Equations aux dérivées partielles / Partial differential equations

L'existence d'un minimiseur global symétrique sur $I\!R^{n-1}$ implique l'existence d'un contre-exemple à une conjecture de De Giorgi dans $I\!R^n$

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Résumé. Pour les solutions d'équations elliptiques semi-linéaire, nous mettons en évidence le lien qui existe entre les minimiseurs globaux sur $I\!R^{n-1}$ et l'existence de solutions monotones non triviales sur $I\!R^n$. Nous prouvons que l'existence d'un minimiseur global symétrique sur $I\!R^{n-1}$ implique l'existence d'une solution monotone non plane sur $I\!R^n$ qui est minimiseur global. De plus nous prouvons que toute solution monotone sur $I\!R^n$ est minimiseur global si et seulement si ses limites inférieure et supérieure le sont aussi.

The existence of a symmetric global minimizer on $I\!R^{n-1}$ implies the existence of a counter-example to a conjecture of De Giorgi in $I\!R^n$

Abstract. For solutions of semilinear elliptic equations, we show the link that exists between global minimizers on $I\!R^{n-1}$ and the existence of nontrivial monotonic solutions on $I\!R^n$. We prove that the existence of a symmetric global minimizer on $I\!R^{n-1}$ implies the existence of a nonplanar monotonic solution on $I\!R^n$ which is a global minimizer. Moreover we prove that every monotonic solution on $I\!R^n$ is a global minimizer if and only if its up and down limits are global minimizers.

Version Française Abrégée.

I. INTRODUCTION

Dans cette note nous considérons des solutions bornées $u \in C^2(I\!R^n)$ de l'équation suivante

$$\Delta u = F'(u) \text{ sur } I\!R^n \tag{1}$$

pour des nonlinéarités $F \in C^{1,1}(I\!R)$.

Définition 1 (Minimiseurs globaux sur $I\!R^n$)
Une fonction u est dite minimiseur global sur $I\!R^n$ si pour tout domaine borné regulier $\Omega \subset I\!R^n$ nous avons

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \le \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

pour toute fonction $v \in C^{0,1}(\overline{\Omega})$ telle que

$$v = u \text{ sur } \partial\Omega$$

II. RESULTATS PRINCIPAUX

Théorème 1 *Considérons une fonction $F \in C^{1,1}(I\!R)$ satisfaisant*

$$\begin{cases} F \geq 0 & sur \quad I\!R \\ F > 0 & sur \quad (-1,1) \quad et \quad F(-1) = F(1) = 0 \end{cases}$$

Supposons l'existence d'un minimiseur global v sur $I\!R^{n-1}$ (au sens de la définition 1) satisfaisant $|v| < 1$ et les symétries suivantes:

$$v(x_1, ..., -x_i, ..., x_{n-1}) = v(x_1, ..., x_i, ..., x_{n-1}) \qquad pour\ i = 1, ..., n-1$$

Alors pour chaque $\gamma \in \left(0, \sqrt{2F(v(0))}\right)$ il existe une fonction $u \in C^2(I\!R^n)$ solution de (1) satisfaisant $|u| \leq 1$ et

$$\partial_{x_n} u > 0 \ sur\ I\!R^n \tag{2}$$

et telle que pour un $\lambda \in I\!R$, l'ensemble $\{u = \lambda\}$ n'est pas un hyperplan.
De plus cette solution u est un minimiseur global sur $I\!R^n$, satisfait

$$\partial_{x_n} u(0) = \gamma \quad et \quad u(0) = v(0)$$

et les symétries:

$$u(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u(x_1, ..., x_i, ..., x_{n-1}, x_n) \qquad pour\ i = 1, ..., n-1 \tag{3}$$

Théorème 2 (Minimalité des solutions monotones)
Soit une fonction $F \in C^{1,1}(I\!R)$ et u une solution bornée de (1) satisfaisant (2). Nous notons $x' = (x_1, ..., x_{n-1})$ et introduisons

$$\begin{cases} \overline{u}(x') = \lim_{x_n \to +\infty} u(x', x_n) \\ \underline{u}(x') = \lim_{x_n \to -\infty} u(x', x_n) \end{cases} \tag{4}$$

Alors u est un minimiseur global sur $I\!R^n$ si et seulement si $\overline{u}$ et $\underline{u}$ sont des minimiseurs globaux sur $I\!R^{n-1}$.

