline{x}_{N+1})}{\varepsilon}\right)\right) dz \\ &\quad + \int_{|z|>2/\varepsilon} J(z)\left(\frac{U^\varepsilon(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi^\varepsilon(\bar{t},\overline{X})}{\varepsilon}\right) dz = T_1 + T_2 + T_3 + T_4 \end{aligned}$$

where

$$\begin{aligned} T_1 &= \int_{|z|\leq R_0} J(z)\left(\frac{\phi(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1}) - \phi(\bar{t},\overline{x},\overline{x}_{N+1}) - \nabla\phi(\bar{t},\overline{x})\cdot\varepsilon z}{\varepsilon}\right) dz \\ T_2 &= \int_{\varepsilon R_0\leq|x|\leq 2} (\phi(\bar{t},\overline{x}+x,\overline{x}_{N+1}) - \phi(\bar{t},\overline{x},\overline{x}_{N+1}) - \nabla_x\phi(\bar{t},\overline{x},\overline{x}_{N+1})\cdot x)\mu(dx) \\ T_3 &= \int_{|z|\leq 2/\varepsilon} J(z)\left(V\left(\overline{\tau},\overline{y}+z,\frac{F(\bar{t},\overline{x}+\varepsilon z,\overline{x}_{N+1})}{\varepsilon}\right) - V\left(\overline{\tau},\overline{y},\frac{F(\bar{t},\overline{x},\overline{x}_{N+1})}{\varepsilon}\right)\right) dz \\ T_4 &= \int_{|x|>2} \left(U^\varepsilon(\bar{t},\overline{x}+x,\overline{x}_{N+1}) - \phi(\bar{t},\overline{X}) - \varepsilon V(\overline{\tau},\overline{Y}) - k_\varepsilon\varepsilon\right)\mu(dx). \end{aligned}$$

Estimate of T_1.

$$|T_1| \leq \int_{|z|\leq R_0} J(z)\,\varepsilon\frac{\|D^2\phi\|_\infty}{2}|z|^2 dz = \mathcal{O}(\varepsilon).$$

Estimate of T_2. We claim that:

$$T_2 = \int_{|x|\leq 2} (\phi(t_0, x_0 + x, x^0_{N+1}) - \phi(t_0, x_0, x^0_{N+1}) - \nabla\phi(t_0, x_0, x^0_{N+1}) \cdot x)\mu(dx) + o_r(1).$$

Remark first that:

$$\int_{|x|\leq \varepsilon R_0} (\phi(\bar{t}, \overline{x} + x, \overline{x}_{N+1}) - \phi(\bar{t}, \overline{x}, \overline{x}_{N+1}) - \nabla_x\phi(\bar{t}, \overline{x}, \overline{x}_{N+1}) \cdot x)\mu(dx) = o_\varepsilon(1) = o_r(1).$$

Now use the continuity of the map

$$(t, x) \mapsto \quad \textstyle\int_{|z|\leq 2}(\phi(t, x + z, x_{N+1}) - \phi(t, X) - \nabla\phi(t, X) \cdot z)\,\mu(dz);$$

it follows from the continuity of the integrand and of the bound:

$$|\phi(t, x + z, x_{N+1}) - \phi(t, X) - \nabla_x\phi(t, X) \cdot z| \leq \frac{\|D^2\phi\|_\infty}{2}|z|^2.$$

Estimate of T_3. We claim that

$$T_3 = M[V(\overline{\tau}, \cdot, \overline{y}_{N+1})](\overline{y}) + o_r(1).$$

Let us define

$$T_3^\varepsilon := \int_{\mathbb{R}^N} J(z)\left(V\left(\overline{\tau}, \overline{y} + z, \frac{F(\bar{t}, \overline{x} + \varepsilon z, \overline{x}_{N+1})}{\varepsilon}\right) - V\left(\overline{\tau}, \overline{y}, \frac{F(\bar{t}, \overline{x}, \overline{x}_{N+1})}{\varepsilon}\right)\right) dz$$

Then, on the one hand we have

$$|T_3 - T_3^\varepsilon| \leq 2||V||_\infty \int_{|z|\leq 2/\varepsilon} J(z)\ dz = o_\varepsilon(1).$$

On the other hand,

$$\begin{aligned}
|T_3^\varepsilon - M[V(\overline{\tau}, \cdot, \overline{y}_{N+1})](\overline{y})| &\leq 4||V||_\infty\mu(\mathbb{R}^N \setminus B_{R_r}) \\
&+ \int_{|z|\leq R_r} \frac{D_K}{\varepsilon}|\phi(\bar{t}, \overline{x} + \varepsilon z, \overline{x}_{N+1}) - \phi(\bar{t}, \overline{x}, \overline{x}_{N+1}) - \varepsilon p \cdot z)|\, J(z)\, dz \\
\leq o_r(1) + \int_{|z|\leq R_r} \frac{D_K}{\varepsilon}[|\phi(\bar{t}, \overline{x} + \varepsilon z, \overline{x}_{N+1}) &- \phi(\bar{t}, \overline{x}, \overline{x}_{N+1}) - \varepsilon\nabla_x\phi(\bar{t}, \overline{x}, \overline{x}_{N+1}) \cdot z)| + Cr|\varepsilon z|]J(z)\, dz \\
&\leq o_r(1) + C(\varepsilon(R_r)^2 + rR_r)
\end{aligned}$$

for $R_r \to +\infty$ as $r \to 0$. Now choose $\varepsilon \leq r^2$ and R_r such that $rR_r \to 0$. We conclude that:

$$|T_3^\varepsilon - M[V(\overline{\tau}, \cdot, \overline{y}_{N+1})](\overline{y})| \leq o_r(1).$$

