An Unstable Elliptic Free Boundary Problem arising in Solid Combustion

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Abstract

We prove a regularity result for the unstable elliptic free boundary problem

$$\Delta u = -\chi_{\{u>0\}} \tag{0.1}$$

related to traveling waves in a problem arising in solid combustion.
The maximal solution and every local minimizer of the energy are regular, that is, $\{u = 0\}$ is locally an analytic surface and $u|_{\overline{\{u>0\}}}, u|_{\overline{\{u<0\}}}$ are locally analytic functions. Moreover we prove a partial regularity result for solutions that are non-degenerate of second order: here $\{u = 0\}$ is analytic up to a set of Hausdorff dimension $n - 2$. We discuss possible singularities.

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1 Introduction

We prove a regularity result for the unstable elliptic free boundary problem

$$\Delta u = -\chi_{\{u>0\}} \ . \tag{1.2}$$

The problem (1.2) is related to traveling wave solutions in solid combustion with ignition temperature: let us consider the solid combustion system

$$\begin{aligned} &\partial_t \theta - \Delta \theta = g(\eta)\chi_{\{\theta>\theta_0\}}, \\ &\partial_t \eta = g(\eta)\chi_{\{\theta>\theta_0\}} \end{aligned} \tag{1.3}$$

where θ is the temperature, θ_0 is the ignition temperature and $\eta \in (0,1)$ is the fractional conversion. Although an ignition temperature has no meaning for gas flames, it has been recently rediscovered and used in combustion synthesis (see for example [20],[4],[2] and [17]). The reaction kinetics suggested in [20, p.1462] is $g(\eta) = k_0 \ \chi_{\{\eta<1\}}$, but the particular form of g is not of importance for what follows. Solving the ODE we obtain for $u := \theta - \theta_0$

$$\partial_t u - \Delta u = c\chi_{\{u>0\}} \ , \tag{1.4}$$

where c is a non-negative memory term depending on (t, x) as well as the history $\{u(s, x), s < t\}$. Although c is important when considering the large-time behavior of u, we may consider (1.4) to be a perturbation of

$$\partial_t u - \Delta u = \chi_{\{u>0\}} \tag{1.5}$$

when we are interested in transient or local phenomena at the ignition front $\partial\{u < 0\}$. Actually the traveling pulses in our model correspond well to the fingering phenomenon for burned regions observed in solid combustion experiments (see for example [24]).
We are interested in traveling wave solutions. As we deal in the present paper mostly with regularity issues, we may drop the drift term resulting from the time derivative of the traveling wave.
An equation similar to our elliptic one arises in the composite membrane problem (see [10], [9], [6]). Another application is the shape of self-gravitating rotating fluids describing stars (see [7, equation (1.26)]).
From a mathematical point of view, (1.5) is the equation of the parabolic obstacle problem with inverted sign, and (1.2) is the equation of the elliptic obstacle problem with inverted sign. The change of sign changes the character of the problem *drastically* in that it changes the *stable* obstacle problem into an *unstable problem.* In (1.2) and (1.5), we find examples of non-uniqueness, bifurcation phenomena etc.
As surprisingly many known free boundary problems turn out to be stable problems, this means, unfortunately, that many known methods in free boundary problems do not

apply here. Examples of PDE techniques that do not work are, apart from all one-phase methods, the Bernstein technique, the Alt-Caffarelli-Friedman monotonicity formula ([3]) and the differential inequality technique of Cazenave-Lions ([8]).

Our main result is that the maximal solution and every local minimizer of the energy are regular, that is, $\{u = 0\}$ is locally an analytic surface and $u|_{\overline{\{u>0\}}}, u|_{\overline{\{u<0\}}}$ are locally analytic functions (Theorem 8.1).

The surprise is that – in contrast to the usual procedure – we obtain $C^{1,1}$-regularity of local minimizers *by proving regularity of the free boundary first!*

For general solutions that are non-degenerate of second order, we prove a partial regularity result: here $\{u = 0\}$ is smooth up to a set of Hausdorff dimension $n - 2$ (Proposition 6.3). We discuss the behavior at possible singularities.

In case of a non-degenerate minimal solution in two dimensions we also obtain that $\{u = 0\}$ consists of Lipschitz arcs meeting at right angles in at most finitely many singularities.

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2 Notation

Throughout this article $\mathbf{R}^n$ will be equipped with the Euclidean inner product $x \cdot y$ and the induced norm $|x|$. We define e_i as the i-th unit vector in $\mathbf{R}^n$, and $B_r(x^0)$ will denote the open n-dimensional ball of center x^0, radius r and volume $r^n \omega_n$. We shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in \Omega \,:\, u(x) > 0\}$, $\{x_n > 0\} := \{x \in \mathbf{R}^n \,:\, x_n > 0\}$ etc. and occasionally we shall employ the decomposition $x = (x_1, \dots, x_n)$ of a vector $x \in \mathbf{R}^n$. We will use the k-dimensional Hausdorff measure $\mathcal{H}^k$ approximated by $\mathcal{H}^{k,\delta}$ which we define as the H^δ_k of [14]. When considering a set A, χ_A shall stand for the characteristic function of A, while ν shall typically denote the outward normal to a given boundary.

3 Existence and Non-Degeneracy

Lemma 3.1 (Existence of a maximal and a minimal solution)
Let Ω be a bounded domain of class $C^{2,\alpha}$ in $\mathbf{R}^n$, and assume that the Dirichlet boundary data $u_D \in C^{2,\alpha}(\bar\Omega)$.
Then there exists a maximal solution *u with the following properties: $u \in W^{2,p}(\Omega)$ for every $p \in [1, +\infty)$, $u = u_D$ on $\partial\Omega$ and $u \geq v$ for every subsolution $v \in W^{2,n}(\Omega)$ of (1.2) in $\Omega' \subset \Omega$ satisfying $v \leq u$ on $\partial\Omega'$.*
There also exists a minimal solution *with analogous properties.*

Proof: We prove the existence of the maximal solution: We consider a regularization of the equation from above,

$$\Delta v = -\bar\beta_\epsilon(v),$$

where $\bar\beta_\epsilon \in C^\infty(\mathbf{R}), \bar\beta_\epsilon(z) \geq \chi_{\{z>0\}} =: \beta(z)$ in $\mathbf{R}$ and $\bar\beta_\epsilon \downarrow \beta$ as $\epsilon \downarrow 0$. By Perron's method, there exists a maximal solution u_ϵ in the above sense for each $\epsilon > 0$. By $W^{2,p}$-estimates the family $(u_\epsilon)_{\epsilon\in(0,1)}$ is bounded in $W^{2,p}(\Omega)$. Moreover, for any subsolution v of (1.2) and $0 < \tilde\epsilon < \epsilon$ we obtain

$$\Delta v \geq -\beta(v) \geq -\bar\beta_\epsilon(v) \text{ and } \Delta u_{\tilde\epsilon} = -\beta_{\tilde\epsilon}(u_{\tilde\epsilon}) \geq -\bar\beta_\epsilon(u_{\tilde\epsilon})$$

so that v and $u_{\tilde\epsilon}$ are subsolutions of the ϵ-equation. Consequently $v \leq u_\epsilon$ and $u_{\tilde\epsilon} \leq u_\epsilon$. Using the fact that $D^2u = 0$ a.e. on $\{u = 0\}$, it follows that $u(x) := \lim_{\epsilon\to 0} u_\epsilon(x)$ is the maximal solution.

