On the rate of convergence in periodic homogenization of scalar first-order ordinary differential equations

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Abstract

In this paper, we study the rate of convergence in periodic homogenization of scalar ordinary differential equations. We provide a quantitative error estimate between the solutions of a first-order ordinary differential equation with rapidly oscillating coefficients, and the solution of the limiting homogenized equation. As an application of our result, we obtain an error estimate for the solution of some particular linear transport equations.

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1 Introduction

1.1 Homogenization of an ODE

In this paper, we consider the solutions of the following first-order ordinary differential equation:

$$\begin{cases} u^\epsilon_t = f\left(\dfrac{u^\epsilon}{\epsilon}, \dfrac{t}{\epsilon}, u^\epsilon, t\right), \quad t > 0, \\ u^\epsilon(0) = u_0, \end{cases} \tag{1.1}$$

where $\epsilon > 0$, u^ϵ_t stands for $\frac{du^\epsilon}{dt}$ or equivalently $\partial_t u^\epsilon$, and u_0 is a real number. We are interested in the rate of convergence of the solution u^ϵ to its limit in the framework of periodic homogenization. We employ the following assumptions on the function f:

- (A1) **Regularity:** the function $f : \mathbb{R}^4 \to \mathbb{R}$ is a bounded Lipschitz continuous function with $\alpha = Lip(f)$ its Lipschitz constant, and $\beta = \|f\|_{L^\infty(\mathbb{R}^4)}$;
- (A2) **Periodicity:** for any $(v, \tau, u, t) \in \mathbb{R}^4$, we have:

$$f(v + l, \tau + k, u, t) = f(v, \tau, u, t) \quad \text{for any} \quad (l, k) \in \mathbb{Z}^2;$$

- (A3) **Monotonicity:** for any $(v, \tau, t) \in \mathbb{R}^3$,

$$\text{the function } u \longmapsto f(v, \tau, u, t) \text{ is non-increasing.}$$

Let us make short comments on these assumptions. Remark that assumption (A1) ensures the existence and uniqueness of the solution u^ϵ of (1.1) via the Cauchy-Lipschitz theorem. Moreover, the assumed boundedness of f is not a restrictive condition while we work on any finite time interval $[0, T]$. The monotonicity assumption (A3)

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may seem unnecessary at a first glance, but will be indeed useful to guarantee the uniqueness of the solution to the homogenized equation (see Proposition 1.4). Moreover, assumption (A3) will play a crucial role to establish the rate of convergence of u^ϵ to its limit u^0 (see for instance Section 4).

In order to define the homogenized equation, we will use the following proposition:

Proposition 1.1 ***(Definition and properties of the effective slope $\overline{f}$).*** *Fix $(u,t) \in \mathbb{R}^2$. Then there exists $\lambda \in \mathbb{R}$ such that for any initial data $u_0 \in \mathbb{R}$, the solution $v \in C^1([0,\infty);\mathbb{R})$ of the following ordinary differential equation:*

$$\begin{cases} v_\tau = f(v,\tau,u,t), \quad \tau > 0, \\ v(0) = u_0, \end{cases}$$

satisfies

$$\frac{v(\tau)}{\tau} \to \lambda \quad as \quad \tau \to \infty. \tag{1.2}$$

Let us set the effective slope:

$$\overline{f}(u,t) = \lambda. \tag{1.3}$$

Then the following holds:

$$\begin{cases} \overline{f} : \mathbb{R}^2 \to \mathbb{R} \text{ is continuous.} \\ \text{For any } t \geq 0, \text{ the map } u \mapsto \overline{f}(u,t) \text{ is non-increasing.} \end{cases} \tag{1.4}$$

Let us mention that, for some specific functions f, explicit formulas for $\overline{f}$ can be obtained (see for instance [23], and the examples below).

Example 1.2 *For $f(v,\tau,u,t) = -u + \cos(2\pi v)$, we have $\overline{f}(u,t) = \left(\int_0^1 \frac{dv}{-u+\cos(2\pi v)} \right)^{-1}$ for $u > 1$, and*

$$\overline{f}(u,t) \sim -c\sqrt{u-1} \quad as \quad u \to 1^+,$$

for some constant $c > 0$.

Example 1.3 *For $f(v,\tau,u,t) = -u + |\sin(2\pi v)|$, we have, for some constant $c > 0$:*

$$\overline{f}(u,t) \sim \frac{c}{|\log|u||} \quad as \quad u \to 0^-. \tag{1.5}$$

Example 1.2 shows in particular that even for analytic f, the function $\overline{f}$ could be non-Lipschitz. Example 1.3 shows a case where $\overline{f}$ is not Hölder continuous when f is Lipschitz continuous. The proof of Example 1.3 will be given in the Appendix. At this stage, we can write the homogenized equation associated to equation (1.1) as follows:

$$\begin{cases} u^0_t = \overline{f}(u^0,t), \quad t > 0, \\ u^0(0) = u_0. \end{cases} \tag{1.6}$$

Even if $\overline{f}$ may not be Lipschitz continuous in u, we can show the existence and uniqueness of the solution of (1.6), taking advantage of the monotonicity of $\overline{f}(u^0,t)$ in u^0. Indeed, we have:

Proposition 1.4 ***(Existence and uniqueness).*** *Under assumption (1.4) on $\overline{f}$, there exists a unique solution $u^0 \in C^1([0,\infty);\mathbb{R})$ of (1.6).*

It is worth noticing that assumption (1.4) satisfied by $\overline{f}$ in the homogenized equation is our motivation to make assumption (A3) on f. We can now state our main result:

Theorem 1.5 ***(Error estimate for ODEs).*** *Under assumptions (A1)-(A2)-(A3), if u^ϵ is the solution of (1.1), and u^0 is the solution of the homogenized equation (1.6), then for every $C > 0$, $\epsilon > 0$, and every $T \geq C\epsilon|\log\epsilon|$, we have the following estimate:*

$$\|u^\epsilon - u^0\|_{L^\infty(0,T)} \leq \frac{cT}{|\log\epsilon|}, \tag{1.7}$$

where $c > 0$ is a positive constant only depending on C and on α, β defined in assumption (A1).

Such a result for a general monotone system of ODEs seems to be completely open. The above estimate in $\frac{1}{|\log\epsilon|}$ is in fact related to the behavior of $\overline{f}$ in Example 1.3, which is the worst possible regularity of $\overline{f}$. Moreover, it is possible to show that under the condition $T \geq C\epsilon|\log\epsilon|$, inequality (1.7) is sharp, see the following example whose proof will be given in the Appendix:

Example 1.6 *Let $f(v,\tau,u,t) = g(v+\tau) - 1$ with a 1-periodic function g satisfying*

$$g(w) = |w - 1/2| \quad \textit{for} \quad w \in [0,1]. \tag{1.8}$$

In this case $\overline{f}(u,t) = -1$. Let us choose the initial data $u_0 = 0$. Then for any $\delta > 0$, we have the following estimate between the solution u^ϵ to (1.1) and u^0 to (1.6):

$$u^\epsilon(t) - u^0(t) \sim \frac{t}{2\delta|\log\epsilon|} \quad \textit{for} \quad t = \delta\epsilon|\log\epsilon|.$$

Remark 1.7 *It is worth mentioning that assumption (A1) could be replaced by the weaker assumption:*

- (A1)' **Regularity:** *the function $f : \mathbb{R}^4 \to \mathbb{R}$ is a bounded continuous function such that for every $\tau \in \mathbb{R}$, the function $f(.,\tau,.,.)$ is Lipschitz continuous.*

