Justification of the nonlinear Kirchhoff-Love theory of plates as the application of a new singular inverse method

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Abstract

In the framework of isotropic homogeneous nonlinear elasticity for a St. Venant-Kirchhoff material, we consider a three-dimensional plate of thickness ε and periodic in the two other directions. Using a new method that we call *Singular Inverse Method*, we prove the existence of a rescaled solution uniformly in ε for small forces, and by the way we prove the rigorous convergence of this rescaled solution to the solution of the nonlinear Kirchhoff-Love plate model. We also state a 3d-2d error estimate.

1 Introduction

We consider solutions u to the system of three-dimensional nonlinear elasticity

$$\mathcal{A}^\varepsilon(u) = F \tag{1.1}$$

over a rescaled periodic plate

$$\Omega = \omega \times (-1,1) \quad \text{with} \quad \omega = \mathbf{R}^2/\mathbf{Z}^2$$

Here u is the rescaled displacement of the plate, F are the rescaled exterior forces, and ε denotes the thickness of the unrescaled original plate. We will give the precise expressions of all these quantities in section 2. For $\varepsilon > 0$, equation (1.1) is a second order quasilinear elliptic system on the three components of the displacement u with three

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nonlinear boundary conditions. For $\varepsilon = 0$, this system degenerates in a system of equations which can be posed on ω, and which is called the nonlinear Kirchhoff-Love plate model. This limit system is composed of a semilinear elliptic system of order 2 on the first two components of the displacement u coupled with a semilinear elliptic equation of order 4 on the third component of the displacement. A special case of this model are the *von Kármán plate equations.* It is in particular known (see Ciarlet [5]) that the solutions for the problem with $\varepsilon > 0$ formally converges as $\varepsilon \to 0$ to the solutions of the problem at $\varepsilon = 0$.
Let us cite P.G. Ciarlet [5]:

"A natural idea consists in trying to construct, for ε small enough, a three-dimensional solution $u(\varepsilon)$ of problem $\mathcal{P}(\varepsilon)$ "close to" a two-dimensional solution u^0 of the limit problem $\mathcal{P}(0)$. (...) One further step would consists of comparing the three-dimensional and the two-dimensional solutions, thus generalizing to the nonlinear case the known convergence of the linear case."

Following this idea we present here a study in the particular case of plates with periodic boundary conditions.

Let us recall that using the classical inverse function theorem in approriate Banach spaces it is possible to prove the existence of solutions in the following cases (see Ciarlet [6], Paumier [29]):
1) for $\varepsilon = 0$ and small data $|F| \leq R_0$
2) for each $\varepsilon \neq 0$ and corresponding small data $|F| \leq R_\varepsilon$.

But the difficulty with this classical approach is that R_ε tends to zero as ε goes to zero. In particular it seems impossible to give a rigorous convergence result as ε goes to zero (see Fig. 1). In the present article we prove the following result (see theorem 2.2 for a precise statement of this result):

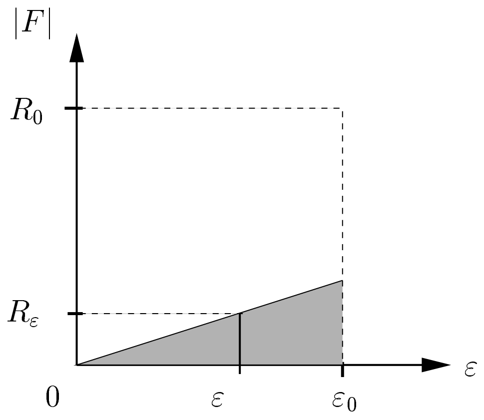

Figure 1: The limit of the classical approach : the dark triangle

Theorem (Existence, uniqueness, 3d-2d error estimate)
There exist some constants $\varepsilon_0, R_0, M > 0$ such that for each $\varepsilon \in [0, \varepsilon_0]$, and $|F| \leq R_0$, there exists a solution u of (1.1). Moreover if we denote by u^0 the solution for $\varepsilon = 0$, then we have the error estimate:

$$||u - u^0|| \leq \varepsilon M \tag{1.2}$$

Finally every solution satisfying (1.2) is unique.

This result is based on a new method that we call the *Singular Inverse Method* which allows to get existence results even in the case where the problem is too singular to apply the usual inverse function theorem. In our case we want to find a branch of solutions u to

$$\mathcal{A}^\varepsilon(u) = F$$

as the parameter ε varies from $\varepsilon = 0$ for which we have the particular solution

$$\mathcal{A}^0(u^0) = F$$

The fact that the operator $\mathcal{A}^\varepsilon$ has a behaviour like $1/\varepsilon$ as $\varepsilon \to 0$ forbides the application of the usual inverse function theorem. Moreover it is well known that this problem presents some loss of derivatives in the limit $\varepsilon \to 0$ which can be seen on the passage of equations of order 2 for $\varepsilon > 0$ to order 4 for $\varepsilon = 0$, even in the case of linear elasticity.

Techniques like the Nash-Moser inverse function theorem on the one hand (see the result of Paumier [28] with an additional assumption on the inversibility of the linearized operator) or singular perturbations theories like those in Lions [18] on the other hand seem unapplicable without additional assumptions. Finally a fixed point theorem even if it should exists in this context seems difficult to state.

Let us underline that our method gives an error estimate, only with assumptions on the forces, contrarily for instance to the fundamental work of F. John [12, 13, 14], where the strain were assumed bounded in L^∞. Moreover let us remark that the existence result of a three-dimensional solution proved by our method for positive ε, is not contained in the general and powerfull existence theory known for polyconvex energy (see J. Ball [2]), because our energy is not polyconvex, as it is well-known (see Le Dret, Raoult [16]). Before to continue, let us mention the work of Mielke [20] where a nonlinear model of beam has been justified by an original method, namely the Central Manifold theory.

Our Singular Inverse Method consists in a mixing between the requirement of a fixed point theorem and the flexibility of the classical inverse function theorem.

The Singular Inverse Method to solve $\mathcal{A}^\varepsilon(u) = F$:

Step 1 : (Existence for $\varepsilon = 0$)
For $\varepsilon = 0$, the solution u^0 of $\mathcal{A}^0(u^0) = F$ can be obtained by the usual inverse function theorem from the particular solution $u^0 = 0$ for $F = 0$ for every $|F| \leq R_0$.

Step 2 : (Existence for $\varepsilon > 0$)
For each $\varepsilon > 0$, the a priori solution u can be obtained for $|F| \leq R_\varepsilon$ by the usual inverse function theorem from the particular solution $u = 0$ for $F = 0$. (At this step, we have no control on R_ε as ε goes to 0.)

Step 3 : (Inversibility of the linearized operator)
There exists a constant $c_0 = c_0(R_0) > 0$, such that for every $M > 0$, for every $|F| \leq R_0$,

there exists $\varepsilon_0 = \varepsilon_0(M, R_0) > 0$, such that for every function u of the set

$$K_M^\varepsilon = \Big\{u, \quad ||u - u^0|| \leq \varepsilon M\Big\}$$

and for every $\varepsilon \in (0, \varepsilon_0]$, we have

$$(D_u \mathcal{A}^\varepsilon(u) \cdot w, w) \quad \geq \quad c_0 |w|^2 \tag{1.3}$$

which proves that $D_u \mathcal{A}^\varepsilon(u)$ is inversible.
Let us remark that the norm $||\cdot||$ on u is here stronger that the norm $|\cdot|$ on w.

