Towards a counter-example to a conjecture of De Giorgi in high dimensions ?

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1 Introduction

In this paper we consider entire solutions to semilinear elliptic equations. We show that solutions that are monotone in one direction are energy minimizers and we discuss existence and uniqueness. Our main result is that the existence of a symmetric global minimizer on $I\!R^{n-1}$ implies the existence of a counter-example to the conjecture of De Giorgi on $I\!R^n$. This is the analogue in the context of semilinear equations of the result

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of Bombieri, de Giorgi and Giusti [18] on existence of nonplanar minimal graphs in $I\!R^n$ for $n \geq 9$. Our result is, however, a partial one because it relies on the existence of a symmetric global minimizer, which has not yet been proved. To find such a global minimizer is analogous to the problem in the theory of minimal surfaces of finding the Simons cone.

1.1 Some analogies between minimal surfaces and a conjecture of De Giorgi

We recall De Giorgi's conjecture and the analogy between minimal surfaces and solutions of semilinear elliptic equations.

Conjecture 1.1 (De Giorgi [30])
Let us consider a solution $u \in C^2(I\!R^n)$ of

$$\Delta u = u^3 - u$$

such that

$$|u| \leq 1, \ \partial_n u > 0$$

in the whole $I\!R^n$. Is it true that, for every $\lambda \in I\!R$, the sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$?

In the table, we present some analogies between some well known facts on minimal surfaces (see [41]) and natural conjectures on solutions to semilinear elliptic partial differential equations.

The conjecture of De Giorgi $C5$ was proved by Ghoussoub and Gui [38] in dimension $n = 2$ and by Ambrosio and Cabré [4] in dimension $n = 3$. This conjecture is still open in dimensions $n \geq 4$.
Conjectures $C1, C7$ are true: $C1$ is true by the sliding method, $C7$ was proved by Barlow, Bass and Gui [10].
Conjecture $C2$ was first proved under some assumptions by Alberti, Ambrosio and Cabré [2] (see theorem 1.5). Conjecture $C2$ is also true under the assumptions of theorem 1.3 which is proved in the present document. On the contrary conjecture $C2$

is not true without any additional assumptions.

Conjecture $C4$ is still unknown. Nevertheless we prove in theorem 1.6 that under some simple assumptions $C4 \Longrightarrow C6$. For minimal surfaces, conjecture C6 and C4 correspond to the well known results proved by Bombieri, De Giorgi and Giusti [18].

N	Minimal Surfaces (Results)	Semilinear Elliptic PDE (Conjectures)
1	uniqueness of graph solutions (with graph $\subset \mathbb{R}^n$)	uniqueness of x_n- monotonic solutions (defined on cylinders $\omega \times (a,b) \subset \mathbb{R}^n$)
2	by definition minimize the energy	entire monotonic solutions are global minimizers
3	minimal surfaces in $\mathbb{R}^n$ are hyperplanes if $n \leq 7$	global minimizers on $\mathbb{R}^n$ are planar if $n \leq 7$
4	nonhyperplane minimal sets exist in $\mathbb{R}^n$ for $n \geq 8$ (Simons cone)	nonplanar global minimizers exist on $\mathbb{R}^n$ for $n \geq 8$
5	minimal graphs in $\mathbb{R}^n$ are hyperplanes if $n \leq 8$	monotonic solutions on $\mathbb{R}^n$ are planar if $n \leq 8$ (De Giorgi conjecture)
6	nonhyperplane minimal graphs exist in $\mathbb{R}^n$ for $n \geq 9$	nonplanar monotonic solutions exists on $\mathbb{R}^n$ for $n \geq 9$
7	Lipschitz graphs in $\mathbb{R}^n$ are hyperplanes in all dimensions	monotonic solutions on $\mathbb{R}^n$ with Lipschitz 0-level set are planar in all dimensions (Barlow Bass and Gui result)
8	flat enough minimal surface in B_R is a graph in $B_{\frac{R}{2}}$ independently on R	with flat enough 0-level set in B_R, every solution is a graph in $B_{\frac{R}{2}}$ independently on R

1.2 On the minimality of monotonic solutions

In this document we consider bounded solutions u to the following equation

$$\Delta u = F'(u) \text{ on } I\!R^n$$

where $F \in C^{1,1}(I\!R)$. In particular every such solution is a critical point of the energy

$$E_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u)$$

where Ω can be any bounded domain in $I\!R^n$. In order to extend the definition of minimizers of this energy to unbounded domains, we introduce the following definition

Definition 1.2 (Global minimizers on unbounded domains)
Let $\Omega_0 \subset I\!R^n$ be a domain (not necessary bounded). We say that a function u is a global minimizer on Ω_0 if for every smooth bounded domain $\Omega \subset\subset \Omega_0$ we have

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \leq \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

for every function $v \in C^{0,1}(\overline{\Omega})$ such that

$$v = u \text{ on } \partial\Omega$$

Then we have

Theorem 1.3 (Minimality of monotonic solutions)
Let a function $F \in C^{1,1}(I\!R)$ and u a bounded solution of

$$\Delta u = F'(u) \text{ in } I\!R^n \tag{1.1}$$

satisfying

$$\partial_n u > 0 \text{ in } I\!R^n \tag{1.2}$$

Let

$$\begin{cases} \overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n) \\ \underline{u}(x') = \lim_{x_n \to -\infty} u(x', x_n) \end{cases}$$

Then u is a global minimizer on $I\!R^n$ if and only if $\overline{u}$ and $\underline{u}$ are global minimizers on $I\!R^{n-1}$.

Proposition 1.4 (Global minimizers are unique local minimizers)
Let us assume that u is a global minimizer on $I\!R^n$. Then u is locally the unique minimizer of the energy under its own boundary conditions, this means that for every bounded domain $\Omega \subset I\!R^n$ and every function $v \in C^{0,1}(\overline{\Omega})$ we have

$$\left(\begin{array}{c} v = u \ \ on\ \partial\Omega \\ \\ \displaystyle\int_\Omega \frac{1}{2}|\nabla v|^2 + F(v) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \end{array}\right) \Longrightarrow (v = u \ \ on\ \Omega)$$

Let us recall the following result:

Theorem 1.5 (Alberti, Ambrosio, Cabré [2])
Let u be a bounded entire solution of (1.1) satisfying (1.2), and let $\Omega \subset I\!R^n$ be a smooth bounded domain. Then

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \leq \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

for every function $v \in C^{0,1}(\overline{\Omega})$ such that $v = u$ on $\partial\Omega$ and

$$\underline{u}(x') \leq v(x', x_n) \leq \overline{u}(x') \quad \text{ for all } x = (x', x_n) \in \Omega$$

Theorem 1.5 was proved using results on calibrations for scalar functionals of the Calculus of Variations. We also give here a different proof of this result based on well known maximum principle results.

1.3 Existence of a symmetric global minimizer in $I\!R^{n-1}$ implies the existence of a counter-example to the conjecture of De Giorgi in $I\!R^n$

We are now interested in the way to build a "counter-example"to the conjecture of De Giorgi in dimension $n \geq 9$. In this direction we have

Theorem 1.6 (A strategy to build a counter-example)
For the particular nonlinearity $F(v) = \frac{1}{4}(v^2-1)^2$, let us assume the existence of a

global minimizer v in $I\!R^{n-1}$ (in the sense of definition 1.2) which satisfies $|v| < 1$ and the following symmetries:

$$v(x_1, ..., -x_i, ..., x_{n-1}) = v(x_1, ..., x_i, ..., x_{n-1}) \quad \text{for } i = 1, ..., n-1 \tag{1.3}$$

Then there exists a function $u \in C^2(I\!R^n)$ solution of

$$\Delta u = u^3 - u \text{ on } I\!R^n$$

with

$$|u| \leq 1 \quad \text{and} \quad \partial_n u > 0$$

in the whole $I\!R^9$, such that for one $\lambda \in I\!R$, the set $\{u = \lambda\}$ is not an hyperplane.