1.2 Application to the homogenization of linear transport equations

For $x = (x_1, x_2) \in \mathbb{R}^2$, let us consider a vector field $a^\epsilon = (a_1^\epsilon, a_2^\epsilon)$ defined as follows

$$\begin{cases} a_1^\epsilon(x_1,x_2) = -f\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}, x_1, x_2\right) \\ a_2^\epsilon(x_1,x_2) = 1, \end{cases} \tag{1.9}$$

with a function f satisfying (A1)-(A2)-(A3). We consider the viscosity solution $V^\epsilon(t,x)$ of the following linear transport equation:

$$\begin{cases} V_t^\epsilon + a^\epsilon \cdot \nabla V^\epsilon = 0 & \text{on} \quad (0,\infty)\times\mathbb{R}^2 \\ V^\epsilon(0,x) = V_0(x) & \text{on} \quad \mathbb{R}^2, \end{cases} \tag{1.10}$$

where $V_0 : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz continuous function. The existence and uniqueness of a viscosity solution V^ϵ of (1.10) is ensured since $a^\epsilon \in W^{1,\infty}(\mathbb{R}^2)$ and V_0 is Lipschitz continuous (see for instance [3]). The expected homogenized equation associated to (1.10) is:

$$\begin{cases} V_t^0 + \overline{a} \cdot \nabla V^0 = 0 & \text{on} \quad (0,\infty)\times\mathbb{R}^2 \\ V^0(0,x) = V_0(x) & \text{on} \quad \mathbb{R}^2, \end{cases} \tag{1.11}$$

with the vector field $\overline{a} = (\overline{a}_1, \overline{a}_2)$ defined as:

$$\begin{cases} \overline{a}_1(x_1,x_2) = -\overline{f}(x_1,x_2), & \text{with } \overline{f} \text{ given by (1.3).} \\ \overline{a}_2(x_1,x_2) = 1. \end{cases} \tag{1.12}$$

As a consequence of Theorem 1.5, we will show in Section 6 the following result:

Theorem 1.8 ***(Error estimate for linear transport equations).*** *Under the previous assumptions, there exists a Lipschitz continuous function V^0 which is a viscosity solution of (1.11), such that for any $C > 0$, $\epsilon > 0$, and $T \geq C\epsilon|\log \epsilon|$, the solution V^ϵ of (1.10) satisfies:*

$$\|V^\epsilon - V^0\|_{L^\infty(\mathbb{R}^2\times(0,T))} \leq \frac{c'T}{|\log \epsilon|} \tag{1.13}$$

where $c' = cLip(V_0)$, and $c > 0$ is the constant given in Theorem 1.5.

Choosing the initial condition $V_0(x) = x_1$, we can easily deduce from Example 1.6 that inequality (1.13) is also sharp for $T \geq C\epsilon|\log \epsilon|$. Here, in this application, the vector field a^ϵ is quite special. The interested reader could be referred to [14] for some other examples of vector fields in 2D, where a homogenization result is presented without any rate of convergence. In [26], the author gives some non-explicit error estimates for linear transport equations in the particular case of periodic vector field a^ϵ. However, these error estimates obtained in [26] may depend strongly on the irrationality of the rotation number ω_0 associated to the vector field a^ϵ (where ω_0 is nothing else than $-\overline{f}$ in our application). On the contrary, estimate (1.13) only depends on some bounds of the data of the problem, and are completely uniform with respect to the rotation number.

Remark that when $f(v, \tau, u, t)$ is independent of u and t, we have a much better estimate:

Theorem 1.9 ***(Better error estimate).*** *Under the assumptions of Theorem 1.8, if $f(v, \tau, u, t)$ is independent of u and t, then we have:*

$$\|V^\epsilon - V^0\|_{L^\infty(\mathbb{R}^2\times(0,T))} \leq c''\epsilon, \quad \forall T \geq 0,$$

where $c'' = \xi Lip(V_0)$, with ξ (given in Proposition 2.1) only depends on β defined in assumption (A1).

The proof of Theorem 1.9 will also be given in Section 6.

1.3 Brief review of the literature

The pioneering work (via the theory of viscosity solutions) to periodic homogenization was established in [17]. Starting from [17], the homogenization theory for Hamilton-Jacobi equations has received a considerable interest. There is a huge literature that we cannot cite in details, but the interested reader can for instance see [1, 5, 4, 11, 18, 15, 16] and the references therein. Another aspect concerning homogenization of SDEs (stochastic differential equations) has also been studied by several authors (see for instance [20, 13, 6, 24]). These problems are related to our problem when the SDE reduces to an ODE.

To our knowledge, the question of estimating the rate of convergence in homogenization of PDEs has not been widely tackled up elsewhere in the literature. We can cite [7] for several error estimates concerning the rate of convergence of the approximation scheme to the effective Hamiltonian. We can also cite the work in [8] about the rate of convergence in periodic homogenization of first-order stationary Hamilton-Jacobi equations, where an error estimate in $\epsilon^{1/3}$ is obtained for Hamilton-Jacobi equations with Lipschitz effective Hamiltonian.

For the problems of homogenization of ODEs, we refer the reader to [23]. We also refer the reader to [2, 9, 10, 19, 21, 22, 25] for problems on homogenization of nonlinear first-order ODEs and/or the associated linear transport equations. As mentioned above, we refer the reader to [26] for some other error estimates for linear transport equations.

1.4 Organization of the paper

The paper is organized as follows. In Section 2, we present the proof of an ergodicity result (Proposition 2.1) that defines $\overline{f} = \lambda$. We also present the proofs of Propositions 1.1 and 1.4. In Section 3, we give a result of stability of λ under additive perturbation (Proposition 3.1). A basic error estimate (Proposition 4.1) is presented in Section 4. Section 5 is devoted to show our main result of estimating the rate of convergence (Theorem 1.5). In Section 6, we give an application to the case of linear transport equations (Theorems 1.8 and 1.9). We end up in Section 7 with an Appendix where we give the proof of Examples 1.3 and 1.6.

2 Ergodicity and preliminary facts

In this section we present the proof of Propositions 1.1 and 1.4. We first start with the following ergodicity result which is a particular case of [12, Proposition 4.2]. However, we give the proof in our particular case for the sake of completeness.

Proposition 2.1 ***(Ergodicity).*** *Let $g(v,\tau):\mathbb{R}^2\to\mathbb{R}$ be a function satisfying:*

- *(H1)* **Regularity:** *g is Lipschitz continuous with $\|g\|_{L^\infty(\mathbb{R}^2)}\le\beta$;*
- *(H2)* **Periodicity:** *$g(v+l,\tau+k)=g(v,\tau)$ for any $(l,k)\in\mathbb{Z}^2$, $(v,\tau)\in\mathbb{R}^2$.*

Let v be the solution of the following equation

$$\begin{cases} v_\tau = g(v,\tau), & \tau>0,\\ v(0)=v_0, \end{cases} \tag{2.1}$$

then there exist a constant $\lambda\in\mathbb{R}$ (independent of the initial data v_0) such that for every $\tau,\tau'\ge 0$, we have:

$$|v(\tau)-v(\tau')-\lambda(\tau-\tau')|\le\xi \quad \text{with} \quad \xi=1+2\beta. \tag{2.2}$$

Remark 2.2 *Under our assumptions, it is possible (see [12, Theorem 1.5]) to show the existence of a hull function $h:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfying:*

$$\begin{cases} h(\tau+1,x)=h(\tau,x)\\ h(\tau,x+1)=h(\tau,x)+1\\ h_x\ge 0, \end{cases}$$

such that $U(\tau,x)=h(\tau,x+\lambda\tau)$, with λ given in Proposition 2.1, is a viscosity solution of:

$$\begin{cases} U_\tau = g(U,\tau)\\ U(\tau=0,x)=x. \end{cases}$$

This function h may be discontinuous, but its existence suggests that

$$v(\tau)=h(\tau,\lambda\tau) \tag{2.3}$$

is formally a classical of (2.1). The expression (2.3) allows to understand estimate (2.2) and also suggests that $v(\tau)-\lambda\tau$ could be quasi-periodic in some cases. We emphasize the fact that in the proof of Proposition 2.1, we do not use any notion of hull function, but propose a completely independent proof.

Proof of Proposition 2.1. For any $T>0$, define the two quantities:

$$\lambda^+(T)=\sup_{\tau\ge 0}\frac{v(\tau+T)-v(\tau)}{T} \quad \text{and} \quad \lambda^-(T)=\inf_{\tau\ge 0}\frac{v(\tau+T)-v(\tau)}{T}.$$

These quantities are finite since $v_\tau=g$ is bounded. The proof is divided into three steps.

Step 1: Estimate of $|\lambda^+-\lambda^-|$.