I. INTRODUCTION
Let us recall the

Conjecture 1 (De Giorgi [5])
Consider a solution $u \in C^2(I\!R^n)$ of $\Delta u = u^3 - u$ such that $|u| \leq 1$, $\partial_{x_n} u > 0$ in the whole $I\!R^n$. Is it true that, for every $\lambda \in I\!R$, the sets $\{u = \lambda\}$ are hyperplanes, at least if $n \leq 8$?

In this note we consider bounded solutions $u \in C^2(I\!R^n)$ to the following equation

$$\Delta u = F'(u) \text{ in } I\!R^n \tag{5}$$

for more general nonlinearities $F \in C^{1,1}(I\!R)$.

Definition 1 (Global minimizers on $I\!R^n$)
We say that a function u is a global minimizer on $I\!R^n$ if for every smooth bounded domain $\Omega \subset I\!R^n$,

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \le \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

for every function $v \in C^{0,1}(\overline{\Omega})$ such that

$$v = u \text{ on } \partial\Omega$$

Remark 1 *It is possible to prove that if u is a global minimizer and if v satisfies (5) on a smooth bounded domain $\Omega \subset I\!R^n$ such that $v = u$ on $\partial\Omega$, then $v = u$ on Ω.*

II. MAIN RESULTS

Theorem 1 *Consider a function $F \in C^{1,1}(I\!R)$ such that*

$$\begin{cases} F \ge 0 \quad \text{on} \quad I\!R \\ F > 0 \quad \text{on} \quad (-1,1) \quad \text{and} \quad F(-1) = F(1) = 0 \end{cases}$$

Assume the existence of a global minimizer v on $I\!R^{n-1}$ (in the sense of definition 1) which satisfies $|v| < 1$ and the following symmetries:

$$v(x_1, ..., -x_i, ..., x_{n-1}) = v(x_1, ..., x_i, ..., x_{n-1}) \quad \text{for } i = 1, ..., n-1$$

Then for each $\gamma \in \left(0, \sqrt{2F(v(0))}\right)$ there exists a function $u \in C^2(I\!R^n)$ solution to (5) satisfying $|u| \le 1$ and

$$\partial_{x_n} u > 0 \text{ in } I\!R^n \tag{6}$$

and such that for one $\lambda \in I\!R$, the set $\{u = \lambda\}$ is not an hyperplane.
Moreover this solution u is a global minimizer on $I\!R^n$, satisfies

$$\partial_{x_n} u(0) = \gamma \quad \text{and} \quad u(0) = v(0)$$

and the symmetries:

$$u(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u(x_1, ..., x_i, ..., x_{n-1}, x_n) \quad \text{for } i = 1, ..., n-1 \tag{7}$$

Remark 2 *The proof of Theorem 1 shows the existence of a compact connected set of global minimizers on $I\!R^n$ that connects $v(x_1, ..., x_{n-1})$ to a solution $u_0(x_n)$ on $I\!R^n$ whose level sets are hyperplanes.*

Remark 3 *Theorem 1 is analogous to the well-known example of Bombieri, De Giorgi, and Giusti of a minimal graph in $I\!R^9$ that is not a hyperplane, which was constructed from the closely related minimal cone of Simons in $I\!R^8$. The proof in our situation is, however, considerably simpler.*

Theorem 2 (Minimality of monotonic solutions)
Let a function $F \in C^{1,1}(\mathbb{R})$ and u a bounded solution to (5) satisfying (6). Denote $x' = (x_1, ..., x_{n-1})$ and

$$\begin{cases} \overline{u}(x') = \lim\limits_{x_n \to +\infty} u(x', x_n) \\ \underline{u}(x') = \lim\limits_{x_n \to -\infty} u(x', x_n) \end{cases} \tag{8}$$

Then u is a global minimizer on $\mathbb{R}^n$ if and only if $\overline{u}$ and $\underline{u}$ are global minimizers on $\mathbb{R}^{n-1}$.