We do not know if this solution satisfies $\lim_{x_n \to +\infty} u(x', x_n) = 1$ and $\lim_{x_n \to -\infty} u(x', x_n) = -1$. We do not know neither if such a symmetric global minimizer v as assumed in theorem 1.6 exists. We expect that we can find such a minimizer in dimension $n = 8$, only depending on $|X|$, $|Y|$ with $X = (x_1, ..., x_4)$, $Y = (x_5, ..., x_8)$.
We can now state a generalization of theorem 1.6 for more general nonlinearities F.

Theorem 1.7 *Let us consider a function $F \in C^{1,1}(I\!R)$ such that*

$$\begin{cases} F \geq 0 & \text{on} \quad I\!R \\ F > 0 & \text{on} \quad (-1, 1) \quad \text{and} \quad F(-1) = F(1) = 0 \end{cases}$$

Let us assume the existence of a global minimizer v on $I\!R^{n-1}$ (in the sense of definition 1.2) which satisfies $|v| < 1$ and the following symmetries:

$$v(x_1, ..., -x_i, ..., x_{n-1}) = v(x_1, ..., x_i, ..., x_{n-1}) \quad \text{for } i = 1, ..., n-1 \tag{1.4}$$

Then for each $\gamma \in (0, \sqrt{2F(v(0))})$ there exists a function $u \in C^2(I\!R^n)$ solution to

$$\Delta u = F'(u) \text{ on } I\!R^n$$

such that

$$|u| \leq 1 \text{ and } \partial_n u > 0$$

in the whole $\mathbb{R}^n$*, and for one* $\lambda \in \mathbb{R}$*, the set* $\{u = \lambda\}$ *is not an hyperplane. Moreover this solution* u *is a global minimizer on* $\mathbb{R}^n$*, satisfies*

$$\partial_n u(0) = \gamma \quad \textit{and} \quad u(0) = v(0)$$

and the symmetries:

$$u(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u(x_1, ..., x_i, ..., x_{n-1}, x_n) \quad \textit{for } i = 1, ..., n-1 \tag{1.5}$$

Remark 1.8 *Looking at the proof of theorem 1.7, we can see that it gives the existence of a compact connected set of global minimizers on* $\mathbb{R}^n$ *which connects* $u_0(x_n)$ *to* $v(x')$ *on* $\mathbb{R}^n$*.*

Remark 1.9 *The same result holds if we change the symmetries (1.4) by other symmetries which imply* $\nabla v(0) = 0$*.*

Remark 1.10 *Looking at the proof, we can only consider functions* $F \in C^{1,1}([-1, 1])$ *such that*

$$F > 0 \quad \textit{on} \quad (-1, 1) \quad \textit{and} \quad F(-1) = F(1) = F'(-1) = F'(1) = 0$$

Then local perturbations of global minimizers have to be restricted to functions w *which satisfy* $|w| \leq 1$*.*

Using the proofs of theorems 1.6-1.7 we can more generally state the following result

Proposition 1.11 *If* $v_1 \leq v_2$ *are two global minimizers on* $\mathbb{R}^n$*, then either*
i) there exists a global minimizer u *solution to*

$$\Delta u = F'(u) \textit{ on } \mathbb{R}^n$$

with

$$\partial_n u > 0$$

and

$$v_1 < u < v_2$$

or
ii) for each $\lambda \in (v_1(0), v_2(0))$*, there exists a global minimizer* v_λ *on* $\mathbb{R}^{n-1}$ *such that* $v_\lambda(0) = \lambda$ *and*

$$v_1 \leq v_\lambda < v_{\lambda'} \leq v_2 \quad \textit{if } \lambda < \lambda'$$

2 Proof of theorem 1.5

Let us consider a bounded solution u of

$$\Delta u = F'(u) \text{ on } I\!R^n \tag{2.1}$$

satisfying

$$\partial_n u > 0 \text{ on } I\!R^n \tag{2.2}$$

Assume that $F \in C^2(I\!R)$. Then

$$\begin{cases} \overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n) \\ \underline{u}(x') = \lim_{x_n \to -\infty} u(x', x_n) \end{cases} \tag{2.3}$$

are bounded solutions of (2.1) on $I\!R^{n-1}$ such that

$$\underline{u} < \overline{u}$$

Then we have the following result (whose corollary is theorem 1.5)

Proposition 2.1 *If the function u satisfies the above assumptions, then u is locally the unique solution of equation (2.1) in the set of functions v satisfying $\underline{u} \le v \le \overline{u}$. This means that for every smooth bounded domain $\Omega \subset I\!R^n$, if we consider a solution $v \in C^{0,1}(\overline{\Omega})$ of*

$$\begin{cases} \Delta v = F'(v) \ \text{ on } \Omega \\ v = u \ \text{ on } \partial\Omega \end{cases}$$

such that

$$\underline{u} \le v \le \overline{u} \tag{2.4}$$

then

$$v = u \ \text{ on } \Omega$$

We first need the following two lemmata. In particular in the first lemma we recall (without proofs) some well known results in the spirit of the Maximum Principle ([15]). The second lemma will be proved later.

Lemma 2.2 (Maximum Principle Tools)
Let a domain $\Omega \subset I\!R^n$, and a function $w \in C^0(\overline{\Omega})$ satisfying

$$\begin{cases} \Delta w - c(x)w \leq 0 \ \ on\ \Omega \\ w \geq 0 \ \ on\ \Omega \end{cases}$$

where c is a bounded function. Then we have
i) **Strong Maximum Principle:**

$$either\ w > 0\ \ or\ w \equiv 0\ \ on\ \Omega$$

ii) **Harnack Inequality:**
Let a ball $B_R \subset \Omega$. Then there exists a constant $C = C(|c|_{L^\infty(\Omega)}, R)$ such that

$$\sup_{B_{\frac{R}{2}}} w \leq C \inf_{B_{\frac{R}{2}}} w$$

iii) **Hopf Lemma:**
Let $B_r(X_1) \subset \Omega$ with $X_0 \in \partial B_r(X_1)\ \cap\ \partial\Omega$ and

$$w(X_0) = 0$$

Then

$$either \quad w \equiv 0 \quad or \quad \liminf_{\rho \to 0^+}\ \frac{1}{\rho}\, w\left(X_0 + \rho \frac{X_1 - X_0}{|X_1 - X_0|}\right) > 0$$

iv) **Maximum Principle in "narrow domains":**
Let us assume that diam $\Omega \leq d_0$. *There exists $\delta > 0$ depending only on n, d_0 and $|c|_{L^\infty(\Omega)}$ such that if*

$$meas\ \Omega = |\Omega| < \delta$$

then for every $v \in C^0(\overline{\Omega})$

$$\left(\begin{array}{l} \Delta v - c(x)v \leq 0 \ \ on\ \Omega \\ v \geq 0 \ \ on\ \partial\Omega \end{array} \right) \Longrightarrow v \geq 0 \ \ on\ \Omega$$