Lemma 3.2 *There exists a positive constant c_n depending only on the dimension n, such that the maximal solution u with respect to given boundary data satisfies*

$$B_r(x^0) \subset \Omega, \inf_{\partial B_r(x^0)} u > -c_n r^2 \Rightarrow u(x^0) > 0 \, .$$

Thus for any $x^0 \in \{u = 0\}$ and $r < \operatorname{dist}(x^0, \Omega^c)$ we obtain $\inf_{\partial B_r(x^0)} u \leq -c_n r^2$.

Proof: Compare $u(x^0 + rx)/r^2$ for suitable θ to the "stationary pulse" $p(\theta x)/\theta^2$, where

$$p(x) = \begin{cases} (1 - |x|^2)/4, & |x| \leq 1, \\ -\log(|x|^2)/4, & |x| > 1 \end{cases}$$

in the case $n = 2$ and to

$$p(x) = \begin{cases} \frac{1}{2n}(1 - |x|^2), & |x| \leq 1, \\ \frac{1}{n(2-n)}(|x|^{2-n} - 1), & |x| > 1 \end{cases}$$

in the case $n > 2$.

Remark 3.3 *For the case of two dimensions the optimal constant $c_2 = 1/(4e)$.*

Definition 3.4 (Non-degeneracy) *Let u be a solution of (1.2) in Ω, satisfying at $x^0 \in \Omega$*

$$\liminf_{r\to 0} r^{-2} \left(\int_{\partial B_{r_m}(x^0)} u^2 \, d\mathcal{H}^{n-1} \right)^{\frac{1}{2}} > 0 \, . \tag{3.6}$$

Then we call u "non-degenerate of second order at x^0". We call u "non-degenerate of second order" if it is non-degenerate of second order at each point in Ω.

Remark 3.5 *The maximal solution is non-degenerate of second order.*

Lemma 3.6 *Each minimizer u of the energy*

$$E(v) = \int_{B_{r_0}(x^0)} |\nabla v|^2 - 2\max(v, 0)$$

in $K := \{v \in W^{1,2}(B_{r_0}(x^0)) : v = u \text{ on } \partial B_{r_0}(x^0)\}$ satisfies at x^0 the second-order non-degeneracy property (3.6).

Proof: Define for $r \in (0, r_0)$ the solution $v := u(x^0 + rx)/r^2$ and let p be the stationary pulse in $B_1(0)$ with boundary data $\inf_{\partial B_1(0)} v$. Comparing the energy of v to that of $w := \max(p, v)$ we obtain

$$\begin{aligned} 0 &\geq \int_{B_1(0)} |\nabla v|^2 - |\nabla w|^2 - 2\max(v, 0) + 2\max(w, 0) \\ &= \int_{B_1(0)} -(\Delta v + \Delta w)(v - w) + 2(\max(w, 0) - \max(v, 0)) \\ &= \int_{B_1(0)\cap\{v\leq 0\}\cap\{p>0\}} v + p\ . \end{aligned}$$

Now assume that v_m is a sequence of minimizers such that $v_m(0) = 0$ and $\inf_{\partial B_1(0)} v_m \to 0$ as $m \to \infty$. Let $\epsilon \in (0, 1)$ be fixed. In the case that $\mathcal{L}^n(B_{1-\epsilon}(0) \cap \{v_m \leq 0\}) \not\to 0$ as $m \to \infty$, we obtain immediately a contradiction since p_m is for large m on $B_{1-\epsilon}(0)$ uniformly estimated from below by a positive constant depending only on n and ϵ. In the case $\mathcal{L}^n(B_{1-\epsilon}(0) \cap \{v_m \leq 0\}) \to 0$ as $m \to \infty$, we know by $\lim_{m\to\infty} \inf_{B_1(0)} v_m \to 0$ and $\fint_{\partial B_s(0)} v_m \, d\mathcal{H}^{n-1} \leq 0$ for every $s \in (0, 1]$ (which follows from v_m being superharmonic and 0 at the origin) that $\int_{B_1(0)} |v_m| \to 0$ as $m \to \infty$. Consequently

$$0 \leftarrow \int_{B_{1-\epsilon}(0)} \Delta v_m = \int_{B_{1-\epsilon}(0)} -\chi_{\{v_m>0\}} \leq -c_1 < 0 \text{ as } m \to \infty,$$

a contradiction. Thus $u(x^0) = 0$ implies

$$\inf_{B_1(0)} \frac{u(x^0 + r\cdot)}{r^2} \leq -c_2 < 0$$

for all $r \in (0, r_0)$.

Lemma 3.7 *The maximal solution $u_{\max}$ is* the *maximal minimizer of the energy*

$$E(v) = \int_\Omega |\nabla v|^2 - 2\max(v, 0)$$

in $K := \{v \in W^{1,2}(\Omega) : v = u_D \text{ on } \partial\Omega\}$.