Let us recall the following result:

Theorem 3 (Alberti, Ambrosio, Cabré [1])
Let u be a bounded entire solution of (5) satisfying (6), and let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then

$$\int_\Omega \frac{1}{2}|\nabla u|^2 + F(u) \leq \int_\Omega \frac{1}{2}|\nabla v|^2 + F(v)$$

for every function $v \in C^{0,1}(\overline{\Omega})$ such that $v = u$ on $\partial\Omega$ and

$$\underline{u}(x') \leq v(x', x_n) \leq \overline{u}(x') \quad \text{for all } x = (x', x_n) \in \Omega$$

Theorem 3 was proved in [1] using results on calibrations for scalar functionals of the Calculus of Variations. Here our proofs of theorems 2 and 3 are based on the sliding method.

III. SKETCH OF THE PROOFS.

Proof of Theorem 1.
We assume the existence of a function $v(x')$ on $\mathbb{R}^{n-1}$ as defined in the theorem and we will build a function $u(x', x_n)$ on $\mathbb{R}^n$. From the assumptions in the theorem, there exists a unique solution u_0 to

$$u_0'' = F'(u_0) \quad \text{and} \quad u_0' > 0 \quad \text{on} \quad \mathbb{R}$$

with $u_0(-\infty) = -1, u_0(+\infty) = 1, u_0(0) = 0$.
The key idea is to connect on $\mathbb{R}^n$ the two particular solutions $u_0(x_n)$ and $v(x')$ by a continuous curve of solutions on $\mathbb{R}^n$. To this end we introduce the following functions for $t \in [0, 1]$

$$U_t(x', x_n) := u_0\left(t u_0^{-1}(v(x')) + (1-t)x_n\right)$$

which satisfy $\partial_{x_n} U_t \geq 0$ and the same symmetries as v. We will build monotonic solutions on cylinders whose values on the boundary of the cylinder are U_t and then take the limit as the cylinders tend to the whole space $\mathbb{R}^n$.
We claim the following:
Claim : *For each $t \in [0,1]$ there exists a unique global minimizer u_t on $\Sigma = \{x = (x', x_n),\ |x'| < R\}$ which satisfies $|u_t| \leq 1$ and*

$$\begin{cases} \Delta u_t = F'(u_t) \quad on \quad \Sigma \\ u_t = U_t \quad on \quad \partial\Sigma \\ \partial_{x_n} u_t \geq 0 \quad on \quad \Sigma \end{cases}$$

From its uniqueness, we deduce that u_t has the same symmetries as U_t, i.e.

$$u_t(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u_t(x_1, ..., x_i, ..., x_{n-1}, x_n) \quad \text{for} \quad i = 1, ..., n-1$$

Moreover uniqueness implies that the map

$$t \longmapsto \partial_{x_n} u_t(0, ..., 0, h_t) \quad \text{where } h_t \text{ is defined by} \quad u_t(0, ..., 0, h_t) = v(0)$$

is continuous. Furthermore, the values at the endpoints are

$$\begin{cases} \partial_{x_n} u_t(0, ..., 0, h_t) = u_0'(u_0^{-1}(v(0))) = \sqrt{2F(v(0))} & \text{for} \quad t = 0 \\ \partial_{x_n} u_t(0, ..., 0, h_t) = 0 & \text{for} \quad t = 1 \end{cases}$$