Lemma 2.3 *Let a ball $B'_R \subset I\!R^{n-1}$. Then*

$$\lim_{x_n \to +\infty} \sup_{x' \in B'_R} \left(\overline{u}(x') - u(x', x_n)\right) = 0 \tag{2.5}$$

$$\lim_{x_n \to -\infty} \sup_{x' \in B'_R} \left(u(x', x_n) - \underline{u}(x')\right) = 0 \tag{2.6}$$

Proof of lemma 2.3. Let us prove (2.5). The function $w(x) = \overline{u}(x') - u(x', x_n)$ satisfies

$$\begin{cases} \Delta w = c(x)w \text{ on } I\!R^n \\ w \geq 0 \text{ on } I\!R^n \end{cases}$$

where $c(x)$ is bounded as in the proof of proposition 2.1, Step 1. From Harnack inequality (lemma 2.2 ii)) we get by a simple recovering argument that for $B'_R \subset I\!R^{n-1}$

$$\sup_{B'_R \times \{x_n\}} w \leq C_R \inf_{B'_R \times [x_n - R, x_n + R]} w$$

Now we know that for each $x'_0 \in I\!R^{n-1}$, $\lim_{x_n \to +\infty} w(x'_0, x_n) = 0$. Chosing one point $x'_0 \in B'_R$, we deduce (2.5)

$$\lim_{x_n \to +\infty} \sup_{x' \in B'_R} (\overline{u}(x') - u(x', x_n)) = 0$$

The proof of (2.6) is similar. This proves the lemma.

Proof of proposition 2.1. Let us assume that there exists a smooth bounded domain $\Omega \subset I\!R^n$ and a function $v \in C^{0,1}(\overline{\Omega})$ such that

$$\begin{cases} \Delta v = F'(v) \text{ on } \Omega \\ v = u \text{ on } \partial\Omega \end{cases}$$

with

$$\underline{u} \leq v \leq \overline{u} \text{ on } \Omega$$

We will prove in three steps that

$$v \equiv u \text{ on } \Omega$$

Step 1. Construction of a global subsolution.
By the sub and supersolution method, let us build a function $\overline{v}$ as the minimal supersolution above v and u on Ω. In particular we get by construction

$$\begin{cases} \Delta \overline{v} = F'(\overline{v}) \text{ on } \Omega \\ \max(u, v) \leq \overline{v} \leq \overline{u} \text{ on } \Omega \\ \overline{v} = u \text{ on } \partial\Omega \end{cases}$$

Let us assume that for some interior point $X_0 \in \Omega$

$$v(X_0) > u(X_0)$$

Then let us define

$$w = \overline{v} - u$$

which satisfies

$$\begin{cases} \Delta w = c(x)w \text{ on } \Omega \\ w \geq 0 \text{ on } \Omega \\ w(X_0) > 0 \end{cases}$$

where

$$c(x) = \begin{cases} \dfrac{F'(\overline{v}(x')) - F'(u(x))}{\overline{v}(x') - u(x)} & \text{if} \quad \overline{v}(x') \neq u(x) \\ 0 & \text{if} \quad \overline{v}(x') = u(x) \end{cases}$$

Because $F \in C^2(I\!R)$ and $\overline{v}, u$ are bounded, the function c is then bounded and the Strong Maximum Principle (lemma 2.2 i)) implies that $w > 0$ on Ω, i.e.

$$\overline{v} > u \text{ on } \Omega$$

Similarly we can prove that

$$\overline{u} > \overline{v} \text{ on } \Omega$$

Moreover Hopf lemma (lemma 2.2 iii)) implies that

$$\frac{\partial}{\partial n}(\overline{v} - u) < 0 \text{ on } \partial\Omega$$

and then the function

$$\tilde{v}(x) = \begin{cases} \overline{v} \text{ on } \Omega \\ u \text{ on } I\!R^n \backslash \Omega \end{cases}$$

is a subsolution on $I\!R^n$, i.e. satisfies

$$\Delta \tilde{v} \geq F'(\tilde{v}) \text{ on } I\!R^n$$

Step 2. Sliding method.
Let us consider the function

$$u^t(x) = u(x', x_n + t)$$

By lemma 2.3 we know that $u(x', x_n) \to \overline{u}(x', x_n)$ as $x_n \to +\infty$ uniformly for x' in a compact set of $I\!R^{n-1}$. On the other hand by construction of $\tilde{v}$ we have

$$\overline{u} > \tilde{v} \text{ on } \Omega$$

Because Ω is bounded we deduce that for some $t_0 > 0$

$$u^t > \tilde{v} \text{ on } \Omega \text{ for } t \geq t_0$$

Because $\partial_n u > 0$ we get

$$u^t > \tilde{v} \text{ on } I\!R^n \backslash \Omega \text{ for } t \geq t_0$$

Let

$$t^* = \inf\{t \geq 0,\ \forall s \geq t,\ u^s \geq \tilde{v} \text{ on } I\!R^n\}$$

Then

$$u^{t^*} \geq \tilde{v} \text{ on } I\!R^n$$

If $t^* > 0$, then

$$u^{t^*} > \tilde{v} \text{ on } I\!R^n \backslash \Omega$$

and we deduce the existence of an interior point $X_0 \in \Omega$ such that

$$u^{t^*}(X_0) = \tilde{v}(X_0)$$

Using the strong maximum principle (lemma 2.2 i)) as in step 1 we conclude that

$$u^{t^*} \equiv \tilde{v} \text{ on } \Omega$$

And then $u^{t^*} = u$ on $\partial\Omega$. Contradiction because $\partial_n u > 0$.
Therefore $t^* = 0$ and $\tilde{v} = u$, i.e. $\overline{v} = u$ on Ω, and then

$$v \leq u \text{ on } \Omega$$

Step 3. Similarly we can repeat step 1 and 2 with a global supersolution. We then conclude that

$$v \geq u \text{ on } \Omega$$

Then $u = v$ on Ω which proves the proposition.

3 Proof of theorem 1.3 and proposition 1.4

To prove proposition 1.4, we first need the following result

Lemma 3.1 *Let a bounded domain* $\Omega \subset I\!\!R^n$ *such that two functions* $u, v \in C^{0,1}(\overline{\Omega})$ *are minimizers of the same energy*

$$E_\Omega(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 + F(w)$$

with the same boundary condition

$$u = v \ \text{on} \ \partial\Omega$$

Then either $v < u$ *or* $v > u$ *or* $v \equiv u$ *on* Ω.

Proof of lemma 3.1.
Step 1. Let us assume that $\Omega_- := \{v < u\} \neq \emptyset$. We claim that

$$E_{\Omega_-}(v) = E_{\Omega_-}(u)$$

If not, let us assume for example that

$$E_{\Omega_-}(v) < E_{\Omega_-}(u)$$

Now define a new candidate as a minimizer

$$v_- = \begin{cases} v \text{ on } \Omega_- \\ u \text{ on } \Omega \backslash \Omega_- \end{cases}$$

Then we have

$$E_\Omega(v_-) \leq E_{\Omega_-}(v) + E_{\Omega\backslash\Omega_-}(u) < E_\Omega(u)$$

which gives a contradiction with the minimality of u.
Assuming that

$$E_{\Omega_-}(v) > E_{\Omega_-}(u)$$

we get a similar contradiction considering

$$v_+ = \begin{cases} u \text{ on } \Omega_- \\ v \text{ on } \Omega \backslash \Omega_- \end{cases}$$

Step 2. We then have

$$\begin{cases} E_\Omega(v_-) = E_\Omega(u) \\ v_- = u \text{ on } \partial\Omega \\ v_- \leq u \text{ on } \Omega \end{cases}$$

We then conclude from the strong maximum principle (lemma 2.2 i)) that either $v_- \equiv u$ on Ω which is impossible by construction, or $v_- < u$ on Ω which means

$$v < u \text{ on } \Omega$$

Step 3. If we assume that $\Omega_+ := \{v > u\} \neq \emptyset$ we prove similarly that

$$v > u \text{ on } \Omega$$

This ends the proof of the lemma.

