Error estimate for the transition 3d-1d in anisotropic heterogeneous linearized elasticity

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1 Introduction

1.1 Notations

Let ω be a bounded and smooth enough open set of $\mathbf{R}^2$ and $I = (0, L)$. We assume that the origine and the axis are chosen such that

$$\int_\omega x_1 dx' = \int_\omega x_2 dx' = \int_\omega x_1 x_2 dx'$$

where we note $x' = (x_1, x_2)$. We note $x = (x', x_3)$ a point of $\mathbf{R}^3$. We use the greek indices $\alpha, \beta, \gamma, \delta, \ldots$, to take the values $1, 2$ (except for ε which will be used as a quantity converging to zero) and the latin indices $i, j, k, l, \ldots$, to take the values $1, 2, 3$. We use the convention of summation on repeated indices. The notation x^R is used for the vector of $\mathbf{R}^3$ defined by $x^R = (-x_2, x_1, 0)$ and obtained from x by a rotation of the two first components and vanishing

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the last component.

We work on the cylinder $\Omega = \omega \times I$. We note $\Sigma = \partial\omega \times I$ the lateral boundary of Ω and $\Gamma = \omega \times \{0\} \cup \omega \times \{L\}$ the union of the top and bottom faces of the cylinder.

We define the spaces of real valued functions

$$H^1_b(\Omega) = \left\{\phi \in H^1(\Omega) : \phi = 0 \quad \text{on} \quad \Gamma\right\}$$

$$H^1_b(I) = \left\{\phi \in H^1(I) : \phi = 0 \quad \text{on} \quad \partial I\right\}$$

$$H^2_b(I) = \left\{\phi \in H^2(I) : \phi = \frac{d\phi}{dx_3} = 0 \quad \text{on} \quad \partial I\right\}$$

$$H^1_m(\omega) = \left\{\psi \in H^1(\omega) : \int_\omega \psi(x')dx' = 0\right\}$$

We also define the spaces of vectorial functions with values in $\mathbf{R}^3$

$$\left\{\begin{array}{lll} BN_b(\Omega) = \{u : & \exists\zeta_\alpha \in H^2_b(I), \quad u_\alpha(x) = \zeta_\alpha(x_3), \quad \alpha = 1,2 \\ & \exists\zeta_3 \in H^1_b(I), \quad u_3(x) = \zeta_3(x_3) - x_\alpha \frac{d\zeta_\alpha}{dx_3}(x_3)\} \end{array}\right.$$

(where the notation BN is refered to the Bernouilli-Navier displacements),

$$\left\{\begin{array}{ll} R_b(\Omega) = \{v : & \exists c \in H^1_b(I), \quad v_\alpha(x) = c(x_3)x^R_\alpha, \quad \alpha = 1,2 \\ & v_3 \in L^2(I; H^1_m(\omega))\} \end{array}\right.$$

(where the notation R is refered to the two-dimensional rotations),

$$\left\{\begin{array}{ll} RD_2^\perp(\Omega) = \{w : & w_\alpha \in L^2(I; H^1_m(\omega)), \quad \alpha = 1,2 \\ & \int_\omega \left(-x_2 w_1(x', x_3) + x_1 w_2(x', x_3)\right) dx' = 0 \quad \text{a.e.} \quad x_3 \in I, \\ & w_3(x) = 0\} \end{array}\right.$$

(where the notation RD_2 is refered to the rigid two-dimensional displacements).

Given a displacement $\phi \in (\mathcal{D}'(\Omega))^3$, we note $e(\phi)$ its symmetrized gradient, i.e. the symmetric tensor in $(\mathcal{D}'(\Omega))_s^{3\times 3}$ with the following components

$$e_{ij}(\phi) = \frac{1}{2}\left(\frac{\partial \phi_j}{\partial x_i} + \frac{\partial \phi_i}{\partial x_j}\right), \quad i,j = 1,2,3$$

We define

$$\begin{cases} \mathcal{E}(\Omega) = \Big\{ e \in (L^2(\Omega))_s^{3\times 3} : \ \exists (u,v,w) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^{\perp}(\Omega), \\ \qquad\qquad\qquad\qquad e = \begin{pmatrix} e_{\alpha\beta}(w) & e_{\alpha 3}(v) \\ e_{\alpha 3}(v) & e_{33}(u) \end{pmatrix} \Big\} \end{cases}$$

where we have symbolically noted $e_{\alpha\beta}(w)$ for the tensor $\{e_{\alpha\beta}(w) : \alpha, \beta = 1,2\}$ of $\mathbf{R}_s^{2\times 2}$ and $e_{\alpha 3}(v)$ for the vector $\{e_{\alpha 3}(v), \alpha = 1,2\}$ of $\mathbf{R}^2$. We can show (see [16]) that $BN_b(\Omega), R_b(\Omega), RD_2^{\perp}(\Omega)$ are Hilbert spaces for the norms whose square are given by

$$||e_{33}(u)||^2_{L^2(\Omega)}, \quad \sum_{\alpha} ||e_{\alpha 3}(v)||^2_{L^2(\Omega)}, \quad \sum_{\alpha,\beta} ||e_{\alpha\beta}(w)||^2_{L^2(\Omega)}$$

and that $\mathcal{E}(\Omega)$, where the square of its norm is chosen equal to the sum of the square of these norms, is a Hilbert space.
For $\phi \in (\mathcal{D}'(\Omega))^3$, we define the tensor $e^{\varepsilon}(\phi)$ by

$$e^{\varepsilon}(\phi) = \begin{pmatrix} \frac{1}{\varepsilon^2} e_{\alpha\beta}(\phi) & \frac{1}{\varepsilon} e_{\alpha 3}(\phi) \\ \\ \frac{1}{\varepsilon} e_{\alpha 3}(\phi) & e_{33}(\phi) \end{pmatrix}$$

