A new 3d-2d interior error estimate independent on the geometry of a linear elastic plate.

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Abstract

We study how three-dimensional linearized elasticity for thin plates can be approximated by a two-dimensional projection. The classical approach using formal asymptotic expansions in powers of the thickness in the Hellinger-Reissner formulation, only provides error estimates in the H^1 norm for the displacements, assuming at least L^2 regularity for the applied forces, plus additional regularity for some components. Here we make use of elliptic regularity theory. We prove a 3d-2d interior error estimate between the 3d displacement solution and its 2d projection. Moreover the constants involved in our estimate are independent on the particular geometry of the plate. Our approach yields an H^2 error estimate, assuming only L^2 regularity for the applied forces, which is optimal from the point of view of elliptic regularity theory. We also obtain interior $W^{k,p}$ and $C^{k,\alpha}$ error estimates.

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1 Introduction

1.1 Setting of the problem

Let us consider a linear elastic three-dimensional thin plate $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$ where $\omega \subset I\!R^2$ is a bounded open set. We will show a new error estimate in the limit of asymptotically small thickness 2ε, between the three-dimensional displacement and a two-dimensional projection of the three-dimensional displacement. In future works, we will apply the present error estimate to more general linear and nonlinear beam, plate and shell theories.

In the present paper, we are interested in the displacements of the linearly elastic plate Ω^ε

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under the action of exterior volumic forces $f = (f_1, f_2, f_3)$ and surface forces $g = (g_1, g_2, g_3)$. It is classical (cf. Ciarlet [7, 6]) to reformulate the problem, and to search for the rescaled displacements $u = (u_1, u_2, u_3)$ on the rescaled plate

$$\Omega = \omega \times I, \quad \text{where} \quad I = (-1, 1)$$

which are solutions of the rescaled equations of linear elasticity

$$\begin{cases} L^\varepsilon u = -f & \text{on} \quad \omega \times I \\ \\ B^\varepsilon u = g & \text{on} \quad \omega \times \partial I \end{cases} \tag{1.1}$$

where L^ε is a second order elliptic operator, and B^ε is a first order operator. The coefficients of these operators are constant and depend on ε. Since we are interested in interior error estimates, we do not consider particular boundary conditions on $(\partial\omega) \times I$.

We denote by $x' = (x_1, x_2)$ a point of ω and by $x = (x', x_3)$ a point of Ω. We use greek indices $\alpha, \beta, ...$ for values in $\{1, 2\}$, and latin indices $i, j, ...$ for values in $\{1, 2, 3\}$. The quantity $\partial_i u_j$ stands for $\frac{\partial u_j}{\partial x_i}$. More precisely we set $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, $e^\varepsilon_{ij} = \begin{pmatrix} e_{\alpha\beta} & \frac{1}{\varepsilon} e_{\alpha 3} \\ \frac{1}{\varepsilon} e_{\alpha 3} & \frac{1}{\varepsilon^2} e_{33} \end{pmatrix}$, $\sigma_{ij} = \lambda e^\varepsilon_{kk} \delta_{ij} + 2\mu e^\varepsilon_{ij}$ for $\lambda, \mu > 0$, and $\sigma^\varepsilon_{ij} = \begin{pmatrix} \sigma_{\alpha\beta} & \frac{1}{\varepsilon} \sigma_{\alpha 3} \\ \frac{1}{\varepsilon} \sigma_{\alpha 3} & \frac{1}{\varepsilon^2} \sigma_{33} \end{pmatrix}$. Then $(L^\varepsilon u)_j = -\partial_i \sigma^\varepsilon_{ij}$ and $(B^\varepsilon u)_j = \sigma^\varepsilon_{3j}$.

1.2 Why a new error estimate?

Our motivation is the fact that we are able to get a H^2 error estimate (at least for some components) on the difference between the three-dimensional displacement and its two-dimensional projection, and that we do not know how to get similar results for the Kirchhoff-Love model.

Error estimates in H^2, only assuming the volumic forces f in L^2 (for instance for $g = 0$) are natural from the point of view of the regularity theory for the solutions of elliptic equations. Nevertheless the only known results (to our knowledge) are the H^1 error estimates on the difference between the three-dimensional displacements and the solution to the Kirchhoff-Love model, assuming the volumic forces in L^2 (and even more for certain components). This result due to Ciarlet, Destuynder [8] and Destuynder [12] is based on the Hellinger-Reissner formulation which naturally requires different regularities for the x' coordinates and the x_3 coordinates. We also cite the work of Raoult [29] in this direction, and the pioneering work of Shoikhet [30]. Furthermore we mention the work of Dauge, Gruais [9, 10] for interesting estimates with high regularity on the displacements, but requiring more than natural elliptic regularity on the forces. The reader will find additional results in the large litterature on the subject (see for instance Destuynder [13, 14], Destyunder, Salaun [15], John [18], Paumier, Raoult [28], Caillerie [5]). The derivation of the theory of plates can be seen as a dimensional reduction from the three-dimensional elasticity. For the litterature on dimensional reduction see Anzelloti, Baldo, Percivale [4], Maz'ya, Nazarov, Plamenevskij [20, 21], Mielke [24], Vogelius Babuska [31]. Let us mention that this question of dimension reduction for plates has to be placed in the general framework of hierachical models (see the survey Chapter 8 of [17], Fox, Raoult, Simo [16], and Actis, Szabo, Schwab [1]).