Step 4 : (A priori estimate)
Using (1.3), we prove the a priori estimate: $\exists M = M(R_0) > 0$,

$$\forall \varepsilon \in (0, \varepsilon_0], \quad \forall |F| \leq R_0, \quad (u \in K_M^\varepsilon \quad \textit{and} \quad \mathcal{A}^\varepsilon(u) = F) \Longrightarrow \Big(u \in K^\varepsilon_{\frac{M}{2}}\Big) \tag{1.4}$$

Step 5 : (Conclusion)
With help of (1.3), we know that the linearized operator is inversible while $u \in K_M^\varepsilon$. But the solution u can not go outside K_M^ε, because $u \in K^\varepsilon_{\frac{M}{2}}$ by the a priori estimate (1.4) (see Fig. 2). We now conclude a posteriori, that the inverse function theorem applies in step 2, for every F satisfying $|F| \leq R_0$, and for every $\varepsilon \in (0, \varepsilon_0]$. Finally, inequality (1.3) implies the uniqueness of every solution u belonging to K_M^ε.

This method seems applicable in a wide number of cases where the other approachs do not work (for an application of this method in an other context with boundary layers, see [3]).

1.1 Generalizations and perspectives

Using the approach developped in the present paper it is easy to prove the

Theorem 1.1 (2d solutions as limit of 3d solutions)
Every sufficiently smooth solution u^0 of $\mathcal{A}^0(u^0) = F$ which can be obtained by the local inverse function theorem from the particular solution $u^0 = 0$ is the limit of solutions u of $\mathcal{A}^\varepsilon(u) = F$ as ε tends to zero.

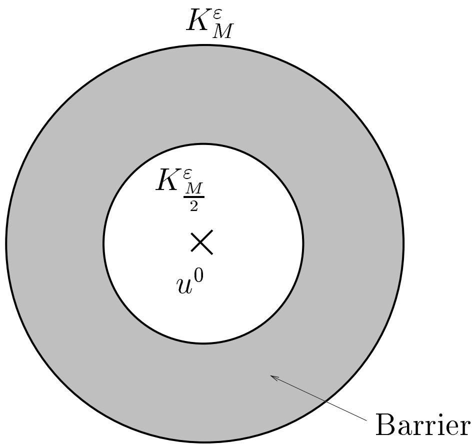

Figure 2: The a priori estimate for the Singular Inverse Method

In the context of elasticity, it would be interesting to apply our approach to the general case of hyperelastic energy or even the case of nonlinear elasticity models which are not derived from an energy. This could be applied to rod, plate and shell models. An other interesting extension would be to deal with the boundaries of a plate that we have avoided here considering periodic plates.

1.2 A brief review of the literature

To better understand the nonlinear Kirchhoff-Love model, let us present a hierarchy of models depending on the assumptions you make on the energy and the displacements. We consider a general plate $\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon)$, where each point x is transformed in $\Phi(x) = x + u(x)$ where u are the displacements. Let us consider the mean energy

$$\overline{\mathcal{E}}^\varepsilon(u) = \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \mathcal{W}(\nabla \Phi)$$

where $\mathcal{W}$ is the isotropic elastic energy density. Then (at least) for different limit models have been identified (see Fox, Raoult, Simo [9]):

i) $\overline{\mathcal{E}}^\varepsilon(u) = O(1)$ and $u = O(1)$

This gives a nonlinear **pure membrane** plate model which involves *quasilinear* elliptic partial differential equations. Both an asymptotic analysis (see Fox, Raoult, Simo [9], Pantz [25, 27]) and a Γ-convergence approach gives the same limit model (see Le Dret, Raoult [17]).

ii) $\overline{\mathcal{E}}^{\varepsilon}(u) = O(\varepsilon^2)$ and $u = O(1)$ is inextensional

This gives a nonlinear **pure flexural** plate model which involves *quasilinear* elliptic partial differential equations. This model has been recently justified by Γ-convergence by Friesecke, James, Müller [10], See also Pantz [26, 27] for a justification by asymptotic analysis.

iii) $\overline{\mathcal{E}}^{\varepsilon}(u) = O(\varepsilon^4)$ and $u = \begin{pmatrix} \varepsilon^2 u_1 \\ \varepsilon^2 u_2 \\ \varepsilon u_3 \end{pmatrix}$

This gives the **nonlinear Kirchhoff-Love** plate model (see Ciarlet [5] for a derivation by asymptotic analysis, and the original work of Ciarlet, Destuynder [8]) which involves *semilinear* elliptic partial differential equations. See also the Γ-convergence justificaton of this model by Friesecke, James, Müller [11]. This model contains both membrane and flexural effects. The famous von Kármán plate equations can be seen as a particular case of these equations. This model exhibits bifurcations phenomena.

iv) $\overline{\mathcal{E}}^{\varepsilon}(u) = O(\varepsilon^{4+2\delta})$ and $u = \begin{pmatrix} \varepsilon^{2+\delta} u_1 \\ \varepsilon^{2+\delta} u_2 \\ \varepsilon^{1+\delta} u_3 \end{pmatrix}$ for some $\delta > 0$.

This gives the **Kirchhoff-Love** plate model which involves *linear* elliptic partial differential equations, and has been extensively studied (see the book of Ciarlet [7] and the references therein, see also Kirchhoff [15] and Love [19]). See also the work of Anzelloti, Baldo, Percivale [1] for a related approach by Γ-convergence.

From our point of view, in the reality the problem is really three-dimensional, the thickness of the plate is fixed, and the plate is submitted to given exterior forces. For these particular data, the equilibrium of the plate is given by certain displacements. Now, depending on the amplitude of the three-dimensional displacement, the three-dimensional solution should be well approximated by one two-dimensional solution of a two-dimensional reduced model (and in good cases, we have in mind one of the previous models). In particular if the displacements are not two small, the linear Kirchhoff-Love theory is not sufficient, and a good two-dimensional approximation of the three-dimensional solution requires the use of the nonlinear Kirchhoff-Love model.

1.3 Organization of the article

In section 2 we give the precise statement of our main result which is proved in section 3. In section 4 we present a priori estimates and in section 5 we give the proof of the coercivity of the hessian of the energy. In an Appendix are given the proofs of two technical lemmata used in section 5.