Proof: For any $v \in K$ we get

$$0 \geq \int_\Omega \max(v,0) - \max(u_{\max},0) + u_{\max}\chi_{\{v>0\}} - v\chi_{\{u_{\max}>0\}}$$

$$= \int_\Omega (\chi_{\{v>0\}} + \chi_{\{u_{\max}>0\}})(u_{\max} - v) + 2(\max(v,0) - \max(u_{\max},0))$$

$$= \int_\Omega |\nabla u_{\max}|^2 - |\nabla v|^2 + 2\max(v,0) - 2\max(u_{\max},0) \; .$$

4 Monotonicity Formula and Frequency Lemma

A powerful tool is now a monotonicity formula introduced in [22] by one of the authors for a class of semilinear free boundary problems. For the sake of completeness let us state the unstable case here:

Theorem 4.1 (Monotonicity formula) *Suppose that u is a solution of (1.2) in Ω and that $B_\delta(x^0) \subset \Omega$. Then for all $0 < \rho < \sigma < \delta$ the function*

$$\Phi_{x^0}(r) := r^{-n-2} \int_{B_r(x^0)} \Big(|\nabla u|^2 \; - \; 2\max(u,0) \Big)$$

$$- \; 2\, r^{-n-3} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1} \; ,$$

defined in $(0,\delta)$, satisfies the monotonicity formula

$$\Phi_{x^0}(\sigma) \; - \; \Phi_{x^0}(\rho) \; = \; \int_\rho^\sigma r^{-n-2} \int_{\partial B_r(x^0)} 2 \left(\nabla u \cdot \nu - 2\, \frac{u}{r} \right)^2 \, d\mathcal{H}^{n-1} \, dr \; \geq 0 \quad .$$

We also need the following frequency lemma, which has been proven in [21, Lemma 4.1]:

Lemma 4.2 (Frequency lemma) *Let $\alpha - 1 \in \mathbf{N}$, let $w \in W^{1,2}(B_1(0))$ be a harmonic function in $B_1(0)$ and assume that $D^j w(0) = 0$ for $0 \leq j \leq \alpha - 1$.*

$$\textit{Then} \; \int_{B_1(0)} |\nabla w|^2 \; - \; \alpha \int_{\partial B_1(0)} w^2 \, d\mathcal{H}^{n-1} \; \geq \; 0 \; ,$$

and equality implies that w is homogeneous of degree α in $B_1(0)$.

5 Classification of Blow-up Limits

A result related to the following classification of blow-up limits is contained in [6, Theorem 2.5]. Note however that the proofs of the related parts are largely different.

Proposition 5.1 (Classification of blow-up limits with fixed center) *Let u be a solution of (1.2) in Ω and let us consider a point $x^0 \in \Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$.*
1) In the case $\Phi_{x^0}(0+) = -\infty$, $\lim_{r\to 0} r^{-3-n} \int_{\partial B_r(x^0)} u^2 d\mathcal{H}^{n-1} = +\infty$, and for $S(x^0, r) := \left(r^{1-n} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}$ each limit of

$$\frac{u(x^0 + rx)}{S(x^0, r)}$$

as $r \to 0$ is a homogeneous harmonic polynomial of degree 2.
2) In the case $\Phi_{x^0}(0+) \in (-\infty, 0)$,

$$u_r(x) := \frac{u(x^0 + rx)}{r^2}$$

is bounded in $W^{1,2}(B_1(0))$, and each limit as $r \to 0$ is a homogeneous solution of degree 2.
3) Else $\Phi_{x^0}(0+) = 0$, and

$$\frac{u(x^0 + rx)}{r^2} \to 0 \text{ in } W^{1,2}(B_1(0)) \text{ as } r \to 0 \, .$$

Proof: In all three cases,

$$\begin{aligned} \partial_r \frac{1}{2} \int_{\partial B_1(0)} u_r^2 \, d\mathcal{H}^{n-1} &= \int_{\partial B_1(0)} u_r \partial_r u_r \, d\mathcal{H}^{n-1} \\ &= \frac{1}{r} \int_{\partial B_1(0)} u_r (\nabla u_r \cdot x - 2u_r) \, d\mathcal{H}^{n-1} \\ &= \frac{1}{r} \left(\int_{B_1(0)} - \max(u_r, 0) + |\nabla u_r|^2 \; - \; 2 \int_{\partial B_1(0)} u_r^2 \, d\mathcal{H}^{n-1} \right) \\ &= \frac{1}{r} \left(\Phi_{x^0}(r) \; + \; \int_{B_1(0)} \max(u_r, 0) \right) . \end{aligned}$$

In particular,

$$\int_{B_1(0)} \max(u_r, 0) \geq M := -\Phi_{x^0}(0+) \text{ implies } \partial_r \int_{\partial B_1(0)} u_r^2 \, d\mathcal{H}^{n-1} \geq 0 \, . \tag{5.7}$$

Observe now that

$$\int_{\partial B_1(0)} \max(-u_r, 0)^2 \, d\mathcal{H}^{n-1} \leq C_1 \left(1 + \int_{\partial B_1(0)} \max(u_r, 0)^2 \, d\mathcal{H}^{n-1} \right) \; : \tag{5.8}$$

supposing towards a contradiction that this is not true, $\int_{\partial B_1(0)} u_r^2 \, d\mathcal{H}^{n-1}$ must be unbounded for a sequence $(r_m)_{m\in\mathbf{N}}$ not satisfying (5.8). We may divide the function of the monotonicity formula 4.1

$$\int_{B_1(0)} |\nabla u_{r_m}|^2 - 2 \int_{\partial B_1(0)} u_{r_m}^2 \, d\mathcal{H}^{n-1} \leq \Phi_{x^0}(1) \; + \; 2 \int_{B_1(0)} \max(u_{r_m}, 0)$$

by $\int_{\partial B_1(0)} \max(-u_{r_m},0)^2 d\mathcal{H}^{n-1}$. As a subsequence $r_m \to 0$, we obtain a weak $L^2(\partial B_1(0))$-limit v of $v_m := u_{r_m}/\left(\int_{\partial B_1(0)} \max(-u_{r_m},0)^2\, d\mathcal{H}^{n-1}\right)^{1/2}$ such that by $\int_{B_1(0)} |u_{r_m}| \le C_2 \left(\int_{\partial B_1(0)} |u_{r_m}|^2\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}$ (which follows from $\min(u_{r_m},0)^2$ being subharmonic and u_{r_m} being superharmonic and 0 at the origin)

$$\int_{B_1(0)} |\nabla v_m|^2 - 2\int_{\partial B_1(0)} v_m^2\, d\mathcal{H}^{n-1} \ \le\ o(1) \text{ as } m \to \infty\ , \tag{5.9}$$

and v is in $B_1(0)$ a non-positive harmonic function satisfying $v(0) = 0$, implying by the strong maximum principle that $v = 0$ in $B_1(0)$. That poses a contradiction to the (by (5.9)) strong $L^2(\partial B_1(0))$-convergence of v_m and the fact that

$$\liminf_{m\to\infty} \int_{\partial B_1(0)} v_m^2\, d\mathcal{H}^{n-1} > 0\ .$$