Estimate of T_4. Remark that for any $x \in B_{R_\delta}$ (with R_δ chosen later):

$$U^\varepsilon(\bar{t}, \overline{x} + x, \overline{x}_{N+1}) - \phi(\bar{t}, \overline{X}) - \varepsilon V(\overline{\tau}, \overline{Y}) \leq U^+(t_0, x_0 + x, x^0_{N+1}) - \phi(t_0, X_0) + o_\varepsilon(1) + o_r(1).$$

Keeping in mind that $\phi(t_0, X_0) = U^+(t_0, X_0)$, we can chose R_δ big enough so that:

$$T_4 \leq \int_{|z|\leq R_\delta}\{\dots\} + \int_{|z|\geq R_\delta}\{\dots\} \leq \int_{|x|>2}\left(U^+(t_0, x_0 + x, x^0_{N+1}) - U^+(t_0, X_0)\right)\mu(dx) + \delta.$$

Combining all these estimates yields (41) with L_0 given in (35). □

5 Approximate cell problems

In this section, we approximate the Hamiltonian of (1) in such a way that: first, the modified solutions are approximate solutions and are sub- or supersolutions of the exact problem; secondly, they are bounded uniformly with respect to the approximation; thirdly they are Lipschitz continuous with respect to (τ, Y) where the Lipschitz constant depend on the approximation.

Our first technical choice is to consider the non-local normal speed of the dislocation (see Section 2.1) and the "correction" term as variables c and h. Next, in order to ensure that modified solutions are good approximate solutions, $H_K^{\delta,+}$ is constructed so that it coincides with the exact Hamiltonian on a ball of radius K centered at the origin. To ensure the Lipschitz continuity of the solution, the Hamiltonian is modified in such a way that it is constant outside a starshaped compact set $\Omega_K^{\delta,+}$. To finish with, in order to be sure that the solution of the approximate problem is a supersolution of the exact one, the approximate Hamiltonian $H_K^{\delta,+}$ is constructed above the exact one on $\Omega_K^{\delta,+}$.

We modify $H(c,h,q) := c|q| + h$ in order to ensure that it is coercive w.r.t. $Q = (q, q_{N+1})$.

Lemma 4 (Approximate Hamiltonian)**.** *There exist a 0-starshaped compact set $\Omega_K^{\delta,+} \subset \mathbb{R}^{N+1}$ containing the ball $B_K(0)$ and two piecewise linear nondecreasing functions γ_K, T_K (truncature functions) with $\gamma_K \geq 0$, such that the approximate Hamiltonian:*

$$H_K^{\delta,+}(c,h,Q) = T_K(H_K^\delta(c,Q) + h) \quad \textit{where} \quad H_K^\delta(c,Q) = (c + \gamma_K(|q|))|q| + \delta|q_{N+1}|$$

satisfies for any $c \in [-C_K, C_K]$ and $h \in [-G, G]$,

$$H_K^{\delta,+}(c,h,Q) \begin{cases} = c|q| + \delta|q_{N+1}| + h & \textit{if } |Q| \leq K, \\ \geq H_K^\delta(c,Q) + h \geq c|q| + h & \textit{if } Q \in \Omega_K^{\delta,+}, \\ = M_K^{\delta,+} & \textit{if } Q \notin \Omega_K^{\delta,+} \end{cases} \tag{42}$$

for some constant $M_K^{\delta,+}$.

Proof. First, we consider two other constants $R_K^0, R_K^1 > 0$, we define for $c, r \in \mathbb{R}$:

$$\gamma_K(r) = \begin{cases} 0 & \text{if } r \leq R_K^0, \\ r - R_K^0 & \text{if } R_K^0 \leq r \leq R_K^1, \\ R_K^1 - R_K^0 & \text{if } r \geq R_K^1 \end{cases}$$

and we introduce for $\delta > 0$:

$$H_K(c,q) = [c + \gamma_K(|q|)]|q| \quad \text{and} \quad H_K^\delta(c,Q) = H_K(c,q) + \delta|q_{N+1}|.$$