1.2 Setting of the problem

Let A be a tensor of the forth order, strongly coercive and with $L^\infty(\Omega)$ coefficients, i.e. which satisfies

$$A_{ijkl} \in L^\infty(\Omega), \quad i,j,k,l = 1,2,3$$

$$A_{ijkl} = A_{jikl} = A_{ijlk}, \quad i,j,k,l = 1,2,3$$

$$A_{ijkl}(x)\overline{e}_{kl}\overline{e}_{ij} \geq m \sum_{i,j} |\overline{e}_{ij}|^2, \quad \text{p.p.} x \in \Omega, \quad \forall \overline{e} \in \mathbf{R}_s^{3\times 3}$$

for $m > 0$, and let

$$h \in \left(L^2(\Omega)\right)_s^{3\times 3}$$

We study the variational problem

$$\begin{cases} \displaystyle\int_\Omega A(x)e^\varepsilon(u^\varepsilon)e^\varepsilon(\phi)dx = & \displaystyle\int_\Omega he^\varepsilon(\phi)dx \\ u^\varepsilon \in \left(H_b^1(\Omega)\right)^3, & \forall\phi \in \left(H_b^1(\Omega)\right)^3 \end{cases}$$

1.3 Main result

It is proved in [16] that the unique solution u^ε of the problem strongly converges in $\left(H_b^1(\Omega)\right)^3$ to a function u where $u \in BN_b(\Omega)$ is uniquely determined by the quantity $e_{33}(u)$. Moreover $e^\varepsilon(u^\varepsilon)$ strongly converges in $(L^2(\Omega))_s^{3\times 3}$ to $e = \begin{pmatrix} e_{\alpha\beta}(w) & e_{\alpha 3}(v) \\ e_{\alpha 3}(v) & e_{33}(u) \end{pmatrix}$ with $v \in R_b(\Omega), w \in RD_2^\perp(\Omega)$ and where $e \in \mathcal{E}(\Omega)$ is the unique solution of the problem

$$\int_\Omega Ae\overline{e}\ dx = \int_\Omega h\overline{e}\ dx, \quad \forall \overline{e} \in \mathcal{E}(\Omega)$$

Our main result is the following error estimate

Theorem 1.1 *We assume that $v_3, w_\alpha \in H_b^1(\Omega)$ and that $(h - Ae)_{33} \in H_b^2(I; L^2(\omega))$, $(h - Ae)_{\alpha 3} \in H_b^1(I; L^2(\omega))$. There exists a constant $C = C(m, |A|_{L^\infty(\Omega)}) > 0$ such that*

$$\left|\left|e^\varepsilon\left(u^\varepsilon - \left(u + \varepsilon v + \varepsilon^2 w\right)\right)\right|\right|_{L^2(\Omega)} \leq \varepsilon CM$$

where

$$M = ||v||_{(H^1(\Omega))^3} + ||w||_{\left(H_b^1(\Omega)\right)^3} + ||(h - Ae)_{33}||_{H_b^2(I;L^2(\omega))} + ||(h - Ae)_{\alpha 3}||_{H_b^1(I;L^2(\omega))}$$

In particular we deduce

$$||u^\varepsilon - u||_{H^1(\Omega)} \leq \varepsilon C M$$

Remark 1.2 *This result can be extended more easily to the case of a cylinder $\Omega = \omega \times I$ with a periodic interval $I : I = \mathbf{R} \backslash L\mathbf{Z}$ for some $L > 0$.*

2 Proof of theorem 1.1

In what follows, it will be possible to state an error estimate, because we have made the had hoc assumptions to control the boundary layers (assumptions of theorem 1.1).

Let us recall that the problem for fixed $\varepsilon > 0$

$$\int_\Omega A e^\varepsilon(u^\varepsilon) e^\varepsilon(\phi) = \int_\Omega h e^\varepsilon(\phi), \quad \forall \phi \in \left(H^1_b(\Omega)\right)^3 \tag{2.1}$$

degenerates as $\varepsilon \to 0$ in

$$\int_\Omega A e \overline{e} = \int_\Omega h \overline{e}, \quad \forall \overline{e} \in \mathcal{E}(\Omega)$$

Then we naturally introduce the quantity

$$\hat{u}^\varepsilon = u + \varepsilon v + \varepsilon^2 w$$

which satisfies

$$e^\varepsilon(\hat{u}^\varepsilon) = e + \varepsilon \begin{pmatrix} 0 & e_{\alpha 3}(w) \\ e_{\alpha 3}(w) & e_{33}(v) \end{pmatrix}$$

The last term is not in $L^2(\Omega)$ if we only assume $v \in R_b(\Omega)$ and $w \in RD_2^\perp(\Omega)$. On the contrary if we assume more regularity (i.e. $v_3, w_\alpha \in H^1(I; L^2(\omega))$, the

last term is then bounded by $\varepsilon\left(||v_3||^2_{H^1(I;L^2(\omega))}+\sum_\alpha ||w_\alpha||^2_{H^1(I;L^2(\omega))}\right)^{\frac{1}{2}}$.
Now we have

$$\int_\Omega Ae^\varepsilon(u^\varepsilon-\hat{u}^\varepsilon)e^\varepsilon(\phi)=\int_\Omega\left(h-Ae^\varepsilon(\hat{u}^\varepsilon)\right)e^\varepsilon(\phi)$$

Setting

$$z^\varepsilon=u^\varepsilon-\hat{u}^\varepsilon,\quad \rho=h-Ae$$

we can rewrite

$$\int_\Omega Ae^\varepsilon(z^\varepsilon)e^\varepsilon(\phi)=\int_\Omega \rho e^\varepsilon(\phi)-\int_\Omega A\left(e^\varepsilon(\hat{u}^\varepsilon)-e\right)e^\varepsilon(\phi)$$