Even if the 3d-2d limit is not strictly speaking a problem of singular perturbations, the techniques used by the previous authors are close to the singular perturbation theory developed since the sixties (see Lions [19]). In particular it is well known that singular perturbations exhibit a loss of derivative phenomenon.

We cite Jacques-Louis Lions on this subject ([19], p.92):

Les estimations données (...) sont dans l'espace H^1. En fait en utilisant les résultats classiques de régularité des problèmes aux limites elliptiques (...), les solutions u (...) sont dans des espaces plus petits (de Sobolev, de Schauder, etc.). Quelles sont les estimations d'erreur dans ces espaces?

One goal of this article is to give a possible answer to this question, not to singular perturbations strictly speaking, but to a close problem, namely the 3d-2d limit in linear elastic plate theory.

1.3 A two-dimensional projection

Given any function $h = (h_1, ..., h_{11})$ defined on ω, we define its "three-dimensional extension" by

$$P_0(h) = \begin{pmatrix} h_1 + x_3 h_4 + \dfrac{x_3^2}{2!} h_7 + \dfrac{x_3^3}{3!} h_{10} \\ \\ h_2 + x_3 h_5 + \dfrac{x_3^2}{2!} h_8 + \dfrac{x_3^3}{3!} h_{11} \\ \\ h_3 + x_3 h_6 + \dfrac{x_3^2}{2!} h_9 \end{pmatrix}$$

Let us denote by $\mathcal{P}$ the space of all such functions:

$$\mathcal{P} = \left\{ P_0(h), \quad h \in H^2(\omega) \right\}$$

We define the two-dimensional projection $\mathrm{Proj}_{|\mathcal{P}}(u)$, where $\mathrm{Proj}_{|\mathcal{P}}$ is any continuous projector from $L^2([-1,1])$ on $I\!R^{11}$, which is naturally extended to a projector from $H^2(\Omega)$ on $\mathcal{P}$. We will give some estimates on

$$w = u - \mathrm{Proj}_{|\mathcal{P}}(u)$$

Let us give an example of such projection operator. To this end let us consider four functions $d_i \in L^2([-1,1]), i = 0, ..., 3$, satisfying for $i, j = 0, ..., 3$:

$$\int_{-1}^{1} dx_3 \; d_i(x_3) \frac{(x_3)^j}{j!} \quad = \quad \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}$$

For $u \in H^2(\Omega)$, we then define

$$\begin{cases} h_k(x') = \displaystyle\int_{-1}^{1} dx_3 \; u_k(x', x_3) d_0(x_3) & \text{for} \quad k = 1, 2, 3 \\ h_k(x') = \displaystyle\int_{-1}^{1} dx_3 \; u_{k-3}(x', x_3) d_1(x_3) & \text{for} \quad k = 4, 5, 6 \\ h_k(x') = \displaystyle\int_{-1}^{1} dx_3 \; u_{k-6}(x', x_3) d_2(x_3) & \text{for} \quad k = 7, 8, 9 \\ h_k(x') = \displaystyle\int_{-1}^{1} dx_3 \; u_{k-9}(x', x_3) d_3(x_3) & \text{for} \quad k = 10, 11 \end{cases}$$

and then define

$$\mathrm{Proj}_{|\mathcal{P}}(u) = P_0(h)$$

1.4 Error estimates

We are then able to prove the following H^2 error estimate (at least for certain components), which is a corollary of a more general result (theorem 2.3):

Theorem 1.1 **(H^2 error estimate)**
We asume that the open set ω_ℓ is periodic of size ℓ, i.e.

$$\omega_\ell = \ell \left(I\!R^2 \backslash \mathbf{Z}^2 \right)$$

and we define $\Omega_\ell = \omega_\ell \times (-1, 1)$. To simplify we assume that the surfacic forces satisfy $g \equiv 0$ and $f \in L^2(\Omega_\ell)$. For $\lambda, \mu > 0$ fixed, there exists a constant $C = C(\lambda, \mu) > 0$ such that every *solution u of (1.1) with $\omega = \omega_\ell$, we have the following estimates on $w = u - Proj_{|\mathcal{P}}(u)$:*

$$\left.\begin{array}{l} \left|e^\varepsilon_{ij}(w)\right|_{L^2(\Omega_\ell)} \\ \\ \left|\partial_3\left(e^\varepsilon_{ij}(w)\right)\right|_{L^2(\Omega_\ell)} \\ \\ \varepsilon\left|\partial_\alpha\left(e^\varepsilon_{ij}(w)\right)\right|_{L^2(\Omega_\ell)} \end{array}\right\} \quad \leq \quad C\varepsilon\left(\sum_\alpha |f_\alpha|_{L^2(\Omega_\ell)} + \varepsilon|f_3|_{L^2(\Omega_\ell)}\right)$$

In particular the constant C is independent on the size ℓ of the open set Ω_ℓ. Similarly we get $W^{k,p}$, $C^{k,\alpha}$ estimates for the general case with $g \not\equiv 0$.

As a corollary of a uniform $W^{2,p}$ version of this result (namely theorem 2.4), we have

Corollary 1.2 (L^∞ bounds in the periodic case)
Consider a periodic plate (i.e. $\omega = \ell(I\!R^2 \backslash \mathbf{Z}^2)$, for some $\ell > 0$) and $g \equiv 0$. Then there exists a constant $C > 0$ only depending on λ, μ such that if $f \in L^\infty(\Omega)$, then the function $w = u - Proj_{|\mathcal{P}}(u)$ satisfies

$$\left|e^\varepsilon_{ij}(w)\right|_{L^\infty(\Omega)} \quad \leq \quad C\varepsilon\left(\sum_\alpha |f_\alpha|_{L^\infty(\Omega)} + \varepsilon\,|f_3|_{L^\infty(\Omega)}\right)$$

This result gives a particularly interesting L^∞ bound on the stress in the plate. This has to be put in connection for instance with the work of Paumier [27] for periodic plates.

