Uniform Elliptic Estimate for an Infinite Plate in Linear Elasticity.

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Abstract

We present a new study of linear elasticity for an infinite three-dimensional plate of finite thickness $\Omega = I\!R^2 \times (-1,1)$. We first caracterize the kernel of the operator of elasticity as polynomials which can be build from the kernel of the classical Kirchhoff-Love model of plate. Using this characterization we get optimal uniform elliptic estimates $W^{k,p}, C^{k,\alpha}$ on the solution as a function of the exterior forces. We also give an interior estimate.

AMS Classification: 35B10, 35B40, 35B45, 35E20, 35J45, 35Q72.

Keywords: Linear elasticity, Kirchhoff-Love plate theory, Elliptic estimates, Schauder estimates, L^p estimates, Polynomial approximation, Group representation theory.

1 Introduction

In this paper we are interested in uniform elliptic estimates for the system of equations of linear elasticity. We study in details the particular case of an infinite plate.

1.1 General framework

Before to present our results, let us put them in a general framework. We first recall some well known results on elliptic systems with constant coefficients. For a smooth open set Ω of $I\!R^n$, we consider a linear second order elliptic system

$$Lu = f \quad \text{on} \quad \Omega \tag{1.1}$$

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where $Lu = \sum_{i,j,k} a_{ij}^k \frac{\partial^2 u_k}{\partial x_i \partial x_j}$. We assume a first order boundary condition:

$$Bu = g \quad \text{on} \quad \partial\Omega \tag{1.2}$$

where $Bu = \sum_{i,j,k} \nu_j b_{ij}^k \frac{\partial u_k}{\partial x_i}$ and ν is the exterior unit normal to the boundary $\partial\Omega$. To consider a well-posed problem, we assume that the boundary condition (1.2) is a supplementing condition in the sense of [33]. The coefficients a_{ij}^k, b_{ij}^k are assumed constant vectors.

In the particular case of a half space $\Omega = \{x_n > 0\}$ we have for instance the well-known Schauder estimate

$$[D^2 u]_{\alpha;\Omega} \quad \leq \quad C\left([f]_{\alpha;\Omega} + [Dg]_{\alpha;\partial\Omega}\right) \tag{1.3}$$

Here for a general function v defined on an open subset A we note the Hölder seminorm for $\alpha \in (0,1)$

$$[v]_{\alpha;A} = \sup_{x,y\in A, \quad x\neq y} \frac{|v(x)-v(y)|}{|x-y|^\alpha}$$

More generally we recall the norms

$$|v|_{k+\alpha;A} = \sum_{j=0}^{k} |D^j v|_{\alpha;A} \qquad \text{with} \quad |v|_{\alpha;A} = |v|_{0;A} + [v]_{\alpha;A} \quad \text{and} \quad |v|_{0;A} = \sup_{x\in A} |v(x)|$$

Let us mention a few references about some methods to prove Schauder estimates: for the method of Singular integrals, see for instance S. Agmon, A. Douglis, L. Nirenberg [1, 2], C.B. Morrey [33] and D. Gilbarg, N.S. Trudinger [21]; for the Trudinger mollification, see N. Trudinger [37, 38], and Y.-Z. Chen, L.-C. Wu [7]; for the use of Campanato spaces, see C.B. Morrey [32], S. Campanato [6], M. Giaquinta [19, 20]; for Polynomial approximations, see M.V. Safonov [35], N.V. Krylov [36]; for the Scaling approach, see L. Simon [36]; for Smoothing operators, see L. Hörmander, Appendix A of [23].

The literature on elliptic estimates usually focuses on estimates on the whole space or on the half space. And up to our knowledge, there is not so much work in the direction of uniform estimates for more general geometries like for instance the case of a slab $\Omega = \{-1 < x_n < 1\}$. More generally when the open set Ω is not invariant by scaling, there is no hope to get an estimate similar to (1.3) for general elliptic systems (see remark 4.3 for a counter-example, and theorem 7.1 for an example). In particular when $\Omega = I\!R^k \times \omega$ where ω is a smooth bounded open set of $I\!R^{n-k}$, we still have the following estimate:

$$|u|_{2+\alpha;B_1\cap\Omega} \quad \leq \quad C\left(|f|_{\alpha;B_2\cap\Omega} + |g|_{1+\alpha;B_2\cap\partial\Omega} + |u|_{0;B_2\cap\Omega}\right)$$

Here we are interested in similar inequalities, but without the norm on u on the right hand side of the inequality (see in particular the works on weighted Sobolev spaces on cylinders like for instance Kozlov, Maz'ya [24], Maz'ya, Nazarov, Plamenevskij [25, 26]).
In its simplest form, we make the following

Conjecture 1.1 (Schauder Estimate for Elliptic Systems on Infinite Cylinders) *Let us consider a solution u to system (1.1)-(1.2) on $\Omega = I\!R^k \times \omega$ where ω is a smooth bounded open set in $I\!R^{n-k}$. If $u \in C_0^\infty(\overline{\Omega})$ (with compact support in $\overline{\Omega}$), then there exists $d \in (0, +\infty)$ such that*

$$\mathcal{N}^d_{2+\alpha}(u) \quad \leq \quad C\left(|Lu|_{\alpha;\Omega} + |Bu|_{1+\alpha;\partial\Omega}\right)$$

where the seminorm $\mathcal{N}^d_{2+\alpha}$ is given by

$$\mathcal{N}^d_{2+\alpha}(u) = \sup_{x\in\overline{\Omega}} \quad \inf_{P\in\mathcal{P}_d} \quad |u-P|_{2+\alpha;B_1(x)\cap\Omega}$$

Here $\mathcal{P}_d$ denotes the kernel of the system caracterized by

$$\mathcal{P}_d = \left\{v \in C^2(\overline{\Omega}), \quad Lv = Bv = 0 \quad \text{and} \quad \exists C > 0, \quad |v(x)| \leq C(1+|x|)^d\right\}$$

Let us remark that this conjecture is true with $d = 2$ for the Laplace operator on a slab with Neumann boundary conditions. More precisely we have:

Theorem 1.2 *For $\Omega = I\!R^{n-1} \times (-1, 1)$, $L = \Delta$, $B = \dfrac{\partial}{\partial x_n}$, conjecture 1.1 is true with $d = 2$.*

Here the degree $d = 2$ is quite natural. The goal of this article is to prove this conjecture in the particular case of the elliptic system of linear elasticity for a three-dimensional plate $\Omega = I\!R^2 \times (-1, 1)$, is true for $d = 4$, but false for $d \leq 3$. This reveals a striking difference between elliptic equations and systems.

Before to present our results, let us recall that a lot of work has been done on thin elastic plates. In particular some Sobolev estimates have been obtained. Among other works, let us cite [8, 9, 10, 11, 16, 17].

1.2 Results for the system of linear elasticity

We consider a solution $u = (u_1, u_2, u_3)$ on the slab $\Omega = I\!R^2 \times (-1, 1)$. In the case of isotropic homogeneous linear elasticity, we have:

$$Lu = \left\{ \begin{array}{l} (\lambda + 2\mu)\partial_{11}u_1 + \mu(\partial_{22}u_1 + \partial_{33}u_1) + (\lambda + \mu)(\partial_{12}u_2 + \partial_{13}u_3) \\ \\ (\lambda + 2\mu)\partial_{22}u_2 + \mu(\partial_{11}u_2 + \partial_{33}u_2) + (\lambda + \mu)(\partial_{21}u_1 + \partial_{23}u_3) \\ \\ (\lambda + 2\mu)\partial_{33}u_3 + \mu(\partial_{11}u_3 + \partial_{22}u_3) + (\lambda + \mu)(\partial_{31}u_1 + \partial_{32}u_2) \end{array} \right| \quad \text{on} \quad \Omega$$

$$Bu = \left\{ \begin{array}{l} \mu(\partial_3 u_1 + \partial_1 u_3) \\ \\ \mu(\partial_3 u_2 + \partial_2 u_3) \\ \\ \lambda(\partial_1 u_1 + \partial_2 u_2) + (\lambda + 2\mu)\partial_3 u_3 \end{array} \right| \quad \text{on} \quad \partial\Omega$$

where $\lambda, \mu > 0$ are Lamé constants.