Let $\delta>0$ be an arbitrary constant. From the definition of $\lambda^\pm(T)$, there always exists $\tau^\pm$ such that

$$\left|\lambda^\pm(T)-\frac{v(\tau^\pm+T)-v(\tau^\pm)}{T}\right|\le\delta. \tag{2.4}$$

Denote by $\lfloor t\rfloor$ and $\lceil t\rceil$ as the floor and the ceil integer parts of the real number t respectively. We consider

$$k=\lfloor\tau^- - \tau^+\rfloor,\quad \widetilde{\tau}^-=\tau^- - k \quad \text{and} \quad l=\lceil v(\widetilde{\tau}^-)-v(\tau^-)\rceil, \tag{2.5}$$

and consider $w(\tau) = v(\tau + k) + l$. Using (2.1) and (H2), we check that w is a solution of the following equation

$$\begin{cases} w_\tau = g(w, \tau), & \tau > 0 \\ w(\widetilde{\tau}^-) = v(\tau^-) + l. \end{cases} \tag{2.6}$$

We remark, from (2.5), that

$$\tau^+ \leq \widetilde{\tau}^- < \tau^+ + 1, \tag{2.7}$$

and

$$v(\widetilde{\tau}^-) \leq w(\widetilde{\tau}^-) < v(\widetilde{\tau}^-) + 1. \tag{2.8}$$

From (2.1), (2.6) and (2.8), the comparison principle for ODEs gives in particular

$$v(\tau) \leq w(\tau) \quad \text{for all} \quad \tau \geq \widetilde{\tau}^-. \tag{2.9}$$

If we suppose $T \geq 1$, we obtain from (2.7) that $\tau^+ + T \geq \widetilde{\tau}^-$ and hence (using (2.9)):

$$v(\tau^+ + T) \leq w(\tau^+ + T). \tag{2.10}$$

Direct computations give

$$\begin{aligned} (v(\tau^- + T) - v(\tau^-)) - (v(\tau^+ + T) - v(\tau^+)) \quad &= \quad (w(\widetilde{\tau}^- + T) - w(\tau^+ + T)) + (v(\widetilde{\tau}^-) - w(\widetilde{\tau}^-)) \\ &\quad + (w(\tau^+ + T) - v(\tau^+ + T)) + (v(\tau^+) - v(\widetilde{\tau}^-)), \end{aligned}$$

where, from (2.8) and (2.10), we deduce that $(v(\tau^- + T) - v(\tau^-)) - (v(\tau^+ + T) - v(\tau^+)) \geq -1 - 2\|g\|_\infty$. This inequality, together with (2.4) show that $0 \leq \lambda^+(T) - \lambda^-(T) \leq \frac{1+2\|g\|_\infty}{T} + 2\delta$, and since this is true for any $\delta > 0$, we obtain

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{1 + 2\|g\|_\infty}{T}.$$

However, in the case where $T \leq 1$, we always have $\frac{|v(\tau+T)-v(\tau)|}{T} \leq \|g\|_\infty \leq \frac{\|g\|_\infty}{T}$ and therefore

$$0 \leq \lambda^+(T) - \lambda^-(T) \leq \frac{2\|g\|_\infty}{T},$$

then

$$|\lambda^+(T) - \lambda^-(T)| \leq \frac{\xi}{T} \quad \text{for every } T > 0. \tag{2.11}$$

Step 2: Existence of the limit of $\lambda^\pm(\mathbf{T})$ as $T \to \infty$.

First, if we compute $\lambda^+(PT)$ for $P \in \mathbb{N} \setminus \{0\}$, we get:

$$\lambda^+(PT) = \sup_{\tau > 0} \frac{1}{P} \left[\sum_{i=1}^{P} \frac{v(\tau + iT) - v(\tau + (i-1)T)}{T} \right] \leq \lambda^+(T).$$

Similarly, we get $\lambda^-(PT) \geq \lambda^-(T)$. Consider T_1, $T_2 > 0$ such that $T_1 P = T_2 Q$ for some $P, Q \in \mathbb{N} \setminus \{0\}$. Using this and (2.11), we have

$$\lambda^+(T_2) \geq \lambda^+(T_2 Q) = \lambda^+(T_1 P) \geq \lambda^-(T_1 P) \geq \lambda^-(T_1) \geq \lambda^+(T_1) - \frac{\xi}{T_1},$$

similarly we have $\lambda^+(T_2) - \lambda^+(T_1) \leq \frac{\xi}{T_2}$, then

$$|\lambda^+(T_1) - \lambda^+(T_2)| \leq \max\left(\frac{\xi}{T_1}, \frac{\xi}{T_2}\right). \tag{2.12}$$

By the same arguments as above we can get

$$|\lambda^-(T_1) - \lambda^-(T_2)| \le \max\left(\frac{\xi}{T_1}, \frac{\xi}{T_2}\right). \tag{2.13}$$

Recall that (2.12) and (2.13) are true when T_1/T_2 is rational. By an approximation argument, joint with the continuity of $\lambda^\pm$, it is easy to see that this is still true when T_1/T_2 is any positive real number. Moreover, the identities (2.12) and (2.13) give that the sequence $(\lambda^\pm(T))_T$ is a Cauchy sequence as $T \to \infty$, and hence it has a limit:

$$\lim_{T\to\infty} \lambda^\pm(T) = \lambda, \tag{2.14}$$

which is the same limit because of (2.11). From (2.14), (2.12) and (2.13), inequality (2.2) directly follows.

Step 3: Independence of v_0.

The fact that λ is independent of v_0 follows directly from the comparison principle and inequality (2.2). □

We can now present the proof of Propositions 1.1 and 1.4.

Proof of Proposition 1.1. Using inequality (2.2) of Proposition 2.1, we can easily see that (1.2) directly follows. It remains to show (1.4). We argue in two steps.

Step 1: Monotonicity of $\overline{f}$.

Let $u \le \tilde{u}$. Call $\lambda = \overline{f}(u,t)$ and $\tilde{\lambda} = \overline{f}(\tilde{u},t)$. Let v and $\tilde{v}$ be the solutions of:

$$\begin{cases} v_\tau = f(v,\tau,u,t), \quad \tau > 0, \\ v(0) = u_0, \end{cases}$$

and

$$\begin{cases} \tilde{v}_\tau = f(\tilde{v},\tau,\tilde{u},t), \quad \tau > 0, \\ \tilde{v}(0) = u_0, \end{cases}$$

respectively. Assume without loss of generality that $u_0 = 0$. Using (A3), we deduce that $f(\tilde{v},\tau,\tilde{u},t) \le f(\tilde{v},\tau,u,t)$. Hence, the comparison principle gives:

$$\tilde{v}(\tau) \le v(\tau) \quad \text{for every} \quad \tau \ge 0. \tag{2.15}$$

From inequality (2.2) of Proposition 2.1, we have:

$$|v(\tau) - \lambda\tau| \le \xi \quad \text{and} \quad |\tilde{v}(\tau) - \tilde{\lambda}\tau| \le \xi \quad \text{for all} \quad \tau \ge 0. \tag{2.16}$$

We then easily conclude that $\tilde{\lambda} \le \lambda$ as a consequence of (2.15) and (2.16).

Step 2: Continuity of $\overline{f}$.

We refer the reader to Proposition 3.2 which implies in particular the continuity of $\overline{f}$. The main idea of the proof is to apply a perturbation argument using the inequality $|\lambda^\pm(T) - \lambda| \le \frac{\xi}{T}$. □

Proof of Proposition 1.4.

Global existence. This is a direct consequence of the Cauchy-Péano theorem, using in particular the continuity of f (see 1.4).