Because we have assumed that $v_3, w_\alpha\in H^1_b(I;L^2(\omega))$, it is possible by a classical density argument to choose $\phi=z^\varepsilon$, which gives

$$\int_\Omega Ae^\varepsilon(z^\varepsilon)e^\varepsilon(z^\varepsilon)=\int_\Omega \rho e^\varepsilon(z^\varepsilon)-\int_\Omega A\left(e^\varepsilon(\hat{u}^\varepsilon)-e\right)e^\varepsilon(z^\varepsilon)$$

Then

$$||e^\varepsilon(z^\varepsilon)||_{L^2(\Omega)}\leq\frac{1}{m}\left\{\frac{|\int_\Omega \rho e^\varepsilon(z^\varepsilon)|}{||e^\varepsilon(z^\varepsilon)||_{L^2(\Omega)}}+\varepsilon\,|A|_{L^\infty(\Omega)}\left(||v_3||^2_{H^1(I;L^2(\omega))}+\sum_\alpha ||w_\alpha||^2_{H^1(I;L^2(\omega))}\right)^{\frac{1}{2}}\right\}$$

To finish the proof of theorem 1.1 we only have to estimate the term

$$\int_\Omega \rho e^\varepsilon(z^\varepsilon)$$

This is a straightforward application of the following result, which will be proved in the rest of the article

Proposition 2.1 *Let $z\in\left(H^1(\Omega)\right)^3$ and $\rho\in\left(L^2(\Omega)\right)^{3\times3}_s$ with $\rho_{\alpha3}\in H^1_b(I;L^2(\omega))$, $\rho_{33}\in H^2_b(I;L^2(\omega))$. There exists a constant $C=C(\Omega)$ such that if*

$$\int_\Omega \rho\overline{e}=0,\quad \forall\overline{e}\in\mathcal{E}(\Omega)$$

Then

$$\left|\int_\Omega \rho e^\varepsilon(z)\right|\leq C\varepsilon\,||e^\varepsilon(z)||_{L^2(\Omega)}\left(||\rho_{33}||_{H^2_b(I;L^2(\omega))}+||\rho_{\alpha3}||_{H^1_b(I;L^2(\omega))}\right)$$

We can apply this proposition to z^ε which belongs to $(H^1(\Omega))^3$ because of our assumptions on v_3, w_α in theorem 1.1. This ends the proof of theorem 1.1.

3 Proof of proposition 2.1

3.1 Reexpression using cancellations

In the following we will prove the lemma

Lemma 3.1 *Let $z \in (H^1(\Omega))^3$ and $\rho \in (L^2(\Omega))_s^{3\times 3}$ with $\rho_{\alpha 3}, \rho_{33} \in H_b^1(I; L^2(\omega))$. If*

$$\int_\Omega \rho\overline{e} = 0, \quad \forall \overline{e} \in \mathcal{E}(\Omega)$$

then

$$\int_\Omega \rho e^\varepsilon(z) = I + II$$

where

$$I = -\frac{1}{2}\varepsilon^{-1}\int_\Omega z_\alpha^2\left(\partial_3\rho_{\alpha 3}\right), \quad with \quad z_\alpha^2 = z_\alpha - \frac{\int_\omega z_\alpha}{\int_\omega 1} - x_\alpha^R\frac{\int_\omega < z_\beta, x_\beta^R >}{\int_\omega |x^R|^2}$$

$$II = -\int_\Omega z_3^1\left(\partial_3\rho_{33}\right), \quad with \quad z_3^1 = z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha\frac{d}{dx_3}\left(\frac{\int_\omega z_\alpha}{\int_\omega 1}\right)\right)$$

3.1.1 Preliminary : functional decomposition

Lemma 3.2 *If $z \in (H_b^2(\Omega))^3$, then $\exists!(z^0, z^1, z^2) \in BN_b(\Omega)\times R_b(\Omega)\times RD_2^\perp(\Omega)$ such that*

$$z = z^0 + z^1 + z^2$$

Proof of lemma 3.2

We only have to remark that

$$\begin{pmatrix} z_\alpha \\ \\ z_3 \end{pmatrix} = \begin{pmatrix} \frac{\int_\omega z_\alpha}{\int_\omega 1} \\ \\ \frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3}\left(\frac{\int_\omega z_\alpha}{\int_\omega 1}\right) \end{pmatrix} + \begin{pmatrix} x_\alpha^R \frac{\int_\omega <z_\beta, x_\beta^R>}{\int_\omega |x^R|^2} \\ \\ z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3}\left(\frac{\int_\omega z_\alpha}{\int_\omega 1}\right)\right) \end{pmatrix} + \begin{pmatrix} z_\alpha^2 \\ \\ 0 \end{pmatrix}$$

where

$$z_\alpha^2 = z_\alpha - \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} + x_\alpha^R \frac{\int_\omega < z_\beta, x_\beta^R >}{\int_\omega |x^R|^2}\right)$$

This shows the existence of a triplet $(z^0, z^1, z^2) \in BN_b(\Omega) \times R_b(\Omega) \times RD_2^\perp(\Omega)$ such that $z = z^0 + z^1 + z^2$. The uniqueness comes from the unique decomposition of z_α. This one is a consequence of the orthogonality of the components $z_\alpha^0, z_\alpha^1, z_\alpha^2$ for the scalar product of $L^2(\Omega)$.