We prove the conjecture for this particular system:

Theorem 1.3 *There exists a constant $C > 0$ such that for every $u \in C_0^\infty(\overline{\Omega})$ we have*

$$\mathcal{N}^4_{2+\alpha}(u) \quad \leq \quad C\left(|Lu|_{\alpha;\Omega} + |Bu|_{1+\alpha;\partial\Omega}\right)$$

This result is optimal in the sense that

Theorem 1.4 *For each $n \in \mathbf{N}$, there exists $u^n \in C_0^\infty(\overline{\Omega})$ such that*

$$\mathcal{N}^3_{2+\alpha}(u^n) > n\left(|Lu^n|_{\alpha;\Omega} + |Bu^n|_{1+\alpha;\partial\Omega}\right)$$

We also have interior estimates (see theorem 6.1) and a L^p version of these estimates.

A corollary of theorem 1.3 and of the characterization of the kernel is the following

Theorem 1.5 *For any function $h = (h_1, ..., h_{11})$ defined on $I\!R^2$ we define*

$$P_0(h) = \begin{pmatrix} h_1 + x_3 h_4 + \dfrac{x_3^2}{2!} h_7 + \dfrac{x_3^3}{3!} h_{10} \\ \\ h_2 + x_3 h_5 + \dfrac{x_3^2}{2!} h_8 + \dfrac{x_3^3}{3!} h_{11} \\ \\ h_3 + x_3 h_6 + \dfrac{x_3^2}{2!} h_9 \end{pmatrix}$$

Let us denote by $\mathcal{P}$ the space of all such functions:

$$\mathcal{P} = \left\{ P_0(h), \quad h \in C_0^\infty(I\!R^2) \right\}$$

Then for any function $u \in C_0^\infty(\overline{\Omega})$, we have

$$\left| u - Proj_{|\mathcal{P}}(u) \right|_{2+\alpha;\Omega} \quad \leq \quad C\left(|Lu|_{\alpha;\Omega} + |Bu|_{1+\alpha;\partial\Omega} \right)$$

where $Proj_{|\mathcal{P}}$ is any continuous projector from $C^{2+\alpha}([-1,1])$ on $I\!R^{11}$, which is naturally extended from $C^{2+\alpha}(\overline{\Omega})$ on $\mathcal{P}$.

This last result can be put in relation on the one hand with Naghdi models of plates (see Destuynder [15]) and on the other hand with director models as in Mielke [27].
One interpretation of theorems 1.3 and 1.5 is a dimension reduction phenomena. For some related questions see [5, 28, 29, 30, 31, 39].

1.3 Organization of the article

We prove our main theorem 1.3 in section 2 and give more general $W^{k,p}$, $C^{k,\alpha}$ estimates in section 3. We prove theorem 1.4 in section 4. Proposition 2.1 was a key argument to prove theorem 1.3, and this proposition is proved in section 5. In section 6 we present some extension of the previous results: we establish interior estimates for a finite plate with boundaries. In section 7 we give the proof of theorem 1.2. In an Appendix, we have rejected the precise characterization of the kernel of the operator of elasticity (which originally has been found explicitly by the author using the group representation theory) and some other technical tools on weighted Sobolev spaces. We finally give an example of a linear operator with non-polynomial kernel.

2 Schauder estimate: proof of theorem 1.3

The proof of theorem 1.3 uses the following proposition which will be proved later:

Proposition 2.1 **(L^∞ bounds)**
There exist a constant $C > 0$ and $\varepsilon \in (0,1)$ such that for every $u \in C_0^\infty(\overline{\Omega})$ we have

$$\exists P \in \mathcal{P}_4, \quad \forall R \geq 1, \quad |u - P|_{L^\infty(B_R \cap \Omega)} \quad \leq \quad C R^{4+\varepsilon} \mathcal{N}^4_{2+\alpha}(u)$$

Lemma 2.2

$$\mathcal{P}_{4+\varepsilon} = \mathcal{P}_4$$

Proof of lemma 2.2

This lemma is a consequence of theorem 8.1.

We will also use the straightforward

Lemma 2.3

$$\exists B_1 \subset I\!R^3, \quad \mathcal{N}^4_{2+\alpha}(u) \quad \leq \quad C \inf_{P\in\mathcal{P}_4} \quad |u-P|_{2+\alpha;B_1\cap\Omega}$$

Proof of theorem 1.3

Here we follow a classical argument which can for instance be found in Morrey [33].

If the inequality to prove is false, then there exists a sequence $(u^n)_n \in C_0^\infty(\overline{\Omega})$ such that

$$\mathcal{N}^4_{2+\alpha}(u^n) = 1 \quad \text{and} \quad |Lu^n|_{\alpha;\Omega}, |Bu^n|_{1+\alpha;\partial\Omega} \longrightarrow 0$$

From lemma 2.3, up to translate u^n we can still assume

$$1 = \mathcal{N}^4_{2+\alpha}(u^n) \quad \leq C \quad \inf_{P\in\mathcal{P}_4} \quad |u^n - P|_{2+\alpha;B_1(0)\cap\Omega}$$

From proposition 2.1, up to substract to u^n an element of $\mathcal{P}_4$, we can assume that uniformly in n

$$|u^n|_{L^\infty(B_R\cap\Omega)} \quad \leq \quad CR^{4+\varepsilon} \quad \text{for} \quad R\geq 1$$

Let us recall that the following Schauder estimate still holds

$$|u^n|_{2+\alpha;B_1\cap\Omega} \quad \leq \quad C\left(|Lu^n|_{\alpha;B_2\cap\Omega} + |Bu^n|_{\alpha;B_2\cap\partial\Omega} + |u^n|_{0;B_2\cap\Omega}\right)$$

We deduce that up to consider a subsequence we have

$$u^n \longrightarrow u^\infty \quad \text{in} \quad C^{2+\beta}_{loc}(\overline{\Omega}) \quad \text{for every} \quad \beta\in(0,\alpha) \tag{2.1}$$

We deduce that

$$Lu^\infty = Bu^\infty = 0 \quad \text{and} \quad |u^\infty|_{L^\infty(B_R\cap\Omega)} \quad \leq \quad CR^{4+\varepsilon} \quad \text{for} \quad R\geq 1$$

We deduce that $u^\infty \in \mathcal{P}_{4+\varepsilon} = \mathcal{P}_4$, and then up to substract u^∞ to u^n we can assume $u^\infty = 0$. Now because $u^\infty = 0$ in (2.1) we have $|u^n|_{0;B_2\cap\Omega} \longrightarrow 0$. This implies that the right hand side of the following inequality tends to zero:

$$1 = \mathcal{N}^4_{2+\alpha}(u^n) \quad \leq \quad C\left(|Lu^n|_{\alpha;B_2\cap\Omega} + |Bu^n|_{\alpha;B_2\cap\partial\Omega} + |u^n|_{0;B_2\cap\Omega}\right)$$

Contradiction. This ends the proof of the theorem.

We see that the main difficulty is to prove proposition 2.1 which will be done in section 5.