Proof of proposition 1.4. Let us assume that u is a global minimizer on $\mathbb{R}^n$, but locally not unique, i.e. there exist a bounded domain $\Omega \subset \mathbb{R}^n$ and a function $v \in C^{0,1}(\overline{\Omega})$, $v \not\equiv u$ such that

$$v = u \text{ on } \partial\Omega$$

and

$$E_\Omega(v) = E_\Omega(u)$$

We will prove a contradiction in two steps.
Step 1. By lemma 3.1, let us assume that $v < u$ on Ω (the case $v > u$ can be treated similarly). From Hopf lemma (lemma 2.2 iii)), we deduce that the function

$$w = u - v$$

satisfies

$$\liminf_{\rho \to 0^+} \; \frac{1}{\rho} \, w\left(X_0 + \rho \frac{X_1 - X_0}{|X_1 - X_0|}\right) > 0 \tag{3.1}$$

for $B_r(X_1) \subset \Omega$ with $X_0 \in \partial B_r(X_1) \;\cap\; \partial\Omega$.
Step 2. Let us consider a ball B_R which strictly contains Ω:

$$B_R \supset\supset \Omega$$

Then the extension $\tilde{v}$ of v to B_R defined by

$$\tilde{v} = \begin{cases} v \text{ on } \Omega \\ u \text{ on } B_R \backslash \Omega \end{cases}$$

satisfies

$$E_\Omega(\tilde{v}) = E_\Omega(u)$$

and then is a minimizer on B_R and satisfies the Euler-Lagrange equation

$$\Delta \tilde{v} = F'(\tilde{v}) \text{ on } B_R$$

By the standard elliptic estimates, we duduce that $\tilde{v} \in C^2(B_R)$ which is in contradiction with (3.1) and the fact that $u \in C^2(B_R)$.

To prove theorem 1.3, we first need the following lemma which will be proved later:

Lemma 3.2 *Let a domain $\Omega_0 \subset I\!\!R^{n-1}$ (not necessary bounded). Let a function w defined on Ω_0. Then w is a global minimizer on $\Omega_0 \times I\!\!R$ if and only if w is a global minimizer on Ω_0.*

Proof of lemma 3.2.
i) w global minimizer on $\Omega_0 \Longrightarrow w$ global minimizer on $\Omega_0 \times I\!\!R$.
Let us consider a smooth bounded domain $\Omega \subset \Omega_0 \times I\!\!R$, and let us assume that w is not a minimizer of the energy on Ω, i.e. there exists a function $v \in C^{0,1}(\overline{\Omega})$, $v \not\equiv w$ such that

$$v(x', x_n) = w(x') \text{ on } \partial\Omega$$

and

$$E_\Omega(v) < E_\Omega(w) := \int_\Omega \frac{1}{2}|\nabla w|^2 + F(w) \tag{3.2}$$

Then we consider a cylindrical open set $\Omega_R := \Omega'_R \times (-R, R)$ which contains Ω:

$$\Omega_R \supset\supset \Omega$$

where $\Omega'_R \subset\subset \Omega_0$ is a smooth bounded domain. We define the extension v_R of the function v to Ω_R by

$$v_R(x) = \begin{cases} v \text{ on } \Omega \\ w \text{ on } \Omega_R \backslash \Omega \end{cases}$$

In particular v_R satisfies

$$v_R(x', x_n) = w(x') \text{ on } \partial\Omega_R$$

As a consequence of (3.2) we get

$$E_{\Omega_R}(v_R) \leq E_{\Omega_R \setminus \Omega}(w) + E_\Omega(v) < E_{\Omega_R}(w)$$

i.e.

$$\int_{-R}^{R} dx_n \int_{\Omega'_R} \frac{1}{2}|\nabla v_R|^2 + F(v_R) < \int_{-R}^{R} dx_n \int_{\Omega'_R} \frac{1}{2}|\nabla w|^2 + F(w) \tag{3.3}$$

This implies that at least one section $v_{R,t}$ for some $t \in (-R, R)$

$$v_{R,t}(x') := v_R(x', x_n = t)$$

satisfies

$$E_{\Omega'_R}(v_{R,t}) < E_{\Omega'_R}(w)$$

Moreover

$$v_{R,t} = w \text{ on } \partial\Omega'_R$$

which by the minimality of w on $\Omega'_R \subset I\!R^{n-1}$ gives a contradiction.
ii) w global minimizer on $\Omega_0 \times I\!R \Longrightarrow w$ global minimizer on Ω_0 .
Let us assume that w is not a global minimizer on Ω_0, i.e. there exists a smooth bounded domain $\Omega' \subset\subset \Omega_0$ and a function $v \in C^{0,1}(\overline{\Omega'})$ such that

$$E_{\Omega'}(v) < E_{\Omega'}(w) \quad \text{and} \quad v = w \quad \text{on} \quad \partial\Omega'$$

Then let $\psi \in \mathcal{D}(I\!R)$ such that

$$\begin{cases} \psi = 1 & \text{on} \quad (-t, t) \\ \psi = 0 & \text{on} \quad I\!R \setminus (-t-1, t+1) \end{cases}$$

We then define

$$u(x', x_n) = \psi(x_n)v(x') + (1 - \psi(x_n))w(x')$$

and let $\Omega \subset \Omega' \times I\!R$ be any smooth bounded domain such that

$$\Omega \supset \Omega' \times (-t-1, t+1)$$

Because we can find a constant C such that

$$\left|E_{\Omega'\times((-t-1,-t)\cup(t,t+1))}(w) - E_{\Omega'\times((-t-1,-t)\cup(t,t+1))}(u)\right| \leq C$$

we get

$$E_\Omega(w) - E_\Omega(u) \geq -C + 2t(E_{\Omega'}(w) - E_{\Omega'}(v)) > 0$$

which is striclty positive for t large enough. Because $u = w$ on $\partial\Omega$, we then get a contradiction with the fact that w is a minimizer on $\Omega \subset \Omega_0 \times I\!R$.
This ends the proof of the lemma.

Proof of theorem 1.3.
i) $\overline{u}, \underline{u}$ *global minimizers* $\Longrightarrow$ u *global minimizer.*
Let us consider a smooth bounded domain $\Omega \subset I\!R^n$ and a function v which minimizes the energy

$$E_\Omega(v) := \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

among all functions in $C^{0,1}(\overline{\Omega})$ with

$$v = u \text{ on } \partial\Omega$$

It is easy to see that such a function v exists. We will prove that

$$v \equiv u \text{ on } \Omega$$

To this end let us introduce

$$\Omega_+ = \{x \in \Omega,\ v > \overline{u}\}$$

$$\Omega_- = \{x \in \Omega,\ v < \underline{u}\}$$

Because we have

$$\underline{u} < u < \overline{u}$$

we get

$$\Omega_+ \subset\subset \Omega$$

$$\Omega_- \subset\subset \Omega$$

and a natural candidate to minimize the energy is then

$$\tilde{v} = \begin{cases} \underline{u} \text{ on } \Omega_- \\ v \text{ on } \Omega \backslash (\Omega_- \cup \Omega_+) \\ \overline{u} \text{ on } \Omega_+ \end{cases}$$

Applying lemma 3.2 to the function $\overline{u}$ (resp. $\underline{u}$), we get that $\overline{u}$ (resp. $\underline{u}$) is a global minimizer on $\mathbb{R}^n$. From proposition 1.4, we deduce that $\overline{u}$ (resp. $\underline{u}$) is the unique minimizer of the energy under its own boundary conditions on Ω_+ (resp. on Ω_-), and then

$$E_\Omega(v) > E_\Omega(\tilde{v}) \quad \text{if } \Omega_- \neq \emptyset \text{ or } \Omega_+ \neq \emptyset$$

By the minimality of v we conclude that $\Omega_- = \Omega_+ = \emptyset$ and then

$$\underline{u} \leq v \leq \overline{u}$$

We then apply proposition 2.1 which proves that $v = u$ on Ω.
ii) u global minimizer $\Longrightarrow \overline{u}, \underline{u}$ global minimizers.
By definition of $\overline{u}, \underline{u}$ it is clear that there are global minimizers on $\mathbb{R}^n$. Then lemma 3.2 implies that $\overline{u}, \underline{u}$ are also global minimizers on $\mathbb{R}^{n-1}$.
This ends the proof of the theorem.