We give now a corollary of our approach (a particular case of theorem 2.3):

Theorem 1.3 (Interior H^2 error estimates)
Consider a plate $\Omega = \omega \times I$ where ω is a bounded open set of $I\!R^2$. We define the following interior open set

$$\Omega_d = \omega_d \times I, \quad \textit{where} \quad \omega_d = \left\{x' \in I\!R^2, \quad \textit{dist}\left(x', I\!R^2\backslash\omega\right) > d\right\}$$

To simplify we assume that $g \equiv 0$ and $f \in L^2(\Omega)$. Then there exist three constants $C, c, d_0 > 0$ only depending on λ, μ such that for every $d \geq d_0$, for every solution $u \in H^2(\Omega)$ of (1.1) on Ω, we have the following estimates on $w = u - Proj_{|\mathcal{P}}(u)$:

$$\left.\begin{array}{l} \left|e^\varepsilon(w)\right|_{L^2\left(\Omega_{\frac{d}{\varepsilon}}\right)} \\ \\ \left|\partial_3 e^\varepsilon(w)\right|_{L^2\left(\Omega_{\frac{d}{\varepsilon}}\right)} \\ \\ \varepsilon\left|\partial_\alpha e^\varepsilon(w)\right|_{L^2\left(\Omega_{\frac{d}{\varepsilon}}\right)} \end{array}\right\} \quad \leq \quad C\left(\begin{array}{c} \varepsilon\left(\sum_\alpha |f_\alpha|_{L^2(\Omega)} + \varepsilon\,|f_3|_{L^2(\Omega)}\right) \\ \\ + \quad e^{-c\frac{d}{\varepsilon}}\left(|u_3|_{L^2(\Omega)} + \varepsilon\sum_\alpha |u_\alpha|_{L^2(\Omega)}\right) \end{array}\right)$$

Remark 1.4 *Let us remark that the prefactor ε in the expression $\varepsilon\left|\partial_\alpha e^\varepsilon(w)\right|_{L^2\left(\Omega_{\frac{d}{\varepsilon}}\right)}$ does not allow to get a true error estimate in general. Nevertheless, in the particular case where $f_\alpha = 0$ we recover a full H^2 error estimate which justifies the title H^2 interior error estimates. This is due to the following equality $\partial_{ij}w_k = \partial_i e_{jk}(w) + \partial_j e_{ik}(w) - \partial_k e_{ij}(w)$, which allows to recover all L^2 estimates on the second derivatives of w.*

This last result can be put in relation on the one hand with Naghdi model of plates (see Destuynder [12]) and on the other hand with the director model as in Mielke [22], although our approach is different. The exponential factor is also reminiscent of the Saint-Venant Principle (see Mielke [23]).

1.5 Organization of the article

In section 2, we reduce the problem to the case $\varepsilon = 1$ and give some general 3d-2d interior error estimates (for the comparison with the 2d projection) which are proved in section 3. These results are based on some basic estimates which are proved in section 4.

2 Some general interior error estimates

2.1 Preliminaries

2.1.1 The scaling in ε

First of all, let us remark that if we set $\overline{x} = (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, x_3)$ and

$$\overline{u}_\alpha(\overline{x}) = \epsilon^2 u_\alpha(x) \qquad \overline{\partial}_\alpha = \epsilon\partial_\alpha \qquad \overline{f}_\alpha(\overline{x}) = \epsilon^4 f_\alpha(x) \qquad \overline{g}_\alpha(\overline{x}) = \epsilon^4 g_\alpha(x)$$

$$\overline{u}_3(\overline{x}) = \epsilon u_3(x) \qquad \overline{\partial}_3 = \partial_3 \qquad \overline{f}_3(\overline{x}) = \epsilon^5 f_3(x) \qquad \overline{g}_3(\overline{x}) = \epsilon^5 g_3(x)$$

then

$$\begin{cases} L^\varepsilon u = -f & \text{on} \quad \omega \times I \\ B^\varepsilon u = g & \text{on} \quad \omega \times \partial I \end{cases}$$

is equivalent to

$$\begin{cases} L^1 \overline{u} = -\overline{f} & \text{on} \quad \frac{\omega}{\varepsilon} \times I \\ B^1 \overline{u} = \overline{g} & \text{on} \quad \frac{\omega}{\varepsilon} \times \partial I \end{cases}$$

This remark reduces the proofs to the particular case $\varepsilon = 1$.