2 Statement of the main result

We consider a periodic plate of thickness 2ε

$$\Omega^\varepsilon = \omega \times (-\varepsilon, \varepsilon) \quad \text{with} \quad \omega = \mathbf{R}^2/\mathbf{Z}^2$$

We consider exterior volumic forces f^ε and surfacic forces g^ε, and we note

$$F^\varepsilon = (f^\varepsilon, g^\varepsilon)$$

We recall the nonlinear elastic energy (for $\lambda, \mu > 0$)

$$\mathcal{E}_{F^\varepsilon}(u^\varepsilon) = \int_{\Omega^\varepsilon} \frac{\lambda}{2}\left(E_{kk}(u^\varepsilon)\right)^2 + \mu\left(E_{kl}(u^\varepsilon)E_{lk}(u^\varepsilon)\right) - f_i^\varepsilon u_i^\varepsilon - \int_{\omega\times\{+\varepsilon\}} g_i^\varepsilon u_i^\varepsilon + \int_{\omega\times\{-\varepsilon\}} g_i^\varepsilon u_i^\varepsilon \tag{2.1}$$

where

$$E_{ij}(w) = \frac{1}{2}(\partial_i w_j + \partial_j w_i + \partial_i w_k \partial_j w_k)$$

Here we have used the Einstein convention of sumation on repeated indices. We are interested in rescaled quantities defined on $\Omega = \omega \times (-1, 1)$ and we denote from now on by $u, F = (f, g)$ the quantities defined on Ω related to the corresponding quantities $u^\varepsilon, F^\varepsilon$ on Ω^ε and defined by

$$\begin{array}{lll} u_\alpha^\varepsilon = \varepsilon^2 u_\alpha & f_\alpha^\varepsilon = \varepsilon^2 f_\alpha & g_\alpha^\varepsilon = \varepsilon^3 g_\alpha \\ u_3^\varepsilon = \varepsilon u_3 & f_3^\varepsilon = \varepsilon^3 f_3 & g_3^\varepsilon = \varepsilon^4 g_3 \end{array} \tag{2.2}$$

where the quantities indexed by ε are evaluated at $x^\varepsilon = (x_1, x_2, \varepsilon x_3) \in \Omega^\varepsilon$ and the quantities without index ε are evaluated at $x = (x_1, x_2, x_3) \in \Omega$. The greek index takes the values $1, 2$. We refer to the book of Ciarlet [5] for a physical justification of this choice of the scaling in ε.

We formally see that u^ε is a critical point of the energy $\mathcal{E}_{F^\varepsilon}$ if and only if u satisfies the Euler-Lagrange equation that we note in a compact form:

$$\mathcal{A}^\varepsilon(u) = F \quad \text{on} \quad \Omega \tag{2.3}$$

where $\mathcal{A}^\varepsilon$ is a quasilinear differential operator with nonlinear boundary conditions.

We recall the following formal asymptotics:

Theorem 2.1 (Ciarlet, Destuynder [8]; Ciarlet [5]; Formal asymptotics) *Every solution of (2.3) formally converges to a function u^0 given by*

$$u^0 = \begin{cases} u^0_\alpha = \zeta_\alpha(x') - x_3\partial_\alpha\zeta_3(x') \\ \\ u^0_3 = \zeta_3(x') \end{cases} \tag{2.4}$$

where $x' = (x_1, x_2)$ and ζ is a solution to the nonlinear Kirchhoff-Love plate equations on ω:

$$\begin{cases} -\partial_\beta n_{\alpha\beta}(\zeta) = h_\alpha \\ \\ \dfrac{8\mu(\lambda+\mu)}{3(\lambda+2\mu)}\Delta^2\zeta_3 - \partial_\beta\left(n_{\alpha\beta}(\zeta)\partial_\alpha\zeta_3\right) = h_3 \\ \\ \textit{where} \quad n_{\alpha\beta}(\zeta) = \dfrac{4\lambda\mu}{\lambda+2\mu}E^0_{\gamma\gamma}(\zeta)\delta_{\alpha\beta} + 4\mu E^0_{\alpha\beta}(\zeta) \end{cases} \tag{2.5}$$

and

$$E^0_{\alpha\beta}(\zeta) = \frac{1}{2}(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \partial_\alpha\zeta_3\partial_\beta\zeta_3)$$
$$h_\alpha = IF_\alpha, \quad h_3 = IF_3 + \partial_\alpha I(x_3F_\alpha)$$

where the operator I is defined by

$$IF_i(x') = \int_{-1}^{+1} f_i(x', x_3)dx_3 + [g_i(x', x_3)]_{x_3=-1}^{x_3=+1}$$

Every function u^0 as defined in theorem 2.1 will be called a solution of the equation

$$\mathcal{A}^0(u^0) = F \quad \text{on} \quad \Omega \tag{2.6}$$

where we have introduced the abstract notation $\mathcal{A}^0$ to preserve an expression similar to $\mathcal{A}^\varepsilon$ in (2.3).

In particular the existence of a solution formally requires the equilibrium of exterior forces:

$$\int_\Omega f + \int_{\partial\Omega} g = 0 \tag{2.7}$$

Moreover the solution u is defined up to addition of a constant on each components. We formally get a unique solution if we require the normalization condition:

$$\int_\Omega u = 0 \tag{2.8}$$

Here one need to impose (2.8) just because of the translation invariance of the original problem. A similar remark is true for the solution u^0 for $\varepsilon = 0$.

Notations

For a function h we note $D^l h$ the set of all partial derivatives with total order equal to $l \in \mathbf{N}$:

$$D^l h = \left\{ \partial_1^{l_1} \partial_2^{l_2} \partial_3^{l_3} h, \quad l_i \geq 0, \quad i = 1, 2, 3, \quad l_1 + l_2 + l_3 = l \right\}$$

and the tangential derivatives:

$$D_T^l h = \left\{ \partial_1^{l_1} \partial_2^{l_2} h, \quad l_\alpha \geq 0, \quad \alpha = 1, 2, \quad l_1 + l_2 = l \right\}$$

For $k \geq 0, 1 \leq p \leq +\infty$, we say that $F = (f, g) \in W^{k,p}$ if and only if $f \in W^{k,p}(\Omega)$ and $g \in W^{k,p}(\omega)$ and we define the corresponding norm

$$|F|_{k,p} = \sum_{l=0}^{k} |D^l f|_{L^p(\Omega)} + |D_T^l g|_{L^p(\omega)}$$

For $k = 0$, we note $|F|_p := |F|_{0,p}$. We define

$$\nabla^\varepsilon u = \left(\frac{e_{33}(u)}{\varepsilon^2}, \frac{e_{13}(u)}{\varepsilon}, \frac{e_{23}(u)}{\varepsilon}, e_{12}(u), e_{11}(u), e_{22}(u), e_{\bar{13}}(u), e_{\bar{23}}(u), e_{\bar{12}}(u) \right)$$

where

$$e_{ij}(u) = \frac{1}{2} \left(\partial_i u_j + \partial_j u_i \right) \quad \text{and} \quad e_{\bar{ij}}(u) = \frac{1}{2} \left(\partial_i u_j - \partial_j u_i \right) = -e_{\bar{ji}}(u)$$

Moreover given a function u^0, we note $u^2 = u^2[u^0]$ the function

$$u^2[u^0] = \begin{cases} u^2_\alpha = 0 \\ \\ u^2_3 = Tz \end{cases}$$

where z is given by:

$$z = -\left(E^2_{33}(u^0) + \frac{\lambda}{\lambda + 2\mu} E^0_{\beta\beta}(u^0)\right) \tag{2.9}$$

$$E^0_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3), \qquad E^2_{ij}(u) = \frac{1}{2}\partial_i u_\gamma \partial_j u_\gamma$$

and for a general function h:

$$Th = \int_0^{x_3} h(x', s) ds$$

We are now in position to give the precise statement of our main result

Theorem 2.2 (Existence, uniqueness, 3d-2d error estimate)
Let $\Omega = \left(\mathbf{R}^2/\mathbf{Z}^2\right) \times (-1, 1)$. There exists $\varepsilon_0, R_0, M > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$ and all F satisfying the equilibrium equation (2.7) such that $|F|_{4,2} \leq R_0$, there exists a solution $u \in W^{3,2}(\Omega)$ of

$$\mathcal{A}^\varepsilon(u) = F$$

beyong the solutions satisfying the normalizing condition (2.8). Moreover if we denote by u^0 the solution for $\varepsilon = 0$ satisfying the normalizing condition (2.8), and $v = u^0 + \varepsilon^2 u^2[u^0]$, then we have the error estimate

$$|\nabla^\varepsilon(u - v)|_{2,2} \leq \varepsilon M \tag{2.10}$$

Finally every solution u satisfying (2.10)-(2.8) is unique.