Note also that in the case $\Phi_{x^0}(0+) > -\infty$, for small r and $\tilde r \in (r/2, r)$

$$\begin{aligned} &1 > \Phi_{x^0}(r) - \Phi_{x^0}(\tilde r) = \textstyle\int_{\tilde r}^r 2s \int_{\partial B_1(0)} (\partial_s u_s)^2\, d\mathcal{H}^{n-1}\, ds \\ &\ge \textstyle\int_{\partial B_1(0)} (u_r - u_{\tilde r})^2 d\mathcal{H}^{n-1}\ . \end{aligned} \tag{5.10}$$

Combining (5.10), (5.8) and (5.7) we see that in the case $\Phi_{x^0}(0+) > -\infty$,

$$\int_{\partial B_1(0)} u_r^2 \ge \tilde M \text{ implies } \partial_r \int_{\partial B_1(0)} u_r^2\, d\mathcal{H}^{n-1} \ge 0\ . \tag{5.11}$$

But then $\int_{\partial B_1(0)} u_r^2\, d\mathcal{H}^{n-1}$ has to be bounded in the case $\Phi_{x^0}(0+) > -\infty$.
From the monotonicity formula Theorem 4.1 we infer that in the case $\Phi_{x^0}(0+) > -\infty$, each limit u_0 of u_r in $B_1(0)$ is a homogeneous solution of degree 2.
Moreover, in the case $\Phi_{x^0}(0+) \ge 0$, we obtain that each limit u_0 satisfies

$$0 \le \Phi_{x^0}(0+)\ =\ \int_{B_1(0)} |\nabla u_0|^2 - 2\max(u_0,0)\ -\ 2\int_{\partial B_1(0)} u_0^2\, d\mathcal{H}^{n-1}\ =\ -\int_{B_1(0)} \chi_{\{u_0>0\}}\ .$$

It follows that $u_0 \equiv 0$ in $B_1(0)$ and that $\Phi_{x^0}(0+) = 0$.
Last, in the case $\Phi_{x^0}(0+) = -\infty$ we obtain that

$$\lim_{r\to 0} S(x^0,r)/r^2 = \lim_{r\to 0}\left(r^{-3-n}\int_{\partial B_r(x^0)} u^2\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} = +\infty\ .$$

Taking a subsequence

$$w_m(x) := \frac{u(x^0 + r_m x)}{S(x^0, r_m)}$$

that converges weakly in $L^2(\partial B_1(0))$ to w_0 and setting $T_m := S(x^0, r_m)/{r_m}^2$, we infer from the monotonicity formula Theorem 4.1 for large m that

$$\int_{B_1(0)} |\nabla u_{r_m}|^2 \le \Phi_{x^0}(r_0)\ +\ \int_{B_1(0)} 2\max(u_{r_m},0)\ +\ 2\int_{\partial B_1(0)} u_{r_m}^2\, d\mathcal{H}^{n-1}\ .$$

Division by T_m^2 yields

$$\int_{B_1(0)} |\nabla w_m|^2 \le T_m^{-2}\Phi_{x^0}(r_0) \;+\; T_m^{-1}\int_{B_1(0)} 2\max(w_m,0) \;+\; 2\int_{\partial B_1(0)} w_m^2 \, d\mathcal{H}^{n-1} \; .$$

Since $|\Delta w_m| \le 1$ in $B_1(0)$, it follows that

$$\int_{B_1(0)} |\nabla w_0|^2 \le 2\int_{\partial B_1(0)} w_0^2 \, d\mathcal{H}^{n-1} \; , \tag{5.12}$$

that $w_0(0) = |\nabla w_0(0)| = 0$ and that w_0 is harmonic in $B_1(0)$. From Lemma 4.2 we infer that w_0 is a homogeneous harmonic polynomial of degree 2.

Lemma 5.2 *In two dimensions, the only solution of (1.2) that is homogeneous of degree* 2, *is the trivial solution* 0.

Proof: Let u be a solution of (1.2) on $\mathbf{R}^2$ that is homogeneous of degree 2. Passing to the ODE $y'' + 4y = -\chi_{\{y>0\}}$, each component of $\{u < 0\}$ must be a cone of opening $\pi/2$ and each component of $\{u > 0\}$ must for some $\tau \in (0, +\infty)$ be a cone of opening $|\arcsin(1/(4\tau))| < \pi/2$, a contradiction.

Remark 5.3 *Let u be a solution in a neighborhood of $x^0 \in \Omega$. If the point x^0 is non-degenerate of second order, then all points in some open neighborhood of x^0 are non-degenerate of second order.*

Proof: $\Phi_{x^0}(0+) < 0$ implies by upper semicontinuity of

$$x \mapsto \Phi_x(0+)$$

that every point $x \in \{u = 0\}$ in some open neighborhood of x^0 satisfies $\Phi_x(0+) < 0$ and is therefore non-degenerate of second order.

6 Partial Regularity

A result related to the following Corollary has been independently obtained in [18, Theorem 1.1].

Corollary 6.1 (Partial regularity in two dimensions) *Let $n = 2$ and let u be a solution of (1.2) in Ω that is non-degenerate of second order. Then for each $K \subset\subset \Omega$, the singular set $K \cap \{u = 0\} \cap \{\nabla u = 0\}$ contains at most finitely many points.*

Proof: Suppose this is not true. Then there is a sequence $\Omega \cap \{u = 0\} \cap \{\nabla u = 0\} \ni x^m \to x^0 \in \Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$. Take a blow-up limit u_0 with respect to the fixed center x^0 such that $\partial B_1(0)$ contains a point of $\{u_0 = 0\} \cap \{\nabla u_0 = 0\}$. By Proposition 5.1 and Lemma 5.2 we know that u_0 is a homogeneous harmonic polynomial of degree 2. This is a contradiction, since for a homogeneous harmonic polynomial of degree 2 in two dimensions the set $\{u_0 = 0\} \cap \{\nabla u_0 = 0\} = \{0\}$.

Lemma 6.2 *Let u be a solution of (1.2) in Ω that is non-degenerate of second order, let $x^0 \in \Omega \cap \{u=0\} \cap \{\nabla u = 0\}$, and let u_0 be a blow-up limit of*

$$u_m(x) := \frac{u(x^0 + r_m x)}{S(x^0, r_m)}$$

in sense of Proposition 5.1. Then for each compact set $K \subset \mathbf{R}^n$ and each open set $U \supset K \cap S_0$ there exists $m_0 < \infty$ such that $S_m \cap K \subset U$ for $m \geq m_0$; here $S_0 := \{u_0 = 0\} \cap \{\nabla u_0 = 0\}$ and $S_m := \{u_m = 0\} \cap \{\nabla u_m = 0\}$.