3.2 Proof of lemma 3.1

For a displacement $z \in H^1(\Omega)$, we formally write

$$z = z^0 + z^1 + z^2$$

and

$$\int_\Omega \rho e^\varepsilon(z) = \sum_{ijk} \int_\Omega \rho_{ij} e_{ij}^\varepsilon(z^k)$$

where each term $\int_\Omega \rho_{ij} e_{ij}^\varepsilon(z^k)$ is precisely defined as follows :

Terms in $\rho_{\alpha\beta}$

$$\varepsilon^2 \int_\Omega \rho_{\alpha\beta} e_{\alpha\beta}^\varepsilon(z^0) = \int_\Omega \rho_{\alpha\beta} \frac{1}{2}\left(\partial_\alpha \frac{\int_\omega z_\beta}{\int_\omega 1} + \partial_\beta \frac{\int_\omega z_\alpha}{\int_\omega 1}\right) = 0$$

$$\varepsilon^2 \int_\Omega \rho_{\alpha\beta} e_{\alpha\beta}^\varepsilon(z^1) = \int_\Omega \rho_{\alpha\beta} \frac{\int_\omega <z, x^R>}{\int_\omega |x^R|^2} \frac{1}{2}\left(\partial_\alpha(x_\beta^R) + \partial_\beta(x_\alpha^R)\right) = 0$$

$$\varepsilon^2 \int_\Omega \rho_{\alpha\beta} e_{\alpha\beta}^\varepsilon(z^2) = 0$$

because $e(z^2) \in \mathcal{E}(\Omega)$, because $z^2 = \begin{pmatrix} z_\alpha - \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} + x^R_\alpha \frac{\int_\omega <z, x^R>}{\int_\omega |x^R|^2} \right) \\ \\ 0 \end{pmatrix}$ is an element of $RD_2^\perp(\Omega)$

Terms in $\rho_{\alpha 3}$

$$\varepsilon \int_\Omega \rho_{\alpha 3} e^\varepsilon_{\alpha 3}(z^0) = \int_\Omega \rho_{\alpha 3} e_{\alpha 3} \begin{pmatrix} \frac{\int_\omega z_\beta}{\int_\omega 1} \\ \\ \frac{\int_\omega z_3}{\int_\omega 1} - x_\beta \frac{d}{dx_3} \left(\frac{\int_\omega z_\beta}{\int_\omega 1} \right) \end{pmatrix} = 0$$

$$\varepsilon \int_\Omega \rho_{\alpha 3} e^\varepsilon_{\alpha 3}(z^1) = 0$$

because $z^1 = \begin{pmatrix} x^R_\alpha \frac{\int_\omega <z, x^R>}{\int_\omega |x^R|^2} \\ \\ z_3 - \left(\frac{\int_\omega z^\varepsilon_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z^\varepsilon_\alpha}{\int_\omega 1} \right) \right) \end{pmatrix} = \begin{pmatrix} 0 \\ \\ z_3 - \frac{\int_\omega z^\varepsilon_3}{\int_\omega 1} \end{pmatrix} + \begin{pmatrix} x^R_\alpha \frac{\int_\omega <z, x^R>}{\int_\omega |x^R|^2} \\ \\ -x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z^\varepsilon_\alpha}{\int_\omega 1} \right) \end{pmatrix}$ where the first term is an element of $R_b(\Omega)$, and the second term vanishes $e_{\alpha 3}$.

$$\varepsilon \int_\Omega \rho_{\alpha 3} e^\varepsilon_{\alpha 3}(z^2) = \int_\Omega \rho_{\alpha 3} \tfrac{1}{2} \partial_3 z^2_\alpha$$

because $z^2 = \begin{pmatrix} z^\varepsilon_\alpha - \left(\frac{\int_\omega z^\varepsilon_\alpha}{\int_\omega 1} + x^R_\alpha \frac{\int_\omega <z, x^R>}{\int_\omega |x^R|^2} \right) \\ 0 \end{pmatrix}$.

Terms in ρ_{33}

Formally we would like to write

$$\int_\Omega \rho_{33} e^\varepsilon_{33}(z^0) = \int_\Omega \rho_{33} \partial_3 \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right)$$

There is a difficulty from the regularity point of view because $\frac{d^2}{dx_3^2} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \in H^{-1}(I)$. This difficulty can be avoided, if we change the definition of the term

$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^0)$. By assumptions we know that $\rho_{33} \in H_b^2(I; L^2(\omega))$, but at this stage we only use the fact that $\rho_{33} \in H_b^1(I; L^2(\omega)) = H_0^1(I; L^2(\omega))$ and define

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^0) = \int_I \left(\int_\omega \rho_{33}\right) \partial_3 \left(\frac{\int_\omega z_3}{\int_\omega 1}\right) - < \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1}\right) >_{H_0^1(I)\times H^{-1}(I)}$$

Similarly we have formally

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^1) = \int_\Omega \rho_{33} \partial_3 \left(z_3^\varepsilon - \left(\frac{\int_\omega z_3^\varepsilon}{\int_\omega 1} - x_\alpha \frac{d}{dx_3}\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right)\right)\right)$$

that we define as

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^1) = \int_\Omega \rho_{33} \partial_3 \left(z_3^\varepsilon - \frac{\int_\omega z_3^\varepsilon}{\int_\omega 1}\right) + < \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right) >_{H_0^1(I)\times H^{-1}(I)}$$

Finally

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^2) = 0$$

because by definition $z_3^2 = 0$.

More remarks on the case ρ_{33}

We now show the

Lemma 3.3

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^0) = 0$$

Proof of lemma 3.3

By construction we have

$$\int_\Omega \rho_{33} e_{33}(\overline{u}) = 0, \quad \forall \overline{u} = \left\{ \begin{array}{l} \zeta_\alpha(x_3) \\ \zeta_3(x_3) - x_\alpha \frac{d}{dx_3}\zeta_\alpha(x_3) \end{array} \right. \in BN_b(\Omega)$$

i.e.

$$\int_I \left(\int_\omega \rho_{33}\right) \frac{d}{dx_3}\zeta_3 + \left(\int_\omega x_\alpha \rho_{33}\right) \frac{d^2}{dx_3^2}\zeta_\alpha = 0$$

for all $\zeta_3 \in H_b^1(I), \zeta_\alpha \in H_b^2(I)$.