Uniqueness. Assume that there exists $u^1 \in C^1([0,\infty);\mathbb{R})$ another solution of (1.6). Define $k(t) = |u^0(t) - u^1(t)|$, we compute (with the sign function $\operatorname{sgn}(x) = x/|x|$ if $x \neq 0$):

$$\begin{aligned} k_t(t) &= (u_t^0(t) - u_t^1(t)) \operatorname{sgn}(u^0(t) - u^1(t)) \\ &= (\overline{f}(u^0(t),t) - \overline{f}(u^1(t),t)) \operatorname{sgn}(u^0(t) - u^1(t)) \leq 0, \end{aligned}$$

where for the last line we have used the monotonicity of $\overline{f}$ (see (1.4)). This immediately implies that $u^0 = u^1$. □

3 A stability result for the effective slope $\overline{f}$

In this section, we will show a stability result for the term λ given by Proposition 2.1 under a perturbation of (2.1) of the form:

$$\begin{cases} v_t = g_\gamma(v,t) = g(v,t) \pm \gamma, & t > 0 \\ v(0) = v_0, \end{cases} \tag{3.1}$$

where $\gamma > 0$ is a small real number. More precisely, we have the following proposition:

Proposition 3.1 ***(Stability result).*** *Take $0 < \gamma < 1$. Let λ_γ be the effective slope given by Proposition 2.1, which is associated to equation (3.1), and let λ_0 be the one corresponds to $\gamma = 0$ in (3.1). Then we have the following estimate:*

$$|\lambda_\gamma - \lambda_0| \leq \frac{\bar{\xi}}{|\log \gamma|} \quad \text{with} \quad \bar{\xi} = (3 + 2\xi)(1 + 2L), \tag{3.2}$$

and $L = \left\| \frac{\partial g}{\partial v} \right\|_{L^\infty(\mathbb{R}^2)}$.

Proof. We assume, for the sake of simplicity, that $g_\gamma = g + \gamma$. In this case, it is easy to check that $\lambda_\gamma \geq \lambda_0$. The other case with $g_\gamma = g - \gamma$ is treated similarly. We first transform our ODE problem into a PDE one by setting v^γ as the solution of the following equation:

$$\begin{cases} v_t^\gamma(t,x) = g(v^\gamma(t,x),t) + \gamma, & \text{in} \quad (0,\infty) \times \mathbb{R} \\ v^\gamma(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{3.3}$$

The proof is divided into three steps.

Step 1: A control on v_x^γ.

Using comparison principle arguments for (3.3), it is easily checked that $v^\gamma(t,.)$ is a non-decreasing function satisfying $v^\gamma(t,x+1) = v^\gamma(t,x) + 1$. We want to control $v_x^\gamma(t,.)$ for any t. For this reason, we proceed as follows. Define for $z \geq 0$:

$$\eta(t,x) = v^\gamma(t,x+z) - ze^{Lt}, \quad t > 0,\ x \in \mathbb{R}.$$

We compute

$$\begin{aligned} \eta_t(t,x) &= v_t^\gamma(t,x+z) - zLe^{Lt} \\ &= g(\eta(t,x) + ze^{Lt}, t) - zLe^{Lt} + \gamma \\ &\leq g(\eta(t,x),t) + \gamma, \end{aligned}$$

which proves that η is a sub-solution of (3.3) with $\eta(0,x) = v^\gamma(0,x+z) - z = v^\gamma(0,x)$, and therefore, by the comparison principle, we obtain

$$\eta(t,x) = v^\gamma(t,x+z) - ze^{Lt} \leq v^\gamma(t,x)$$

hence for any $t \geq 0$, we have $0 \leq v^\gamma(t,x+z) - v^\gamma(t,x) \leq ze^{Lt}$, then $v^\gamma(t,x)$ is Lipschitz continuous in the variable x, satisfying:

$$0 \leq v_x^\gamma(t,x) \leq e^{Lt} \quad \text{for } t \geq 0 \text{ and a.e. } x \in \mathbb{R}. \tag{3.4}$$

In a similar way, we can obtain a positive bound from below on v_x^γ, and finally get

$$e^{-Lt} \leq v_x^\gamma(t,x) \leq e^{Lt}. \tag{3.5}$$

Step 2: An upper bound of v^γ.

We seek to find an upper bound of v^γ by constructing an explicit super-solution of (3.3) with suitable initial data, and comparing it with v^γ. For this purpose, let

$$w(t,x) = v^0(t, x + c_1\gamma t), \quad (t,x) \in (0,\infty) \times \mathbb{R}, \tag{3.6}$$

where v_0 is equal to v^γ for $\gamma = 0$, and c_1 is positive constant to be precised later. We calculate:

$$\begin{array}{rcl} w_t(t,x) & = & v_t^0(t, x + c_1\gamma t) + c_1\gamma v_x^0(t, x + c_1\gamma t) \\ & = & g(w(t,x), t) + c_1\gamma v_x^0(t, x + c_1\gamma t), \end{array}$$

where from (3.5), we deduce that

$$w_t(t,x) \geq g(w(t,x), t) + c_1\gamma e^{-Lt}. \tag{3.7}$$

Take $c_1 = e^{LT}$ for some fixed $T > 0$. Then using (3.7), we get $w_t(t,x) \geq g(w(t,x),t) + \gamma$ for any $t \in [0,T]$. Hence w is a super-solution of (3.3) over $[0,T]$ whose initial condition $w(0,x) = v^\gamma(0,x)$, which finally gives:

$$w(t,x) \geq v^\gamma(t,x) \quad \forall t \in [0,T],\ x \in \mathbb{R}. \tag{3.8}$$

Step 3: Conclusion.

We will now show the error estimate (3.2). To this end, we will estimate both sides of inequality (3.8) involving λ_0 and λ_γ. Firstly, using (2.2) and (3.5), we compute:

$$\begin{array}{rcl} |v^0(t, x + e^{LT}\gamma t) - v^0(0,x)| & \leq & |v^0(t, x + e^{LT}\gamma t) - v^0(t,x)| + |v^0(t,x) - v^0(0,x)| \\ & \leq & e^{Lt}e^{LT}\gamma t + \lambda_0 t + \xi. \end{array}$$

We take this inequality for $t = T$ and $x = 0$, we get

$$w(T,0) = v^0(T, e^{LT}\gamma T) \leq \gamma T e^{2LT} + \lambda_0 T + \xi. \tag{3.9}$$

Secondly, using similar arguments, and the fact that $\gamma < 1$, we obtain $|v^\gamma(T,0) - v^\gamma(0,0) - \lambda_\gamma T| \leq 2 + \xi$, hence

$$v^\gamma(T,0) \geq \lambda_\gamma T - (2 + \xi). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), it follows that

$$(\lambda_\gamma - \lambda_0)T \leq \gamma T e^{2LT} + 2(1 + \xi) \tag{3.11}$$

Using (3.11), we deduce that:

$$|\lambda_\gamma - \lambda_0| \leq \gamma e^{2LT} + \frac{2(1+\xi)}{T}. \tag{3.12}$$

Since the variable T was arbitrary chosen, let T satisfies $\gamma T e^{2LT} = 1$ and therefore $T \geq \frac{|\log \gamma|}{1+2L}$. From (3.12), the result directly follows. □

An immediate consequence of Proposition 3.1 is the following:

Proposition 3.2 ***(Modulus of continuity of $\overline{f}$).*** *The function $\overline{f}(u,t)$ given by (1.3) satisfies for any $(u,t) \in \mathbb{R}^2$, and for all $|v| + |s| < \frac{1}{\alpha}$:*

$$|\overline{f}(u+v, t+s) - \overline{f}(u,t)| \leq \frac{\bar{\xi}}{|\log \alpha(|v| + |s|)|}, \tag{3.13}$$

where α is given in assumption (A1).

Remark that estimate (3.13) is optimal in view of Example 1.3.

4 Basic error estimate

We start this section by considering a discrete scheme associated to the ODE (1.6). Namely, for a given v^0 (which may be chosen equal to u_0 or may be different), and for a time step $\Delta t > 0$, we define the sequence $(v^k)_{k\in\mathbb{N}}$ as follows:

$$v^{k+1} = v^k + \lambda_k \Delta t, \quad \lambda_k = \overline{f}(v^k, k\Delta t), \quad k \in \mathbb{N}. \tag{4.1}$$

In this section we give a local error estimate between the solution u^ϵ of (1.1) and the sequence v^k. To be more precise, we will show the following proposition:

Proposition 4.1 ***(Basic error estimate).*** *Under assumptions (A1), (A2) and (A3) on the function f, let $\Delta t > 0$ be small enough (depending only on α and β), and take*

$$e_i = |u^\epsilon(i\Delta t) - v^i|, \quad i = 0, 1.$$

Then we have:

$$e_1 \le e_0 + c\epsilon + \frac{c\Delta t}{|\log c\Delta t|}, \tag{4.2}$$

where $c = c(\alpha, \beta) > 0$ is a positive constant.