4 Proof of theorem 1.7

To prove theorem 1.7 we will need many lemmata. We start with the following

Lemma 4.1 *Let u and v be two global minimizers on $\Sigma = B'_R \times \mathbb{R}$ such that*

$$u \geq v \quad on \quad \partial\Sigma$$

satisfying moreover

$$\lim_{\lambda \to +\infty} \sup_{\substack{x' \in B'_R \\ |x_n| \geq \lambda}} \left(|u - v| + |\nabla (u - v)| \right) = 0$$

Then either $u < v$ or $u > v$ or $u = v$.

Proof of lemma 4.1.
For $h > 0$ let us introduce

$$\Sigma_h := \Sigma \cap \{|x_n| < h\}$$

And let us define

$$\tilde{u} = \begin{cases} u & \text{on} \quad \{u \geq v\} \\ v & \text{on} \quad \{u < v\} \end{cases}$$

Step 1
Let us consider

$$\liminf_{h \to +\infty} \quad (E_{\Sigma_h}(\tilde{u}) - E_{\Sigma_h}(u)) = \underline{a} \in \overline{I\!R}$$

Let us assume that $\underline{a} < 0$. Then there exists a constant $C > 0$ and a sequence $\{h_k\}_{k \in \mathbf{N}}$ such that $h_k \to +\infty$ and

$$E_{\Sigma_{h_k}}(\tilde{u}) - E_{\Sigma_{h_k}}(u) \leq -C < 0$$

Now let $\psi_k \in \mathcal{D}(I\!R)$ such that

$$\begin{cases} \psi = 1 & \text{on} \quad (-h_k, h_k) \\ \psi = 0 & \text{on} \quad I\!R \backslash (-h_k - 1, h_k + 1) \end{cases}$$

And let

$$\tilde{u}_k(x) = \psi_k(x_n)\tilde{u}(x) + (1 - \psi_k(x_n))u(x)$$

Then we can find a smooth bounded domain $\Omega_k \subset \Sigma$ such that

$$\Omega_k \supset\supset \Sigma_{h_k+1}$$

Then

$$E_{\Omega_k}(\tilde{u}_k) - E_{\Omega_k}(u) = E_{\Sigma_{h_k+1}}(\tilde{u}_k) - E_{\Sigma_{h_k+1}}(u) \leq c_k - C$$

where c_k is a constant such that

$$\left| E_{\Sigma_{h_k+1} \backslash \Sigma_{h_k}}(\tilde{u}_k) - E_{\Sigma_{h_k+1} \backslash \Sigma_{h_k}}(u) \right| \leq c_k$$

But because $|u - v| + |\nabla (u - v)| \to 0$ as $|x_n|$ tends to $+\infty$, we get

$$c_k \to 0$$

and then

$$E_{\Omega_k}(\tilde{u}_k) - E_{\Omega_k}(u) < -C/2$$

for k large enough. Because $\tilde{u}_k = u$ on $\partial\Omega_k$, we get a contradiction with the minimality of u. Therefore

$$\underline{a} \geq 0$$

Step 2
We now consider

$$\limsup_{h\to+\infty} \quad (E_{\Sigma_h}(\tilde{u}) - E_{\Sigma_h}(u)) = \overline{a} \in \overline{I\!R}$$

We simply remark that

$$E_{\Sigma_h}(\tilde{u}) - E_{\Sigma_h}(u)) = -(E_{\Sigma_h}(\tilde{v}) - E_{\Sigma_h}(v))$$

where

$$\tilde{v} = \begin{cases} v & \text{on} \quad \{u \geq v\} \\ u & \text{on} \quad \{u < v\} \end{cases}$$

We can then apply step 1 to $\tilde{v}$ and v and we get

$$\overline{a} \leq 0$$

We then have

$$\lim_{h\to+\infty} \quad (E_{\Sigma_h}(\tilde{u}) - E_{\Sigma_h}(u)) = 0 \tag{4.1}$$

Step 3
Let $\phi \in \mathcal{D}(\Sigma)$ and $h_0 > 0$ such that

$$\mathrm{supp}\,(\phi) \subset \Sigma_{h_0} \tag{4.2}$$

Then for every $h \geq h_0$:

$$\begin{array}{rcl} E_{\Sigma_{h_0}}(\tilde{u}+\phi) - E_{\Sigma_{h_0}}(\tilde{u}) & = & E_{\Sigma_h}(\tilde{u}+\phi) - E_{\Sigma_h}(\tilde{u}) \\ & = & (E_{\Sigma_h}(\tilde{u}+\phi) - E_{\Sigma_h}(u)) + ((E_{\Sigma_h}(\tilde{u}) - E_{\Sigma_h}(u)) \end{array}$$

As in step 1 we deduce that

$$\liminf_{h\to+\infty}(E_{\Sigma_h}(\tilde{u}+\phi) - E_{\Sigma_h}(u)) \geq 0$$

and then by (4.1) we have

$$E_{\Sigma_{h_0}}(\tilde{u}+\phi) - E_{\Sigma_{h_0}}(\tilde{u}) \geq 0$$

Now because it is true for every $\phi \in \mathcal{D}(\Sigma)$ with h_0 satisfying (4.2), we deduce that $\tilde{u}$ has to statisfy the Euler-Lagrange equation:

$$\Delta\tilde{u} = F'(\tilde{u}) \quad \text{on} \quad \Sigma$$

Because $\tilde{u} \geq u$ we deduce from the strong maximum principle (lemma 2.2 i)) that either $\tilde{u} > u$ or $\tilde{u} = u$, i.e.

$$v > u \quad \text{or} \quad v \leq u$$

Similarly we get that

$$u > v \quad \text{or} \quad u \leq v$$

This proves the lemma.

Lemma 4.2 *Let us consider global minimizers u on $\Sigma = B'_R \times I\!\!R$ solutions to the following equations:*

$$\left\{ \begin{array}{l} \Delta u = F'(u) \ \textit{on}\ \Sigma \\ u = g \ \textit{on}\ \partial\Sigma \end{array} \right.$$

where $g \in C^1(\partial\Sigma)$ satisfies

$$\partial_n g > 0 \ \textit{on}\ \partial\Sigma_R$$

If we assume the following limits at infinity

$$\left\{ \begin{array}{l} \lim\limits_{x_n \to +\infty} \sup\limits_{x' \in B'_R} \left(|1 - u(x',x_n)| + |\nabla u(x',x_n)|\right) = 0 \\ \lim\limits_{x_n \to -\infty} \sup\limits_{x' \in B'_R} \left(|-1 - u(x',x_n)| + |\nabla u(x',x_n)|\right) = 0 \end{array} \right.$$

then these solutions u are unique.