We can rewrite it as

$$\int_I \left(\int_\omega \rho_{33}\right) \frac{d}{dx_3}\zeta_3 + < \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2}\zeta_\alpha >_{H_0^1(I)\times H^{-1}(I)} = 0$$

because $\rho_{33} \in H_b^1(I;L^2(\omega)) = H_0^1(I;L^2(\omega))$. In particular we can choose

$$\zeta_3(x_3) = \frac{\int_\omega z_3}{\int_\omega 1}$$

Now by density, it is possible to find a sequence of functions $\zeta_\alpha^n \in H_b^2(I)$ which converges in $H^1(I)$ to $\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right) \in H_b^1(I)$. As a consequence $\frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha^n}{\int_\omega 1}\right)$ converges in $H^{-1}(I)$ to $\frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right) \in H^{-1}(I)$. By continuity of the duality braket, we deduce that

$$0 =< \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2}\zeta_\alpha^n >_{H_0^1(I)\times H^{-1}(I)} \quad \longrightarrow \quad < \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right) >_{H_0^1(I)\times H^{-1}(I)}$$

Therefore

$$\int_\Omega \rho_{33} e_{33}^\varepsilon(z^0) = \int_I \left(\int_\omega \rho_{33}\right) \partial_3 \left(\frac{\int_\omega z_3}{\int_\omega 1}\right) - < \left(\int_\omega x_\alpha \rho_{33}\right), \frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha}{\int_\omega 1}\right) >_{H_0^1(I)\times H^{-1}(I)} = 0$$

which ends the proof of lemma 3.3.

Conclusion

From the previous computations we deduce that

$$\begin{aligned} \int_\Omega \rho e^\varepsilon(z) &= \int_\Omega \rho_{\alpha 3} e_{\alpha 3}^\varepsilon(z^2) + \int_\Omega \rho_{33} e_{33}^\varepsilon(z^1) \\ &= I + II \end{aligned}$$

and

$$I = \frac{\varepsilon^{-1}}{2}\int_\Omega \rho_{\alpha 3}\partial_3 z_\alpha^2 = -\frac{\varepsilon^{-1}}{2}\int_\Omega z_\alpha^2 \left(\partial_3 \rho_{\alpha 3}\right)$$

where the boundary term of the integration by parts is zero because $\rho_{\alpha 3} \in H_b^1(I;L^2(\Omega))$;

$$II = \int_\Omega \rho_{33}\partial_3\left(z_3^\varepsilon - \frac{\int_\omega z_3^\varepsilon}{\int_\omega 1}\right) + < \int_\omega x_\alpha \rho_{33}, \frac{d^2}{dx_3^2}\left(\frac{\int_\omega z_\alpha^\varepsilon}{\int_\omega 1}\right) >_{H_0^1(I)\times H^{-1}(I)}$$

Then

$$II = -\int_\Omega (\partial_3\rho_{33}) \left\{ z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right) \right\}$$

where the boundary term of the integration by parts is zero because $\rho_{33} \in H_b^1(\Omega)$. This ends the proof of lemma 3.1.

3.3 Proof of proposition 2.1

Term I

Because $z^2 \in RD_2^\perp(\Omega)$, and because $||e_{\alpha\beta}(z^2)||_{L^2(\Omega)}$ is the norm of z^2 in the Hilbert space $RD_2^\perp(\Omega)$, we deduce that

$$\sum_\alpha \left|\left|z_\alpha^2\right|\right|_{L^2(\Omega)} \le C \left|\left|e_{\alpha\beta}\left(z^2\right)\right|\right|_{(L^2(\Omega))_s^{2\times 2}}$$

From the relation $e_{\alpha\beta}\left(\frac{z^2}{\varepsilon^2}\right) = e^\varepsilon_{\alpha\beta}(z)$ we then deduce the following estimate on $I = -\frac{\varepsilon^{-1}}{2}\int_\Omega z_\alpha^2 \left(\partial_3\rho_{\alpha 3}\right)$

$$|I| \le C\varepsilon \, ||\partial_3\rho_{\alpha 3}||_{L^2(\Omega)} \left|\left|e^\varepsilon_{\alpha\beta}(z)\right|\right|_{L^2(\Omega)}$$

Term II

Because $\rho_{33} \in H_b^2(\Omega)$, we can rewrite

$$II = - < (\partial_3\rho_{33}), \left\{ z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right) \right\} >_{H_0^1(I;L^2(\omega))\times H^{-1}(I;L^2(\omega))}$$

We now apply a partial Korn's inequality (remark 4.2), which gives

$$\begin{aligned} |II| &\le C\, ||\partial_{33}\rho_{33}||_{L^2(\Omega)} \Big(||e_{\alpha\beta}(z)||_{L^2(\Omega)} + ||e_{\alpha 3}(z)||_{L^2(\Omega)} \Big) \\ &\le C\, ||\partial_{33}\rho_{33}||_{L^2(\Omega)} \left(\varepsilon^2 \left|\left|e^\varepsilon_{\alpha\beta}(z)\right|\right|_{L^2(\Omega)} + \varepsilon\, ||e^\varepsilon_{\alpha 3}(z)||_{L^2(\Omega)} \right) \\ &\le C\varepsilon\, ||\partial_{33}\rho_{33}||_{L^2(\Omega)} \, ||e^\varepsilon(z)||_{L^2(\Omega)} \end{aligned}$$

This ends the proof of proposition 2.1.

4 A partial Korn's inequality

We will prove the following result. Let us recall that $\Omega = \omega \times I$ where $I = (0, L)$.