Proof: Suppose towards a contradiction that $S_m \cap (K - U) \ni x^m \to \bar{x}$ as $m \to \infty$. Then $\bar{x} \in \{u_0 = 0\} \cap \{\nabla u_0 = 0\} \cap (K - U)$, contradicting the assumption $U \supset K \cap S_0$.

Proposition 6.3 (Partial regularity in higher dimensions) *Let u be a solution of (1.2) in Ω that is non-degenerate of second order. Then the Hausdorff dimension of the set $S = \Omega \cap \{u=0\} \cap \{\nabla u = 0\}$ is less than or equal to $n-2$.*

Proof: Suppose that $s > n-2$ and that $\mathcal{H}^s(S) > 0$. Then we may use [14, Proposition 11.3], Lemma 6.2 as well as [14, Lemma 11.5] at $\mathcal{H}^s$-a.e. point of S to obtain a blow-up limit u_0 with the properties mentioned in Proposition 5.1, satisfying $\mathcal{H}^{s,\infty}(S_0) > 0$ for $S_0 := \{u_0 = 0\} \cap \{\nabla u_0 = 0\}$. According to Proposition 5.1 there are two possibilities:
1) $\Phi_{x^0}(0+) = -\infty$ and u_0 is a homogeneous harmonic polynomial of degree 2. But for such a polynomial $\mathcal{H}^{n-2}(\{u_0 = 0\} \cap \{\nabla u_0 = 0\}) < +\infty$ and we obtain a contradiction. Thus the second possibility has to apply:
2) for some $\alpha \in (0, +\infty)$, αu_0 is a solution of (1.2) on $\mathbf{R}^n$ that is homogeneous of degree 2. In this case we proceed with the dimension reduction: By [14, Lemma 11.2] we find a point $\bar{x} \in S_0 - \{0\}$ at which the density in [14, Proposition 11.3] is estimated from below. Now each blow-up limit u_{00} with respect to $\bar{x}$ (and with respect to a subsequence $m \to \infty$ such that the limit superior in [14, Proposition 11.3] becomes a limit) again satisfies the properties of Proposition 5.1; in addition, we obtain from the homogeneity of u_0 as in Lemma 3.1 of [23] that the rotated u_{00} is constant in the direction of the n-th unit vector. Defining $\bar{u}$ as the restriction of this rotated u_{00} to $\mathbf{R}^{n-1}$, it follows therefore that $\mathcal{H}^{s-1}(\{\bar{u} = 0\} \cap \{\nabla \bar{u} = 0\}) > 0$.
Repeating the whole procedure $n-2$ times we obtain a nontrivial homogeneous solution $u^\star$ of degree 2 in $\mathbf{R}$, satisfying $\mathcal{H}^{s-(n-2)}(\{u^\star = 0\} \cap \{\nabla u^\star = 0\}) > 0$, a contradiction.

7 Lipschitz arcs in two dimensions

In this section we show that the zero-set of the minimal solution consists in every second-order non-degenerate part of Ω of finitely many Lipschitz arcs which end – if so at all – in quadruple junctions, meeting at right angles.

In order to do the analysis, we have to prove uniform Lipschitz regularity close to singular points. The difficulty is that convergence to the blow-up limit is *not uniform* at singular points. In [19] we used a *novel* intersection-comparison approach to obtain that close to the singular point the free boundary is *uniformly the union of two graphs.* In our case it turns out that the classical intersection-comparison method (also called zero-number technique or lap-number technique) is sufficient, when combined with a very elementary implicit function theorem argument. The proof of the following theorem is inspired by [5].

Theorem 7.1 (Unique blow-up limit) *Let $n = 2$, let u be the minimal solution of (1.2) in Ω and suppose that u is non-degenerate of second order at $x^0 \in \Omega \cap \{u = 0\} \cap \{\nabla u = 0\}$. Then, as $0 < r \to 0$, and $S(x^0, r) := \left(r^{1-n} \int_{\partial B_r(x^0)} u^2 \, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}$,*

$$u_r(x) := \frac{u(x^0 + rx)}{S(x^0, r)}$$

converges to p where $(p \circ U)(x) = (x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}$ for some rotation U.

Proof: First, by Proposition 5.1, for any $\tilde{\epsilon} > 0$ there is $\tilde{\rho} > 0$ such that

$$\operatorname{dist}(u_r, M_g) < \tilde{\epsilon} \text{ for } r < \tilde{\rho} \; ;$$

here $M_g^* := \{(x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}\}, M_g := \{q \;:\; q \circ V(x) = (x_1^2 - x_2^2)/\|x_1^2 - x_2^2\|_{L^2(\partial B_1(0))}$ for some rotation $V\}$, and $\operatorname{dist}(u_r, M_g) := \inf_{q \in M_g} \sup_{x \in B_1(0)} |u_r(x) - q(x)|$. Denote by U_θ the counterclockwise rotation of positive angle θ. If the statement of the theorem does not hold, then – by uniform continuity of $t \mapsto u_{\exp(-t)}$ – there exists a sequence $r_m \downarrow 0$ and rotations U_{θ_1} and U_{θ_2} satisfying $|\theta_1 - \theta_2| = \epsilon \in (0, \pi)$ as well as

$$\operatorname{dist}(u_{r_{2m}} \circ U_{\theta_1}, M_g^*) \le \tilde{\epsilon} \text{ and } \operatorname{dist}(u_{r_{2m+1}} \circ U_{\theta_2}, M_g^*) \le \tilde{\epsilon}$$

for $m = 0, 1, 2, \ldots$

Note that we may assume $\theta_1 > \theta_2$. Now let $U = U_{\frac{\theta_1+\theta_2}{2}}, \omega = (\theta_1 - \theta_2)/2 \in (0, \pi/2)$ and define

$$\phi(r, \theta) := \frac{u(x^0 + rU(\cos\theta, \sin\theta))}{S(x^0, r)} \; .$$