3 Other Estimates

Using our approach it is easy to prove estimates with more regularity. For instance with obvious notations we have:

$$\mathcal{N}^d_{k+2+\alpha}(u) \quad \leq \quad C_k\left(|Lu|_{k+\alpha;\Omega} + |Bu|_{k+1+\alpha;\partial\Omega}\right)$$

On the other hand we can prove L^p-estimates. For an open subset A and a function v we set the norms on the Banach space $W^{k,p}(A)$:

$$||v||_{k,p;A} = \sum_{j=0}^{k} ||D^j v||_{p;A} \quad \text{and} \quad ||v||_{p;A} = \left(\int_A |v|^p\right)^{\frac{1}{p}}$$

We also define the uniform norms on the Banach space $W^{k,p}_{unif}(A)$:

$$||v||^{unif}_{k,p;A} = \sum_{j=0}^{k} ||D^j v||^{unif}_{p;A} \quad \text{with} \quad ||v||^{unif}_{p;A} = \sup_{x\in A} ||v||_{p;B_1(x)\cap A}$$

Then we have easily

Theorem 3.1 *There exists a constant $C > 0$ such that for every $u \in C^\infty_0(\Omega)$ we have*

$$\mathcal{N}^4_{2,p}(u) \quad \leq \quad C\left(||Lu||^{unif}_{p;\Omega} + ||Bu||^{unif}_{1,p;\partial\Omega}\right)$$

where

$$\mathcal{N}^4_{2,p}(u) = \sup_{x\in\Omega} \quad \inf_{P\in\mathcal{P}_4} ||u - P||_{2,p;B_1(x)\cap\Omega}$$

4 A counter-example : proof of theorem 1.4

4.1 Preliminaries on symmetry

For a scalar function v defined on $\overline{\Omega}$, let us define the symmetric and antisymmetric parts with respect to x_3:

$$v^s(x) = \frac{1}{2}\left(v(x_1,x_2,x_3) + v(x_1,x_2,-x_3)\right) \quad \text{and} \quad v^a(x) = \frac{1}{2}\left(v(x_1,x_2,x_3) - v(x_1,x_2,-x_3)\right)$$

For a vector function $u = (u_1, u_2, u_3)$ we define the following *symmetric* and *antisymmetric* parts:

$$u^S = (u_1^s, u_2^s, u_3^a) \quad \text{and} \quad u^A = (u_1^a, u_2^a, u_3^s)$$

4.2 Characterization of the kernel

We define the following operators acting on functions $h = (h_1, h_2)$ defined on $I\!R^2$.

$$\mathrm{div}' u = \partial_1 u_1 + \partial_2 u_2$$

$$\Delta' = \partial_{11} + \partial_{22}$$

We will prove in the Appendix the following result (see Dauge, Gruais, Rössle [12, 13, 14] for a different approach)

Theorem 4.1 (Characterization of the kernel)
We have

$$\mathcal{P}_d = \mathcal{P}_d^S \oplus \mathcal{P}_d^A$$

Every element of $\mathcal{P}_d^A$ can be written

$$P^A(h) = \begin{pmatrix} -x_3\partial_1 h + a(x_3)\partial_1\Delta' h \\ \\ -x_3\partial_2 h + a(x_3)\partial_2\Delta' h \\ \\ h + \dfrac{\lambda}{\lambda+2\mu}\dfrac{x_3^2}{2!}\Delta' h \end{pmatrix}$$

where

$$a(x_3) = \left(\frac{3\lambda+4\mu}{\lambda+2\mu}\right)\frac{x_3^3}{3!} - 2\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)x_3$$

and h is a polynomial in (x_1, x_2) which satisfies

$$\Delta'^2 h = 0 \quad \textit{on} \quad I\!R^2$$

Similarly, every element of $\mathcal{P}_d^S$ can be written

$$P^S(h) = \begin{pmatrix} h_1 + \dfrac{\lambda}{\lambda+2\mu}\dfrac{x_3^2}{2!}\partial_1 div' h \\ \\ h_2 + \dfrac{\lambda}{\lambda+2\mu}\dfrac{x_3^2}{2!}\partial_2 div' h \\ \\ -\dfrac{\lambda}{\lambda+2\mu} x_3 div' h \end{pmatrix}$$

where h_1, h_2 are polynomials in (x_1, x_2) which satisfy

$$\Delta' h_\alpha + \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\partial_\alpha div' h = 0 \quad on \quad I\!R^2$$

4.3 Proof of theorem 1.4

Let us recall that

$$\mathcal{N}^3_{2+\alpha}(u) = \sup_{x\in\overline{\Omega}} \inf_{P\in\mathcal{P}_3} |u-P|_{2+\alpha;B_1(x)\cap\Omega}$$

where $\mathcal{P}_3$ is the kernel of polynomials of degree less or equal to 3 in (x_1, x_2).

For a general function $h(x_1, x_2)$, it is easy to compute

$$BP^A(h) = \begin{cases} c(x_3^2-1)\cdot\partial_1\Delta' h \\ c(x_3^2-1)\cdot\partial_2\Delta' h \\ b_3\cdot\Delta'^2 h \end{cases}$$

$$LP^A(h) = \begin{cases} l_1\cdot\partial_1\Delta'^2 h \\ l_2\cdot\partial_2\Delta'^2 h \\ l_3\cdot\Delta'^2 h \end{cases}$$

where c is a constant and b_3, l_1, l_2, l_3 are polynoms in x_3.

Let us assume that theorem 1.4 is false. Then

$$\forall u \in C_0^\infty(\overline{\Omega}), \quad \mathcal{N}^3_{2+\alpha}(u) \quad \leq \quad C\left(|Lu|_{\alpha;\Omega} + |Bu|_{1+\alpha;\partial\Omega}\right)$$

In particular for any function $h(x_1, x_2) \in C_0^\infty(I\!R^2)$, we have

$$\mathcal{N}^3_{2+\alpha}(P^A(h)) \quad \leq \quad C\left(|LP^A(h)|_{\alpha;\Omega} + |BP^A(h)|_{1+\alpha;\partial\Omega}\right)$$

Looking at the term $\frac{x_3^2}{2!}\Delta' h$ in $P^A(h)$ we get with $g = \Delta' h$

$$\sup_{x'\in I\!R^2}\inf_p |g-p|_{2+\alpha;B_1(x')} \quad \leq \quad C\,|\Delta' g|_{1+\alpha;I\!R^2}$$

where the infinmum is taken over polynomials p of degree less or equal to 1. In particular we deduce

$$|D^2 g|_{\alpha;\mathbb{R}^2} \quad \leq \quad C\,|\Delta' g|_{1+\alpha;\mathbb{R}^2}$$

By scaling we get

$$|D^2 g|_{0;\mathbb{R}^2}+\varepsilon^\alpha\left[D^2 g\right]_{\alpha;\mathbb{R}^2} \quad \leq \quad C\left(|\Delta' g|_{0;\mathbb{R}^2}+\varepsilon^\alpha\left[\Delta' g\right]_{\alpha;\mathbb{R}^2}+\varepsilon\,|\nabla'\Delta' g|_{0;\mathbb{R}^2}+\varepsilon^{1+\alpha}\left[\nabla'\Delta' g\right]_{\alpha;\mathbb{R}^2}\right)$$

where the constant C is independent on ε. At the limit $\varepsilon = 0$ we have for any function $g = \Delta' h$ with $h \in C_0^\infty(\mathbb{R}^2)$

$$|D^2 g|_{0;\mathbb{R}^2} \quad \leq \quad C|\Delta' g|_{0;\mathbb{R}^2}$$

This inequality is known to be false for general functions $g \in C_0^\infty(\mathbb{R}^2)$. It is sufficient to consider the limit case (in regularity) $g = q(x_1, x_2)\ln(x_1^2 + x_2^2)$ where q is a homogeneous harmonic polynomial of degree 2, and to reduce the problem to this limit case with help of mollifier and cut-off functions.
This contradiction ends the proof of theorem 1.4.