2.1.2 Notation (case $\varepsilon = 1$)

We note

$$Lu = \left\{ \begin{array}{l} (\lambda+2\mu)\partial_{11}u_1 + \mu(\partial_{22}u_1 + \partial_{33}u_1) + (\lambda+\mu)(\partial_{12}u_2 + \partial_{13}u_3) \\ \\ (\lambda+2\mu)\partial_{22}u_2 + \mu(\partial_{11}u_2 + \partial_{33}u_2) + (\lambda+\mu)(\partial_{21}u_1 + \partial_{23}u_3) \\ \\ (\lambda+2\mu)\partial_{33}u_3 + \mu(\partial_{11}u_3 + \partial_{22}u_3) + (\lambda+\mu)(\partial_{31}u_1 + \partial_{32}u_2) \end{array} \right| \quad \text{on} \quad \Omega$$

$$Bu = \left\{ \begin{array}{l} \mu(\partial_3 u_1 + \partial_1 u_3) \\ \\ \mu(\partial_3 u_2 + \partial_2 u_3) \\ \\ \lambda(\partial_1 u_1 + \partial_2 u_2) + (\lambda+2\mu)\partial_3 u_3 \end{array} \right| \quad \text{on} \quad \partial\Omega$$

We recall that

$$e(u) = (e_{ij}(u))_{ij}\,, \quad \text{where} \quad e_{ij}(u) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right)$$

Given a function $\zeta(x') = (\zeta_1(x'), \zeta_2(x'), \zeta_3(x'))$ for $x' = (x_1, x_2)$, we introduce the following 2d approximation of a 3d displacement in the plate:

$$U(\zeta) = \left(\begin{array}{lcl} \zeta_\alpha - x_3\partial_\alpha\zeta_3 & + & \left(\frac{\lambda}{\lambda+2\mu}\frac{x_3^2}{2}\partial_\alpha \mathrm{div}'\zeta + a(x_3)\partial_\alpha\Delta'\zeta_3\right) \\ \\ \zeta_3 & + & \frac{\lambda}{\lambda+2\mu}\left(-x_3\mathrm{div}'\zeta + \frac{x_3^2}{2}\Delta'\zeta_3\right) \end{array} \right)$$

where

$$\mathrm{div}'\zeta = \partial_1\zeta_1 + \partial_1\zeta_2, \quad \Delta' = \partial_1^2 + \partial_2^2, \quad a(x_3) = \left(\frac{3\lambda+4\mu}{\lambda+2\mu}\right)\frac{x_3^3}{3!} - 2\left(\frac{\lambda+\mu}{\lambda+2\mu}\right)x_3$$

We recall the expression of the Kirchhoff-Love operator:

$$M^0\zeta = \left\{ \begin{array}{ll} M^0_\alpha\zeta & = \mu\left(\Delta'\zeta_\alpha + \left(\frac{3\lambda+2\mu}{\lambda+2\mu}\right)\partial_\alpha \mathrm{div}'\zeta\right) \\ \\ M^0_3\zeta & = \frac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta'^2\zeta_3 \end{array} \right.$$

The interesting property of the reconstructed displacement $U(\xi)$ is that it is useful to build solutions in the kernel of the operator of elasticity:

Proposition 2.1 (On the kernel of the operator of elasticity, Dauge, Gruais, Rössle [11]; Monneau [25])
If ξ is a solution of the Kirchhoff-Love equations on ω, i.e. satisfies

$$M^0\xi = 0 \quad \text{on} \quad \omega$$

then

$$\begin{cases} L(U(\xi)) = 0 & on \quad \omega \times I \\ \\ B(U(\xi)) = 0 & on \quad \omega \times \partial I \end{cases}$$

Reciprocically, if for the infinite slab we introduce the kernel

$$\mathcal{P}_\infty = \left\{ v \in C^2(\mathbb{R}^2 \times [-1,1]), \quad Lv = Bv = 0, \quad \exists C, p > 0, \quad |v(x)| \leq C(1+|x|)^p \right\},$$

then we have the following result:

Proposition 2.2 ([25])
If $v \in \mathcal{P}_\infty$, then there exists a (polynomial) solution ξ of

$$M^0 \xi = 0 \quad on \quad \mathbb{R}^2$$

such that $v = U(\xi)$.

2.2 Global estimates

We can now state the following general result which implies theorem 1.3:

Theorem 2.3 (Global $W^{k,p}$ interior error estimates, case $\varepsilon = 1$)
Let $1 < p < +\infty$, $k \in \mathbb{N}$ and the plate $\Omega = \omega \times I$ where ω is a bounded open set of $\mathbb{R}^2$. We define the interior open set

$$\Omega_d = \omega_d \times I, \quad where \quad \omega_d = \left\{ x' \in \mathbb{R}^2, \quad dist\left(x', \mathbb{R}^2 \backslash \omega\right) > d \right\}$$

Then there exist three constants $C, c, d_0 > 0$ only depending on k, p, λ, μ such that for every $d \geq d_0$ and for every solution $u \in W^{k+2,p}(\Omega)$ of

$$\begin{cases} Lu = -f & on \quad \omega \times I \\ \\ Bu = g & on \quad \omega \times \partial I \end{cases}$$

for $w = u - Proj_{|\mathcal{P}}(u)$, we have

$$|w|_{W^{k+2,p}(\Omega_d)} \leq C \left(\begin{array}{l} |f|_{W^{k,p}(\Omega)} + |g|_{W^{k+1-\frac{1}{p},p}(\omega \times \partial I)} \\ \\ +e^{-cd} \; |u|_{L^p(\Omega)} \end{array} \right) \tag{2.2}$$

2.3 Uniform estimates

We also get some uniform a priori estimates, for which we need to introduce the following notation.