For each $0 < r < r_0$, the function $\phi(r, \cdot)$ defines a function on the unit circle $[-\pi, \pi)$. Inspired by applications of the Aleksandrov reflection (see for example [12], [13], [5]) we consider now

$$\xi(r, -\theta) := \phi(r, \theta) - \phi(r, -\theta).$$

Observe that $\xi(r, 0) = \xi(r, \pi) = 0$. In what follows we will prove that $\frac{\partial \xi}{\partial \theta}(r_{2m}, 0) < 0$ and $\frac{\partial \xi}{\partial \theta}(r_{2m+1}, 0) > 0$ for large m. The comparison principle (applied to the minimal solutions $S(x^0,r)\phi(r,\theta)$ and $S(x^0,r)\phi(r,-\theta)$ in the two-dimensional domain $[0,r_0)\times(0,\pi)$ with respect to the original coordinates x_1 and x_2), tells us now that the connected component of $\{\xi<0\}$ touching $(r_{2m},0)$ intersects $\{r_0\}\times(0,\pi)$, and that the connected component of $\{\xi>0\}$ touching $(r_{2m+1},0)$ intersects $\{r_0\}\times(0,\pi)$. It follows that $\{r_0\}\times(0,\pi)$ contains infinitely many connected components of $\{\xi>0\}$ and $\{\xi<0\}$. On the other hand we know that, provided that r_0 has been chosen small enough, $u_{r_0}\circ U_{\theta_1}$ is close to $(x_1^2-x_2^2)/\|x_1^2-x_2^2\|_{L^2(\partial B_1(0))}$, so $\xi(r_0,\theta)$ is close to $\zeta(\theta):=c_1(\cos(\omega+\theta)^2-\sin(\omega+\theta)^2-\cos(\omega-\theta)^2+\sin(\omega-\theta)^2)=2c_1(\cos(\theta+\omega)^2-\cos(\theta-\omega)^2)$ in $C^1([0,\pi])$. As the zeroes of ζ are all non-degenerate, it is not possible that $\xi(r_0,\cdot)$ has infinitely many zeroes.

Therefore, in order to finish the proof, we have to show that $\frac{\partial\xi}{\partial\theta}(r_{2m},0)<0$ and $\frac{\partial\xi}{\partial\theta}(r_{2m+1},0)>0$ for large m. As $\xi(r_{2m},\cdot)$ is close to ζ in $C^1([0,\pi])$ and $\xi(r_{2m+1},\cdot)$ is close to $-\zeta$ in $C^1([0,\pi])$ we need only calculate $\zeta'(0)=-8c_1\cos(\omega)\sin(\omega)\le c_2(\omega)<0$. So for $\tilde\epsilon$ and r_0 sufficiently small (depending on ϵ) we obtain a contradiction.

Corollary 7.2 (Lipschitz arcs) *Let $n=2$, let u be the minimal solution of (1.2) in Ω and suppose that u is non-degenerate of second order at $x^0\in\Omega\cap\{u=0\}\cap\{\nabla u=0\}$. Then $\{u=0\}$ consists in an open neighborhood of x^0 of four Lipschitz arcs meeting at right angles.*

Proof: By Theorem 7.1 we know that for small $\delta>0$ and $r\in(0,\delta)$, the u_r of Theorem 7.1 satisfies

$$|\partial_2(u_r\circ U)|\ge c_1 \text{ in } B_2(0)\cap\{|x_2|\ge 1/8\}$$
$$\text{and } B_2(0)\cap\{u_r\circ U=0\}\subset B_{1/16}(\{|x_1|=|x_2|\})\ .$$

Thus $(B_2(0)-B_{1/8}(0))\cap\{u_r\circ U=0\}$ is the union of four C^1-graphs $g_r^j (j=1,2,3,4)$ in the x_2-direction, satisfying

$$\|(g_r^j)'\|_{C^0(I_j)}\le\frac{1}{c_1}\|\nabla u_r\|_{C^0(B_2(0))}\le C_2\ ,$$

where $I_j=[\pm1/2,\pm1)$. Rescaling yields the statement of the corollary.

8 Regularity of Local Minimizers

Theorem 8.1 (Regularity of local minimizers) *Let u be a minimizer of the energy*

$$E(v)=\int_{B_{r_0}(x^0)}|\nabla v|^2-2\max(v,0)$$

in $K:=\{v\in W^{1,2}(B_{r_0}(x^0)):v=u \text{ on } \partial B_{r_0}(x^0)\}$. Then the free boundary $\partial\{u>0\}$ is locally in Ω an analytic surface and $u|_{\overline{\{u>0\}}}, u|_{\overline{\{u<0\}}}$ are locally in Ω analytic functions.

Remark 8.2 *By Lemma 3.7 this implies the same regularity for the maximal solution.*

Remark 8.3 *The theorem also implies that local minimizers are locally in Ω of class $C^{1,1}$. Usually regularity of the solution is proved* before *proving regularity of the free boundary, but here we do it the other way around.*

We start with some preliminary results.

Lemma 8.4 *Let $n \geq 2$, let u be a minimizer of the energy*

$$E(v) = \int_{B_{r_0}(x^0)} |\nabla v|^2 - 2\max(v, 0)$$

in $K := \{v \in W^{1,2}(B_{r_0}(x^0)) : v = u \text{ on } \partial B_{r_0}(x^0)\}$, and suppose that $\nabla u(x) \neq 0$ on $\{u = 0\} \cap (B_{r_0}(x^0) \setminus \{x^0\})$. Then for $w \in C_0^\infty(B_{r_0}(x^0) \setminus \overline{B_\delta(x^0)})$ we have

$$0 \leq \delta^2 E(u) \cdot (w)(w) = \int_{B_{r_0}(x^0)} |\nabla w|^2 - \int_{\{u=0\} \cap B_{r_0}(x^0)} \frac{1}{|\nabla u|} w^2 \, .$$

Proof: We define

$$E_\epsilon(u) = \int_{B_{r_0}(x^0)} \frac{1}{2}|\nabla u|^2 - \gamma_\epsilon(u)$$

where $\gamma_\epsilon(u)$ is an approximation of $\max(u, 0)$ such that $\gamma_\epsilon''(u) = 1/\epsilon$ if $u \in (0, \epsilon)$ and zero otherwise. We see that for $w \in C_0^\infty(B_{r_0}(x^0) \setminus \overline{B_\delta(x^0)})$ we have

$$\frac{1}{t^2}\left(E_\epsilon(u + tw) - E_\epsilon(u) - t\delta E_\epsilon(u)(w)\right) = A_\epsilon^t$$

where

$$A_\epsilon^t = \frac{1}{t^2} \int_{B_{r_0}(x^0)} \frac{t^2}{2}|\nabla w|^2 - (\gamma_\epsilon(u + tw) - \gamma_\epsilon(u) - t\gamma_\epsilon'(u)w)$$

can be rewritten as

$$A_\epsilon^t = \int_{B_{r_0}(x^0)} \left(\frac{1}{2}|\nabla w|^2 - \int_0^1 d\alpha \int_0^\alpha d\sigma \; \gamma_\epsilon''(u + \sigma tw)\; w^2\right)$$

$$= \int_{B_{r_0}(x^0)} \frac{1}{2}|\nabla w|^2 - \int_0^1 d\alpha \int_0^\alpha \frac{1}{\epsilon} \int_{B_{r_0}(x^0) \cap \{0 < u + \sigma tw < \epsilon\}} w^2 \, .$$