Remark 4.2 *Simlarly, for a general function $h(x_1, x_2)$, it is easy to compute*

$$BP^S(h) = \begin{cases} 0 \\ 0 \\ b_3 \cdot \Delta'\, div' h \end{cases}$$

and

$$LP^S(h) = \begin{cases} \mu\left(\Delta' h_1 + \dfrac{3\lambda+2\mu}{\lambda+2\mu}\partial_1\, div' h\right) + \lambda\dfrac{x_3^2}{2!}\partial_1\Delta'\, div' h \\[2ex] \mu\left(\Delta' h_2 + \dfrac{3\lambda+2\mu}{\lambda+2\mu}\partial_2\, div' h\right) + \lambda\dfrac{x_3^2}{2!}\partial_2\Delta'\, div' h \\[2ex] l_3 \cdot \Delta'\, div' h \end{cases}$$

where b_3, l_3 are polynoms in x_3. Moreover we can prove that there exists a sequence of symmetric vectors $u^n \in C_0^\infty(\overline{\Omega})$ such that $\mathcal{N}^1_{2+\alpha}(u^n) > n\left(|Lu^n|_{\alpha;\Omega} + |Bu^n|_{1+\alpha;\partial\Omega}\right)$

Remark 4.3 *Let us remark that the classical Schauder estimate (1.3) fails for the system of linear elasticity on Ω.*

To see it, it is sufficient to consider a cut-off function $\psi \in C_0^\infty(I\!R^2)$ satisfying $\psi = 1$ on $B_1(0) \subset I\!R^2$. We define for $\varepsilon > 0$

$$u^\varepsilon(x) = \psi(\varepsilon^2 x_1, \varepsilon^2 x_2) P^A \left(\left(\frac{\sin(\varepsilon x_1)}{\varepsilon} \right)^2 \left(\frac{\sin(\varepsilon x_2)}{\varepsilon} \right) \right)$$

Then we have

$$\frac{[D^2 u^\varepsilon]_{\alpha;\Omega}}{[Lu^\varepsilon]_{\alpha;\Omega} + [DBu^\varepsilon]_{\alpha;\partial\Omega}} \longrightarrow +\infty \quad \textit{as} \quad \varepsilon \longrightarrow 0$$

5 Proof of proposition 2.1

We will prove the following result which implies proposition 2.1.

Proposition 5.1 *There exists $\varepsilon \in (0,1)$ such that for every function $u \in C_0^2(\overline{\Omega})$, we have*

$$\exists P \in \mathcal{P}_{2,4}, \quad \left\{ \begin{array}{lcl} \left|(u-P)^S\right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{2+\varepsilon}\mathcal{N} \\ \\ \left|(u-P)^A\right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{4+\varepsilon}\mathcal{N} \end{array} \right.$$

where

$$\mathcal{N} = \sup_{x\in\Omega} \quad \inf_{P\in\mathcal{P}_{2,4}} \quad |u-P|_{W^{2,\infty}(B_1(x)\cap\Omega)}$$

and $\mathcal{P}_{2,4} \subset \mathcal{P}_4$ is defined by

$$\mathcal{P}_{2,4} = \left\{ v \in C^2(\overline{\Omega}), \quad Lv = Bv = 0, \quad \exists C > 0, \quad |v^S(x)| \leq C(1+|x|)^2, \quad |v^A(x)| \leq C(1+|x|)^4 \right\}$$

Proof of proposition 5.1: the antisymmetric part
From the expression of $P^A(h)$ we naturally introduce the projection $T^A u$ of any function u:

$$T^A u = \left(\begin{array}{c} -x_3 (T_1^A u)_1 + a(x_3)(T_3^A u)_1 \\ \\ -x_3 (T_1^A u)_2 + a(x_3)(T_3^A u)_2 \\ \\ T_0^A u + \dfrac{\lambda}{\lambda+2\mu} \dfrac{x_3^2}{2!} T_2^A u \end{array} \right)$$

where $T_i^A u$ only depend on (x_1, x_2).
On way to define precisely a projection operator T^A, is to set $d_0(x_3) = 1$, $d_1(x_3) = -x_3$, $d_2(x_3) = \frac{\lambda}{\lambda+2\mu}\frac{x_3^2}{2!}$, $d_3(x_3) = a(x_3)$, and to choose four functions $k_i(x_3), i = 0, ..., 3$ such that

$$\int_{-1}^1 dx_3 \ k_i(x_3) d_i(x_3) = \delta_{ij}, \quad \text{for} \quad i,j = 0, ..., 3$$

Then we set

$$\begin{cases} (T_i^A u)_\alpha(x_1,x_2) = \int_{-1}^{1} dx_3 \; k_i(x_3) u_\alpha(x_1,x_2,x_3), \quad i=1,3; \quad \alpha=1,2 \\ \\ (T_i^A u)(x_1,x_2) = \int_{-1}^{1} dx_3 \; k_i(x_3) u_3(x_1,x_2,x_3), \quad i=0,2 \end{cases}$$

We moreover remark that the expression of $P^A(h)$ exhibits the following sequence of operators:

$$\begin{pmatrix} h \\ 0 \end{pmatrix} \longmapsto^{M_0^A} \nabla' h \longmapsto^{M_1^A} \begin{pmatrix} \Delta' h \\ 0 \end{pmatrix} \longmapsto^{M_2^A} \nabla'\Delta' h \longmapsto^{M_3^A} \begin{pmatrix} \Delta'^2 h \\ 0 \end{pmatrix}$$

where $R(M_i^A) \supset Ker\left(M_{i+1}^A M_{i+2}^A ... M_3^A\right)$. Here $M_0^A = M_2^A = \nabla' = \begin{pmatrix} \partial_1 \\ \partial_2 \end{pmatrix}$ and $M_1^A = M_3^A = \begin{pmatrix} \mathrm{div}' \\ \mathrm{curl}' \end{pmatrix}$ where $\mathrm{div}' h = \partial_1 h_1 + \partial_2 h_2$, $\mathrm{curl}' h = \partial_1 h_2 - \partial_2 h_1$.

We then introduce the following quantity

$$\mathcal{N}^A = \sum_{i=0}^{3} \left| M_i^A T_i^A u - T_{i+1}^A u \right|_{L^\infty(I\!R^2)}$$

with $T_4^A \equiv 0$. To estimate $T^A u$, we now use the following lemma (which will be proved in the Appendix)

Lemma 5.2 *For $n=2$ and any $p>n$, $\alpha > \frac{n}{p}$ such that $\alpha - \frac{n}{p} \notin \mathbf{N}$, we consider the space (with $r = (x_1^2+x_2^2)^{\frac{1}{2}}$)*

$$W^{k,p}_{-\alpha} = \left\{ u, \quad \frac{D^l u}{(1+r)^{k-l+\alpha}} \in L^p(I\!R^n), l \in [0,k] \right\}$$

with the norm

$$|u|_{W^{k,p}_{-\alpha}} = \sum_{l=0}^{k} \left| \frac{D^l u}{(1+r)^{k-l+\alpha}} \right|_{L^p(I\!R^n)}$$

For $M^A = M_0^A$ or M_1^A, there exists a constant $C>0$ such that

$$\left(M^A h \in W^{0,p}_{-\alpha} \right) \Longrightarrow \left(\exists P, \quad deg(P) \leq \left[1+\alpha - \frac{n}{p} \right], \quad M^A P = 0, \quad |h-P|_{W^{1,p}_{-\alpha}} \leq C |M^A h|_{W^{0,p}_{-\alpha}} \right)$$

Here P are polynomials of degree less or equal to $\left[1+\alpha-\frac{n}{p}\right]$ such that $M^A P = 0$.

Applying four times this lemma successively with $\alpha, \alpha+1, \alpha+2, \alpha+3$, we get for some constant $C = C(\alpha,p) > 0$:

$$\exists P(x_1,x_2), \quad \deg(P) \leq 4, \quad \Delta'^2 P = 0, \quad |T^A u - P^A(P)|_{W^{1,p}_{-(\alpha+3)}} \quad \leq \quad C\mathcal{N}^A \qquad (5.2)$$

where we have used the fact that for a general function $|f|_{W^{0,p}_{-\alpha}} \leq C|f|_{L^\infty(I\!R^n)}$.