The scaling in ε needed in the definition of ∇^ε, is just a reflection of the fact that we need a special attention to the terms $e_{i3}(u)$, because for $\varepsilon = 0$, we have $e_{i3}(u^0) = 0$ for $i = 1, 2, 3$.

Estimate (2.10) is obtained following the Nirenberg's translations method (see Brezis [4]) and is based on Korn's inequality.

Remark 2.3 *One bad drawbak of the application of the Singular Inverse Method in this particular case is the high level of regularity required on the forces F in comparison to the expected regularity of the solution u. To avoid technicalities, we have not tried to optimize this regularity, in particular with respect to x' and x_3.*
An other way to avoid the loss of regularity would be to consider a limit model different from the nonlinear Kirchhoff-Love theory, in the spirit of Monneau [21].

3 Proof of the main result: theorem 2.2

On the one hand, let us remark that the classical inverse function theorem gives us a locally unique solution u^0 of $\mathcal{A}^0(u^0) = F$ for $|F|_{4,2} \leq R_0$ for some R_0 small enough fixed from now on. On the other hand the classical inverse function theorem can be applied to equation

$$\mathcal{A}^\varepsilon(u) = F \tag{3.1}$$

while $D_u\mathcal{A}^\varepsilon$ is inversible. Let us recall that (3.1) is the Euler-Lagrange equation of the energy

$$\mathcal{E}^\varepsilon_F(u) := \frac{1}{\varepsilon^5}\mathcal{E}_{F^\varepsilon}(u^\varepsilon)$$

where u, F are related to $u^\varepsilon, F^\varepsilon$ by (2.2). As a consequence we have

$$(D_u\mathcal{A}^\varepsilon(u)\cdot w, w) = \mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(w,w) \tag{3.2}$$

We see that the coercivity of the hessian of the energy gives a measure of the inversibility of the linearized operator of elasticity. Keeping this remark in mind, for $|F|_{4,2} \leq R_0$ we introduce the set

$$K^\varepsilon_M(F) = \left\{u \in W^{3,2}(\Omega), \quad |\nabla^\varepsilon(u-v)|_{2,2} \leq \varepsilon M\right\}$$

where $v = u^0 + \varepsilon^2 u^2[u^0]$. For the radius R_0 that we have fixed previously, and using Korn's inequality we will prove the

Proposition 3.1 (Coercivity of the hessian)
There exist $c_0(R_0) > 0$ and $\varepsilon'_0 = \varepsilon'_0(M, R_0) > 0$ such that for every $|F|_{4,2} \leq R_0$, if $u \in K^\varepsilon_M(F)$ with $\varepsilon \in (0, \varepsilon'_0]$, then

$$\forall w \in H^1(\Omega), \quad \mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(w,w) \quad \geq \quad c_0|\nabla^\varepsilon w|^2_2 \tag{3.3}$$

To get this estimate, a L^∞ bound on ∇u is required, that is why we work with $u \in W^{3,2}(\Omega)$.

Remark 3.2 *Let us remark that inequality (3.3) is false (see Monneau [22]) for* $u \in \left\{u \in W^{3,2}(\Omega), \quad |\nabla u|_{2,2} \leq \varepsilon M\right\}$ *(case* $F = 0 = u^0 = v$*). This explains why we need to consider* ∇^ε *in place of* ∇.

We will prove later the following a priori estimate

Proposition 3.3 (A priori estimate)
There exist $M = M(R_0) > 0$ *and* $\varepsilon_0 = \varepsilon_0(M, R_0) \leq \varepsilon_0'(M, R_0)$ *such that if* $\varepsilon \in (0, \varepsilon_0]$ *then for every* $|F|_{4,2} \leq R_0$*:*

$$(\mathcal{A}^\varepsilon u = F \quad \textit{and} \quad u \in K^\varepsilon_M(F)) \Longrightarrow \left(u \in K^\varepsilon_{\frac{M}{2}}(F)\right)$$

Uniqueness
Let u^1 and u^2 two solutions of (3.1) belonging to $K^\varepsilon_M(F)$. Taking the difference of the partial differential equations, multiplying by $w = u^2 - u^1$ and integrating by part we get

$$0 = \int_0^1 dt \quad \mathcal{E}_0^{\varepsilon\prime\prime}(u^1 + tw) \cdot (w, w) \geq c_0 |\nabla^\varepsilon w|_2^2$$

which ends the proof of the uniqueness of solutions satisfying the normalizing condition (2.8).

Existence
Let us recall that (by the usual inverse function theorem) we have the existence of a solution u^0 for $\varepsilon = 0$ and for every force F satisfying $|F|_{4,2} \leq R_0$. Now we work with a fixed $\varepsilon \in (0, \varepsilon_0]$ and consider a fixed force F satisfying $|F|_{4,2} \leq R_0$. Then we define the set

$$I = \left\{ t \in [0,1], \quad \left| \begin{array}{l} \exists u_t \in K^\varepsilon_M(tF) \\ \\ \mathcal{A}^\varepsilon(u_t) = tF \\ \\ \partial_u \mathcal{A}^\varepsilon(u_t) \text{ is inversible} \end{array} \right. \right\}$$

We have

i) $I \neq \emptyset$.

This is a consequence of the usual inverse function theorem which proves the existence of an interval $[0, \delta) \subset I$ for some $\delta > 0$.

ii) I is open.

Proposition 3.3 proves that if $t_0 \in I$, then $u_{t_0} \in K^{\varepsilon}_{\frac{M}{2}}(t_0 F)$, and from the usual inverse function theorem, this proves the existence of $u_t \in K^{\varepsilon}_M(tF)$ for $|t - t_0|$ small enough.

iii)) I is closed.

If there exists a sequence of points $t^n \in I$ such that $t^n \to t_0 \in [0, 1]$, then (up to extraction of a subsequence) u_{t^n} weakly converges to a function $u_{t_0} \in K^{\varepsilon}_M(t_0 F)$ which satisfies $\mathcal{A}^{\varepsilon}(u_{t_0}) = t_0 F$. The delicate fact is that $\partial_u \mathcal{A}^{\varepsilon}(u_{t_0})$ is inversible. This is a consequence of proposition 3.1 and (3.2).

The conclusion is that $I = [0, 1]$ and that there exists a solution u of $\mathcal{A}^{\varepsilon}(u) = F$ with $u = u_t$ for $t = 1$.

This ends the proof of theorem 2.2.

The following sections will be devoted to the proofs of propositions 3.1 and 3.3.