The proof of the above proposition will be presented later in this section. In what follows in this section and in Section 5, we will assume that the arguments of the various logarithms are all less than 1.

Lemma 4.2 ***(Refined basic error estimate).*** *Under the same hypothesis of Proposition 4.1, let*

$$e_i^+ = \max(0, u^\epsilon(i\Delta t) - v^i), \quad e_i^- = \min(0, u^\epsilon(i\Delta t) - v^i), \quad i = 0, 1$$

and let $d_1^+ = \sup_{t\in[0,\Delta t]} (\max(0, u^\epsilon(t) - v^0))$, $d_1^- = \inf_{t\in[0,\Delta t]} (\min(0, u^\epsilon(t) - v^0))$. Then we have:

$$e_1^+ \le e_0^+ + (2+\xi)\epsilon + \frac{\bar{\xi}\Delta t}{|\log \alpha(|d_1^-| + \Delta t)|} \tag{4.3}$$

and

$$e_1^- \ge e_0^- - (2+\xi)\epsilon - \frac{\bar{\xi}\Delta t}{|\log \alpha(d_1^+ + \Delta t)|}, \tag{4.4}$$

with $\xi = 1 + 2\beta$ and β defined in (A1).

Proof. In order to get estimates (4.3), (4.4), the main idea is to freeze the last two arguments of f, and to use some comparison arguments. We start by estimating the term $f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, t\right)$ from below. Since $u^\epsilon(t) \le v^0 + d_1^+$ for $t \in [0, \Delta t]$, we deduce, using (A3), that

$$f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, t\right) - f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right) \ge f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0 + d_1^+, t\right) - f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right),$$

and hence, from (A1), we get

$$f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, t\right) \ge f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right) - \alpha(d_1^+ + \Delta t). \tag{4.5}$$

Using similar arguments, we can also show

$$f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, u^\epsilon, t\right) \le f\left(\frac{u^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right) + \alpha(|d_1^-| + \Delta t). \tag{4.6}$$

We know from the definition of e_0^+ and e_0^- that:

$$v^0 + e_0^- \le u^\epsilon(0) \le v^0 + e_0^+. \tag{4.7}$$

Let $\overline{w}^\epsilon$ and $\underline{w}^\epsilon$ be the solutions of the following ODEs:

$$\begin{cases} \overline{w}^\epsilon_t = f\left(\frac{\overline{w}^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right) - \alpha(d_1^+ + \Delta t), \quad t > 0 \\ \overline{w}^\epsilon(0) = v^0 + e_0^- \end{cases} \tag{4.8}$$

and

$$\begin{cases} \underline{w}^\epsilon_t = f\left(\frac{\underline{w}^\epsilon}{\epsilon}, \frac{t}{\epsilon}, v^0, 0\right) + \alpha(|d_1^-| + \Delta t), \quad t > 0 \\ \underline{w}^\epsilon(0) = v^0 + e_0^+, \end{cases} \tag{4.9}$$

respectively. From (4.5), (4.6) and (4.7), we deduce, using the comparison principle, that:

$$\overline{w}^\epsilon \leq u^\epsilon \leq \underline{w}^\epsilon \quad \text{on} \quad [0, \Delta t]. \tag{4.10}$$

Applying Proposition 2.1 to the functions $\overline{w}^\epsilon$ and $\underline{w}^\epsilon$, we know that there exists two real numbers $\overline{\lambda}_0$ and $\underline{\lambda}_0$ such that for $t \in [0, \Delta t]$ (assuming $\alpha(d_1^+ + \Delta t) \leq 1$ and $\alpha(|d_1^-| + \Delta t) \leq 1$):

$$|\overline{w}^\epsilon(t) - \overline{w}^\epsilon(0) - \overline{\lambda}_0 t| \leq (2 + \xi)\epsilon \quad \text{and} \quad |\underline{w}^\epsilon(t) - \underline{w}^\epsilon(0) - \underline{\lambda}_0 t| \leq (2 + \xi)\epsilon. \tag{4.11}$$

Inequalities (4.10) and (4.11) give:

$$v^0 + e_0^- + \overline{\lambda}_0 t - (2 + \xi)\epsilon \leq u^\epsilon(t) \leq v^0 + e_0^+ + \underline{\lambda}_0 t + (2 + \xi)\epsilon, \quad t \in [0, \Delta t]. \tag{4.12}$$

Using Proposition 3.1, we obtain, for $\lambda_0 = \overline{f}(v^0, 0)$, that

$$0 \geq \overline{\lambda}_0 - \lambda_0 \geq -\frac{\bar{\xi}}{|\log \alpha(d_1^+ + \Delta t)|} \tag{4.13}$$

and

$$0 \leq \underline{\lambda}_0 - \lambda_0 \leq \frac{\bar{\xi}}{|\log \alpha(|d_1^-| + \Delta t)|}. \tag{4.14}$$

The above two inequalities, together with (4.7) and (4.12) give the result. □

Two immediate corollaries of the above lemma are the following:

Corollary 4.3 ***(Refined estimates involving $e_i^\pm$).*** *Under the same hypothesis of Lemma 4.2, we have:*

$$e_1^+ \leq e_0^+ + (2 + \xi)\epsilon + \frac{\bar{\xi}\Delta t}{|\log \alpha_1(|e_0^-| + \Delta t)|} \tag{4.15}$$

and

$$e_1^- \geq e_0^- - (2 + \xi)\epsilon - \frac{\bar{\xi}\Delta t}{|\log \alpha_1(e_0^+ + \Delta t)|}, \tag{4.16}$$

where $\alpha_1 = \alpha_1(\alpha, \beta) > 0$ is a positive constant.

Proof. We have $d_1^+ \leq e_0^+ + \beta\Delta t$ and $|d_1^-| \leq |e_0^-| + \beta\Delta t$ (recall that $\beta = \|f\|_{L^\infty(\mathbb{R}^4)}$). Therefore, the result can be deduced from (4.3) and (4.4). □

Corollary 4.4 ***(Refined estimates with continuous time).*** *Under the same hypothesis of Proposition 4.1, for every $t \in [0, \Delta t]$, define the continuous function $\bar{e}$ by:*

$$\bar{e}(t) = u^\epsilon(t) - (v^0 + \lambda_0 t). \tag{4.17}$$

Also define $e^+(t) = \max(0, \bar{e}(t))$ and $e^-(t) = \min(0, \bar{e}(t))$. Then we have for $0 \leq t_1 < t_2 \leq \Delta t$ and $c = c(\alpha, \beta) > 0$:

$$e^+(t_2) \leq e^+(t_1) + c\epsilon + \frac{c(t_2 - t_1)}{|\log c(|e^-(t_1)| + (t_2 - t_1))|} \tag{4.18}$$

and

$$e^-(t_2) \geq e^-(t_1) - c\epsilon - \frac{c(t_2 - t_1)}{|\log c(e^+(t_1) + (t_2 - t_1))|}. \tag{4.19}$$

Proof. We apply the same proof (word by word) of Lemma 4.2 and Corollary 4.3, with the origin 0 shifted to t_1 and the time step Δt replaced by $t_2 - t_1$. □

Now we are ready to prove Proposition 4.1.

Proof of Proposition 4.1. From the definition (4.17) of $\bar{e}$, we know that

$$e_0 = |\bar{e}(0)| \quad \text{and} \quad e_1 = |\bar{e}(\Delta t)|.$$

The case where $e_1 = \bar{e}(\Delta t) = 0$ is obvious. Four cases could be considered.

Case 1. *($\bar{e}(0) \geq 0$ and $\bar{e}(\Delta t) > 0$).* In this case we have $e_0 = e_0^+$, $e_1 = e_1^+$ and $e_0^- = 0$. Therefore (4.2) is an immediate consequence of (4.15).

Case 2. *($\bar{e}(0) \leq 0$ and $\bar{e}(\Delta t) < 0$).* Similar to Case 1.