For $1 < p < +\infty$, we define the following norms

$$|u|_{L^p_{\mathrm{unif}}(\Omega)} = \sup_{x\in\Omega} |u|_{L^p(B_1(x)\cap\Omega)}$$

and more generally

$$|u|_{W^{k,p}_{\mathrm{unif}}(\Omega)} = \sup_{x\in\Omega} |u|_{W^{k,p}(B_1(x)\cap\Omega)}$$

We also define the semi-norms

$$\mathcal{N}_{W^{k,p}_{\mathrm{unif}}(\Omega)}(u) = \sup_{x\in\Omega} \quad \inf_{\xi\in\Xi(x')} |u - U(\xi)|_{W^{k,p}(B_1(x)\cap\Omega)}$$

where

$$\Xi(x') = \left\{\xi \in C^\infty(B_2(x')), \quad M^0\xi = 0 \quad \text{on} \quad B_2(x')\right\}$$

Then we have

Theorem 2.4 (Uniform $W^{k,p}$ interior error estimates, case $\varepsilon = 1$)
Let $1 < p < +\infty$, $k \in \mathbb{N}$ and the plate $\Omega = \omega \times I$ where ω is a (possibly unbounded) open set of $\mathbb{R}^2$. We define the interior open set

$$\Omega_d = \omega_d \times I, \quad \textit{where} \quad \omega_d = \left\{x' \in \mathbb{R}^2, \quad \mathit{dist}\left(x', \mathbb{R}^2\backslash\omega\right) > d\right\}$$

Then there exist three constants $C, c, d_0 > 0$ only depending on k, p, λ, μ such that for every $d \geq d_0$, for every solution $u \in W^{k+2,p}_{loc}(\Omega)$ of

$$\begin{cases} Lu = -f & \textit{on} \quad \omega \times I \\ \\ Bu = g & \textit{on} \quad \omega \times \partial I \end{cases}$$

we have

$$\mathcal{N}_{W^{k+2,p}_{unif}(\Omega_d)}(u) \quad \leq \quad C \begin{pmatrix} |f|_{W^{k,p}_{unif}(\Omega)} + |g|_{W^{k+1-\frac{1}{p},p}_{unif}(\omega\times\partial I)} \\ \\ +e^{-cd}\ \mathcal{N}_{W^{k+2,p}_{unif}(\Omega)}(u) \end{pmatrix} \tag{2.3}$$

This implies the

Corollary 2.5 (L^∞ Error estimate for a periodic plate, case $\varepsilon = 1$)
For $\Omega = \omega \times I$ with ω periodic, there exists a constant $C = C(\lambda, \mu) > 0$ such that (in the special case $g = 0$) the function $w = u - Proj_{|\mathcal{P}}(u)$ satisfies

$$|e(w)|_{L^\infty(\Omega)} \quad \leq \quad C|f|_{L^\infty(\Omega)}$$

Similarly for a general function v defined on Ω we note the Hölder seminorm for $\alpha \in (0,1)$

$$[v]_{\alpha;\Omega} = \sup_{x,y\in\Omega,\quad x\neq y} \frac{|v(x)-v(y)|}{|x-y|^{\alpha}}$$

More generally we recall the norms

$$|v|_{k+\alpha;\Omega} = \sum_{j=0}^{k} |D^j v|_{\alpha;\Omega} \qquad \text{with} \quad |v|_{\alpha;\Omega} = |v|_{0;\Omega} + [v]_{\alpha;\Omega} \quad \text{and} \quad |v|_{0;\Omega} = \sup_{x\in\Omega} |v(x)|$$

We also introduce the following seminorm

$$\mathcal{N}_{k+\alpha;\Omega} = \sup_{x\in\Omega} \quad \inf_{\xi\in\Xi(x')} |u - U(\xi)|_{k+\alpha;B_1(x)\cap\Omega}$$

Then we can also state the $C^{k+\alpha}$ version of theorem 2.4, whose proof is similar and will be dropped (see Monneau [25] for similar estimates):

Theorem 2.6 (Uniform $C^{k+\alpha}$ interior error estimates, case $\varepsilon = 1$)
Let $\alpha \in (0,1)$, $k \in I\!N$ and the plate $\Omega = \omega \times I$ where ω is a (possibly unbounded) open set of $I\!R^2$. We define the interior open set

$$\Omega_d = \omega_d \times I, \quad \textit{where} \quad \omega_d = \left\{x' \in I\!R^2, \quad \textit{dist}\left(x', I\!R^2\backslash\omega\right) > d\right\}$$

Then there exist three constants $C, c, d_0 > 0$ only depending on k, α, λ, μ such that for every $d \geq d_0$, for every solution $u \in C^{k+2+\alpha}(\Omega)$ of

$$\left\{\begin{array}{l} Lu = -f \quad \textit{on} \quad \omega \times I \\ \\ Bu = g \quad \textit{on} \quad \omega \times \partial I \end{array}\right.$$

we have

$$\mathcal{N}_{k+2+\alpha;\Omega_d}(u) \quad \leq \quad C \left(\begin{array}{l} |f|_{k+\alpha;\Omega} + |g|_{k+1+\alpha;\omega\times\partial I} \\ \\ +e^{-cd}\mathcal{N}_{k+2+\alpha;\Omega}(u) \end{array}\right)$$

3 Proofs of a basic error estimate

Here we will prove the following basic error estimate, which will be used in the proof of the general interior error estimate.