Remark 3.4 *An other direction of research would be to find a direct proof that the energy $\mathcal{E}^{\varepsilon}_0(u)$ has the shape of a locally strictly convex well in a neighborhood of u^0 in a norm strong enough, if the hessian $\mathcal{E}^{0\,\prime\prime}_0(u^0)$ is positive for $\varepsilon = 0$.*

4 A priori estimate on the displacement

4.1 Preliminary on Korn's inequality

Let us recall that the classical Korn's inequality

$$|\nabla \overline{v}|_2^2 + |\overline{v}|_2^2 \quad \leq \quad C \left(\Sigma_{i,j} |e_{ij}(\overline{v})|_2^2 + |\overline{v}|_2^2 \right)$$

takes here the following particular form (see [6] proof of theorem 6.3-4, [24] p 240):

Lemma 4.1 (Korn's inequality)

$$\exists C > 0, \quad \forall \overline{v} \in H^1(\Omega), \quad |\nabla \overline{v}|_2^2 + \left| \overline{v} - \frac{1}{|\Omega|} \int_\Omega \overline{v} \right|_2^2 \quad \leq \quad C\ \Sigma_{i,j} |e_{ij}(\overline{v})|_2^2$$

4.2 Strategy of the proof of the a priori estimate

The main idea of the proof is the following.
We consider solutions u of the equation

$$\mathcal{A}^\varepsilon(u) = F \tag{4.1}$$

For a function v (to chose later) we note

$$F_v = \mathcal{A}^\varepsilon(v) \tag{4.2}$$

Then computing the difference between (4.1) and (4.2), multiplying by $w = u - v$ and integrating by part we get

$$\int (F - F_v) \cdot w \quad = \quad (\mathcal{E}_0^{\varepsilon\prime}(u) - \mathcal{E}_0^{\varepsilon\prime}(v)) \cdot w \quad = \quad \int_0^1 dt \quad \mathcal{E}_0^{\varepsilon\prime\prime}(v + tw) \cdot (w, w) \quad \geq \quad c_0 |\nabla^\varepsilon w|_2^2$$

On the other hand

$$\int (F - F_v) \cdot w \quad \leq \quad |F - F_v|_2 |w|_2 \quad \leq \quad C |F - F_v|_2 |\nabla^\varepsilon w|_2$$

where the last inequality comes from Korn's inequality (lemma 4.1). Consequently:

$$|\nabla^\varepsilon w|_2 \quad \leq \quad \frac{C}{c_0} |F - F_v|_2$$

If there would exist a function v such that $|F - F_v| = o(1)$ as $\varepsilon \to 0$, then we would have

$$|\nabla^\varepsilon w|_2 = o(1) \quad \text{as} \quad \varepsilon \to 0 \tag{4.3}$$

The situation is more complicated, and it is impossible to find such a function v independent on ε to get (4.3). To avoid this difficulty we chose $v = u^0 + \varepsilon^2 u^2$ which still satisfies $|F - F_v| \neq o(1)$ but has the advantage for a good choice of u^0, u^2 to imply

$$\left| \int (F - F_v) \cdot w \right| \quad \leq \quad o(1) \cdot |\nabla^\varepsilon w|_2$$

which is sufficient to get (4.3). Moreover using the method of Nirenberg's translations, we get a priori estimates on higher derivatives of w.

4.3 Notations

We use the generic notation C for constants which only depend on the Lamé coefficients λ, μ, and the bound R_0 on $|F|_{4,2}$. These constants satisfy in particular the following rules: $C + C = C$, $C \cdot C = C$, $\dfrac{1}{C} = C$.

For simplicity we will note $U = \nabla^\varepsilon u,\ V = \nabla^\varepsilon v,\ W = \nabla^\varepsilon w$ for $w = u - v$, and

$$U^2 - V^2 = (U - V, U \otimes U - V \otimes V)$$

is the row vector with components $(U - V)$ and $(U \otimes U - V \otimes V)$, and

$$U^3 - V^3 = (U - V, U \otimes U - V \otimes V, U \otimes U \otimes U - V \otimes V \otimes V)$$

4.4 Details of the a priori estimate

Let us recall that we assume $u \in K^\varepsilon_M(F)$ which implies by proposition 3.1:

$$\forall \overline{v} \in H^1(\Omega), \quad \mathcal{E}^{\varepsilon}_0{}''(u) \cdot (\overline{v}, \overline{v}) \quad \geq \quad c_0 |\nabla^\varepsilon \overline{v}|_2^2$$

To simplify the notations we first rewrite the energy as:

$$\mathcal{E}^\varepsilon_F(u) = \int_\Omega \tilde{\mathcal{W}}^\varepsilon(\nabla u) - \int_\omega I(F_j u_j)$$

where $\tilde{\mathcal{W}}^\varepsilon$ depends on ε (and we reserve the notation $\mathcal{W}^\varepsilon$ for $\mathcal{W}^\varepsilon(\nabla^\varepsilon u) = \tilde{\mathcal{W}}^\varepsilon(\nabla u)$). The first variation of the energy is

$$\mathcal{E}^{\varepsilon}_F{}'(u) \cdot \overline{v} = \int_\Omega \partial_i \overline{v}_j A^\varepsilon_{ij}(u) - \int_\omega I(F_j \overline{v}_j) \tag{4.4}$$

where $A^\varepsilon_{ij}(u) = \partial_{\nabla_i u_j} \tilde{\mathcal{W}}^\varepsilon(\nabla u)$. A straightforward computation gives the following Euler-Lagrange equations of $\mathcal{E}^\varepsilon_F$:

$$\begin{cases} \partial_i A^\varepsilon_{ij}(u) = -f_j \text{ on } \Omega \\ \\ A^\varepsilon_{3j}(u) = g_j \text{ on } \partial\Omega \end{cases} \tag{4.5}$$

with

$$\begin{aligned}
\varepsilon^4 A^\varepsilon_{33}(u) = \quad & (\lambda+2\mu)(1+e_{33})E^0_{33} \\
& +\varepsilon^2\left\{(1+e_{33})((\lambda+2\mu)E^2_{33}+\lambda E^0_{\gamma\gamma})+2\mu(\partial_\gamma u_3)E^0_{3\gamma}\right\} \\
& +\varepsilon^4\left\{\lambda(1+e_{33})E^2_{\gamma\gamma}+2\mu(\partial_\gamma u_3)E^2_{3\gamma}\right\}
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2 A^\varepsilon_{\rho 3}(u) = \quad & 2\mu(1+e_{33})E^0_{3\rho}+\lambda(\partial_\rho u_3)E^0_{33} \\
& +\varepsilon^2\left\{2\mu(1+e_{33})E^2_{3\rho}+(\partial_\gamma u_3)(\lambda(E^2_{33}+E^0_{\nu\nu})\delta_{\gamma\rho}+2\mu E^0_{\gamma\rho})\right\}+O(\varepsilon^4)
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2 A^\varepsilon_{3\alpha}(u) = \quad & 2\mu E^0_{3\alpha}+(\lambda+2\mu)(\partial_3 u_\alpha)E^0_{33} \\
& +\varepsilon^2\left\{2\mu E^2_{3\alpha}+(\partial_3 u_\alpha)\left((\lambda+2\mu)E^2_{33}+\lambda E^0_{\gamma\gamma}\right)+2\mu(\partial_\gamma u_\alpha)E^0_{3\gamma}\right\}+O(\varepsilon^4)
\end{aligned}$$

$$\begin{aligned}
\varepsilon^2 A^\varepsilon_{\rho\alpha}(u) = \quad & \lambda E^0_{33}\delta_{\alpha\rho} \\
& +\varepsilon^2\left\{\lambda(E^2_{33}+E^0_{\nu\nu})\delta_{\alpha\rho}+2\mu E^0_{\alpha\rho}+2\mu(\partial_3 u_\alpha)E^0_{3\rho}+\lambda(\partial_\gamma u_\alpha)E^0_{33}\delta_{\gamma\rho}\right\}+O(\varepsilon^4)
\end{aligned}$$