The end of the proof comes from the following lemma

Lemma 5.3

$$\exists P(x_1,x_2), \quad deg(P)\le 4, \quad \Delta'^2 P=0, \quad |T^A u - P^A(P)|_{L^\infty(B_R)} \quad \le \quad C R^{4+\varepsilon}\mathcal{N}^A$$

Proof of lemma 5.3

We use the fact that for a general function

$$\left|\frac{f}{(1+r)^{1+\alpha-\frac{n}{p}}}\right|_{L^\infty(I\!R^n)} \quad \le \quad C|f|_{W^{1,p}_{-\alpha}}$$

which is a consequence of the classical Morrey estimate: $|f(x)-f(y)|\le C|x-y|^{1-\frac{n}{p}}|\nabla f|_{L^p(I\!R^n)}$ applied to $\frac{f}{(1+r)^\alpha}$ and Poincaré-Wirtinger inequality to control $f(0)$. As a consequence of (5.2) we get for some p large and $\alpha-\frac{n}{p}$ small that for $\varepsilon=\alpha-\frac{n}{p}$ and

$$\forall R\ge 1, \quad \frac{1}{R^{4+\varepsilon}}|T^A u - P^A(P)|_{L^\infty(B_R)} \quad \le \quad C_\varepsilon \mathcal{N}^A$$

Proof of proposition 5.1: the symmetric part

From the expression of $P^S(h)$ we naturally introduce the projection T^S of any function u:

$$T^S u = \left(\begin{array}{l} (T_0^S u)_1 + \dfrac{\lambda}{\lambda+2\mu}\dfrac{x_3^2}{2!}(T_2^S u)_1 \\ \\ (T_0^S u)_2 + \dfrac{\lambda}{\lambda+2\mu}\dfrac{x_3^2}{2!}(T_2^S u)_2 \\ \\ -\dfrac{\lambda}{\lambda+2\mu}x_3 T_1^S u \end{array}\right)$$

where $T_i^S u$ only depend on (x_1,x_2). We moreover remark that the expression of $P^S(h)$ exhibits the following operator:

$$M^S h = \Delta' h_\alpha + \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\partial_\alpha \mathrm{div}' h$$

We then introduce the following quantity

$$\mathcal{N}^S = \left|M^S T_0^S u\right|_{L^\infty(I\!R^2)} + \sum_{i=0}^{2}\left|M_i^S T_i^S u - T_{i+1}^S u\right|_{L^\infty(I\!R^2)} + \left|M_1^S M_0^S T_0^S u - T_2^S u\right|_{L^\infty(I\!R^2)}$$

where $M_0^S=\mathrm{div}'$, $M_1^S=\nabla'$, $M_2^S=\begin{pmatrix}\mathrm{div}'\\ \mathrm{curl}'\end{pmatrix}$, and $T_3^S\equiv 0$. To estimate $T^S u$, we now use the following lemma (which will be proved in the Appendix)

Lemma 5.4 *With the same notations as in lemma 5.2, there exists a constant $C > 0$ such that*

$$\left(M^S h \in W^{0,p}_{-\alpha}\right) \Longrightarrow \left(\exists P, \quad deg(P) \leq \left[2+\alpha-\frac{n}{p}\right], \quad M^S P = 0, \quad |h-P|_{W^{2,p}_{-\alpha}} \leq C|M^S h|_{W^{0,p}_{-\alpha}}\right)$$

Here P are polynomials of degree less or equal to $\left[2+\alpha-\frac{n}{p}\right]$ such that $M^S P = 0$.

As previously we get for some universal constant $C = C(\alpha, p) > 0$:

$$\exists P(x_1,x_2), \quad \deg(P) \leq 2, \quad M^S P = 0, \quad \left\{\begin{array}{lcl} |T_0^S u - P|_{L^\infty(B_R)} & \leq & CR^{2+\varepsilon}\mathcal{N}^S \\ |M_0^S\left(T_0^S u - P\right)|_{L^\infty(B_R)} & \leq & CR^{1+\varepsilon}\mathcal{N}^S \end{array}\right.$$

and then using the definition of $\mathcal{N}^S$ we get

$$|T_1^S u - M_0^S P|_{L^\infty(B_R)} \quad \leq \quad CR^{1+\varepsilon}\mathcal{N}^S$$

For $v = T_0^S u - P$, we have

$$|M_1^S M_0^S v|_{W^{0,p}_{-\alpha}} \quad \leq \quad C\mathcal{N}^S$$

which implies (still using the definition of $\mathcal{N}^S$)

$$|T_2^S v|_{W^{0,p}_{-\alpha}} \quad \leq \quad C\mathcal{N}^S$$

Moreover we have

$$\left|M_2 T_2^S v\right|_{W^{0,p}_{-\alpha}} \quad \leq \quad C\mathcal{N}^S$$

This proves by classical elliptic estimates and usual imbeddings that

$$\left|T_2^S v\right|_{L^\infty(B_R)} \quad \leq \quad CR^{\varepsilon}\mathcal{N}^S$$

We finally have proved the following

Lemma 5.5

$$\exists P(x_1,x_2), \quad deg(P) \leq 2, \quad M^S P = 0, \quad |T^S u - P^S(P)|_{L^\infty(B_R)} \quad \leq \quad CR^{2+\varepsilon}\mathcal{N}^S$$

This ends the proof the proposition for the symmetric part.

End of the proof of proposition 5.1

The proposition follows from

$$\left|u - \left(T^S u + T^A u\right)\right|_{L^\infty(I\!R^2)} + \mathcal{N}^S + \mathcal{N}^A \quad \leq \quad C\mathcal{N}$$

6 Interior estimates

For this section let us note the infinite plate by $\Omega_\infty = I\!R^2 \times (-1,1)$ in place of Ω which will be reserved for a finite plate.

Theorem 6.1 *Let $\Omega = \omega \times (-1,1)$ where $\omega \subset I\!R^2$ is any open set (possibly unbounded). Let us denote by $dist(x',\omega)$ the distance of a point $x' \in I\!R^2$ to ω and*

$$\Omega_d = \omega_d \times (-1,1) \quad where \quad \omega_d = \left\{ x' \in I\!R^2, dist(x',\omega) < d \right\}$$

For a general set $A \subset \Omega_\infty$, we define

$$\mathcal{N}_{2+\alpha;A}(u) = \sup_{x\in A} \inf_h |u - P(h)|_{2+\alpha; B_1(x)\cap A}$$

where $h = (h_1,h_2,h_3)$, $P(h) = P^S(h_1,h_2) + P^A(h_3)$, and

$$\left\{ \begin{array}{l} M^S h = 0 \\ \Delta'^2 h_3 = 0 \end{array} \right| \quad on \quad B_2(x')$$

where we recall that $(M^S h)_\alpha = \Delta' h_\alpha + \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\partial_\alpha div' h$. Then there exist constants $C, c > 0$ such that for any function $u \in C^{2+\alpha}(\overline{\Omega}_d)$, we have for $d \geq 0$

$$\mathcal{N}_{2+\alpha;\Omega}(u) \quad \leq \quad C\left(|Lu|_{\alpha;\Omega_d} + |Bu|_{1+\alpha;\Omega_d} + e^{-cd}\mathcal{N}_{2+\alpha;\Omega_d}(u) \right)$$

The proof of this theorem is based on the following proposition (which is a variant of proposition 5.1)

Proposition 6.2 *Let $u \in C^{2+\alpha}(\overline{\Omega})$. Then there exists $h \in C^\infty(\omega)$ which is a solution of*

$$\left\{ \begin{array}{l} M^S h = 0 \\ \Delta'^2 h_3 = 0 \end{array} \right| \quad on \quad \omega$$

such that we have the estimates

$$\left\{ \begin{array}{lcl} \left| (u - P(h))^S \right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{2+\varepsilon}\mathcal{N}_{2+\alpha;\Omega}(u) \\ \\ \left| (u - P(h))^A \right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{4+\varepsilon}\mathcal{N}_{2+\alpha;\Omega}(u) \end{array} \right.$$

for some constants $C > 0$ and $\varepsilon \in (0,1)$ only depending on λ, μ.