Lemma 3.1 (Basic error estimate)
For $x = (x', x_3)$, we note $C_r(x) = \overline{B}_r(x') \times [-1,1]$. Then we define

$$M(x) = \inf_{\xi\in\Xi(x')} |u - U(\xi)|_{W^{2,p}(C_1(x))}$$

and for some $r_0 > 0$ we define

$$N(x) = |f|_{L^p(C_{r_0}(x))} + |g|_{W^{1-\frac{1}{p},p}(B_{r_0}(x')\times\partial I)}$$

Then for every $\theta \in (0,1)$ there exists $C > 0, r_0 > 1$ such that for every x:

$$M(x) \leq CN(x) + \theta \sup_{z\in C_{r_0}(x)} M(z)$$

3.1 Preliminaries on the symmetry

For a scalar function v defined on $\overline{\Omega}$, let us define the symmetric and antisymmetric parts with respect to x_3:

$$v^s(x) = \frac{1}{2}\left(v(x_1,x_2,x_3) + v(x_1,x_2,-x_3)\right) \quad \text{and} \quad v^a(x) = \frac{1}{2}\left(v(x_1,x_2,x_3) - v(x_1,x_2,-x_3)\right)$$

For a vector function $u = (u_1,u_2,u_3)$ we define the following *symmetric* and *antisymmetric* parts:

$$u^S = (u_1^s, u_2^s, u_3^a) \quad \text{and} \quad u^A = (u_1^a, u_2^a, u_3^s)$$

In particular we can easily check that

$$\begin{cases} L(u^S) = -f^S \\ \\ B(u^S) = g^A \end{cases} \quad \text{and} \quad \begin{cases} L(u^A) = -f^A \\ \\ B(u^A) = g^S \end{cases}$$

To prove the basic estimates, we need the following proposition

Proposition 3.2 (Polynomial control on the solution)
Let $1 < p < +\infty$ and $\Omega = \omega \times I$. There exists $\delta \in (0,1)$ and a constant $C > 0$ only depending on p, λ, μ, such that for every function $u \in W^{2,p}_{loc}(\Omega)$, there exists $\xi \in C^\infty(\omega)$ solution of

$$M^0\xi = 0 \quad on \quad \omega$$

such that for $R \geq 1$

$$\begin{cases} \left|(u-U(\xi))^S\right|_{L^p_{unif}(B_R\cap\Omega)} & \leq & CR^{2+\delta}\mathcal{N}_{W^{2,p}_{unif}(\Omega)}(u) \\ \\ \left|(u-U(\xi))^A\right|_{L^p_{unif}(B_R\cap\Omega)} & \leq & CR^{4+\delta}\mathcal{N}_{W^{2,p}_{unif}(\Omega)}(u) \end{cases}$$

The proof of proposition 3.2 is a simple adaptation of the proof of proposition 6.2 of [25].

3.2 Proof of the basic error estimate: lemma 3.1

We introduce the set of polynomials:

$$\mathcal{P}_{q,r} = \left\{ v \in C^2(I\!R^2 \times [-1,1]), \quad Lv = Bv = 0 \quad \exists C > 0, \quad \begin{array}{l} |v^S(x)| \leq C(1+|x|)^q \\ |v^A(x)| \leq C(1+|x|)^r \end{array} \right\}$$

To simplify the notations, let us consider the case $g \equiv 0$. The general case is similar. Moreover, from the invariance by translations we can take $x = 0$, and we note $C_r = C_r(0) = B_r(0) \times [-1,1]$ where $B_r(0) \subset I\!R^2$.

We will prove the following inequality which implies lemma 3.1:

$$(3.4) \qquad \begin{array}{l} \forall \theta \in (0,1), \quad \exists r_0 > 1, \quad \exists C > 0, \\ \inf_{\xi \in \Xi(x')} |u - U(\xi)|_{W^{2,p}(C_1(x))} \quad \leq \quad C\,|f|_{L^p_{\mathrm{unif}}(C_{r_0}(x))} + \theta\, \mathcal{N}_{W^{2,p}_{\mathrm{unif}}(C_{r_0}(x)))}(u) \end{array}$$

Let us assume that inequality (3.4) is false. Then we can find a $\theta \in (0,1)$ and a sequence of solutions $(u^n)_n$ on $C_{r_0^n}$ with sequences

$$r_0^n, C_n \longrightarrow +\infty$$

such that

$$\inf_{\xi \in \Xi(0)} |u^n - U(\xi)|_{W^{2,p}(C_1)} \quad > \quad C_n\,|f^n|_{L^p_{\mathrm{unif}}(C_{r_0^n})} + \theta \mathcal{N}_{W^{2,p}_{\mathrm{unif}}(C_{r_0^n})}(u^n)$$

Up to multiply the solutions by a constant we can assume that the left hand side is equal to 1:

$$(3.5) \qquad \inf_{\xi \in \Xi(0)} |u^n - U(\xi)|_{W^{2,p}(C_1)} = 1$$

In particular we deduce that

$$|f^n|_{L^p_{\mathrm{unif}}(C_{r_0^n})} \longrightarrow 0$$

$$\mathcal{N}_{W^{2,p}_{\mathrm{unif}}(C_{r_0^n}))}(u^n) \quad \leq \quad \theta^{-1}$$

With help of proposition 3.2 we can find a sequence $(h^n)_n$ with bounds on $v^n = u^n - U(h^n)$. This proves that $v^n \to v^\infty$ locally on compact sets where v^∞ satisfies $Lv^\infty = Bv^\infty = 0$ and the bounds given by proposition 3.2. In particular we deduce that $v^\infty \in \mathcal{P}_{2+\delta,4+\delta} = \mathcal{P}_{2,4}$ (from Proposition 2.2) and then there exists a polynomial k such that $v^\infty = U(k)$. As a consequence, up to substract $U(h^n + k)$ to u^n, we can assume that

$$u^n \longrightarrow 0 \quad \text{locally in } L^p \text{ on compact sets}$$

Now let us recall that we have the classical "interior" elliptic estimate (see Morrey [26]; Agmon, Douglis, Nirenberg [2, 3])

$$|u^n|_{W^{2,p}(C_1)} \quad \leq \quad C\left(|f^n|_{L^p(C_2)} + |u^n|_{L^p(C_2)}\right)$$

which proves that

$$u^n \longrightarrow 0 \quad \text{locally in } W^{2,p} \text{ on compact sets} \tag{3.6}$$

We finally realize that strong convergences (3.6) on u^n is in contradiction with equality (3.5) on u^n. This ends the proof of inequality (3.4), and consequenlty of lemma 3.1.