Theorem 4.1 (A partial Korn's inequality)
There exists a constant $C = C(\omega) > 0$ such that if $z \in \left(H^1(\Omega)\right)^2$ then

$$\left\| z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right) \right\|_{H^{-1}(I;H^1(\omega))} \le C \left(\begin{array}{c} \sum_{\alpha,\beta} ||e_{\alpha\beta}(z)||_{L^2(I;H^{-1}(\omega))} \\ + \sum_\alpha ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \end{array} \right) \tag{4.2}$$

Remark 4.2 *The following inequality*

$$\left\| z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right) \right\|_{H^{-1}(I;L^2(\omega))} \le C \left(\sum_{\alpha,\beta} ||e_{\alpha\beta}(z)||_{L^2(\Omega)} + \sum_\alpha ||e_{\alpha 3}(z)||_{L^2(\Omega)} \right) \tag{4.3}$$

is a straightforward consequence of the partial Korn's inequality (4.2). Moreover this last inequality is easier to prove and sufficient to state the main result of this article (theorem 1.1).

4.1 Preliminary remarks

A natural question is to ask if a similar inequality like the following is true?

$$\left\| z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right) \right\|_{L^2(\Omega)} \le C \left(\sum_{\alpha,\beta} ||e_{\alpha\beta}(z)||_{L^2(\Omega)} + \sum_\alpha ||e_{\alpha 3}(z)||_{L^2(\Omega)} \right) \tag{4.4}$$

Althougth this last inequality can seem more natural at a first glance, inequality (4.4) is false because of the following counter-example.

Let us consider a scalar function $\phi(x)$ defined on Ω. Then we set

$$z_\alpha = \partial_\alpha \phi, \quad z_3 = -\partial_3 \phi$$

We have in particular

$$e_{\alpha 3}(z) = 0, \quad e_{\alpha\beta}(z) = \partial_{\alpha\beta}\phi$$

Moreover for $z_3^1 = z_3 - \left(\frac{\int_\omega z_3}{\int_\omega 1} - x_\alpha \frac{d}{dx_3} \left(\frac{\int_\omega z_\alpha}{\int_\omega 1} \right) \right)$, we have

$$\begin{array}{rcl} z_3^1 & = & \partial_3 \phi - \left(\frac{\int_\omega \partial_3 \phi}{\int_\omega 1} - x_\alpha \frac{\int_\omega \partial_{\alpha 3}\phi}{\int_\omega 1} \right) \\ & = & \partial_3 \left(\phi - \left(\frac{\int_\omega \phi}{\int_\omega 1} - x_\alpha \frac{\int_\omega \partial_\alpha \phi}{\int_\omega 1} \right) \right) \end{array}$$

In particular we deduce that $||z_3^1||_{L^2(\Omega)}$ is not bounded by $||e_{\alpha\beta}(z)||_{L^2(\Omega)} + ||e_{\alpha 3}(z)||_{L^2(\Omega)}$.
More generally this counter-example shows that $||e_{\alpha\beta}(z)||_{L^2(\Omega)} + ||e_{\alpha 3}(z)||_{L^2(\Omega)}$ can not bound the difference between z_3 and a projection $\tilde{z}_3$ of z_3, except in the trivial case where $z_3 - \tilde{z}_3 = 0$.

4.2 Proof of theorem 4.1

Let us recall that we note

$$z_\alpha^2 = z_\alpha - d_\alpha(x_3) - x_\alpha^R c(x_3)$$

where

$$d_\alpha(x_3) = \frac{\int_\omega z_\alpha}{\int_\omega 1}, \quad c(x_3) = \frac{\int_\omega < x_\beta^R, z_\beta >}{\int_\omega |x^R|^2}$$

Now we compute

$$\begin{array}{rcl} \partial_\alpha z_3 & = & -\partial_3 z_\alpha + 2e_{\alpha 3}(z) \\ & = & -\partial_3 \left(z_\alpha^2 + d_\alpha(x_3) + x_\alpha^R c(x_3) \right) + 2e_{\alpha 3}(z) \end{array}$$

This implies

$$\partial_\alpha \left(z_3 + x_\beta \partial_3 d_\beta(x_3)\right) + x_\alpha^R \partial_3 c(x_3) = -\partial_3 z_\alpha^2 + 2e_{\alpha 3}(z)$$

We then take the two-dimensional curl and get

$$2\partial_3 c(x_3) = -\partial_3 \left(\partial_1 z_2^2 - \partial_2 z_1^2\right) + 2\left(\partial_1 e_{23}(z) - \partial_2 e_{13}(z)\right)$$

In particular we deduce that

$$\begin{aligned} ||\partial_3 c(x_3)||_{H^{-1}(I;H^{-1}(\omega))} &\leq \tfrac{1}{2} ||\partial_3 \left(\partial_1 z_2^2 - \partial_2 z_1^2\right)||_{H^{-1}(I;H^{-1}(\omega))} \\ &\quad + ||\partial_1 e_{23}(z) - \partial_2 e_{13}(z)||_{H^{-1}(I;H^{-1}(\omega))} \\ &\leq \tfrac{1}{2} \textstyle\sum_\alpha ||\partial_3 z_\alpha^2||_{H^{-1}(I;L^2(\omega))} + \sum_\alpha ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \end{aligned}$$