Case 3. *($\bar{e}(0) < 0$ and $\bar{e}(\Delta t) > 0$).* In this case $e_1 = e^+(\Delta t)$. Let the time t_{-+} be defined as follows:

$$t_{-+} = \max\{t \in (0, \Delta t); \quad \bar{e}(t) = 0\}.$$

Using inequality (4.18) with $t_1 = t_{-+}$, $t_2 = \Delta t$ and $\underline{\Delta} t = \Delta t - t_{-+} \leq \Delta t$, we get:

$$e_1 \leq c\epsilon + \frac{c\underline{\Delta} t}{|\log c\underline{\Delta} t|} \leq e_0 + c\epsilon + \frac{c\Delta t}{|\log c\Delta t|}$$

where we have used the fact that $e^+(t_1) = e^-(t_1) = 0$, and hence (4.2) follows.

Case 4. *($\bar{e}(0) > 0$ and $\bar{e}(\Delta t) < 0$).* Similar to Case 3. □

5 Estimate of the rate of convergence

This section is entirely devoted to the proof of Theorem 1.5. Let $T > 0$ and let $\Delta t > 0$ be such that

$$n\Delta t = T, \quad n \in \mathbb{N}, \tag{5.1}$$

where n to be chosen large enough (the choice of Δt will be given later). In order to estimate $\|u^\epsilon - u^0\|_{L^\infty(0,T)}$, we add and subtract v, the continuous piecewise linear function passing through the points $(k\Delta t, v^k)$, $k = 0 \cdots n$. In other words

$$v(t) = v^k + (t - k\Delta t)\lambda_k, \quad k\Delta t \leq t \leq (k+1)\Delta t, \quad k = 0 \cdots n-1,$$

where v^k and λ_k are defined in (4.1). We start by stating the following corollary that generalizes Proposition 4.1.

Corollary 5.1 ***(Basic error estimate in continuous time).*** *Under assumptions (A1), (A2) and (A3), let $T > 0$, and $\Delta t > 0$ given by (5.1). Define the function $e(t)$ by:*

$$e(t) = |u^\epsilon(t) - v(t)|, \quad t \in [0, T]. \tag{5.2}$$

For $k = 0 \cdots n-1$, call $e_k = e(k\Delta t)$. Then for $k\Delta t \leq t \leq (k+1)\Delta t$, we have:

$$e(t) \leq e_k + c\epsilon + \frac{c\Delta t}{|\log c\Delta t|}, \tag{5.3}$$

where $c = c(\alpha, \beta) > 0$ is a positive constant (the same given by Proposition 4.1).

Proof. We simply apply Proposition 4.1 with the origin 0 shifted to $k\Delta t$ and with the time step Δt replaced by $\delta t = t - k\Delta t \leq \Delta t$. □

At this stage, we can show an inequality similar to (5.3) with $e(t)$ and e_k replaced respectively by the functions

$$e^0(t) = |u^0(t) - v(t)| \quad \text{and} \quad e_k^0 = e^0(k\Delta t), \tag{5.4}$$

where u^0 is the solution of (1.6). Indeed, we have the following proposition:

Proposition 5.2 ***(Basic error estimate for the homogenized ODE).*** *Let $\overline{f}$ be the function given by (1.3), and enjoying the properties given by Propositions 1.1 and 3.2. Let $T > 0$, and $\Delta t > 0$ given by (5.1). Then for $k\Delta t \leq t \leq (k+1)\Delta t$, $k = 0 \cdots n-1$, we have:*

$$e^0(t) \leq e_k^0 + \frac{c\Delta t}{|\log c\Delta t|} \tag{5.5}$$

where e^0, e_k^0 are given by (5.4), and $c = c(\alpha, \beta) > 0$ is a positive constant.

Sketch of the proof. Although the proof is an adaptation of the proof of inequality (5.3) (it suffices to deal with $\overline{f}(u,t)$ instead of $f(v,\tau,u,t)$, and to take $\epsilon = 0$), we will indicate the main points where it slightly differs. The crucial idea is that, on the one hand, the monotonicity of the function $f(v,\tau,u,t)$ with respect to the variable u (see assumption (A3)) is replaced by the monotonicity of $\overline{f}(u,t)$ with respect to u (see (1.4)). On the other hand, the fact that f is Lipschitz continuous (see assumption (A1)) is replaced by the fact that $\overline{f}$ satisfies a modulus of continuity (see Proposition 3.2), and the fact that $\overline{f}(u,t) = \lambda(u,t)$.

Let us go into the details. In fact, the proof of Lemma 4.2 can be adapted where inequalities (4.5) and (4.6) are replaced by

$$\overline{f}(u^0,t) \geq \overline{f}(v^0,0) - \frac{\bar{\xi}}{|\log \alpha(d_1^{0,+} + \Delta t)|}, \quad \text{with} \quad d_1^{0,+} = \sup_{t\in[0,\Delta t]} (\max(0, u^0(t) - v^0)) \tag{5.6}$$

and

$$\overline{f}(u^0,t) \leq \overline{f}(v^0,0) + \frac{\bar{\xi}}{|\log \alpha(|d_1^{0,-}| + \Delta t)|}, \quad \text{with} \quad d_1^{0,-} = \inf_{t\in[0,\Delta t]} (\min(0, u^0(t) - v^0)) \tag{5.7}$$

respectively. Here we have used the monotonicity of $\overline{f}$, and inequality (3.13) given by Proposition 3.2. Having (5.6) and (5.7) in hands, the sub- and super-solution $\overline{w}^\epsilon$, $\underline{w}^\epsilon$ defined by (4.8) and (4.9) are replaced by $\overline{w}^0$, $\underline{w}^0$ solutions of

$$\begin{cases} \overline{w}_t^0 = \overline{f}(v^0,0) - \dfrac{\bar{\xi}}{|\log \alpha(d_1^{0,+} + \Delta t)|} \\ \overline{w}^0(0) = v^0 + e^{0,-} \quad \text{with} \quad e^{0,-} = \min(0, u^0(0) - v^0) \end{cases}$$

and

$$\begin{cases} \underline{w}_t^0 = \overline{f}(v^0,0) + \dfrac{\bar{\xi}}{|\log \alpha(|d_1^{0,-}| + \Delta t)|} \\ \underline{w}^0(0) = v^0 + e^{0,+} \quad \text{with} \quad e^{0,+} = \max(0, u^0(0) - v^0) \end{cases}$$

respectively, and we have

$$\overline{w}^0(t) - \overline{w}^0(0) - \overline{\lambda}_0^0 t = 0 \quad \text{and} \quad \underline{w}^0(t) - \underline{w}^0(0) - \underline{\lambda}_0^0 t = 0 \tag{5.8}$$

with

$$\overline{\lambda}_0^0 = \overline{w}_t^0 = \lambda_0 - \frac{\bar{\xi}}{|\log \alpha(d_1^{0,+} + \Delta t)|} \quad \text{and} \quad \underline{\lambda}_0^0 = \underline{w}_t^0 = \lambda_0 + \frac{\bar{\xi}}{|\log \alpha(|d_1^{0,-}| + \Delta t)|}, \tag{5.9}$$

where we recall the reader that $\lambda_0 = \overline{f}(v^0,0)$. Remark that (5.8) replaces (4.11), while (5.9) gives (as a replacement of (4.13) and (4.14)):

$$0 \geq \overline{\lambda}_0^0 - \lambda_0 = -\frac{\bar{\xi}}{|\log \alpha(d_1^{0,+} + \Delta t)|} \quad \text{and} \quad 0 \leq \underline{\lambda}_0^0 - \lambda_0 = \frac{\bar{\xi}}{|\log \alpha(|d_1^{0,-}| + \Delta t)|}.$$

At this point, the rest of the proof proceeds in a very similar way as the proof of (5.3) with $\epsilon = 0$, and a possible changing of the constants but always depending on α and β. $\square$

Now we are ready to present the proof of our main result.