Proof of theorem 6.1

Theorem 6.1 is a consequence of the following inequality

$$\forall \theta \in (0,1), \quad \exists d_0 > 0, \qquad \exists C > 0,$$

$$\mathcal{N}_{2+\alpha;\Omega}(u) \quad \leq \quad C\left(|Lu|_{\alpha;\Omega_{d_0}} + |Bu|_{1+\alpha;\Omega_{d_0}}\right) + \theta\mathcal{N}_{2+\alpha;\Omega_{d_0}}(u) \tag{6.1}$$

If this inequality is false, then we can find a $\theta \in (0,1)$ and a sequence of solutions $(u^n)_n$ on $\Omega^n = \omega^n \times (-1,1)$ and sequences

$$d_0^n, C_n \longrightarrow +\infty$$

such that

$$1 = \mathcal{N}_{2+\alpha;\Omega^n}(u^n) \quad \geq \quad C_n\left(|Lu^n|_{\alpha;\Omega^n_{d_0^n}} + |Bu^n|_{1+\alpha;\Omega^n_{d_0^n}}\right) + \theta\mathcal{N}_{2+\alpha;\Omega^n_{d_0^n}}$$

With help of proposition 6.2 we can find a sequence h^n with bounds on $v^n = u^n - P(h^n)$ which proves that $v^n \to v^\infty$ locally on compact sets. As in the proof of theorem 1.3, we have $v^\infty \in \mathcal{P}_{2,4}$ and up to substract v^∞ to v^n, we can assume that $v^\infty \equiv 0$. Now if we choose the origine such that

$$1 = \inf_h |v^n - P(h)|_{2+\alpha;B_1(0)} \quad \geq \quad \sup_{x\in\Omega} \inf_h |v^n - P(h)|_{2+\alpha;B_1(x)}$$

we see that the classical interior estimate is

$$1 = \inf_h |v^n - P(h)|_{2+\alpha;B_1(0)} \quad \leq \quad C\left(|f^n|_{\alpha;B_2(0)} + |g^n|_{1+\alpha;B_2(0)} + |v^n|_{L^1(B_2(0))}\right) \tag{6.2}$$

because $LP(h) = BP(h) = 0$ on $B_2(0)$. Now the contradiction comes from the fact that the right hand side of (6.2) tends to zero as n tends to infinity. This ends the proof of theorem 6.1.

Remark 6.3 *As a consequence of (6.1), we get a more precise result:*

$$\mathcal{N}_{2+\alpha;\Omega}(u) \quad \leq \quad C\left(\int_0^d ds\; e^{-cs}\left(|Lu|_{\alpha;\Omega_s} + |Bu|_{1+\alpha;\Omega_s}\right) + e^{-cd}\mathcal{N}_{2+\alpha;\Omega_d}(u)\right)$$

Proof of proposition 6.2

We simply remark that

$$\begin{cases} \mathcal{N}^A := \sum_{i=0}^3 \left|M_i^A T_i^A u - T_{i+1}^A u\right|_{L^\infty(\omega)} \quad \leq \quad C\mathcal{N}_{2+\alpha;\Omega}(u) \\ \\ \mathcal{N}^S := \left|M^S T_0^S u\right|_{L^\infty(\omega)} + \sum_{i=0}^2 \left|M_i^S T_i^S u - T_{i+1}^S u\right|_{L^\infty(\omega)} + \left|T^S u - P^S\left(T_0^S u\right)\right|_{L^\infty(\Omega)} \quad \leq C\mathcal{N}_{2+\alpha;\Omega}(u) \end{cases}$$

Extending these quantities on the whole space $I\!R^2$ as $L^\infty(I\!R^2)$ quantities, we can apply the proof of proposition 5.1. This gives the existence of a function $k = (k_1, k_2, k_3)$ defined on $I\!R^2 \times (-1,1)$ such that

$$k = T^A(k) + P^S(T_0^S k)$$

such that $w = Tu - P(k)$ satisfies

$$\left\{ \begin{array}{l} \left(M_i^A T_i^A - T_{i+1}^A\right) w = 0 \\ \\ (M^S T_0^S) w = 0 \end{array} \right| \quad \text{on} \quad \omega$$

which implies

$$\left\{ \begin{array}{l} T^A w = P^A(T_0^A w) \\ \\ \Delta'^2(T_0^A w) = 0 \\ \\ M^S(T_0^S w) = 0 \end{array} \right| \quad \text{on} \quad \omega$$

Moreover there exists $P \in \mathcal{P}_{2,4}$ such that

$$\left\{ \begin{array}{lcl} \left|(k-P)^S\right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{2+\varepsilon}\mathcal{N}_{2+\alpha;\Omega}(u) \\ \\ \left|(k-P)^A\right|_{L^\infty(B_R\cap\Omega)} & \leq & CR^{4+\varepsilon}\mathcal{N}_{2+\alpha;\Omega}(u) \end{array} \right.$$

The antisymmetric part

We can write

$$\begin{array}{rcl} T^A u & = & T^A w + T^A k \\ \\ & = & T^A(w+P) + T^A(k-P) \\ \\ & = & P^A\left(T_0^A(w+P)\right) + (k-P)^A \end{array}$$

We set

$$h_3 = T_0^A(w+P)$$

The symmetric part

We have

$$\begin{aligned} T^S u &= P^S\left(T_0^S u\right) + \left(T^S u - P^S\left(T_0^S u\right)\right) \\ &= P^S\left(T_0^S u\right) + q \\ &= P^S\left(T_0^S (w)\right) + P^S\left(T_0^S k\right) + q \\ &= P^S\left(T_0^S (w+P)\right) + P^S\left(T_0^S (k-P)\right) + q \\ &= P^S\left(T_0^S (w+P)\right) + (k-P)^S + q \end{aligned}$$

where $q = T^S u - P^S\left(T_0^S u\right)$ satisfies by definition

$$|q|_{L^\infty(\mathbb{R}^2)} \leq \mathcal{N}^S$$

We set

$$h_\alpha = \left(T_0^S (w+P)\right)_\alpha$$

Conclusion

Now setting

$$h = (h_\alpha, h_3)$$

we get

$$\begin{cases} \left|(Tu - P(h))^S\right|_{L^\infty(B_R\cap\Omega)} & \leq \quad CR^{2+\varepsilon}\mathcal{N}^S \\ \left|(Tu - P(h))^A\right|_{L^\infty(B_R\cap\Omega)} & \leq \quad CR^{4+\varepsilon}\mathcal{N}^A \end{cases}$$

The result follows from

$$\left|u - \left(T^S u + T^A u\right)\right|_{L^\infty(\mathbb{R}^2)} + \mathcal{N}^S + \mathcal{N}^A \quad \leq \quad C\mathcal{N}$$

This ends the proof of proposition 6.2.

7 Proof of theorem 1.2

Proof of theorem 1.2

We consider a solution u of

$$\begin{cases} \Delta u = f \quad \text{on} \quad \Omega \\ \dfrac{\partial u}{\partial x_n} = g \quad \text{on} \quad \partial\Omega \end{cases}$$

Using the notation $x' = (x_1, ..., x_{n-1})$, we define $\overline{u}(x') = \frac{1}{2}\int_{-1}^{2} ds\ u(x', s)$. Then by integration of the equation we get

$$\Delta\overline{u} = \frac{1}{2}\int_{-1}^{2} ds\ f(x', s) - \frac{1}{2}\left(g(x', 1) - g(x', -1)\right) =: \overline{f}$$

Moreover for every $y' \in I\!R^{n-1}$, using a Taylor expansion, we get

$$\overline{u}(x') = P_y(x') + \frac{1}{2}\frac{\Delta\overline{u}(y')}{(n-1)}|x'-y'|^2 + \int_0^1 ds\ \int_0^s d\beta\ {}^t(x'-y')\left\{D^2\overline{u}(y' + \beta(x'-y')) - D^2\overline{u}(y')\right\}(x'-y')$$

where $P_{y'}$ is a polynomial belonging to $\mathcal{P}_2$ defined by (using the notation D' for the gradient with respect to x', and Id for the identity matrix):

$$P_{y'}(x') = \overline{u}(y') + (x'-y')D'\overline{u}(y') + \frac{1}{2}{}^t(x'-y')\left\{D^2\overline{u}(y') - \frac{\Delta\overline{u}(y')}{(n-1)}Id\right\}(x'-y')$$