In particular we get

$$\left\{\begin{array}{l}
\partial_i(A^\varepsilon_{ij}(u)-A^\varepsilon_{ij}(v))=-\tilde{f}_j \\
A^\varepsilon_{3j}(u)-A^\varepsilon_{3j}(v)=\tilde{g}_j
\end{array}\right. \tag{4.6}$$

where

$$\left\{\begin{array}{l}
\tilde{f}_j=f_j+\partial_i A^\varepsilon_{ij}(v) \\
\tilde{g}_j=g_j-A^\varepsilon_{3j}(v)
\end{array}\right.$$

We note $\tau=0$, α or $\alpha\beta$ and $\partial_0 w=w$. Then deriving equation (4.6) with respect to ∂_τ, multipying by $\partial_\tau w$ and integrating we get with $\tilde{F}=(\tilde{f}_j,\tilde{g}_j)$:

$$\int_\Omega \partial_i(\partial_\tau w_j)\partial_\tau\left(A^\varepsilon_{ij}(u)-A^\varepsilon_{ij}(v)\right)=\int_\omega I\left((\partial_\tau\tilde{F}_j)(\partial_\tau w_j)\right) \tag{4.7}$$

Let us note J_τ this quantity. We will estimate this quantity in two different ways.

The first way to estimate J_τ is the following:

Proposition 4.2 (Upper bound on J_τ using the PDE) *With the previous notations we have*

$$J_\tau = \int_\omega d^2x' \left\{ \begin{array}{l} (\partial_\tau I\tilde{F}_\alpha)\,(\partial_\tau w_\alpha(x',0)) + \Big(\partial_\tau(I\tilde{F}_3 + \partial_\alpha I x_3 \tilde{F}_\alpha)\Big)\,(\partial_\tau w_3(x',0)) \\ \\ +I\left\{(\partial_\tau\partial_\alpha \tilde{F}_\alpha) T\left[T(e_{33}\,(\partial_\tau w))\right] + (\partial_\tau \tilde{F}_3) T\,(e_{33}(\partial_\tau w)) + (\partial_\tau \tilde{F}_\alpha) T\,(2e_{\alpha 3}(\partial_\tau w))\right\} \end{array} \right\}$$

Moreover we have:

$$\left|\partial_\tau I\tilde{F}_\alpha\right|_2 + \left|\partial_\tau(I\tilde{F}_3 + \partial_\alpha I x_3\tilde{F}_\alpha)\right|_2 \leq C\varepsilon^2, \quad \left|\partial_\tau \tilde{F}\right|_2 + \left|\partial_\tau\partial_\alpha\tilde{F}_\alpha\right|_2 \leq C$$

and then

$$|J_\tau| \quad \leq \quad C\varepsilon|\partial_\tau W|_2$$

Remark 4.3 *Here the constant C depends on a bound on $|F|_{4,2}$. A bound on $|F|_{3,2}$ would be not sufficient.*

Remark 4.4 *Choosing more carefully the function v such that $w = u - v$, we should get $J_\tau = O(\varepsilon^2)$.*

The second way is the following:

Proposition 4.5 (Lower bound on J_τ using coercivity) *We have:*

$$\begin{aligned} J_0 &\geq c_0|W|_2^2 \\ J_\alpha &\geq c_0|\partial_\alpha W|_2^2 - C|\partial_\alpha W|_2|U^2 - V^2|_2 \\ J_{\alpha\beta} &\geq c_0|\partial_{\alpha\beta}W|_2^2 - C|\partial_{\alpha\beta}W|_2\{|U^2 - V^2|_2 + |U|_\infty|\left(D_T U\right)^2 - \left(D_T V\right)^2|_2\} \end{aligned}$$

We deduce the

Corollary 4.6 (Bound on the tangential derivatives of W)

$$\begin{aligned} |W|_2 &\leq C\varepsilon \\ |\partial_\alpha W|_2 &\leq C\{\varepsilon + |U^2 - V^2|_2\} \\ |\partial_{\alpha\beta} W|_2 &\leq C\{\varepsilon + |U^2 - V^2|_2 + |U|_\infty|\left(D_T U\right)^2 - \left(D_T V\right)^2|_2\} \end{aligned}$$

Proof of proposition 4.2

Here we perform the proof for $\tau = 0$. The proof is similar for $\tau \neq 0$.

Using the definition of the operator I and the fact that $w(x) = w(x', 0) + T(\partial_3 w)$, we get:

$$
\begin{aligned}
J_0 &= \int_\omega \left\{ w_j(x',0) I\tilde{F}_j + I\left(\tilde{F}_j T(\partial_3 w_j)\right)\right\} \\
&= \int_\omega \left\{ w_3(x',0) I\tilde{F}_3 + I\left(\tilde{F}_3 T(e_{33}(w)\right) + w_\alpha(x',0) I\tilde{F}_\alpha + I\left(\tilde{F}_\alpha T\left(2e_{\alpha 3}(w)\right) - \tilde{F}_\alpha T(\partial_\alpha w_3)\right)\right\} \\
&= \int_\omega \left\{ w_3(x',0) I\tilde{F}_3 + I\left(\tilde{F}_3 T(e_{33}(w)\right) + w_\alpha(x',0) I\tilde{F}_\alpha + I\left(\tilde{F}_\alpha T(2e_{\alpha 3}(w)\right) \right. \\
&\quad \left. + I\left(\left(\partial_\alpha \tilde{F}_\alpha\right) T(w_3(x',0) + T\left(e_{33}(w)\right)\right)\right)\right\} \\
&= \int_\omega \left\{ (I\tilde{F}_\alpha) w_\alpha(x',0) + \left(I\tilde{F}_3 + \partial_\alpha I x_3 \tilde{F}_\alpha\right) w_3(x',0) \right. \\
&\quad \left. + I\left((\partial_\alpha \tilde{F}_\alpha) TT\left(e_{33}(w)\right) + \tilde{F}_3 T\left(e_{33}(w)\right) + \tilde{F}_\alpha T\left(2e_{\alpha 3}(w)\right)\right)\right\}
\end{aligned}
$$

Using again the fact that $w(x',0) = w(x) - T(\partial_3 w)$ and the fact that $|w|_2 \leq C|\nabla^\varepsilon w|_2$ we deduce that

$$
|J_0| \quad \leq \quad C|\nabla^\varepsilon w|_2 \left\{ \Sigma_\alpha |I\tilde{F}_\alpha|_2 + |I\tilde{F}_3 + \partial_\beta I x_3 \tilde{F}_\beta|_2 + \varepsilon^2 \left(|\partial_\beta \tilde{F}_\beta|_2 + |\tilde{F}_3|_2\right) + \varepsilon \Sigma_\alpha |\tilde{F}_\alpha|_2 \right\}
$$

But we have $|\tilde{F}|_2 + |\partial_\alpha \tilde{F}_\alpha|_2 \leq C$ and the

Lemma 4.7 (A bound using the 2d solution)

$$
|I\tilde{F}_\alpha|_2 + |I\tilde{F}_3 + \partial_\beta I x_3 \tilde{F}_\beta|_2 \quad \leq \quad C\varepsilon^2
$$

This proves proposition 4.2.