Now for every $\gamma \in \Big(0, u_0'(u_0^{-1}(v(0)))\Big)$, we get by continuity the existence of a $t_\gamma \in (0,1)$ such that

$$\partial_{x_n} u_{t_\gamma}(0, ..., 0, h_{t_\gamma}) = \gamma \quad \text{where} \quad u_{t_\gamma}(0, ..., 0, h_{t_\gamma}) = v(0)$$

Next, the translate of the solution by $(0, ..., 0, h_{t_\gamma})$ is defined by

$$u_R^\gamma(x', x_n) := u_{t_\gamma}(x', x_n + h_{t_\gamma}) \quad \text{on} \quad \{x = (x', x_n),\ |x'| < R\}$$

For a subsequence $R' \to +\infty$, $u_{R'}^\gamma \to u^\gamma$ uniformly locally on compact sets of $I\!R^n$ and the functions u^γ are still global minimizers and satisfy

$$\begin{cases} \Delta u^\gamma = F'(u^\gamma) \quad \text{on} \quad I\!R^n \\ \partial_{x_n} u^\gamma \geq 0 \quad \text{on} \quad I\!R^n \\ \partial_{x_n} u^\gamma(0) = \gamma \quad \text{and} \quad u^\gamma(0) = v(0) \end{cases}$$

The fact that u^γ is not a planar solution comes from the fact u^γ inherits the symmetries of v:

$$u^\gamma(x_1, ..., -x_i, ..., x_{n-1}, x_n) = u^\gamma(x_1, ..., x_i, ..., x_{n-1}, x_n) \quad \text{for} \quad i = 1, ..., n-1.$$

If u^γ were planar, it would be equal to some function

$$u_1(x_n + h) \quad \text{where } h \text{ is defined by} \quad u_1(h) = u^\gamma(0) = v(0)$$

Moreover $u_1(+\infty)$ and $u_1(-\infty)$ would be constant minimizers and then $u_1(+\infty) = 1$, $u_1(-\infty) = -1$ and $u_1(x_n + h) \equiv u_0(x_n + h')$. But this is impossible because $\partial_{x_n} u^\gamma(0) = \gamma < u_0'(u_0^{-1}(v(0)))$. Finally the fact that $\partial_{x_n} u^\gamma > 0$ on $I\!R^n$ is a consequence of the strong maximum principle and the fact that $\partial_{x_n} u^\gamma(0) = \gamma > 0$. This ends the proof the theorem.

Proof of Theorem 3.
Let $\Omega \subset I\!R^n$ a smooth bounded domain. Let $w \leq \overline{u}$ the maximal solution to (5) on Ω with $w = u$ on $\partial\Omega$. Let

$$\tilde{u} = \begin{cases} w & \text{on} \quad \Omega \\ u & \text{on} \quad I\!R^n \backslash \Omega \end{cases}$$

Then $\tilde{u}$ is a subsolution on $I\!R^n$. Applying the sliding method, we can slide u over $\tilde{u}$ and get $\tilde{u} = u$. As a consequence $w = u$ on Ω. Similarly we can prove that the minimal solution w

above $\underline{u}$ on Ω with $w = u$ on $\partial\Omega$ is also $w = u$. It proves the uniqueness of the solutions to (5) on Ω which coincides with u on the boundary. In particular this implies the theorem.

Proof of Theorem 2.
On the one hand, if $\overline{u}$ and $\underline{u}$ are global minimizers, one can prove by a truncation argument that every minimizer w on a smooth bounded domain $\Omega \subset \mathbb{R}^n$ which coincides with u on the boundary necessary satisfies $\underline{u} \leq w \leq \overline{u}$. Applying theorem 3 we conclude that $w = u$.
On the other hand, if u is a global minimizer on $\mathbb{R}^n$, then $\overline{u}$ and $\underline{u}$ are global minimizers on $\mathbb{R}^n$. It is then possible to prove by contradiction that there are also global minimizers on $\mathbb{R}^{n-1}$.

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