From this result, we deduce that for every $y' \in I\!R^{n-1}$ we have

$$|\overline{u} - P_{y'}|_{B_1(y')} \leq C\left(|\overline{f}|_{0;I\!R^{n-1}} + [D^2\overline{u}]_{\alpha;I\!R^{n-1}}\right)$$

Consequently, we see that we can reduce the problem to the special case

$$\int_{-1}^{1} ds\ u(x', s) = 0 \tag{7.1}$$

Under this assumption, the proof follows the proof of theorem 1.3.
More precisely, we assume that there exists a sequence $(u^k)_k$ of functions such that

$$\mathcal{N}^2_{2+\alpha}(u^k) = 1 \quad \text{and} \quad |\Delta u^k|_{\alpha;\Omega}, \left|\frac{\partial u^k}{\partial x_n}\right|_{1+\alpha;\partial\Omega} \longrightarrow 0$$

We get an estimate $|u^k(x)|_{L^\infty(B_R\cap\Omega)} \leq CR^3$ for $R \geq 1$. We extract a subsequence, convergent to u^∞ on compact sets of $\overline{\Omega}$, and check that u^∞ satisfies $\Delta u^\infty = \frac{\partial u^\infty}{\partial x_n} = 0$. We deduce that u^∞ is a polynomial in x' only. But by assumption (7.1), still satisfied by the limit function u^∞, we conclude that $u^\infty \equiv 0$. Then the classical Schauder estimate shows that for some well chosen ball B_2 we have

$$1 = \mathcal{N}^2_{2+\alpha}(u^k) \leq C\left(|\Delta u^k|_{\alpha;B_2\cap\Omega} + \left|\frac{\partial u^k}{\partial x_n}\right|_{1+\alpha;B_2\cap\partial\Omega} + |u^k|_{0;B_2\cap\Omega}\right)$$

We get a contradiction, because the right hand side of the inequality goes to zero. This ends the proof of the theorem.

We complete the study of this case by the following result:

Theorem 7.1 *Let $\Omega = I\!R^{n-1} \times (-1,1)$ and $\alpha \in (0,1)$. Then there exists a constant $C > 0$ such that for every function $u \in C_0^\infty(\overline{\Omega})$, we have with $f = \Delta u$, $g = \dfrac{\partial u}{\partial x_n}$:*

$$[D^2 u]_{\alpha;\Omega} \quad \le \quad C\left([f]_{\alpha;\Omega} + [Dg]_{\alpha;\partial\Omega}\right) \tag{7.2}$$

Proof of theorem 7.1

We first extend f by symmetry and periodicity on the whole space as follows for $x' = (x_1, ..., x_{n-1})$:

$$\tilde{f}(x', x_n) = \begin{cases} f(x', x_n) & \text{if} \quad x_n \in [-1,1] \\ \\ f(x', 2 - x_n) & \text{if} \quad x_n \in [1,3] \end{cases}$$

and

$$\tilde{f}(x', x_n + 4) = \tilde{f}(x', x_n) \quad \text{for all} \quad x \in I\!R^n$$

Then from theorem 8.2 we deduce the existence of a solution u_0 of $\Delta u_0 = f$ on $I\!R^n$. From the standard Schauder estimate (see for instance [36]), we deduce that $[D^2 u_0]_{\alpha;I\!R^n} \le C[f]_{\alpha;I\!R^n}$. Up to substract u_0 to u, this reduces the proof to the case $f = 0$.

We now work in the case $f \equiv 0$. Here we have

$$\begin{cases} \Delta u = 0 \quad \text{on} \quad \Omega \\ \\ \dfrac{\partial u}{\partial x_n} = g \quad \text{on} \quad \partial\Omega \end{cases}$$

Extending on the whole space u in $\tilde{u}$ still by symmetry and periodicity, we get on $I\!R^n$

$$\begin{aligned} \Delta \tilde{u} &= \sum_{k \in \mathbf{Z}} \left(-2g(x', 1)\delta_0(x_n - (1+4k)) + 2g(x', -1)\delta_0(x_n - (-1+4k))\right) \\ &= \frac{\partial h}{\partial x_n} \end{aligned}$$

where we can define

$$h(x) = \begin{cases} 0 & \text{if} \quad x_n \in [-1,1) \\ \\ -2g(x', 1) & \text{if} \quad x_n \in [1,3) \end{cases}$$

and h has the following "periodicity"

$$h(x', x_n + 4) = h(x', x_n) + 2g(x', -1) - 2g(x', 1)$$

We then define for $i = 1, ..., n-1$, $\tilde{u}_i = \dfrac{\partial \tilde{u}}{\partial x_i}$, $h_i = \dfrac{\partial h}{\partial x_i}$. By derivation, we get

$$\Delta \tilde{u}_i = \frac{\partial h_i}{\partial x_n}$$

From the Schauder estimate (see for instance [36]), we get

$$[D\tilde{u}_i]_{\alpha;I\!R^n} \leq C[h_i]_{\alpha;I\!R^n}$$

We deduce that

$$\left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{\alpha;\Omega} \leq C[Dg]_{\alpha;\partial\Omega} \quad \text{for} \quad i = 1, ..., n-1; j = 1, ..., n \tag{7.3}$$

Finally we remark that

$$\frac{\partial^2 u}{\partial x_n^2} = -\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2}$$

and then

$$\left[\frac{\partial^2 u}{\partial x_n^2}\right]_{\alpha;\Omega} \leq C[Dg]_{\alpha;\partial\Omega}$$

Together with (7.3), this proves (7.2) and ends the proof of the theorem.

8 Appendix

8.1 Characterization of the kernel

Theorem 4.1 is a corollary of the following result:

Theorem 8.1 (Characterization of the kernel)
We have

$$\mathcal{P}_d = \mathcal{P}_d^S \oplus \mathcal{P}_d^A$$

More precisely if we note $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$ *and* $\Re, \Im$ *the real and imaginary part of a complex number, we have*

$$\mathcal{P}_d^S = \bigoplus_{n=0}^{d} V_n$$

where

$$V_n = Span\left\{\Re(v^n), \Im(v^n), \Re(\partial_{\overline{z}} v^{n+1}), \Im(\partial_{\overline{z}} v^{n+1})\right\}$$

where

$$v^n = P^S\left((5\lambda + 6\mu)\frac{z^n}{n!} - (3\lambda + 2\mu)\frac{\overline{z}z^{n-1}}{(n-1)!}, \quad -i(5\lambda + 6\mu)\frac{z^n}{n!} - i(3\lambda + 2\mu)\frac{\overline{z}z^{n-1}}{(n-1)!}\right)$$

Similarly

$$\mathcal{P}_d^A = \bigoplus_{n=0}^{d} W_n$$

where

$$W_n = Span\left\{\Re(w^n), \Im(w^n), \Re(\partial_{\overline{z}} w^{n+1}), \Im(\partial_{\overline{z}} w^{n+1})\right\}$$

and

$$w^n = P^A\left(\frac{\overline{z}z^{n-1}}{(n-1)!}\right)$$

Finally we have

$$dim\ V_0 = 2, \quad dim\ V_n = 4 \quad for \quad n \geq 1$$

$$dim\ W_0 = 1, \quad dim\ W_1 = 2, \quad dim\ W_2 = 3, \quad dim\ V_n = 4 \quad for \quad n \geq 4$$

Sketch of the proof of theorem 8.1

1) It is straightforward to check that these polynomials are solutions.