Proof of lemma 4.7

A tedious but straightforward computation allows to check that for the special cloice $v = u^0 + \varepsilon^2 u^2[u^0]$ we have

$$
A^\varepsilon_{ij}(v) = \tilde{A}^0_{ij} + \varepsilon^2 \tilde{A}^{2,\varepsilon}_{ij}
$$

with $\tilde{A}^{2,\varepsilon}_{ij} = O(1)$. Denoting by $n_{\alpha\beta}(u^0) = \dfrac{4\lambda\mu}{\lambda+2\mu} E^0_{\nu\nu}(u^0)\delta_{\rho\alpha} + 4\mu E^0_{\rho\alpha}(u^0)$, we have:

$$\begin{aligned}
\tilde{A}^0_{33} = \ & \frac{\lambda+2\mu}{2}\Big(e_{33}(u^2)\Big)^2 + \lambda E^2_{\gamma\gamma}(u^0) + 2\mu(\partial_\gamma u^0_3)\tilde{A}^0_{3\gamma} \\
& +(\lambda+2\mu)(\partial_3 u^0_\gamma)(\partial_3 u^2_\gamma) + \lambda\Big(e_{\gamma\gamma}(u^2) + (\partial_\gamma u^0_3)(\partial_\gamma u^2_3)\Big) \\
\tilde{A}^0_{\rho 3} = \ & \tilde{A}^0_{3\rho} + \frac{1}{2}(\partial_\gamma u^0_3) n_{\gamma\rho}(u^0) \\
\tilde{A}^0_{3\alpha} = \ & 2\mu\left\{e_{3\alpha}(u^2) + E^2_{3\alpha}(u^0) + \frac{e_{33}(u^2)\partial_\alpha u^0_3}{2}\right\} \\
\tilde{A}^0_{\rho\alpha} = \ & \frac{1}{2} n_{\rho\alpha}(u^0)
\end{aligned}$$

Then we can compute with $h_\beta = IF_\beta$:

$$\begin{aligned}
I\tilde{F}_\beta \ & = h_\beta + \partial_\alpha \int_{-1}^1 A^\varepsilon_{\alpha\beta}(v) \\
& = h_\beta + \partial_\alpha \int_{-1}^1 \frac{1}{2} n_{\alpha\beta}(u^0) + \varepsilon^2 \partial_\alpha \int_{-1}^1 \tilde{A}^{2,\varepsilon}_{\alpha\beta} \\
& = \varepsilon^2 \partial_\alpha \int_{-1}^1 \tilde{A}^{2,\varepsilon}_{\alpha\beta}
\end{aligned}$$

where for the last line we have used equation (2.5) whose u^0 is a solution.

Similarly we compute with $h_3 = IF_3 + \partial_\alpha I x_3 F_\alpha$:

$$\begin{aligned}
I\tilde{F}_3 + \partial_\alpha I x_3 \tilde{F}_\alpha \ & = h_3 + \partial_\alpha \int_{-1}^1 A^\varepsilon_{\alpha 3}(v) + \partial_\alpha \left\{\int_{-1}^1 x_3 \partial_i A^\varepsilon_{i\alpha}(v) + [-x_3 A^\varepsilon_{3\alpha}(v)]^1_{-1}\right\} \\
& = h_3 + \partial_{\alpha\beta} \int_{-1}^1 x_3 A^\varepsilon_{\beta\alpha}(v) + \partial_\alpha \int_{-1}^1 [A^\varepsilon_{\alpha 3}(v) - A^\varepsilon_{3\alpha}(v)] \\
& = h_3 + \partial_{\alpha\beta} \int_{-1}^1 x_3 \frac{1}{2} n_{\beta\alpha}(u^0) + \partial_\alpha \int_{-1}^1 \frac{1}{2}(\partial_\gamma u^0_3) n_{\gamma\alpha}(\zeta) \\
& \quad + \varepsilon^2 \left\{\partial_{\alpha\beta} \int_{-1}^1 x_3 \tilde{A}^{2,\varepsilon}_{\beta\alpha} + \partial_\alpha \int_{-1}^1 [\tilde{A}^{2,\varepsilon}_{\alpha 3} - \tilde{A}^{2,\varepsilon}_{3\alpha}]\right\} \\
& = \varepsilon^2 \left\{\partial_{\alpha\beta} \int_{-1}^1 x_3 \tilde{A}^{2,\varepsilon}_{\beta\alpha} + \partial_\alpha \int_{-1}^1 [\tilde{A}^{2,\varepsilon}_{\alpha 3} - \tilde{A}^{2,\varepsilon}_{3\alpha}]\right\}
\end{aligned}$$

where for the last line we have also used equation (2.5).

This ends the proof of lemma 4.7.

Proof of proposition 4.5

Let us define $\mathcal{W}^\varepsilon$ by

$$\mathcal{W}^\varepsilon(U) = \tilde{\mathcal{W}}^\varepsilon(\nabla u), \quad \text{where} \quad U = \nabla^\varepsilon u$$

Then

$$\mathcal{E}_0^\varepsilon(u) = \int_\Omega \mathcal{W}^\varepsilon(U)$$

and

$$\mathcal{W}^\varepsilon(U) = U_I U_J \mathcal{W}^\varepsilon_{IJ}(U)$$

with

$$\mathcal{W}^\varepsilon_{IJ}(U) = \mathcal{W}^\varepsilon_{IJ,0} + \mathcal{W}^\varepsilon_{IJ,K} U_K + \mathcal{W}^\varepsilon_{IJ,KL} U_K U_L$$

and $\mathcal{W}^\varepsilon_{IJ} = (\mathcal{W}^\varepsilon_{IJ,0}, \mathcal{W}^\varepsilon_{IJ,K}, \mathcal{W}^\varepsilon_{IJ,KL})$ is a polynomial in ε with constant coefficients. Consequently:

$$\mathcal{E}_0^{\varepsilon\prime}(u) \cdot \overline{v} = \int_\Omega \partial_U \mathcal{W}^\varepsilon(U) \cdot \overline{V}$$

where $\overline{V} = \nabla^\varepsilon \overline{v}$. From (4.7) and (4.4) we deduce that

$$J_\tau = \int_\Omega \partial_\tau(U - V) \cdot \partial_\tau[\partial_U \mathcal{W}^\varepsilon(U) - \partial_U \mathcal{W}^\varepsilon(V)]$$

case $\tau = 0$

$$\begin{aligned}
J_0 &= \int_\Omega W \cdot \Big\{ \int_0^1 dt\ \partial_{UU} \mathcal{W}^\varepsilon(V + tW) \cdot W \Big\} \\
&= \int_0^1 dt\ \Big\{ \int_\Omega \partial_{UU} \mathcal{W}^\varepsilon(V + tW) \cdot (W, W) \Big\} \\
&= \int_0^1 dt\ \mathcal{E}_0^{\varepsilon\prime\prime}(v + tw) \cdot (w, w) \\
&\geq c_0 |W|_2^2
\end{aligned}$$