We choose R_K^0 and R_K^1 as follows:

$$R_K^0 = K \text{ and } R_K^1 = R_K^0 + 2C_K \tag{43}$$

so that, on the one hand, $H_K(c,q)$ is coercive w.r.t. q uniformly with respect to $c \in [-C_K, C_K]$ and, on the other hand, $\gamma(|q|) = 0$ if $|q| \leq K$. Next, define for $r \geq 0$ and $\alpha \in \mathbb{R}$:

$$\begin{aligned} h_K(r) &= \sup\{H_K(c,q) + h : c \in [-C_K, C_K], h \in [-G, G], |q| \leq r\}, \\ r_K(\alpha) &= \inf\{r \geq 0 : (|q| \geq r) \Rightarrow (H_K(c,q) + h > \alpha \text{ for any } (c,h) \in [-C_K, C_K] \times [-G, G])\}, \\ h_K^\delta(r) &= h_K(r) + \delta r \end{aligned}$$

and

$$H_K^{\delta,+}(c,Q) = T_K(H_K^\delta(c,Q) + h) \tag{44}$$

where

$$T_K(\alpha) = \begin{cases} \alpha & \text{if } \alpha \leq h_K^\delta(K), \\ h_K^\delta(K) + \mu_K^+(\alpha - h_K^\delta(K)) & \text{if } h_K^\delta(K) \leq \alpha \leq 2h_K^\delta(K), \\ (1 + \mu_K^+)h_K^\delta(K) & \text{if } \alpha \geq 2h_K^\delta(K) \end{cases}$$

with $\mu_K^+ \geq 1$ to be fixed later. Now we can introduce the following compact set of $\mathbb{R}^{N+1}$:

$$\Omega_K^{\delta,+} = \{Q \in \mathbb{R}^{N+1} : |q| \leq r_K(2h_K^\delta(K) - \delta|q_{N+1}|), \delta|q_{N+1}| \leq 2h_K^\delta(K) - \inf H_K + G\}$$

with $\inf H_K = \inf_{|c|\leq C_K, q\in\mathbb{R}^N} H_K(c,q) \leq -C_K K$ (we used (43)) and we can check that

$$B_K(0) \subset \{Q \in \mathbb{R}^{N+1} : \exists (c,h) \in [-C_K, C_K] \times [-G,G], H_K^\delta(c,Q) + h \leq 2h_K^\delta(K)\} \subset \Omega_K^{\delta,+}.$$

Eventually, we define:

$$M_K^{\delta,+} = \sup_{Q\in\Omega_K^{\delta,+}} \{h_K(|q|) + \delta|q_{N+1}|\}.$$

Remark that $M_K^{\delta,+} \geq 2h_K^\delta(K) - \inf H_K + G \geq 2h_K^\delta(K)$ by choosing $q = 0$ and $\delta|q_{N+1}| = 2h_K^\delta(K) - \inf H_K + G$.

Let us check that if we choose $\mu_K^+ \geq 1$ so that:

$$(1+\mu_K^+)h_K^\delta(K) = M_K^{\delta,+}$$

we are sure that (42) holds true.

First, if $|Q| \leq K$, then $H_K^\delta(c,Q) + h \leq h_K^\delta(K)$ and the first property of (42) is proved.

Secondly, if $Q \in \Omega_K^{\delta,+}$, then $H_K^\delta(c,Q) + h \leq M_K^{\delta,+}$ and: either $H_K^{\delta,+}(c,h,Q) = M_K^{\delta,+}$ and we are done; or $H_K^\delta(c,Q) + h \leq 2h_K^\delta(K)$ and in this case, we use the fact that $T_K(\alpha) \geq \alpha$ for $\alpha \leq 2h_K^\delta(K)$.

To finish with, consider $Q \notin \Omega_K^{\delta,+}$. Suppose first that $\delta|q_{N+1}| > 2h_K^\delta(K) - \inf H_K + G$. In this case,

$$H_K^\delta(c,Q) + h \geq \delta|q_{N+1}| + \inf H_K - G > 2h_K^\delta(K)$$

and we are done. In the other case, $|q| > r_K(2h_K^\delta(K) - \delta|q_{N+1}|)$ and we conclude, by definition of r_K that for any $(c,h) \in [-C_K, C_K] \times [-G,G]$,

$$H_K(c,q) + h > 2h_K^\delta(K) - \delta|q_{N+1}|$$

and we can conclude in this case too. □

We can now consider the approximate cell problem:

$$\left\{\begin{array}{lll} \lambda + \partial_\tau V = H_K^{\delta,+}(& c(y) + L + M[V(\tau,\cdot,y_{N+1})](y), & \\ & h(\lambda\tau + P\cdot Y + V, p + \nabla_y V), P + \nabla V) + \beta & \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V(0,Y) = 0 & & \text{on} \quad \mathbb{R}^{N+1} \end{array}\right. \tag{45}$$

and state the existence of Lipschitz continuous approximate supercorrectors.