Now we remark that for each $\alpha = 1, 2$ we have

$$\left|\left|x_\alpha^R \partial_3 c(x_3)\right|\right|_{H^{-1}(I;L^2(\omega))} \leq a_0 \, ||\partial_3 c(x_3)||_{H^{-1}(I;H^{-1}(\omega))}$$

where $a_0 = \sup_\alpha \frac{||x_\alpha^R||_{L^2(\omega)}}{||1||_{H^{-1}(\omega)}}$, because $c(x_3)$ only depends on x_3.
Finally from the relation

$$\partial_\alpha \left(z_3 + x_\beta \partial_3 d_\beta(x_3)\right) = -x_\alpha^R \partial_3 c(x_3) - \partial_3 z_\alpha^2 + 2e_{\alpha 3}(z)$$

and the Poincaré-Wirtinger inequality for a general function $\phi \in H^1(\omega)$

$$\left|\left|\phi - \frac{\int_\omega \phi}{\int_\omega 1}\right|\right|_{H^1(\omega)}^2 \leq C_{PW} \, ||\nabla \phi||_{L^2(\omega)}^2$$

we deduce that

$$
\begin{aligned}
J := \quad & \left\| z_3 + x_\beta \partial_3 d_\beta(x_3) - \frac{\int_\omega z_3}{\int_\omega 1} \right\|^2_{H^{-1}(I;H^1(\omega))} \\
\leq C_{PW} \textstyle\sum_\alpha & \left(\left\| x^R_\alpha \partial_3 c(x_3) \right\|_{H^{-1}(I;L^2(\omega))} + ||\partial_3 z^2_\alpha||_{H^{-1}(I;L^2(\omega))} + 2\, ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \right)^2 \\
\leq C_{PW} \textstyle\sum_\alpha & \left(a_0\, ||\partial_3 c(x_3)||_{H^{-1}(I;H^{-1}(\omega))} + ||\partial_3 z^2_\alpha||_{H^{-1}(I;L^2(\omega))} + 2\, ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \right)^2 \\
\leq C_{PW}(a_0+2)^2 \textstyle\sum_\alpha & \left(\tfrac{1}{2}\, ||\partial_3 z^2_\alpha||_{H^{-1}(I;L^2(\omega))} + ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \right)^2 \\
\leq C_{PW}(a_0+2)^2 \textstyle\sum_\alpha & \left(\tfrac{1}{2}\, ||z^2_\alpha||_{L^2(I;L^2(\omega))} + ||e_{\alpha 3}(z)||_{H^{-1}(I;L^2(\omega))} \right)^2 \\
\leq 2C_{PW}(a_0+2)^2 \textstyle\sum_\alpha & \left(\tfrac{1}{4}\, ||z^2_\alpha||^2_{L^2(I;L^2(\omega))} + ||e_{\alpha 3}(z)||^2_{H^{-1}(I;L^2(\omega))} \right)
\end{aligned}
$$

where we have used the general inequality $(x+y)^2 \leq 2x^2 + 2y^2$. At this stage we can apply the classical Korn's inequality for a general two-dimensional displacement $\psi \in (H^1(\omega))^2$ with $\psi^2_\alpha = \psi_\alpha - \frac{\int_\omega \psi_\alpha}{\int_\omega 1} - x^R_\alpha \frac{\int_\omega < x^R_\beta, \psi_\beta >}{\int_\omega |x^R|^2}$:

$$
\sum_\alpha \left\| \psi^2_\alpha \right\|^2_{L^2(\omega)} \leq C_K \sum_{\alpha,\beta} ||e_{\alpha\beta}(\psi)||^2_{L^2(\omega)}
$$

which gives

$$
J \leq 2C_{PW}(a_0+2)^2 \left(C_K \frac{1}{4} \sum_{\alpha,\beta} ||e_{\alpha\beta}(z)||^2_{L^2(\Omega)} + \sum_\alpha ||e_{\alpha 3}(z)||^2_{H^{-1}(I;L^2(\omega))} \right)
$$

This obviously implies inequality (4.3) and proves remark 4.2.

Now using the following refined Korn's inequality :

Proposition 4.3 (Refined Korn's inequality)
For any function $\psi \in (H^1(\omega))^2$, with $\psi^2_\alpha = \psi_\alpha - \frac{\int_\omega \psi_\alpha}{\int_\omega 1} - x^R_\alpha \frac{\int_\omega < x^R_\beta, \psi_\beta >}{\int_\omega |x^R|^2}$, we have

$$
\sum_\alpha \left\| \psi^2_\alpha \right\|^2_{L^2(\omega)} + \sum_\alpha \left\| \nabla \psi^2_\alpha \right\|^2_{H^{-1}(\omega)} \leq C'_K \sum_{\alpha,\beta} ||e_{\alpha\beta}(\psi)||^2_{H^{-1}(\omega)}
$$

we deduce that

$$J \leq 2C_{PW}(a_0+2)^2 \left(C'_K \frac{1}{4} \sum_{\alpha,\beta} ||e_{\alpha\beta}(z)||^2_{L^2(I;H^{-1}(\omega))} + \sum_{\alpha} ||e_{\alpha 3}(z)||^2_{H^{-1}(I;L^2(\omega))} \right)$$

This ends the proof of the theorem 4.1.

5 Appendix : a refined Korn's inequality

In this appendix we will prove proposition 4.3.

More generally, in any dimension n, this proposition can be stated as follows :

Proposition 5.1 (Refined Korn's inequality in dimension n) *Let $\omega \in \mathbf{R}^n$ be a bounded Lipschitz-continuous domain. Let us define the set $\mathcal{R}$ of rigid motions as the following finite-dimensional vector subspace of $(L^2(\omega))^n$:*

$$\mathcal{R} := \left\{ u \in \left(L^2(\omega)\right)^n, \quad \exists c \in \mathbf{R}^n, \exists A \in \mathbf{R}^{n\times n}_{antisymmetric}, \quad u_i(x) = c_i + \sum_j A_{ij} x_j \right\}$$

For $u \in (L^2(\omega))^n$, we define $u_{\mathcal{R}}$ as the orthogonal projection of u on $\mathcal{R}$ for the scalar product of $(L^2(\omega))^n)$. Then there exists a constant $C'_K > 0$ which only depends on n and ω such that for any $u \in (L^2(\omega))^n$ we have the following inequality on $v = u - u_{\mathcal{R}}$:

$$||v||^2_{(L^2(\omega))^n} + ||Dv||^2_{(H^{-1}(\omega))^n} \quad \leq \quad C'_K \quad ||e(v)||^2_{(H^{-1}(\omega))^{n\times n}_s} \tag{5.5}$$