2) We consider solutions u of $Lu = Bu = 0$ and $|u(x)| \leq C(1+|x|)^d$ on Ω for some constants $C, d > 0$. Then for every $x_3 \in [-1, 1]$, the function $x' = (x_1, x_2) \longmapsto u(x', x_3)$ in is the dual of the Schwarz space, i.e. belongs to $\mathcal{S}'(I\!R^2)$. We can then apply the partial Fourier transform, and define $\hat{u}(\xi', x_3)$. Then we build the vector ${}^tU(\xi', x_3) = \left(\hat{u}_1, \hat{u}_2, \frac{\partial \hat{u}_3}{\partial x_3}, \frac{\partial \hat{u}_1}{\partial x_3}, \frac{\partial \hat{u}_2}{\partial x_3}, \hat{u}_3\right)$. This vector solves the following ODE as a function $U : [-1, 1] \longrightarrow (\mathcal{S}'(I\!R^2))^6$:

$$\frac{dU}{dx_3} = A(\xi')U$$

where $A(\xi')$ is an explicit 6×6 matrix, polynomial in ξ' of total degree less or equal to 2. The explicit solution of this ODE is

$$U(\xi', x_3) = e^{x_3 A(\xi')} U(\xi', 0)$$

The boundary conditions $Bu = 0$ are equivalent to

$$Q(\xi')U(\xi', x_3) = 0 \quad \text{for} \quad x_3 = \pm 1$$

where $Q(\xi')$ is an explicit 3×6 matrix, polynomial in ξ' of degree less or equal to 1. Finally u is a solution if and only if

$$M(\xi')U(\xi', 0) = 0 \quad \text{where} \quad M(\xi') = Q(\xi')\begin{pmatrix} e^{A(\xi')} \\ e^{-A(\xi')} \end{pmatrix}$$

If there exists some $\xi'_0 \in I\!R^2$ such that $M(\xi'_0)U_0 = 0$ for some $U_0 \neq 0$, then the inverse partial Fourier transform of $e^{x_3 A(\xi')} U_0 \delta_0(\xi' - \xi'_0)$ is a bounded solution of $Lv = Bv = 0$. But using a classical cut-off argument, it is easy to check that every bounded solution of these equations is constant, which implies $\xi'_0 = 0$. We deduce that the support of the distribution $U(\xi', 0)$ (and then of $U(\xi', x_3)$) is $\{0\}$, and therefore $u(x', x_3)$ is a polynomial in (x_1, x_2). It is then easy to check that it is necessarily a polynomial in x_3.
3) It is easy to prove by recurrency on the degree n (in x_1, x_2) of the polynomials that $\mathcal{P}_n = \bigoplus_{k=0}^{n} V_k \oplus W_k$. It is in particular sufficient to prove for a polynomial P of degree n in x_1, x_2 that $\dfrac{\partial P}{\partial z}, \dfrac{\partial P}{\partial \overline{z}} \in V_{n-1} \oplus W_{n-1}$ implies $P \in \mathcal{P}_n$.

8.2 On weighted Sobolev spaces

In this subsection we give the proof of lemmata 5.2 and 5.4.

To do this we need the following result

Theorem 8.2 (Amrouche, Girault, Giroire [3],[4])
Let integers $n \geq 1$, $k \in \mathbf{N}$, and real numbers $1 < p < +\infty$, $\alpha \geq 0$. For any function $f \in W^{k,p}_{-\alpha}(I\!R^n)$, there exists a function $h \in W^{k+2,p}_{-\alpha}(I\!R^n)$ such that $\Delta h = f$ and

$$\exists \text{ polynomial } P, \quad deg(P) \leq \left[k + 2 + \alpha - \frac{n}{p} \right], \quad \Delta P = 0, \quad |h - P|_{W^{k+2,p}_{-\alpha}} \leq C \, |\Delta h|_{W^{k,p}_{-\alpha}}$$

while $\alpha - \frac{n}{p} \notin \mathbf{Z}$. Moreover the constant C only depends on n, p, k, α.

Proof of lemma 5.2
The estimate for the gradient is obvious.
For the operator M_1 we want to solve

$$\begin{cases} \mathrm{div}' u = f_1 \\ \mathrm{curl}' u = f_2 \end{cases}$$

The estimate follows from the following relations.
We define v_i given by $\Delta v_i = f_i$ and

$$\begin{cases} w_1 = \partial_1 v_2 - \partial_2 v_1 \\ w_2 = \partial_1 v_1 + \partial_2 v_2 \end{cases}$$

We see that $M(u - w) = 0$ and then $u = w + P$ with a polynomial P of degree 1 solution of $MP = 0$. The estimate on $u - P$ comes from the fact that w is controled by $\nabla' v$ and then by f in the corresponding norms.

Proof of lemma 5.4

For the operator M we want to solve

$$\begin{cases} \Delta' h_1 + b\,\partial_1 \mathrm{div}' h = f_1 \\ \Delta' h_2 + b\,\partial_2 \mathrm{div}' h = f_2 \end{cases}$$

where b is a positive constant. The estimate follows from the following relations. Taking the div' and the curl' we get

$$\begin{cases} \mathrm{div}'\Delta' h = \dfrac{1}{1+b}\mathrm{div}' f \\ \mathrm{curl}'\Delta' h = \mathrm{curl}' f \end{cases}$$

Then we define g_i by $\Delta g_i = f_i$ and define $k = (k_1, k_2)$ by

$$\begin{cases} \mathrm{div}' k = \dfrac{1}{1+b}\mathrm{div}' g \\ \mathrm{curl}' k = \mathrm{curl}' g \end{cases}$$

We get $M_1\Delta'(h-k) = 0$ and then $h = k + P$ with $MP = 0$ with the corresponding control on k by f. This ends the proof.

8.3 On the usefulness of groups

Although it is not presented in detail in this paper, we have used the group representation theory to find the general expression of the polynomial solutions in the kernel (see [22, 18]).

Let us define the generator σ of the rotations with respect to the normal to the plate:

$$\sigma(u) = \begin{pmatrix} -u_2 + x_2\partial_1 u_1 - x_1\partial_2 u_1 \\ u_1 + x_2\partial_1 u_2 - x_1\partial_2 u_1 \\ x_2\partial_1 u_3 - x_1\partial_2 u_3 \end{pmatrix}$$

Then the differential operators $\partial_1, \partial_2, \sigma$ generate a Lie algebra caracterized by

$$\begin{aligned} &[\partial_1, \partial_2] = 0 \\ &[\partial_1, \sigma] = -\partial_2 \\ &[\partial_2, \sigma] = \partial_1 \end{aligned}$$

In particular it is possible to check that polynomials given in theorem 8.1 are eigenfunctions of σ.

8.4 An example of elliptic system where the kernel contains periodic solutions

Let us consider the following system

$$\left\{\begin{array}{l} 3\partial_{11}u_1 + \partial_{22}u_1 + 2\partial_{12}u_2 = 0 \\ 3\partial_{22}u_2 + \partial_{11}u_2 + 2\partial_{12}u_1 = 0 \end{array}\right| \quad \text{on} \quad \Omega := I\!R \times (-1,1) \tag{8.1}$$

with the following boundary conditions

$$\left\{\begin{array}{l} \partial_1 u_2 + \partial_2 u_1 = 0 \\ (1+\delta)^{-1}\partial_1 u_1 + 3\partial_2 u_2 = 0 \end{array}\right| \quad \text{on} \quad \partial\Omega \tag{8.2}$$

For $\delta = 0$ this system reduces to the system of linear elasticity with $\lambda = \mu = 1$ on the strip Ω. Using Partial Fourier Transform, it can be seen that for $\delta > 0$, there exist non-constant x_1-periodic solutions with frequency ξ satisfying:

$$\frac{\sinh(2\xi)}{2\xi} = 1 + \frac{\delta}{\frac{4}{3} + \delta}$$

In this case, we see in particular that we can increase the dimension of the kernel by perturbation. Such kind of behaviour has been remarked for other elliptic equations (see [34]).

Aknowledgments

I am very gratefull to P.G. Ciarlet, M. Dauge, H. Le Dret and F. Murat for stimulating discussions and their interest in my work. I would like to thank F. Alliot for enlighting discussions on weighted Sobolev spaces. I also thank an unknown referee for his careful reading of the manuscript, his valuable suggestions and a proposition of a proof of theorem 7.1. Finally I would like to thank A. Bonnet and H. Berestycki for their encouragements.

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