Proposition 6 (Lipschitz continuous approximate supercorrectors)**.** *Let $p \in \mathbb{R}^N$ and $P = (p,1) \in \mathbb{R}^N \times \mathbb{R}$. Let us consider the truncated Hamiltonian $H_K^{\delta,+}$ defined by (44) for $K > \sqrt{1+|p|^2}$ large enough. For any $\beta \in \mathbb{R}$, there exist real numbers $\lambda_K^{\delta,+}(\beta)$, and solutions $V_K^{\delta,+}$ of (45) with $\lambda = \lambda_K^{\delta,+}(\beta)$ satisfying (26) and*

$$|\lambda_K^{\delta,+}(\beta) - \beta - L|p| - \delta| \leq \|c\|_\infty |p| + G. \tag{46}$$

$\mathbf{W}^{1,\infty}$ a priori bounds on the correctors. *We can construct bounded Lipschitz continuous correctors with:*

$$|V_K^{\delta,+}(\tau,Y)| \quad \leq \quad 6C^+ \tag{47}$$

with $C^+ = \left\lceil \frac{2(\|c\|_\infty + G^* + 1)}{c_0} \right\rceil$ and $c_0 = \inf_{d\in[0,1/2)^N} \int_{\mathbb{R}^N} dz \ \min(J(z+d), J(z-d)) > 0$ and

$$\left\{\begin{array}{lll} \left(P + \nabla V_K^{\delta,+}(\tau,Y)\right) & \in & \Omega_K^{\delta,+}, \\ 0 \quad \leq \quad 1 + \frac{\partial V_K^{\delta,+}}{\partial y_{N+1}}(\tau,Y) & \leq & \frac{2h_K^\delta(K) - \inf H_K + G}{\delta} \end{array}\right. \tag{48}$$

with $\inf H_K = \inf_{|c|\le C_K, q\in\mathbb{R}^N} H_K(c,q) < 0$.

Further properties of the correctors. *The correctors satisfy*

$$\begin{cases} V_K^{\delta,+}(\tau, y, y_{N+1}+1) = V_K^{\delta,+}(\tau, y, y_{N+1}), \\ |V_K^{\delta,+}(\tau, y+k, y_{N+1}) - V_K^{\delta,+}(\tau, y, y_{N+1})| \le 1 \quad \textit{for every} \quad k\in\mathbb{Z}^N. \end{cases} \tag{49}$$

If $p = P/Q$ *with* $P\in\mathbb{Z}^N$ *and* $Q\in\mathbb{N}\setminus\{0\}$*, then following [28], it is possible to build solutions on the whole space-time satisfying*

$$V_K^{\delta,+}(\tau, y+Qk, y_{N+1}) = V_K^{\delta,+}(\tau, y, y_{N+1}) \quad \textit{for every} \quad k\in\mathbb{Z}^N.$$

Proof. To construct such approximate correctors, we first consider the solution of the following initial value problem:

$$\begin{cases} \partial_\tau W = H_K^{\delta,+}(\quad c(y) + L + M[W(\tau,\cdot,y_{N+1})](y), \\ \qquad\qquad\qquad\qquad h(P\cdot Y + W, p+\nabla_y W), P+\nabla W) + \beta \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^{N+1}, \\ W(0,Y) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{50}$$

We perform the proof in two steps. We first construct barriers and get gradient estimates on W for short times; secondly, we control the oscillations of W with respect to space for long times and finally, we control the oscillations with respect to time.

Step 1: existence of a solution and gradient estimates. We first construct barriers for (50) of the form $W^\pm(\tau,Y) = (\beta + L|p| + \delta)\tau \pm C_1\tau$. Since $|P| < K$, one can easily check that if $C_1 = \|c\|_\infty |p| + G$, then $W^\pm$ is a supersolution (resp. subsolution) of (50). It follows that there exists a unique solution W of (50) (by Perron's method, as we did in Theorem 3 for instance).

Let us now recall that we constructed the approximate Hamiltonian $H_K^{\delta,+}$ for speeds $c\in[-C_K;C_K]$. Moreover, we remark that:

$$|M[W(\tau,\cdot,y_{N+1})]| \le \text{osc}\, W(\tau,\cdot) \quad \text{with} \quad \text{osc}\, W(\tau,\cdot) = \max_{Y\in\mathbb{R}^{N+1}} W(\tau,Y) - \min_{Y\in\mathbb{R}^{N+1}} W(\tau,Y)$$

and osc $W(0,\cdot) = 0$. This is the reason why we introduce τ^*, the first time τ such that osc $W(\tau,\cdot) + \|c\|_\infty + |L| \ge C_K$.

We remark that $U(\tau,Y) = W(\tau,Y) + P\cdot Y$ satisfies

$$\begin{cases} \partial_\tau U = H_K^{\delta,+}(c(y) + L + M[U(\tau,\cdot,y_{N+1}) - p\cdot\,](y), h(U,\nabla_y U), \nabla U) + \beta \quad \text{in} \quad \mathbb{R}^+\times\mathbb{R}^{N+1}, \\ U(0,Y) = P\cdot Y \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{51}$$

Remark that $\nabla U(0,Y) = P \in B_K(0) \subset \Omega_K^{\delta,+}$ that is starshaped with respect to the origin and compact. Hence, for any $\xi\in\mathbb{R}^{N+1}$, there exists $M\in\mathbb{R}$ such that

$$\xi\cdot Q > M \Longrightarrow \xi\notin\Omega_K^{\delta,+} \Longrightarrow H_K^{\delta,+}(c,Q) = M_K^{\delta,+}.$$

By adapting [28], we can easily prove that:

Lemma 5 (Gradient estimate)**.** *The solution* U *of* (51) *satisfies:*

$$\xi\cdot\nabla U(0,\cdot) \le M \Longrightarrow \forall t>0, \quad \xi\cdot\nabla U(t,\cdot) \le M.$$

Since $0 \le \partial_{y_{N+1}} U(0,Y) = 1$ for any $Y\in\mathbb{R}^{N+1}$, we can also prove (using the invariance by translation in y_{N+1} of (51)) by following [28]:

Lemma 6 (Monotonicity preserving)**.** *The function* U *is nondecreasing with respect to* y_{N+1}.