Proof of Theorem 1.5. We first note that the constant $c > 0$ may certainly differ from line to line in the proof. We decompose the quantity $|u^\epsilon(t) - u^0(t)|$ into two pieces:

$$|u^\epsilon(t) - u^0(t)| \le \overbrace{|u^\epsilon(t) - v(t)|}^{e(t)} + \overbrace{|u^0(t) - v(t)|}^{e^0(t)}. \tag{5.10}$$

In order to estimate $e(t)$, we iterate inequality (5.3) and we finally obtain for $u^\epsilon(0) = u_0 = v^0$, and $t \in [0, T]$ with $T = n\Delta t$:

$$e(t) \le e_0 + cn\epsilon + \frac{cn\Delta t}{|\log c\Delta t|} \le \frac{c\epsilon T}{\Delta t} + \frac{cT}{|\log c\Delta t|}.$$

The above inequality gives:

$$e(t) \le cT\left(\frac{\epsilon}{\Delta t} + \frac{1}{|\log \Delta t|}\right), \tag{5.11}$$

and, from inequality (5.5) of Proposition 5.2, we can show in the same way as above that we also have:

$$e^0(t) \le \frac{cT}{|\log \Delta t|}. \tag{5.12}$$

Choosing particularly $\Delta t = C\epsilon|\log\epsilon|$, we deduce that $\epsilon \ll \Delta t = C\epsilon|\log\epsilon| \ll T$, and (from (5.11), (5.12)) that:

$$e(t) \le \frac{cT}{|\log\epsilon|} \quad \text{and} \quad e^0(t) \le \frac{cT}{|\log\epsilon|}, \tag{5.13}$$

with c in (5.13) depending on the choice of $C > 0$. finally, inequality (1.7) could now be easily deduced from (5.10) and (5.13). $\square$

6 Application: error estimate for linear transport equations

In this section, as an application of our previous results on ODEs, we give the proof of some error estimates for the homogenization of linear transport equations. Namely, we prove Theorems 1.8 and 1.9. We start with Theorem 1.8 keeping the same notations of Subsection 1.2.

Proof of Theorem 1.8. The proof is divided into four steps. The first three steps are devoted to the definition of the limit solution V^0, and to prove that it is a viscosity solution. The proof of the error estimate is done in the last step.

Step 1: Definition of V^0.

Because the homogenized vector field $\overline{a}$ is not Lipschitz, we define our solution V^0 to (1.11) in an indirect way using the characteristics. Precisely, for $(t, x) \in (0, \infty) \times \mathbb{R}^2$, $x = (x_1, x_2)$, we define $V^0(t, x)$ as follows:

$$V^0(t, x) = V_0(X^0(0; t, x)) \tag{6.1}$$

where the curve $X^0(\tau; t, x) : \tau \in \mathbb{R} \to X^0(\tau; t, x) \in \mathbb{R}^2$,

$$X^0(\tau; t, x) = (X_1^0(\tau; t, x), X_2^0(\tau; t, x)) \tag{6.2}$$

is the solution of the following ODE:

$$\begin{cases} \partial_\tau X^0 = \overline{a}(X^0) \\ X^0(t) = x. \end{cases} \tag{6.3}$$

We will see below that this solution is unique. For the sake of simplicity of notations, we will omit the dependence of X^0 on (t,x) and we will simply write

$$X^0(\tau;t,x) = X^0(\tau).$$

From (6.3) and (1.9), we can easily check that $X_2^0(\tau) = x_2 - t + \tau$, hence using (6.1), we get

$$V^0(t,x) = V_0\left(X_1^0(0), x_2 - t\right)$$

where X_1^0 satisfies:

$$\begin{cases} \partial_\tau X_1^0 = -\overline{f}\left(X_1^0, x_2 - t + \tau\right) \\ X_1^0(t) = x_1. \end{cases} \tag{6.4}$$

In order to show that $X_1^0(0)$ is uniquely defined, we solve (6.4) backwards, in other words, we let

$$\overline{X}_1^0(\tau) = X_1^0(t-\tau).$$

In this case:

$$V^0(t,x) = V_0(\overline{X}_1^0(t), x_2 - t), \tag{6.5}$$

where $\overline{X}_1^0$ satisfies:

$$\begin{cases} \partial_\tau \overline{X}_1^0 = \overline{f}\left(\overline{X}_1^0, x_2 - \tau\right) \\ \overline{X}_1^0(0) = x_1. \end{cases} \tag{6.6}$$

From Proposition 1.4, the solution $\overline{X}_1^0 \in C^1([0,\infty);\mathbb{R})$ is unique and hence $X_1^0(0) = \overline{X}_1^0(t)$ is uniquely determined. Consequently the function V^0 is well defined.

Step 2: V^0 is Lipschitz continuous.

From Step 1, we know that

$$V^0(t,x_1,x_2) = V_0(\overline{X}_1^0(t), x_2 - t) \tag{6.7}$$

with $\overline{X}_1^0$ given by (6.6) also depends on x_1 and x_2. Let the function $Y : \mathbb{R}^3 \to \mathbb{R}$ be defined as follows (with simplified notation showing the dependence on the variables (t,x_1,x_2)):

$$Y(t,x_1,x_2) := \overline{X}_1^0(t).$$

In order to show that V^0 is Lipschitz, it suffices (see (6.7)) to show that Y is Lipschitz. First, it is easily seen from (6.6) that Y is Lipschitz in time t. The Lipschitz continuity with respect to the variable x_1 directly follows from the monotonicity of $\overline{f}$ (see (1.4)), and the comparison principle. In order to show the Lipschitz continuity with respect to x_2, we first give a formal proof by assuming that $\overline{f}$ is smooth, and then we present the main idea that permit to make the proof rigorous. Suppose that

$$\overline{f} \in C^\infty(\mathbb{R}^2;\mathbb{R}) \quad \text{with} \quad |\overline{f}(y,s)| \leq \|\overline{f}\|_\infty.$$

Take

$$\widehat{Y} = \partial_{x_2} Y \quad \text{and} \quad \widetilde{Y} = \partial_t Y,$$

then the above two functions satisfy:

$$\begin{cases} \partial_t \widehat{Y} = (\partial_y \overline{f})\widehat{Y} + \partial_s \overline{f} \\ \widehat{Y}(0,x_1,x_2) = 0, \end{cases} \tag{6.8}$$

and

$$\begin{cases} \partial_t \widetilde{Y} = (\partial_y \overline{f})\widetilde{Y} - \partial_s \overline{f} \\ \widetilde{Y}(0,x_1,x_2) = \overline{f}(x_1,x_2), \end{cases} \tag{6.9}$$

respectively. Let $\overline{Y} = \widehat{Y} + \widetilde{Y}$, we get (from (6.8) and (6.9)):

$$\begin{cases} \partial_t \overline{Y} = (\partial_y \overline{f})\overline{Y} \\ \overline{Y}(0, x_1, x_2) = \overline{f}(x_1, x_2), \end{cases}$$

which gives, because of the monotonicity of $\overline{f}$ (see (1.4)), that the function $t \to |\overline{Y}(t, ., .)|$ is non-increasing. Hence

$$|\overline{Y}(t, x_1, x_2)| \leq |\overline{f}(x_1, x_2)| \leq \|\overline{f}\|_\infty. \tag{6.10}$$

Since $\widetilde{Y} = \partial_t Y$, we know from (6.6) that $|\widetilde{Y}| \leq \|\overline{f}\|_\infty$ where we finally obtain (see (6.10)):

$$|\widehat{Y}| = |\partial_{x_2} Y| \leq 2\|\overline{f}\|_\infty,$$

which shows that Y is Lipschitz continuous in the x_2 variable. In order to make the proof rigorous, it suffices to consider a regular approximation of the function $\overline{f}$ (as for example the convolution with a suitable mollifier sequence) and then to pass to the limit.

Step 3: V^0 is a viscosity solution of (1.11).