The proof of the classical Korn's inequality for functions in H^1 (see Ciarlet [5], and the references therein), is based on the following lemma of J.-L. Lions :

Lemma 5.2 (Lemma of J.L. Lions)
Let $\omega \in \mathbf{R}^n$ a bounded Lipschitz-continuous domain. Then

$$\left\{ v \in H^{-1}(\omega) \quad \textit{and} \quad Dv \in \left(H^{-1}(\omega)\right)^n \right\} \Longrightarrow v \in L^2(\omega)$$

and there exists a constant $c_L > 0$ such that

$$||v||^2_{L^2(\omega)} \leq c_L \left(||v||^2_{H^{-1}(\omega)} + ||Dv||^2_{(H^{-1}(\omega))^n} \right)$$

Here to prove the refined Korn's inequality, we need a generalization of the lemma of J.-L. Lions, which can be found in [1] as a consequence of Nečas' theorem (see [17]) :

Lemma 5.3 (Generalization of the lemma of J.-L. Lions)
Let $\omega \in \mathbf{R}^n$ be a bounded Lipschitz-continuous domain, and $m \in \mathbf{Z}, 1 < r < +\infty$. Then

$$\left\{ v \in \mathcal{D}'(\omega) \quad \textit{and} \quad Dv \in \left(W^{m-1,r}(\omega)\right)^n \right\} \quad \Longrightarrow \quad v \in W^{m,r}(\omega)$$

and there exists a constant $C_N > 0$, depending only on ω,m,r such that

$$||v||_{W^{m,r}(\omega)} \quad \leq \quad C_N \quad \left(||v||_{W^{m-1,r}(\omega)} + ||Dv||_{(W^{m-1,r}(\omega))^n} \right)$$

In the particular case of interest, here $m = -1$, $r = 2$, we will prefer to write the inequality with the square of the norms, and with the constant C_N replaced by $C_L > 0$:

$$||v||^2_{H^{-1}(\omega)} \quad \leq \quad C_L \quad \left(||v||^2_{H^{-2}(\omega)} + ||Dv||^2_{(H^{-2}(\omega))^n} \right) \tag{5.6}$$

Proof of proposition 5.1
Step 1 : $||u||^2_{(L^2(\omega))^n} \quad \leq \quad 9c_L(1+C_L) \left(||u||^2_{(H^{-1}(\omega))^n} + ||e(u)||^2_{H^{-1}(\omega)^{n\times n}_s} \right)$, for every $u \in \left(L^2(\omega)\right)^n$

Applying first the generalization of the lemma of J.L. Lions 5.3 with (5.6), we get that

$$||Du||^2_{(H^{-1}(\omega))^{n\times n}} \quad \leq \quad C_L \quad \left(||Du||^2_{(H^{-2}(\omega))^{n\times n}} + ||D^2u||^2_{(H^{-2}(\omega))^{n\times n\times n}}\right)$$

Now using the general relation between the second derivatives of the displacement and the first derivatives of the deformations $\partial_i\partial_k u_i = \{\partial_j e_{ik}(u) + \partial_k e_{ij}(u) - \partial_i e_{jk}(u)\}$ and the general inequality $||Dv||_{(H^{-2}(\omega))^n} \leq ||v||_{H^{-1}(\omega)}$ for every function $v \in L^2(\omega)$, we get

$$||Du||^2_{(H^{-1}(\omega))^{n\times n}} \quad \leq \quad C_L \quad \left(||u||^2_{(H^{-1}(\omega))^n} + 9||e(u)||^2_{(H^{-1}(\omega))^{n\times n}_s}\right)$$

Finally applying the lemma of J.L. Lions 5.2, we get the expected inequality which ends the proof of step 1.

Step 2 : Proof of inequality (5.5)

We prove inequality (5.5) by contradiction. If inequality (5.5) is false, then we can find a sequence of functions $v^k \in (L^2(\omega))^n$ such that $v^k_{\mathcal{R}} = 0$,

$$||v^k||^2_{(L^2(\omega))^n} + ||Dv^k||^2_{(H^{-1}(\omega))^n} = 1 \tag{5.7}$$

and

$$||e(v^k)||_{(H^{-1}(\omega))^{n\times n}_s} \to 0$$

Because the sequence $(v^k)_k$ is bounded in $(L^2(\omega))^n$, we can extract a subsequence $(v^{k'})_{k'}$ which converges to a function $v \in (L^2(\omega))^n$, weakly in $(L^2(\omega))^n$, and strongly in $(H^{-1}(\omega))^n$ (by compactness of the injection of $L^2(\omega)$ in $H^{-1}(\omega)$, see [13]). In particular the limit v satisfies $e(v) = 0$, and then $v \in \mathcal{R}$. Moreover the limit satisfies $v_{\mathcal{R}} = 0$, because $v^{k'}_{\mathcal{R}} = 0$. Consequently $v = 0$. Now we apply the inequality of step 1 to $v^{k'}$:

$$||v^{k'}||^2_{(L^2(\omega))^n} \quad \leq \quad 9c_L(1 + C_L)\left(||v^{k'}||^2_{(H^{-1}(\omega))^n} + ||e(v^{k'})||^2_{(H^{-1}(\omega))^{n\times n}_s}\right) \tag{5.8}$$

Because $v^{k'}$ strongly converges to its limit $v = 0$ in $(H^{-1}(\omega))^n$, we deduce that the right hand side of (5.8) tends to zero. This implies that $||v^{k'}||^2_{(L^2(\omega))^n}$ and $||Dv^{k'}||^2_{(H^{-1}(\omega))^n} \leq ||v^{k'}||^2_{(L^2(\omega))^n}$ tend to zero, which is in contradiction with (5.7).
This ends the proof of proposition 5.1.

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