These lemmata imply that for any $\tau > 0$ and $Y \in \mathbb{R}^{N+1}$:

$$\begin{cases} P + \nabla W(\tau, Y) \in \Omega_K^{\delta,+}, \\ 0 \leq 1 + \frac{\partial W}{\partial y_{N+1}}(\tau, Y) \leq \frac{2h_K^\delta(K) - \inf H_K + G}{\delta}. \end{cases} \tag{52}$$

Step 2: Control of the oscillations w.r.t. space.
Step 2.1. For a given $k \in \mathbb{Z}^{N+1}$, we set $P \cdot k = l + \alpha$, with $l \in \mathbb{Z}$ and $\alpha \in [0, 1)$. Then we have:

$$l \leq U(0, Y + k) - U(0, Y) \leq l + 1.$$

From the comparison principle for (51) in the class of sublinear functions, and the various invariances by integer translations of the equation, we deduce that W is 1-periodic with respect to y_{N+1} and for all $\tau \geq 0$:

$$l \leq U(\tau, Y + k) - U(\tau, Y) \leq l + 1$$

and then

$$|W(\tau, Y + k) - W(\tau, Y)| \leq 1. \tag{53}$$

Step 2.2. Let us define

$$M(\tau) = \sup_{Y \in \mathbb{R}^{N+1}} W(\tau, Y), \quad m(\tau) = \inf_{Y \in \mathbb{R}^{N+1}} W(\tau, Y),$$
$$q(\tau) := M(\tau) - m(\tau) = \text{osc } W(\tau, \cdot).$$

These three functions are locally Lipschitz continuous. Let us assume that the extrema defining these functions are attained: $M(\tau) = W(\tau, Y^\tau)$, $m(\tau) = W(\tau, Z^\tau)$. If this is not the case, consider an ε-supremum and an ε-infimum and use a variational principle, such as Stegal's one for instance (see [13] for a precise statement). Details are left to the reader.

We adopt the following notations: $Y^\tau = (y^\tau, y_{N+1}^\tau)$ and $Z^\tau = (z^\tau, z_{N+1}^\tau)$. Then we have in the viscosity sense and therefore a.e.:

$$\partial_\tau M \leq H_K^{\delta,+}(c(y^\tau) + L + M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau), h(\dots), P) + \beta \quad \text{with} \quad M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau) \leq 0,$$
$$\partial_\tau m \geq H_K^{\delta,+}(c(z^\tau) + L + M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau), h(\dots), P) + \beta \quad \text{with} \quad M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau) \geq 0.$$

We have for a.e. $\tau \in [0, \tau^*)$:

$$\begin{aligned} \partial_\tau q \quad &\leq \quad H_K^{\delta,+}(c(y^\tau) + L + M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau), h(\dots), P) \\ &\quad\quad -H_K^{\delta,+}(c(z^\tau) + L + M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau), h(\dots), P) \\ &\leq \quad T_K(c(y^\tau) + L + M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau)|p| + \gamma_K(|p|)|p| + h(\dots) + \delta) \\ &\quad\quad -T_K(c(z^\tau) + L + M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau)|p| + \gamma_K(|p|)|p| + h(\dots) + \delta). \end{aligned}$$

Now recall that we chose $K > \sqrt{1 + |p|^2}$ so that $\gamma_K(|p|) = 0$ and we remark that for $\tau \leq \tau^*$:

$$|c(\cdot) + L + M[W(\tau, \cdot)]||p| + h(\dots) + \delta \leq (\|c\|_\infty + |L| + \text{osc } W)|p| + G + \delta \leq C_K K + G + \delta \leq h_K^\delta(K)$$

for $K \geq 1$ since $h_K^\delta(K) = C_K K + G + \delta K$. We conclude that:

$$\begin{aligned} \partial_\tau q \quad &\leq \quad |p| \left\{ c(y^\tau) + M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau) - c(z^\tau) - M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau)) \right\} + 2G^*|p| \\ &\leq \quad 2\|c\|_\infty |p| + 2G^*|p| + |p| \left\{ M[W(\tau, \cdot, y_{N+1}^\tau)](y^\tau) - M[W(\tau, \cdot, z_{N+1}^\tau)](z^\tau)) \right\}. \end{aligned}$$