Proof of lemma 4.2.

Let us assume that there exists two global minimizers u and v on $\Sigma = B'_R \times I\!\!R$ with $u = v = g$ on $\partial\Sigma$. For $t \in I\!\!R$ let us define

$$u^t(x',x_n) = u(x',x_n+t)$$

Because $\partial_n g > 0$ on $\partial\Sigma$, for $t > 0$ we have

$$u^t > v \quad \text{on} \quad \partial\Sigma$$

From lemma 4.1 we deduce that

$$u^t > v \quad \text{on} \quad \Sigma$$

and then in the limit $t \to 0$:

$$u \geq v \quad \text{on} \quad \Sigma$$

Similarly we get

$$v \geq u \quad \text{on} \quad \Sigma$$

which proves the uniqueness of the solution $u = v$. This ends the proof of the lemma.

Let us recall that, in theorem 1.7, we assume the existence of a function $v(x')$ on $\mathbb{R}^{n-1}$. We will need to connect on $\mathbb{R}^n$ the two particular solutions $u_0(x_n)$ and $v(x')$ by a continuous curve of solutions on $\mathbb{R}^n$. To this end we introduce the following functions for $t \in [0,1]$

$$U_t(x', x_n) := u_0\left(tu_0^{-1}(v(x')) + (1-t)x_n\right)$$

which satisfies in particular

$$\begin{cases} U_0(x', x_n) = u_0(x_n) \\ U_1(x', x_n) = v(x') \end{cases}$$

The functions U_t satisfy moreover

$$\partial_n U_t \geq 0$$

and the same symmetries as v, namely (1.5). For each $t \in [0,1]$, we first consider possible solutions u_t satisfying on $\Sigma = B'_R(0) \times \mathbb{R}$

$$\begin{cases} \Delta u_t = F'(u_t) \quad \text{on} \quad \Sigma \\ u_t = U_t \quad \text{on} \quad \partial\Sigma \\ \partial_n u_t \geq 0 \quad \text{on} \quad \Sigma \\ |u_t| \leq 1 \quad \text{on} \quad \Sigma \end{cases} \tag{4.3}$$

We have

Proposition 4.3 (Uniqueness)
For each $t \in [0,1]$, there is at most one solution u_t of (4.3) which is a global minimizer on Σ.

Proposition 4.4 (Existence)
For each $t \in [0,1]$, there exists at least one solution u_t of (4.3) which is a global minimizer on Σ.

Proof of proposition 4.3: (uniqueness)
To simplify the notations let us denote u_t by u. Let us introduce the limits at infinity:

$$\overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n)$$

$$\underline{u}(x') = \lim_{x_n \to -\infty} u(x', x_n)$$

From the monotonicity of u we deduce

$$\underline{u} \leq u \leq \overline{u} \text{ on } \Sigma \tag{4.4}$$

i) In the case $t = 1$, we have

$$u(x', x_n) = v(x') \text{ on } \partial\Sigma$$

In particular $\overline{u}$ satisfies

$$\begin{cases} \Delta\overline{u} = F'(\overline{u}) \text{ on } B'_R \\ \overline{u} = v \text{ on } \partial B'_R \end{cases} \tag{4.5}$$

From the minimality of v and from proposition 1.4 we deduce the uniqueness of the solution to (4.5), i.e.

$$\overline{u} \equiv v$$

Similarly

$$\underline{u} \equiv v$$

and from (4.4) we have

$$u(x', x_n) = v(x') \text{ on } \Sigma$$

ii)When $t \in [0,1)$, $\overline{u}$ satifies

$$\begin{cases} \Delta \overline{u} = F'(\overline{u}) \text{ on } B'_R \\ \overline{u} = 1 \text{ on } \partial B'_R \end{cases} \tag{4.6}$$

Using the fact that the constant 1 is a global minimizer on $I\!R$, we deduce from lemma 3.2 that the constant 1 is also a global minimizer on $I\!R^{n-1}$. We then conclude from proposition 1.4 to the uniqueness of the solution to (4.6), i.e.

$$\overline{u} \equiv 1$$

and then

$$\lim_{x_n \to +\infty} \sup_{x' \in B'_R} (|1 - u(x', x_n)| + |\nabla u(x', x_n)|) = 0$$

where the convergence on the gradient is a consequence of standard elliptic estimates and the smoothness of U_t on the boundary $\partial \Sigma$. Similarly

$$\lim_{x_n \to -\infty} \sup_{x' \in B'_R} (|-1 - u(x', x_n)| + |\nabla u(x', x_n)|) = 0$$

We now apply lemma 4.2 which proves the uniqueness of the solution u on Σ. This ends the proof of the proposition.

Proof of proposition 4.4: (existence)
We prove the existence in two steps. We first build a solution on a cylinder of finite lenght and then get a solution on an infinite cylinder taking a limit when the lenght of the cylinder tends to infinity.

Step 1 : Minimization on finite cylinders.
For $h > 1$, let us consider the bounded cylinder

$$\Sigma_h := B'_R \times (-h, h)$$

where $B'_R = \{x',\ |x'| < R\} \subset I\!R^{n-1}$ with $x' = (x_1, ..., x_{n-1})$. We then consider the minimization of the energy

$$E_{\Sigma_h}(w) = \int_{\Sigma_h} \frac{1}{2}|\nabla w|^2 + F(w)$$

among all functions $w \in H^1(\Sigma_h)$ with the boundary conditions

$$\begin{cases} w = 1 \text{ on } \{x_n = h\} \\ w = -1 \text{ on } \{x_n = -h\} \\ w = U_{t,h} \text{ on } \partial B'_R \times (-h,h) \end{cases} \tag{4.7}$$

where the function $U_{t,h}$ is an approximation of

$$U_t(x',x_n) := u_0\left(tu_0^{-1}\left(v(x')\right) + (1-t)x_n\right)$$

The disadvantage of the function U_t is that it does not satisfy the boundary conditions stated in the two first lines of (4.7). To avoid this difficulty we introduce the approximations $U_{t,h}$ of U_t which satisfies

$$U_{t,h} \in C^{0,1}(\overline{\Sigma_h}) \tag{4.8}$$

and has still the following properties

$$\begin{cases} \partial_n U_{t,h} \geq 0 \quad \text{on} \quad \Sigma_h \\ U_{t,h}(x_1,...,-x_i,...,x_{n-1},x_n) = U_{t,h}(x_1,...,x_i,...,x_{n-1},x_n) \quad \text{for} \quad i = 1,...,n-1 \\ U_{t,h} = U_t \quad \text{on} \quad \Sigma_h \cap \{-h+1 \leq x_n \leq h-1\} \end{cases}$$

We can for example define the function $U_{t,h}$ on $\overline{\Sigma_h}$ by

$$U_{t,h} = \begin{cases} s + (1-s)U_t(x', x_n = h-1) \text{ on } B'_R \times \{h-1+s\} \text{ for } s \in [0,1] \\ U_t \text{ on } B'_R \times (-h+1, h-1) \\ -s + (1-s)U_t(x', x_n = -h+1) \text{ on } B'_R \times \{-h+1-s\} \text{ for } s \in [0,1] \end{cases}$$

From the regularity property (4.8) of $U_{t,h}$ we deduce in particular that

$$E_{\Sigma_h}(U_{t,h}) < +\infty$$

This implies the existence of a minimizer $u_{t,h}$ of the energy, which in particular satisfies the Euler-Lagrange equation

$$\Delta u_{t,h} = F'(u_{t,h}) \text{ on } \Sigma_h$$

with the previous boundary conditions.
Moreover using a sliding method, we will prove (just below) the following result:

Lemma 4.5 *Under the previous assumptions, we have*

$$\partial_n u_{t,h} \geq 0 \text{ on } \Sigma_h$$

Step 2 : The limit on infinite cylinders.
The standard elliptic estimates allow us to take the limit $h \to +\infty$, and give the existence of a solution u_t to

$$\begin{cases} \Delta u_t = F'(u_t) \text{ on } \Sigma := B'_R \times I\!R \\ u_t = U_t \text{ on } \partial\Sigma \\ \partial_n u_t \geq 0 \text{ on } \Sigma \end{cases}$$

Moreover this solution u_t is a global minimizer on Σ. This ends the proof of the proposition.