By the co-area formula, we obtain for small t that

$$A_\epsilon^t \to A_0^t = \int_{B_{r_0}(x^0)} \frac{1}{2}|\nabla w|^2 - \int_0^1 d\alpha \int_0^\alpha d\sigma \int_{\{u+\sigma tw=0\} \cap B_{r_0}(x^0)} \frac{1}{|\nabla(u + \sigma tw)|} w^2$$

as $\epsilon \to 0$. We conclude that

$$0 \leq \frac{1}{t^2}\left(E(u + tw) - E(u) - t\delta E(u)(w)\right) = A_0^t \, .$$

Last, we take the limit $t \to 0$ and obtain

$$0 \le \frac{1}{2}\delta^2 E(u)(w)(w) = \int_{B_{r_0}(x^0)} \frac{1}{2}|\nabla w|^2 - \frac{1}{2}\int_{\{u=0\}\cap B_{r_0}(x^0)} \frac{1}{|\nabla u|} w^2 \ .$$

Lemma 8.5 *If u is a solution in $B_{r_0}(0)$ of satisfying $\nabla u(0) = 0$, then there exists a constant $C < \infty$ such that*

$$|\nabla u(x)| \le C|x| \log 1/|x| \quad \text{in } B_{r_0/2}(0) \ . \tag{8.13}$$

Proof: From Chemin [11] we infer that $\Delta u \in L^\infty$ implies $u \in C^2_*$. Thus $\nabla u \in C^1_*$, and there exists a constant $C < \infty$ such that

$$|\nabla u(x) - \nabla u(y)| \le C|x-y|(1 + \log 1/|x-y|) \ \text{ in } B_{r_0/2}(0)$$

which proves the Lemma.
Proof of Theorem 8.1:
Step 1 (Dimension $n = 2$):
From Lemma 3.6 and Corollary 6.1 we know that locally the free boundary is either a C^1-arc – in which case the gradient is non-zero on the free boundary – or a cross composed of 4 Lipschitz arcs meeting at right angles – in which case the gradient is zero at the center x^1 of the cross. We want to show that such a cross is impossible for local minimizers. We may assume $x^1 = 0$.
From Lemma 8.4 and (8.13), we deduce that for some constant $c_1 > 0$

$$0 \le \int_{B_{r_1}(0)} |\nabla w|^2 - c_1 \int_{\{u=0\}\cap B_{r_1}(0)} \frac{1}{|x| \log 1/|x|} w^2 \tag{8.14}$$

We now consider $w_\delta(x) = \phi(x) - \phi(\frac{x}{\delta})$ where $\phi \in C^\infty_0(B_{r_1}(0))$ such that $\phi = 1$ on $B_{\frac{r_1}{2}}(0)$. It follows that $\int_{B_{r_1}(0)} |\nabla w_\delta|^2 \le C_2 < \infty$, and using for large $i \in \mathbf{N}$ the regularity of $\{u = 0\} \cap (B_{2^{-i}}(0) - B_{2^{-i-1}}(0))$ as well as the closeness to the rotated cross we obtain

$$C_3 \int_{\{u=0\}\cap B_{r_1}(0)} \frac{1}{|x| \log 1/|x|} w_\delta^2 \ge \int_\delta^{\frac{r_1}{2}} dr \ \frac{1}{r \log r} \to +\infty$$

as $\delta \to 0$, a contradiction to the boundedness of w_δ.
Therefore the cross is not a local minimizer, and in dimension $n = 2$ the free boundary is locally in Ω a C^1-arc for each local minimizer.
Step 2 (Dimension $n > 2$):
We proceed by induction.
We assume that we have proved that for local minimizers, the free boundary is smooth up to the dimension $n - 1 \ge 2$.
Now we cannot have an accumulation of singularities in dimension n: Blowing up at a

limit point and blowing up a second time at a singularity $\bar{x} \neq 0$ of the blow-up limit, we would obtain as in the proof of Proposition 6.3 (see also [23, Lemma 3.1, Lemma 3.2]) a local minimizer with a singularity in dimension $n-1$. Thus singularities are isolated, and every blow-up limit of u at each singularity is a harmonic polynomial of degree 2 whose gradient vanishes only at one point of the 0-level set. Thus we still have (8.14), and the free boundary is for large $i \in \mathbf{N}$ on $B_{2^{-i}}(0) - B_{2^{-i-1}}(0)$ close to the zero level set of a homogeneous harmonic polynomial P_i of degree 2 satisfying by Proposition 5.1 $|P_i(x)| \leq C_4|x|^2$ and $|\nabla P_i| \geq c_5|x|$ on $(\mathbf{R}^n - \{0\}) \cap \{P_i = 0\}$ where $C_4 < \infty$ and $c_5 > 0$ do not depend on i.
Now we choose $w_\delta(x) = |x|^{-(\frac{n-2}{2})}(\phi(x) - \phi(\frac{x}{\delta}))$ where $\phi \in C_0^\infty(B_{r_1}(0))$ and $\phi = 1$ on $B_{\frac{r_1}{2}}(0)$. We obtain $\int_{B_{r_1}(0)} |\nabla w_\delta|^2 \leq C_6$, and using the regularity of $\{u = 0\} \cap (B_{2^{-i}}(0) - B_{2^{-i-1}}(0))$ as well as the closeness to $\{P_i = 0\}$ we see that

$$C_7 \int_{\{u=0\} \cap B_{r_1}(0)} \frac{1}{|x| \log 1/|x|} w_\delta^2 \geq \int_\delta^{\frac{r_1}{2}} dr \, \frac{1}{r \log r} \left(\inf_i \int_{\partial B_1(0) \cap \{P_i=0\}} 1 \, d\mathcal{H}^{n-2} \right) \to +\infty$$

as $\delta \to 0$, a contradiction to the boundedness of w_δ.
Therefore the local minimizers have no singularities in dimension n, and the free boundary is locally in Ω a C^1-surface.
Step 3 : Analyticity of the free boundary
We obtain analyticity of the free boundary as well as analyticity of $u|_{\overline{\{u>0\}}}, u|_{\overline{\{u<0\}}}$ as in the proof of Theorem 4.1 in Chapter 6 of [16]. See also [15, Theorem 3.1'].