4 Proof of the general interior error estimate

4.1 Preliminary change of notations

In the first part of the article, it was more convenient to present our interior estimates denoting by ω_d, Ω_d some interior open sets. Here, in the proofs of the theorems, it is more convenient to change these notations, defining some surrounding open sets:

$$\Omega_d = \omega_d \times I, \quad \text{where} \quad \omega_d = \left\{x' \in I\!R^2, \quad \text{dist}\,(x', \omega) < d\right\}$$

4.2 Proof of theorem 2.3

We will prove theorem 2.3 in the case $k = 0$. More generally the case $k > 0$ is similar.

Using several times lemma 3.1, we get

$$\begin{aligned}
M(x) &\leq CN(x) + \theta \sup_{z \in C_{r_0}(x)} M(z) \\
&\leq CN(x) + \theta \sup_{z \in C_{r_0}(x)} \left(CN(z) + \theta \sup_{z' \in C_{r_0}(z)} M(z') \right) \\
&\leq CN(x) + C\theta \sup_{z \in C_{r_0}(x)} N(z) + \theta^2 \sup_{z \in C_{2r_0}(x)} M(z) \\
&\leq C\left(\sum_{k=1}^{K} \theta^{k-1} \sup_{z \in C_{(k-1)r_0}(x)} N(z) \right) + \theta^K \sup_{z \in C_{Kr_0}(x)} M(z)
\end{aligned}$$

In particular we deduce that

$$\begin{aligned}
\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega} M(x) \quad &\leq \quad C\left(\sum_{k=1}^{K}\theta^{k-1}C'(1+k^2)\right)\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega_{Kr_0}} N(x) \\
&\quad +\theta^K C'(1+K^2)\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega_{Kr_0}} M(x) \\
&\leq \quad C\left(\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega_{Kr_0}} N(x) + e^{-cKr_0}\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega_{Kr_0}} M(x)\right)
\end{aligned}$$

for some constant $c>0$ small enough. This implies with $d = Kr_0+1$:

$$\left|u-\mathrm{Proj}_{|\mathcal{P}}(u)\right|_{W^{2,p}(\Omega)} \quad \leq \quad C\left(\begin{array}{l} |f|_{L^p(\Omega_d)} + |g|_{W^{1-\frac{1}{p},p}(\omega_d\times\partial I)} \\ \\ +e^{-cd}\left(\sum_{x\in\mathbf{Z}^2\times\{0\}\cap\Omega_{Kr_0}}\left(\inf_{\xi\in\Xi(x')}|u-U(\xi)|_{W^{2,p}(C_1(x))}\right)\right)\end{array}\right)$$

where we have used the additivity of the Sobolev norms, and

$$|u-\mathrm{Proj}_{|\mathcal{P}}(u)|_{W^{2,p}(C_1(x))} \quad \leq \quad C\inf_{\xi\in\Xi(x')}|u-U(\xi)|_{W^{2,p}(C_1(x))}$$

This last inequality follows from the fact that $U(\xi) = \mathrm{Proj}_{|\mathcal{P}}(U(\xi))$ for any $\xi = \xi(x')$.
We recover the full inequality, replacing on the right hand side the $W^{2,p}$ norm on u by a L^p one. This is possible because of the following general interior estimate for elliptic systems (see Morrey [26]):

$$\inf_{\xi\in\Xi(x')}|u-U(\xi)|_{W^{2,p}(C_1(x))} \leq C\left(\begin{array}{l} |f|_{L^p(C_2(x))} + |g|_{W^{1-\frac{1}{p},p}(B_2(x')\times\partial I)} \\ \\ +\inf_{\xi\in\Xi(x')}|u-U(\xi)|_{L^p(C_2(x))}\end{array}\right) \tag{4.7}$$

We get

$$|w|_{W^{2,p}(\Omega)} \quad \leq \quad C\left(\begin{array}{l} |f|_{L^p(\Omega_d)} + |g|_{W^{1-\frac{1}{p},p}(\omega_d\times\partial I)} \\ \\ +e^{-cd}\left(|u|_{L^p(\Omega_d)}\right)\end{array}\right)$$

which is exactly inequality (2.2), up to the change of notations for Ω_d, ω_d (see subsection 4.1). This ends the proof of the inequality (2.2) of theorem 2.3.

4.3 Proof of theorem 2.4

The proof of inequality (2.3) of theorem 2.4, easily follows (in a very elementar way) from basic error estimates lemma 3.1, by a recurrency on Ω, Ω_{r_0}, Ω_{2r_0}, Ω_{3r_0}, ...
This ends the proof of theorem 2.4.

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References

[1] R. L. Actis, B. A. Szabo, C. Schwab, *Hierarchic models for laminated plates and shells.* Comput. Methods Appl. Mech. Engrg. 172 (1-4), 79-107 (1999).

[2] S. Agmon, A. Douglis, L. Nirenberg, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions. I*, Comm. Pure Appl. Math. **12**, 623-727 (1959).