Then on $[0, \tau^*)$,

$$\partial_\tau q \leq 2(\|c\|_\infty + G^*)|p| + |p| \ \mathcal{L}(\tau)$$

where $\mathcal{L}(\tau) = M[W](\tau,\cdot,y^\tau_{N+1})(y^\tau) - M[W](\tau,\cdot,z^\tau_{N+1})(z^\tau) \le 0$. Let us estimate this quantity from above by a function of q. Let us define $k^\tau \in \mathbb{Z}^{N+1}$ such that $(Z^\tau + k^\tau) - Y^\tau \in [0,1)^{N+1}$, $\tilde{Z}^\tau = Z^\tau + k^\tau$. Now using successively (53) and (52), and the fact that W is 1-periodic with respect to y_{N+1}, we obtain:

$$\begin{aligned}\mathcal{L}(\tau) &\le 1 + \int J(z)(W(\tau, y^\tau + z, y^\tau_{N+1}) - W(\tau, Y^\tau))dz - \int J(z)(W(\tau, \tilde{z}^\tau + z, \tilde{z}^\tau_{N+1}) - W(\tau, Z^\tau)) \\ &\le 2 + \int J(z)(W(\tau, y^\tau + z, y^\tau_{N+1}) - W(\tau, Y^\tau))dz - \int J(z)(W(\tau, \tilde{z}^\tau + z, y^\tau_{N+1}) - W(\tau, Z^\tau)).\end{aligned}$$

Now introduce $c^\tau = \frac{y^\tau + \tilde{z}^\tau}{2}$ and $\delta^\tau = \frac{y^\tau - \tilde{z}^\tau}{2} \in \left[0, \frac{1}{2}\right)^N$ so that $y^\tau = c^\tau + \delta_\tau$ and $\tilde{z}^\tau = c^\tau - \delta^\tau$. Hence,

$$\begin{aligned}\mathcal{L}(\tau) &\le 2 + \int J(z)(W(\tau, c^\tau + z + \delta^\tau, y^\tau_{N+1}) - W(\tau, Y^\tau))dz \\ &\quad - \int J(z)(W(\tau, c^\tau + z - \delta^\tau, y^\tau_{N+1}) - W(\tau, Z^\tau))dz \\ &\le 2 + \int J(z - \delta^\tau)(W(\tau, c^\tau + z, y^\tau_{N+1}) - W(\tau, Y^\tau))dz \\ &\quad - \int J(z + \delta^\tau)(W(\tau, c^\tau + z, y^\tau_{N+1}) - W(\tau, Z^\tau))dz \\ &\le 2 + \int \min(J(z - \delta^\tau), J(z + \delta^\tau))(W(\tau, c^\tau + z) - W(\tau, Y^\tau) - W(\tau, c^\tau + z) + W(\tau, Z^\tau))dz \\ &\le 2 - c_0 q(\tau)\end{aligned}$$

where $c_0 = \inf_{\delta \in [0,1/2)^N} \int_{\mathbb{R}^N} \min\left(J(z-\delta), J(z+\delta)\right) dz > 0$. Therefore we have on $[0, \tau^*)$,

$$q_\tau \le 2(\|c\|_\infty + G^* + 1)|p| - c_0|p|q.$$

From this inequality and the fact that $q(0) = 0$, we deduce that for $\tau \in [0, \tau^*)$, we have

$$0 \le q(\tau) \le C^+.$$

Now, if one chooses

$$C_K > \|c\|_\infty + |L| + C^+, \tag{54}$$

we conclude that $\tau^* = +\infty$ and W satisfies:

$$|W(\tau, Y) - W(\tau, Y')| \le C^+. \tag{55}$$

Step 3: Control of the oscillations in time. For any $T > 0$ we define

$$\lambda^+(T) = \sup_{\tau \ge T} \quad \frac{W(\tau + T, 0) - W(\tau - T, 0)}{2T} \quad \text{and} \quad \lambda^-(T) = \inf_{\tau \ge T} \quad \frac{W(\tau + T, 0) - W(\tau - T, 0)}{2T}$$

which satisfy $\lambda^-(T) \le \lambda^+(T)$. From (55), we get for any $\tau \ge T$:

$$|W(\tau - T, Y) - W(\tau - T, 0)| \le C^+ \tag{56}$$

and we deduce from the comparison principle for (50) that

$$-(C^+ + 1) + 2T(\beta + L|p| + \delta - C_1) \le W(\tau + T, Y) - W(\tau - T, 0) \le (C^+ + 1) + 2T(\beta + L|p| + \delta + C_1)$$

(we used once again the barriers of Step 1) and therefore

$$-\frac{C^+ + 1}{2T} + \beta + L|p| + \delta - C_1 \le \lambda^\pm(T) \le \beta + L|p| + \delta + C_1 + \frac{C^+ + 1}{2T}. \tag{57}$$

By definition of $\lambda^{\pm}(T)$, for any $\alpha > 0$, there exists $\tau^{\pm} \geq T$ such that

$$\left|\lambda^{\pm}(T) - \frac{W(\tau^{\pm}+T,0) - W(\tau^{\pm}-T,0)}{2T}\right| \leq \alpha.$$