For the definition and the study of the theory of viscosity solutions, we refer the reader to [3]. Let us take $\overline{\phi}, \underline{\phi} \in C^1(\mathbb{R}^3; \mathbb{R})$ such that $V^0 - \overline{\phi}$ (resp. $V^0 - \underline{\phi}$) has a local maximum (resp. local minimum) at some point $(\overline{t}, \overline{x}) \in (0, \infty) \times \mathbb{R}^2$ (resp. $(\underline{t}, \underline{x}) \in (0, \infty) \times \mathbb{R}^2$), with $V^0(\overline{t}, \overline{x}) = \overline{\phi}(\overline{t}, \overline{x})$ and $V^0(\underline{t}, \underline{x}) = \underline{\phi}(\underline{t}, \underline{x})$. In order to show that V^0 is a viscosity solution of (1.11), we need to show the following two inequalities:

$$\partial_t \overline{\phi}(\overline{t}, \overline{x}) + (\overline{a} \cdot \nabla \overline{\phi})(\overline{t}, \overline{x}) \leq 0 \tag{6.11}$$

and

$$\partial_t \underline{\phi}(\underline{t}, \underline{x}) + (\overline{a} \cdot \nabla \underline{\phi})(\underline{t}, \underline{x}) \geq 0. \tag{6.12}$$

We only show inequality (6.11). In fact, inequality (6.12) can be proved in exactly the same way. For any $t \in [0, \overline{t}]$ define the function ϕ by

$$\phi : t \to \phi(t) = \overline{\phi}(t, X^0(t; \overline{t}, \overline{x})),$$

where X^0 is defined in (6.3). Let us show an inequality on ϕ in the interval $[\overline{t} - r, \overline{t}]$ for $r > 0$ small enough. Remark that $X^0(\overline{t}; \overline{t}, \overline{x}) = \overline{x}$. Hence, for $t \in [\overline{t} - r, \overline{t}]$, $(t, X^0(t; \overline{t}, \overline{x}))$ is close to $(\overline{t}, \overline{x})$ and therefore (since $V^0 - \overline{\phi}$ has a local maximum at $(\overline{t}, \overline{x})$ with $V^0(\overline{t}, \overline{x}) = \overline{\phi}(\overline{t}, \overline{x})$) we get:

$$\begin{aligned} \phi(t) &= \overline{\phi}(t, X^0(t; \overline{t}, \overline{x})) \geq V^0(t, X^0(t; \overline{t}, \overline{x})) = V_0(X^0(0; t, X^0(t; \overline{t}, \overline{x}))) \\ &= V_0(X^0(0; \overline{t}, \overline{x})) = V^0(\overline{t}, \overline{x}) = \overline{\phi}(\overline{t}, \overline{x}) = \overline{\phi}(\overline{t}, X^0(\overline{t}; \overline{t}, \overline{x})) = \phi(\overline{t}), \end{aligned}$$

where the passage from the first to the second line is due to the fact that the points $(t, X^0(t; \overline{t}, \overline{x}))$ and $(\overline{t}, \overline{x})$ are on the same characteristics. Finally, this implies

$$\partial_t \phi\big|_{t=\overline{t}} \leq 0,$$

which directly gives (6.11).

Step 4: Proof of the error estimate (1.13).

The solution V^ϵ of (1.10) can be written (in analogue with (6.5) and (6.6)) as

$$V^\epsilon(t, x) = V_0(\overline{X}_1^\epsilon(t), x_2 - t) \tag{6.13}$$

where the characteristics $\overline{X}_1^\epsilon$ satisfies:

$$\begin{cases} \partial_\tau \overline{X}_1^\epsilon = f\left(\dfrac{\overline{X}_1^\epsilon}{\epsilon}, \dfrac{x_2 - \tau}{\epsilon}, \overline{X}_1^\epsilon, x_2 - \tau\right) \\ \overline{X}_1^\epsilon(0) = x_1. \end{cases} \tag{6.14}$$

We apply Theorem 1.5, namely inequality (1.7), with u^ϵ and u^0 replaced by $\overline{X}_1^\epsilon$ and $\overline{X}_1^0$ respectivly, we obtain:

$$\left\|\overline{X}_1^\epsilon - \overline{X}_1^0\right\|_{L^\infty(0,T)} \leq \frac{cT}{|\log \epsilon|} \quad \text{for} \quad T \geq C\epsilon|\log\epsilon| \quad \text{with} \quad C>0, \epsilon>0. \tag{6.15}$$

Using (6.5), (6.13) and (6.15), we compute for $(t,x) \in (0,T)\times\mathbb{R}^2$:

$$\begin{aligned} |V^\epsilon(t,x) - V^0(t,x)| &= |V_0(\overline{X}_1^\epsilon(t), x_2 - t) - V_0(\overline{X}_1^0(t), x_2-t)| \\ &\leq Lip(V_0)|\overline{X}_1^\epsilon(t) - \overline{X}_1^0(t)| \\ &\leq Lip(V_0)\frac{cT}{|\log\epsilon|}, \end{aligned}$$

and inequality (1.13) directly follows. □

Proof of Theorem 1.9. Application of Proposition 2.1, and same proof as Theorem 1.8.

7 Appendix: proof of Examples 1.3 and 1.6

Proof of Example 1.3. The function $\overline{f}$ can be expressed as (see for instance [23]):

$$\overline{f}(u,t) = \left(\int_0^1 \frac{dv}{-u + |\sin 2\pi v|}\right)^{-1}.$$

Let $a = -u > 0$, it is easy to check that $\displaystyle\int_0^1 \frac{dv}{a+|\sin 2\pi v|} = 2\int_{|v|\leq 1/4}\frac{dv}{a+|\sin 2\pi v|}$. Take

$$I^a = \int_{|v|\leq 1/4}\frac{dv}{a+|\sin 2\pi v|} \quad \text{and} \quad I^{a,R} = \int_{|v|\leq Ra}\frac{dv}{a+2\pi|v|}.$$

We are interested in the limit $a\to 0$ and $R\to\infty$ with $Ra\to 0$. We compute

$$I^a - I^{a,R} = \overbrace{\int_{Ra\leq|v|\leq 1/4}\frac{dv}{a+|\sin 2\pi v|}}^{A} + \overbrace{\int_{|v|\leq Ra}\frac{2\pi|v| - |\sin 2\pi v|}{(a+2\pi|v|)(a+|\sin 2\pi v|)}dv}^{B},$$

where we have:

$$\begin{cases} B \to 0 \quad \text{as} \quad Ra\to 0 \\ A \leq \dfrac{1}{2}\dfrac{1}{\sin(2\pi Ra)} \sim \dfrac{c}{Ra} \sim c\sqrt{|\log a|}, \end{cases}$$

with $c = \frac{1}{4\pi}$, and we have chosen

$$R = \frac{1}{a\sqrt{|\log a|}}.$$

Now, let $\bar{v} = \frac{v}{a}$, we also compute:

$$I^{a,R} = 2\int_{0\leq\bar{v}\leq R}\frac{d\bar{v}}{1+2\pi\bar{v}} = \frac{1}{\pi}(\log(1+2\pi R)) \sim \frac{1}{\pi}\log R \sim \frac{1}{\pi}|\log a|.$$

Since $A \ll I^{a,R}$, this shows that $I^a \sim I^{a,R}$ and then

$$\overline{f}(u,t) \sim \frac{\pi}{2|\log|u||} \quad \text{as} \quad u\to 0^-,$$

which justifies (1.5). □

Proof of Example 1.6. Since $f(v,\tau,u,t) = g(v+\tau) - 1$, then the function v^ϵ defined by:

$$v^\epsilon(t) = u^\epsilon(t) + t$$

satisfies

$$\begin{cases} v_t^\epsilon = g\left(\dfrac{v^\epsilon}{\epsilon}\right), & t > 0, \\ v^\epsilon(0) = 0. \end{cases} \tag{7.1}$$

From the particular expression (1.8) of g with $g\left(\frac{1}{2}\right) = 0$, we can check that $0 \leq \frac{v^\epsilon(t)}{\epsilon} \leq \frac{1}{2}$ for every $t \geq 0$, which implies that

$$v^\epsilon \to 0 \quad \text{in} \quad L^\infty,$$

then

$$u^\epsilon \to u^0 \text{ in } L^\infty \quad \text{with} \quad u^0(t) = -t,$$

where

$$u_t^0 = \overline{f}(u^0, t) = -1.$$

Moreover, equation (7.1) can be written:

$$v_t^\epsilon = \frac{1}{2} - \frac{v^\epsilon}{\epsilon} \quad \text{with} \quad v^\epsilon(0) = 0,$$

therefore (solving the above equation) we get for $t = \delta\epsilon|\log\epsilon|$

$$\begin{aligned} u^\epsilon(t) - u^0(t) = v^\epsilon(t) &= \frac{\epsilon}{2}(1 - e^{-\frac{t}{\epsilon}}) \\ &= \frac{\epsilon}{2}(1 - e^{-\delta|\log\epsilon|}) \\ &= \frac{\epsilon}{2} - \frac{1}{2}\epsilon^{1+\delta} \sim \frac{\epsilon}{2} \sim \frac{t}{2\delta|\log\epsilon|}, \end{aligned}$$

which terminates the proof of Example 1.6. □

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