9 The Cross Singularity

The reader may wonder whether there exists an example of a singularity for the maximal solution (and thereby a counter-example to the $W^{2,\infty}$-regularity in this unstable problem). We have at this moment no conclusive answer, but the following formal asymptotic expansion suggests that the cross may be a possibility:

Lemma 9.1 (Formal asymptotics)
Let us assume the existence of a solution u in $B_1(0)$ such that u satisfies for

$$x_1 = r\cos\alpha, x_2 = r\sin\alpha$$

$$u(-x_1, x_2) = u(x_1, x_2) = u(x_1, -x_2),$$

and the free boundary in the set $B_1(0) \cap \{x_1 > 0, x_2 > 0\}$ is given by

$$\alpha = \frac{\pi}{4}(1 + \phi(\rho)), \quad \rho = \frac{1}{-\log r}\ .$$

Moreover we assume that

$$u > 0 \quad in \quad 0 < \alpha < \frac{\pi}{4}(1+\phi(\rho)) \quad and \quad u < 0 \quad in \quad \frac{\pi}{4}(1+\phi(\rho)) < \alpha < \pi/2 \ .$$

Then formally ϕ and u satisfy

$$\phi(\rho) = -\frac{\rho}{2} + O(\rho^2)$$

$$and\ u(x) = \frac{1}{2\pi}(x_1^2 - x_2^2)(-\log|x|) + O(|x|^2) \ .$$

Formal proof:
We set

$$u(x_1,x_2) = \begin{cases} r^2(-\log(r))\ z^+(\rho,\theta^+) & \text{for} \quad \theta^+ = \dfrac{\alpha}{1+\phi(\rho)} \qquad \text{if} \quad 0 < \alpha < \dfrac{\pi}{4}(1+\phi(\rho)) \\ \\ -r^2(-\log(r))\ z^-(\rho,\theta^-) & \text{for} \quad \theta^- = \dfrac{\pi/2-\alpha}{1-\phi(\rho)} \qquad \text{if} \quad \dfrac{\pi}{4}(1+\phi(\rho)) < \alpha < \pi/2 \ . \end{cases}$$

By continuity of u and ∇u on the free boundary, we have

$$z^+ = z^- = 0 \quad \text{and} \quad z^+_\theta = \left(\frac{1+\phi(\rho)}{1-\phi(\rho)}\right) z^-_\theta \quad \text{for} \quad \theta^+ = \theta^- = \pi/4 \ .$$

By the symmetries of u we also have

$$z^+_\theta = z^-_\theta = 0 \quad \text{for} \quad \theta^+ = \theta^- = 0 \ .$$

Moreover $z = z^+$ satisfies for $0 < \theta = \theta^+ < \pi/4$:

$$z_{\theta\theta} + 4(1+\phi(\rho))^2 z + (\rho - 4\rho z)\,(1+\phi(\rho))^2 + \rho^2 I[z,\phi] = 0$$

where

$$\begin{aligned} I[z,\phi] = \quad & 4\left((1+\phi(\rho))^2 z_\rho - \theta z_\theta \phi'(\rho)(1+\phi(\rho))\right) \\ + \quad & \rho^2\left((1+\phi(\rho))^2 z_{\rho\rho} + \phi'^2(\rho)(\theta^2 z_{\theta\theta} + 2\theta z_\theta) - (1+\phi(\rho))\left\{2\theta z_{\rho\theta}\phi'(\rho) + \theta z_\theta \phi''(\rho)\right\}\right) , \end{aligned}$$

and for $0 < \theta = \theta^- < \pi/4$, the function $z = z^-$ satisfies a similar equation with ϕ replaced by $-\phi$ and without the term $\rho(1+\phi(\rho))^2$, i.e.

$$z_{\theta\theta} + 4(1-\phi(\rho))^2 z - 4\rho z(1-\phi(\rho))^2 + \rho^2 I[z,-\phi] = 0 \ .$$

Let us now introduce the formal asymptotic expansion

$$\begin{cases} \phi(\rho) = \phi^0 + \rho\phi^1 + \rho^2\phi^2 + ... \\ \\ z^\pm = z^{\pm,0}(\theta) + \rho z^{\pm,1}(\theta) + \rho^2 z^{\pm,2}(\theta) + ... \end{cases}$$

where

$$\phi^0 = 0, \quad z^{\pm,0}(\theta) = A^0 \cos(2\theta) \quad \text{and} \quad A^0 > 0 \quad \text{by assumption.}$$

For the order 0 terms we obtain (for $0 < \theta < \pi/4$)

$$z^{\pm,0}_{\theta\theta} + 4z^{\pm,0} = 0, \quad z^{\pm,0}_\theta(0) = 0, \quad z^{\pm,0}(\pi/4) = 0, \quad z^{+,0}_\theta(\pi/4) = z^{-,0}_\theta(\pi/4)\ ,$$

which is compatible with the assumptions. For the order 1 terms we obtain (for $0 < \theta < \pi/4$)

$$\begin{cases} z^{+,1}_{\theta\theta} + 4z^{+,1} + (8\phi^1 - 4)z^{+,0} + 1 = 0, \qquad z^{-,1}_{\theta\theta} + 4z^{-,1} + (-8\phi^1 - 4)z^{+,0} = 0, \\ \\ z^{\pm,1}_\theta(0) = 0, \quad z^{\pm,1}(\pi/4) = 0, \quad z^{+,1}_\theta(\pi/4) = z^{-,1}_\theta(\pi/4) - 2\phi^1 z^{-,0}_\theta(\pi/4)\ . \end{cases}$$

Thus

$$\begin{aligned} &z^{+,1}(\theta) = A^{+,1}\cos(2\theta) + (1 - 2\phi^1)A^0\theta\sin(2\theta) - \frac{1}{4} \\ &\text{and } z^{-,1}(\theta) = A^{-,1}\cos(2\theta) + (1 + 2\phi^1)A^0\theta\sin(2\theta) \end{aligned}$$

with $\phi^1 = -\frac{1}{2}$, $A^0 = \frac{1}{2\pi}$ and $A^{+,1} - A^{-,1} = \frac{1}{\pi}$.

10 Open Questions

The most urgent remaining questions in this context are, whether the cross singularity can be proven to exist, and whether there are examples of second order degeneracy. Another interesting point is whether methods similar to those used in this paper (possibly combined with arguments as in [1]) can be used to prove regularity in the composite membrane problem (see [6], [10], [9]).

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