Let us define $k \in \mathbb{Z}$ such that $3C^+ \geq W(\tau^- - T, 0) + k - W(\tau^+ - T, 0) > 2C^+$. Then from (56), we deduce that

$$0 < W(\tau^- - T, Y) + k - W(\tau^+ - T, Y) \leq 5C^+.$$

From the invariance by translations in time of (50), from the 1-periodicity of $c(y)$ in y, and from the comparison principle for (50), we deduce that

$$0 \leq W(\tau^- + T, Y) + k - W(\tau^+ + T, Y) \leq 5C^+.$$

Therefore we deduce

$$-5C^+ \leq \big(W(\tau^- + T, Y) - W(\tau^- - T, Y)\big) - \big(W(\tau^+ + T, Y) - W(\tau^+ - T, Y)\big) \leq 5C^+$$

and then

$$|\lambda^+(T) - \lambda^-(T)| \leq 2\alpha + \frac{5C^+}{2T}$$

and because $\alpha > 0$ is arbitrarily small we deduce that

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{5C^+}{2T}. \tag{58}$$

Now let us consider $T_1 > 0$ and $T_2 > 0$ such that $T_2/T_1 = P/Q$ with $P, Q \in \mathbb{N}\setminus\{0\}$. Then we have

$$\lambda^+(T_1) \geq \lambda^+(PT_1) = \lambda^+(QT_2) \geq \lambda^-(QT_2) \geq \lambda^-(T_2) \geq \lambda^+(T_2) - \frac{5C^+}{2T_2}.$$

By symmetry we deduce that

$$|\lambda^+(T_2) - \lambda^+(T_1)| \leq \max\left(\frac{5C^+}{2T_2}, \frac{5C^+}{2T_1}\right) \tag{59}$$

and similarly

$$|\lambda^-(T_2) - \lambda^-(T_1)| \leq \max\left(\frac{5C^+}{2T_2}, \frac{5C^+}{2T_1}\right). \tag{60}$$

Finally the maps $T \longmapsto \lambda^+(T)$ and $T \longmapsto \lambda^-(T)$ are continuous and then the inequalities (59)-(60) are still true even if T_2/T_1 is not rational. Therefore the inequalities (59)-(60) and (58) imply the existence of the following limits

$$\lim_{T\to+\infty} \lambda^+(T) = \lim_{T\to+\infty} \lambda^-(T) = \lambda$$

and we deduce that

$$|\lambda^{\pm}(T) - \lambda| \leq \frac{5C^+}{2T}. \tag{61}$$

Letting $T \to +\infty$ in (57), we get (46).

Step 4: change of unknown function. Consider now $V_K^{\delta,+}(\tau, Y) = V(\tau, Y) = W(\tau, Y) - \lambda\tau$. From (50), we conclude that V satisfies (45). Moreover, combining (61) and (56), we get (47). Estimate (48) is a consequence of (52). Eventually, (53) and the periodicity of W with respect to y_{N+1} yield (49).

Let us remark that the monotonicity of $\beta \mapsto \lambda_K^{\delta,+}(\beta)$ follows from the comparison principle. To conclude the proof of Proposition 6, it remains to prove that the map $\beta \mapsto \lambda_K^{\delta,+}(\beta)$ is continuous. To do so, consider

$\beta_n \to \beta$ and $W^n = W_K^{+,\delta}(\beta_n)$ the solution of (50). Then $W_K^{+,\delta}(\beta_n) \to W^0 = W_K^{+,\delta}(\beta_0)$ locally uniformly in (τ, Y) by stability of viscosity solutions and strong uniqueness for (50). Now write with obvious notations:

$$\left|\lambda^n - \lambda^0\right| = \frac{|(W^n - W^0)(\tau,0) - (V^n - V^0)(\tau,0)|}{\tau} \leq \frac{|(W^n - W^0)(\tau,0)|}{\tau} + \frac{12C^+}{\tau}.$$

Consider any limit of λ^n, let first $n \to +\infty$ and next $\tau \to +\infty$ and conclude that this limit has to be λ^0. The proof is now complete. □

6 Proofs of Propositions 5 and 4 and of Theorem 1

Let us deduce from the construction of the previous section (Proposition 6): the existence of approximate correctors (Proposition 5), the ergodicity of the problem (Theorem 1) and the properties the effective Hamiltonian satisfies (Proposition 4).

Proof of Proposition 5. Let us first remark that (48) and (42), and (54) imply that $V_K^{\delta,+}$ is a supersolution of (23). Let $\tilde{V}_K^{\delta,+}$ denote the solution of (45) with $c(y)$ replaced with $\tilde{c}(y) = -c(-y)$, (L, β) by $(-L, -\beta)$, $h(u,p)$ replaced with $\tilde{h}(u,p) = -h(-u,p)$ and $\tilde{\lambda}_K^{\delta,+}$ be the associated real number. Then, for $\lambda = \lambda_K^{\delta,-} = -\tilde{\lambda}_K^{\delta,+}$, we see that $V_K^{\delta,-}(\tau, Y) = -\tilde{V}_K^{\delta,+}(\tau, -Y)$ is a solution of

$$\begin{cases} \lambda + \partial_\tau V = \tilde{H}_K^{\delta,+}(& c(y) + L + M[W(\tau,\cdot,y_{N+1})](y), \\ & h(\lambda\tau + P\cdot Y + W, p + \nabla_y W), P + \nabla W) + \beta \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V(0,Y) = 0 & \text{on} \quad \mathbb{R}^{N+1} \end{cases} \tag{62}$$