Proof of lemma 4.5. We will prove this lemma using a sliding method. To simplify the notations let us denote $u_{t,h}$ by u and Σ_h by Σ. Let

$$u^t(x', x_n) = u(x', x_n + t) \quad \text{defined on} \quad \Sigma^t := B'_R \times (-h-t, h-t)$$

Let

$$w^t = u^t - u \quad \text{on} \quad \Sigma^t \cap \Sigma$$

which satisfies

$$\Delta w^t - c^t(x) w^t = 0 \quad \text{on} \quad \Sigma^t \cap \Sigma$$

for some bounded function c^t.

Step 1: Initialisation

We have $\Sigma^t \cap \Sigma \subset \{-h \leq x_n \leq -h + \delta_0\}$ for $t \in (2h - \delta_0, 2h)$. As a consequence for δ_0 small enough, the maximum principle in "narrow domains" (lemma 2.2 iv)) applies and proves that

$$\inf_{\Sigma^t \cap \Sigma} w^t \geq \inf_{\partial(\Sigma^t \cap \Sigma)} w^t \geq 0 \tag{4.9}$$

which is nonnegative because $\partial_n u \geq 0$ on the boundary of the cylinder and by construction $-1 \leq u \leq 1$. Then

$$w^t \geq 0 \quad \text{on} \quad \Sigma^t \cap \Sigma \quad \text{for} \quad t \in (2h - \delta_0, 2h)$$

Let

$$t^* = \inf\{t, \forall s \geq t, w^s \geq 0 \quad \text{on} \quad \Sigma^s \cap \Sigma\}$$

By definition

$$w^{t^*} \geq 0 \quad \text{on} \quad \Sigma^{t^*} \cap \Sigma$$

Step 2: Continuation
Let us assume that $t^* > 0$. We want to prove that we can then decrease the parameter t. On the one hand the strong maximum principle implies that

$$w^{t^*} > 0 \quad \text{on} \quad \Sigma^t \cap \Sigma$$

and then for every $\delta > 0$, on the compact set

$$K_\delta := \{x \in \Sigma^{t^*} \cap \Sigma, d(x, \partial(\Sigma^t \cap \Sigma)) \geq \delta\}$$

we have

$$w^{t^*} \geq \delta' > 0 \quad \text{on} \quad K_\delta$$

for some small enough δ' and then

$$w^{t^*-s} \geq \delta'/2 > 0 \quad \text{on} \quad K_\delta \quad \text{for} \quad s \in (0, s^*)$$

for some small enough s^*. Now for δ small enough, the complementar $\mathcal{C}_\delta$ of K_δ defined by $\mathcal{C}_\delta := (\Sigma^t \cap \Sigma) \backslash K_\delta$ has a measure sufficiently small to apply the maximum principle on it (lemma 2.2 iv)). This gives:

$$\inf_{\mathcal{C}_\delta} w^{t^*-s} \geq \inf_{\partial\mathcal{C}_\delta} w^{t^*-s} \geq 0$$

which is nonnegative for the same raisons as in (4.9). Then

$$w^{t^*-s} \geq 0 \quad \text{on} \quad \Sigma^{t^*-s} \cap \Sigma \quad \text{for} \quad s \in (0, s^*)$$

Contradiction with the definition of t^*.
Step 3: Conclusion
Then $t^* = 0$ and

$$u^t \geq u \quad \text{on} \quad \Sigma^t \cap \Sigma \quad \text{for} \quad t \in (0, 2h)$$

i.e.

$$\partial_n u \geq 0 \quad \text{on} \quad \Sigma$$

Coming back to the notation $u_{t,h}$ and Σ_h this proves the lemma.

To prove theorem 1.7, we need a last lemma

Lemma 4.6 *Let F be a $C^{1,1}(\mathbb{R})$ function such that*

$$\begin{cases} F \geq 0 & \textit{on} \quad \mathbb{R} \\ F > 0 & \textit{on} \quad (-1,1) \quad \textit{and} \quad F(-1) = F(1) = 0 \end{cases}$$

Then there exists a unique solution u_0 to

$$u_0'' = F'(u_0) \quad \textit{and} \quad u_0' > 0$$

with $u_0(-\infty) = -1$, $u_0(+\infty) = 1$, $u_0(0) = 0$.

Proof of lemma 4.6.
For $F \in C^2(\mathbb{R})$ this lemma is a consequence of lemma 3.2 in [4]. The proof of lemma 3.2 uses essentially the fact that

$$u_0'^2 = 2F(u_0)$$

It is easy to check that $F \in C^{1,1}(\mathbb{R})$ is sufficient to get the desired result.

Proof of theorem 1.7.
We assume the existence of a function $v(x')$ on $\mathbb{R}^{n-1}$ as defined in the theorem and we will build a function $u(x', x_n)$ on $\mathbb{R}^n$. The key idea is to connect on $\mathbb{R}^n$ the two particular solutions $u_0(x_n)$ and $v(x')$ by a continuous curve of solutions on $\mathbb{R}^n$. To this end we recall that we introduce the following functions for $t \in [0,1]$

$$U_t(x', x_n) := u_0\left(t u_0^{-1}(v(x')) + (1-t)x_n\right)$$

which satisfies in particular

$$\begin{cases} U_0(x', x_n) = u_0(x_n) \\ U_1(x', x_n) = v(x') \end{cases}$$

The functions U_t satisfy moreover

$$\partial_n U_t \geq 0$$

and the same symmetries as v, namely (1.5). We will build monotonic solutions on cylinders whose values on the boundary of the cylinder are U_t and then take the limit as the cylinders tend to the whole space $\mathbb{R}^n$.

From proposition 4.4 we know the existence of a global minimizer u_t on $\Sigma = B'_R(0) \times \mathbb{R}$ which satisfies

$$\begin{cases} \Delta u_t = F'(u_t) \quad \text{on} \quad \Sigma \\ u_t = U_t \quad \text{on} \quad \partial\Sigma \\ \partial_n u_t \geq 0 \quad \text{on} \quad \Sigma \\ |u_t| \leq 1 \quad \text{on} \quad \Sigma \end{cases}$$

Moreover by proposition 4.3, we know that such a function is unique. From this uniqueness, we deduce that u_t has the same symmetries as U_t, i.e.

$$u_t(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u_t(x_1, ..., x_i, ..., x_{n-1}, x_n) \quad \text{for} \quad i = 1, ..., n-1$$

Moreover, standard elliptic estimates imply the compactness of the set of functions $\{u_t\}_{t\in[0,1]}$ for the topology of convergence in C^2-norms on every compact sets of Σ.