[3] S. Agmon, A. Douglis, L. Nirenberg, *Estimates Near the Boundary for Solutions of Elliptic Partial Differential Equations Satisfying General Boundary Conditions. II*, Comm. Pure Appl. Math. **17**, 35-92 (1964).

[4] G. Anzelloti, S. Baldo, D. Percivale, *Dimension reduction in variational problems, asymptotic development in* Γ*-convergence and thin structures in elasticity*, Asymptotic Analysis **9 (1)**, 61-100, (1994).

[5] D. Caillerie, *Thin Elastic and Periodic Plates*, Math. Meth. in the Appl. Sci. **6**, 159-191, (1984).

[6] P.G. Ciarlet, *Plates and Junctions in Elastic Multi-Structures: An Asymptotic Analysis.*, R.M.A. **14**, Masson and Springer-Verlag, Paris and Heidelberg, (1990).

[7] P.G. Ciarlet, *Mathematical Elasticity, Vol II: Theory of Plates*, North-Holland, Amsterdam, (1997).

[8] P.G. Ciarlet, P. Destuynder, *A justification of the two-dimensional plate model*, J. Mécanique **18**, 315-344, (1979).

[9] M. Dauge, I. Gruais, *Développement asymptotique d'ordre arbitraire pour une plaque élastique mince encastrée*, Note C. R. Acad. Sci. Paris, **t. 321**, Série I, p. 375-380, (1995).

[10] M. Dauge, I. Gruais, *Asymptotics of arbitrary order for a thin elastic clamped plate, I. Optimal error estimates*, Asymptotic Analysis **13**, 197-197, (1996).

[11] M. Dauge, I. Gruais, A. Rössle, *The influence of Lateral Boundary Conditions on the Asymptotics in Thin Elastic Plates*, SIAM Journal on Mathematical Analysis **31**, 305-345, (2000).

[12] P. Destuynder, *Sur une justification des modèles de plaques et de coques par les méthodes asymptotiques*, Thèse d'Etat, Université Pierre et Marie Curie, Paris, (1980).

[13] P. Destuynder, *Comparaison entre les modèles tridimensionneles et bidimensionnels de plaques en élasticité.*, R.A.I.R.O. Analyse Numérique **15**, 331-369, (1981).

[14] P. Destuynder, *Estimations d'erreur explicites pour le modèle de plaques de Kirchhoff-Love et Reissner-Mindlin*, C. R. Acad. Sci. Paris **t. 325**, Série I, 233-238, (1997).

[15] P. Destuynder, M. Salaun, *Mathematical Analysis of Thin Plate Models*, Mathematiques & Applications **24**, SMAI, Springer (1996).

[16] D. Fox, A. Raoult, J.C. Simo, *A justification of Nonlinear Properly Invariant Plate Theories*, Arch. Rational Mech. Anal. 124 (1993), 157-199.

[17] *Encyclopedia of computational mechanics*, Vol. 1, Ed. E. Stein, R. Borst, T. Hughes. Chichester : Wiley, (2004).

[18] F. John, *Plane strain problems for a perfectly elastic material of harmonic type*, Comm. Pure Appl. Math. **13**, (1960), 239-296.

[19] J.-L. Lions, *Perturbations Singulières dans les Problèmes aux Limites et en Contrôle Optimal*, Lecture Notes in Math. **323**, Springer-Verlag, Berlin, Heidelberg, New York, (1973).

[20] V. Maz'ya, S. Nazarov, B. Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singular Perturbed Domains, I*, Operator Theory, Advances and Applications **111**, Birkh"auser, (2000).

[21] V. Maz'ya, S. Nazarov, B. Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singular Perturbed Domains, II*, Operator Theory, Advances and Applications **112**, Birkhäuser, (2000).

[22] A. Mielke, *On the Justification of Plate Theories in Linear Elasticity Theory Using Exponential Decay Estimates*, Journal of Elasticity **38**, 165-208, (1995).

[23] A. Mielke, *On Saint-venant's problem for an elastic strip*, Proceedings of the Royal Society of Edimburgh **110A**, 161-181, (1988).

[24] A. Mielke, *Reduction of PDEs on domains with several unbounded directions: A first step towards modulations equations*, Z. angew. Math. Phys. **43**, 449-470, (1992).

[25] M. Monneau, *Uniform elliptic estimate for an infinite plate in linear elasticity*, Comm. Partial Diff. Equations **29** (7-8) 989-1016, (2004).

[26] C.B. Morrey, *Multiple Integrals in the Caculus of Variations*, Springer-Verlag, Berlin-Heidelberg-New York, (1966).

[27] J.-C. Paumier, *Existence and convergence of the expansion in the asymptotic theory of elastic thin plates*, Math. Modelling Numer. Anal. **25** (3), 371-391, (1991).

[28] J.-C. Paumier, A. Raoult, *Asymptotic consistency of the polynomial approximation in the linearized plate theory: application to the Reissner-Mindlin model*, Rapport technique **164** LMC-IMAG Grenoble, (1996).

[29] A. Raoult, *Construction d'un modèle d'évolution de plaques avec terme d'inertie de rotation*, Annali di Matematica Pura ed Applicata **139**, 361-400, (1985).

[30] B.A. Shoikhet, *An energy identity in physically nonlinear elasticity and error estimates of the plate equations*, Prikl. Matem. Mekhan. **40 (2)** (1976) 317-326. English trasnlation J. Appl. Maths. Mechs. (1976) 291-301.

[31] M. Vogelius, I. Babuska, *On a Dimensional reduction method I. The Optimal Selection of Basis Functions*, Mathematics of Computation **37** (155), 31-68 , (1981).