for some function $\tilde{H}_K^{\delta,+}$ such that $V_K^{\delta,-}$ is also a subsolution of

$$\begin{cases} \lambda + \partial_\tau V & = & (c(y) + L + M[V(\tau,\cdot,y_{N+1})](y))|p + \nabla_y V| \\ & & +\tilde{\delta}\left|1 + \frac{\partial V}{\partial y_{N+1}}\right| + h(\lambda\tau + P\cdot Y + V, p + \nabla_y V) + \beta \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^{N+1}, \\ V(0,Y) & = 0 \quad \text{on} \quad \mathbb{R}^{N+1}. \end{cases} \tag{63}$$

with $\tilde{\delta} = -\delta$ and

$$|\lambda_K^{\delta,-}(\beta) - \beta - L|p| + \delta| \leq \|c\|_\infty |p| + G \tag{64}$$

and we have estimates similar to those of Proposition 6.

Let us denote by $V_K^\pm$ the solution $V_K^{\delta,\pm}$ for $\delta = 1/K$ in the previous construction. Notice that it suffices to choose $C = 6C^+$ to get (30). If $\lambda_K^\pm(\beta)$ denotes $\lambda_K^{\delta,\pm}(\beta)$, the comparison principle (for (50) for instance) implies that both functions are nondecreasing w.r.t. β and (46), (64) imply that $\lambda_K^\pm(\beta) \to \pm\infty$ as $\beta \to \pm\infty$.

Let us now prove (29). The proof is very similar to the proof of the continuity of $\lambda_K^{\delta,+}(\beta)$ in β. Since $H_K^{\delta,\pm} \to H$, we know that $W_K^\pm = V_K^\pm + \tau\lambda_K^\pm(\beta) \to W^0$ locally uniformly in (τ, Y). Next write:

$$0 \leq \lambda_K^+ - \lambda_K^- = \frac{W_K^+ - W_K^- - (V_K^+ - V_K^-)}{\tau} \leq \frac{|W_K^+ - W_K^-|}{\tau} + \frac{12C^+}{\tau}.$$

This implies that

$$\limsup \lambda_K^+ \leq \liminf \lambda_K^- \leq \limsup \lambda_K^- \leq \liminf \lambda_K^+$$

and we conclude that they have the same limit (recall (46)).

By using (48), we also deduce the second line of (31) with

$$D_K := |P| + \operatorname{diam} \Omega_K^{\delta,+}.$$

The proof of the fact that $\beta_K^\pm \longrightarrow \beta_0^\pm$ as $K \to +\infty$ is very similar to the proof of (29). □

Proof of Theorem 1. Let us first define:

$$\overline{H}^0(L,p) = \overline{H}(L,p,0) = \lambda(0).$$

Let W (resp. W_K^+, W_K^-) be the solution (resp. supersolution, subsolution) of

$$\begin{cases} \partial_\tau W = (c(y) + L + M[W(\tau,\cdot,y_{N+1})](y))|p+\nabla_y W| + h(P\cdot Y + W, p+\nabla_y W) & \text{in} \quad \mathbb{R}^+\times\mathbb{R}^{N+1}, \\ W(0,Y) = 0 & \text{on} \quad \mathbb{R}^{N+1} \end{cases} \tag{65}$$

where $V_K^\pm = W_K^\pm - \tau\lambda_K^\pm$ and $\lambda_K^\pm = \lambda_K^\pm(0)$ are given by Proposition 5. The comparison principle implies that $W_K^- \le W \le W_K^+$, *i.e.*

$$\lambda_K^-\tau - C \le W(\tau,Y) \le \lambda_K^+\tau + C$$

Now letting $K\to+\infty$ and using (29), we conclude that $|W(\tau,Y)-\lambda(0)\tau| \le C$ and then $W(\tau,Y)/\tau \to \lambda(0)$ as $\tau\to+\infty$ locally uniformly in Y. Using the fact that the solution $w(\tau,y)$ of (3) satisfies $w(\tau,y) = W(\tau,y,0)$ (by an argument similar to the proof of Lemma 2), this proves the Theorem. □

Proof of Proposition 4. From (46), we deduce that

$$|\overline{H}(L,p,\beta) - L|p| - \beta| \le \|c\|_\infty|p| + G$$

and it yields (24), (25). The monotonicity of $\overline{H}$ in L and β follows from the comparison principle. The continuity of $\overline{H}$ w.r.t. (L,p,β) is proved as the continuity of $\lambda_K^{\delta,+}(\beta)$ with respect to β.
Now, because $h(\cdot,0)=0$, we see that $V\equiv 0$ is a corrector for $\lambda = \overline{H}(L,0) = 0$ which proves (27). If moreover we have $h\equiv 0$, and $\int_{[0,1)^N} c = 0$, then let us consider the unique solution v_0 of

$$Mv_0 = -c \quad \text{on} \quad \mathbb{R}^n/\mathbb{Z}^n$$

with zero mean value. When $L=0$, we deduce that for any $p\in\mathbb{R}^N$, the function $V(\tau,Y) = v_0(y)$ is a corrector with $\lambda = \overline{H}(0,p) = 0$, which proves (28), and ends the proof of the Proposition. □

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