For each $t \in [0,1]$ we define the real h_t by

$$u_t(0, ..., 0, h_t) = v(0)$$

Here from the strict monotonicity of u_t for each $t \in [0,1)$ we deduce the uniqueness of h_t. On the contrary h_t for $t = 1$ can be any real number because $u_t(x', x_n) = v(x')$ for $t = 1$. We will prove later the following

Lemma 4.7 *The map* $t \longmapsto \partial_n u_t(0, ..., 0, h_t)$ *is continuous for* $t \in [0,1]$

In particular we have

$$\begin{cases} \partial_n u_t(0, ..., 0, h_t) = u_0'(u_0^{-1}(v(0))) = \sqrt{2F(v(0))} & \text{for} \quad t = 0 \\ \partial_n u_t(0, ..., 0, h_t) = 0 & \text{for} \quad t = 1 \end{cases}$$

Now for every $\gamma \in \left(0, u_0'(u_0^{-1}(v(0)))\right)$, we get by continuity the existence of a $t_\gamma \in (0,1)$ such that

$$\partial_n u_{t_\gamma}(0, ..., 0, h_{t_\gamma}) = \gamma \quad \text{where} \quad u_{t_\gamma}(0, ..., 0, h_{t_\gamma}) = v(0)$$

We then redefine the solution with the origine in $(0,...,0,h_{t_\gamma})$ by

$$u_R^\gamma(x',x_n) := u_{t_\gamma}(x',x_n+h_{t_\gamma}) \quad \text{on} \quad B_R'(0)\times I\!R$$

We can now take the limit $R\to+\infty$, and up to extraction of a convergent subsequence $\{R'\}$ we get that

$$u_{R'}^\gamma \to u^\gamma \quad \text{uniformly locally on compact sets of} \quad I\!R^n$$

and the functions u^γ are still global minimizers and satisfy

$$\begin{cases} \Delta u^\gamma = F'(u^\gamma) \quad \text{on} \quad I\!R^n \\ \partial_n u^\gamma \geq 0 \quad \text{on} \quad I\!R^n \\ \partial_n u^\gamma(0) = \gamma \quad \text{and} \quad u^\gamma(0) = v(0) \end{cases}$$

The fact that u^γ is not a planar solution comes from the fact u^γ inherits the symmetries of v:

$$u^\gamma(x_1,...,-x_i,...,x_{n-1},x_n) = u^\gamma(x_1,...,x_i,...,x_{n-1},x_n) \quad \text{for} \quad i=1,...,n-1$$

and then if u^γ would be planar, it should be equal to some function

$$u_1(x_n+h) \quad \text{where } h \text{ is defined by} \quad u_1(h) = u^\gamma(0) = v(0)$$

Moreover $u_1(+\infty)$ and $u_1(-\infty)$ should be constant minimizers and then $u_1(+\infty)=1$, $u_1(-\infty)=-1$ and $u_1(x_n+h)\equiv u_0(x_n+h')$. But this is impossible because $\partial_n u^\gamma(0) = \gamma < u_0'(u_0^{-1}(v(0)))$. Finally the fact that

$$\partial_n u^\gamma > 0 \quad \text{on} \quad I\!R^n$$

is a consequence of the strong maximum principle (lemma 2.2 i)) and the fact that $\partial_n u^\gamma(0) = \gamma > 0$. This ends the proof the theorem.

Proof of lemma 4.7.
Using the uniqueness and the strict monotonicity of u_t, it is easy to see that the map $t \longmapsto \partial_n u_t(0,...,0,h_t)$ is continuous for $t\in[0,1)$. We will now prove that this map is

continuous up to $t = 1$, i.e. that $\lim_{t \to 1} \partial_n u_t(0, ..., 0, h_t) = 0$. If it is not the case then we can find a sequence (t') of reals which converge to 1 and such that

$$\partial_n u_{t'}(0, ..., 0, h_{t'}) \longrightarrow m \neq 0.$$

Let us define the sequence of functions

$$\tilde{u}_{t'}(x', x_n) = u_{t'}(x', x_n + h_{t'})$$

which are global minimizers on Σ. Then using again the compactness of these new sequence of functions $\tilde{u}_{t'}$ in the topology of convergence in C^2-norms on every compact set of Σ, we deduce, up to extraction of a convergent subsequence, the existence of a limit $\tilde{u}_1$ which is a global minimizer on Σ and satisfies:

$$m = \partial_n \tilde{u}_1(0) \quad \text{and} \quad \tilde{u}_1(0) = v(0)$$

and

$$\begin{cases} \Delta \tilde{u}_1 = F'(\tilde{u}_1) \quad \text{on} \quad \Sigma \\ \\ \tilde{u}_1 = \tilde{U}_1 \quad \text{on} \quad \partial\Sigma \\ \\ \partial_n \tilde{u}_1 \geq 0 \quad \text{on} \quad \Sigma \\ \\ |\tilde{u}_1| \leq 1 \quad \text{on} \quad \Sigma \end{cases}$$

with

$$\tilde{U}_1(x', x_n) = u_0(u_0^{-1}(v(x')) + l), \quad \text{where} \quad l = \lim_{t' \to 1} (1 - t') h_{t'} \in I\!R \cup \{\pm\infty\}$$

Here we use the definition $u_0(u_0^{-1}(v(x')) + \infty) = 1$ and $u_0(u_0^{-1}(v(x')) - \infty) = -1$.
We have now essentially two cases.

i) Case $l = 0$.

From proposition 4.3, we deduce that $\tilde{u}_1(x', x_n) = v(x')$. This implies in particular that $\partial_n \tilde{u}_1(0) = 0$ which is in contradiction with $m \neq 0$.

ii) Case $l \neq 0$.

Let us assume that $l \in (0, +\infty)$ (the same reasoning applies to the cases $l < 0$ and $l = +\infty$). Let us define

$$\underline{\tilde{u}}_1(x') = \lim_{x_n \to -\infty} \tilde{u}_1(x', x_n)$$

Then $\underline{\tilde{u}}_1$ is global minimizer on Σ and by lemma 3.2, it is a minimizer on $B'_R(0)$. Now let us define the open set

$$\Omega_- = \{\underline{\tilde{u}}_1 < v\}$$

Because $l > 0$,

$$v < \tilde{U}_1 = \underline{\tilde{u}}_1 \quad \text{on} \quad \partial B'_R(0) \tag{4.10}$$

Therefore, $\Omega_- \subset\subset B'_R(0)$. Then either $\Omega_- \neq \emptyset$ or else the reasoning of Step 1 in the proof of lemma 3.1 applies and gives:

$$E_{\Omega_-}(v) = E_{\Omega_-}(\underline{\tilde{u}}_1)$$

We then see that the function

$$v_+ = \begin{cases} \underline{\tilde{u}}_1 & \text{on} \quad \Omega_- \\ \\ v & \text{on} \quad B'_R(0) \backslash \Omega_- \end{cases}$$

is a minimizer on $B'_R(0)$ with the same boundary conditions as v. In particular we deduce that v_+ satisfies

$$\begin{cases} \Delta v_+ = F'(v_+) & \text{on} \quad B'_R(0) \\ \\ v_+ \leq \underline{\tilde{u}}_1 & \text{on} \quad B'_R(0) \end{cases}$$

Then the strong maximum principle (lemma 2.2 i)) implies that either $v_+ = \underline{\tilde{u}}_1$ (which is impossible because of (4.10)), or $v < \underline{\tilde{u}}_1$. Consequently we deduce that

$$\tilde{u}_1(0) \geq \underline{\tilde{u}}_1(0) > v(0)$$

which is a contradiction.

This ends the proof of the lemma.

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