case $\partial_\tau = \partial_\alpha$

$$\begin{aligned}
J_\alpha &= \int_\Omega \partial_\alpha W \cdot (\partial_{UU} \mathcal{W}^\varepsilon(U) \cdot \partial_\alpha U - \partial_{UU} \mathcal{W}^\varepsilon(V) \cdot \partial_\alpha V) \\
&= \int_\Omega \partial_{UU} \mathcal{W}^\varepsilon(U) \cdot (\partial_\alpha W, \partial_\alpha W) + (\partial_{UU} \mathcal{W}^\varepsilon(U) - \partial_{UU} \mathcal{W}^\varepsilon(V)) \cdot (\partial_\alpha W, \partial_\alpha V) \\
&\geq c_0 |\partial_\alpha W|_2^2 - C |\partial_\alpha W|_2 |U^2 - V^2|_2
\end{aligned}$$

case $\partial_\tau = \partial_{\alpha\beta}$

$$\begin{aligned}
J_{\alpha\beta} &= \int_\Omega \partial_{\alpha\beta} W \cdot \partial_\alpha \left(\partial_{UU}\mathcal{W}^\varepsilon(U)\cdot\partial_\beta U - \partial_{UU}\mathcal{W}^\varepsilon(V)\cdot\partial_\beta V\right) \\
&= \int_\Omega \partial_{\alpha\beta} W \cdot \{\partial_{UU}\mathcal{W}^\varepsilon(U)\cdot\partial_{\alpha\beta} U - \partial_{UU}\mathcal{W}^\varepsilon(V)\cdot\partial_{\alpha\beta} V \\
&\quad +\partial_{UUU}\mathcal{W}^\varepsilon(U)\cdot\partial_\beta U\cdot\partial_\alpha U - \partial_{UUU}\mathcal{W}^\varepsilon(V)\cdot\partial_\beta V\cdot\partial_\alpha V\} \\
&= \int_\Omega \partial_{\alpha\beta} W \cdot \{\partial_{UU}\mathcal{W}^\varepsilon(U)\cdot\partial_{\alpha\beta} W + (\partial_{UU}\mathcal{W}^\varepsilon(U) - \partial_{UU}\mathcal{W}^\varepsilon(V))\cdot\partial_{\alpha\beta} V \\
&\quad +\partial_{UUU}\mathcal{W}^\varepsilon(U)\cdot(\partial_\alpha U\otimes\partial_\beta U - \partial_\alpha V\otimes\partial_\beta V) \\
&\quad + (\partial_{UUU}\mathcal{W}^\varepsilon(U) - \partial_{UUU}\mathcal{W}^\varepsilon(V))\cdot\partial_\alpha V\cdot\partial_\beta V\} \\
&\geq c_0|\partial_{\alpha\beta} W|_2^2 - C|\partial_{\alpha\beta} W|_2\left\{|U^2-V^2|_2 + |U|_\infty|\left(D_T U\right)^2 - \left(D_T V\right)^2|_2\right\}
\end{aligned}$$

This ends the proof of proposition 4.5.

Estimates on the x_3-derivatives

We start with the

Lemma 4.8 (Bounds on all the derivatives of W)

$$|DW|_2 \leq C\left\{|D_T W|_2 + \Sigma_\alpha\left|\partial_3\left(\frac{e_{\alpha3}(w)}{\varepsilon}\right)\right|_2 + \left|\partial_3\left(\frac{e_{33}(w)}{\varepsilon^2}\right)\right|_2\right\}$$

$$|D^2W|_2 \leq C\left\{|D_T^2 W|_2 + \Sigma_{j,\alpha}\left|\partial_{3j}\left(\frac{e_{\alpha3}(w)}{\varepsilon}\right)\right|_2 + \Sigma_j\left|\partial_{3j}\left(\frac{e_{33}(w)}{\varepsilon^2}\right)\right|_2\right\}$$

Proof of lemma 4.8

We only need to search for estimates on $\partial_3 W, \partial_3 D_T W, \partial_{33} W$. The first inequality comes from the fact that $|\partial_3 W|_2 = |\nabla^\varepsilon\partial_3 w|_2 \leq C\left\{\Sigma_\alpha\left|\frac{e_{\alpha3}(\partial_3 w)}{\varepsilon}\right|_2 + \left|\frac{e_{33}(\partial_3 w)}{\varepsilon^2}\right|_2 + \Sigma_{\alpha,\beta}\left|e_{\alpha\beta}(\partial_3 w)\right|_2\right\}$, and $|e_{\alpha\beta}(\partial_3 w)|_2 \leq C|D_T\nabla^\varepsilon w|_2 = C|D_T W|_2$. The second inequality is proved similarly.

Following the Nirenberg's translations method, to estimate the x_3-derivatives we will use the Euler-Lagrange equations (4.5). Let us recall that

$$A^\varepsilon_{ij}(u) = A^\varepsilon_{ij,I}U_I + A^\varepsilon_{ij,IJ}U_IU_J + A^\varepsilon_{ij,IJK}U_IU_JU_K$$

where the coefficients $A^\varepsilon_{ij} = (A^\varepsilon_{ij,I}, A^\varepsilon_{ij,IJ}, A^\varepsilon_{ij,IJK})$ are rational fractions in ε^2 such that $A^\varepsilon_{ij} = A^{-,\varepsilon}_{ij} + A^{0,\varepsilon}_{ij}$ where $A^{0,\varepsilon}_{ij}$ are polynomials in ε^2 and

$$A^{-,\varepsilon}_{33}(u) = \frac{1}{\varepsilon^2}\left\{(\lambda+2\mu)\frac{e_{33}(u)}{\varepsilon^2} + (\lambda+2\mu)E^2_{33}(u) + \lambda E^0_{\gamma\gamma}(u) + 2\mu(\partial_\gamma u_3)e_{3\gamma}(u)\right\}$$

$$A^{-,\varepsilon}_{\rho 3}(u) = 2\mu\frac{e_{\rho 3}(u)}{\varepsilon^2}$$

$$A^{-,\varepsilon}_{3\alpha}(u) = 2\mu\frac{e_{3\alpha}(u)}{\varepsilon^2}$$

$$A^{-,\varepsilon}_{\rho\alpha}(u) = 0$$

Then we have

$$\partial_3(A^{-,\varepsilon}_{3j}(u) - A^{-,\varepsilon}_{3j}(v)) + \partial_\beta(A^{-,\varepsilon}_{\beta j}(u) - A^{-,\varepsilon}_{\beta j}(v)) = -\tilde{f}_j - \partial_i(A^{0,\varepsilon}_{ij}(u) - A^{0,\varepsilon}_{ij}(v)) \qquad (4.8)$$

We deduce that for $j = \alpha$:

$$\left|\partial_3\left(\frac{e_{3\alpha}(w)}{\varepsilon^2}\right)\right|_2 \le C\{1 + |D(U^3 - V^3)|_2\}$$

$$\left|\partial_{k3}(\frac{e_{3\alpha}(w)}{\varepsilon^2})\right|_2 \le C\{1 + |D^2(U^3 - V^3)|_2\}$$

Similarly, if we multiply $A^\varepsilon_{ij}(u)$ by ε, and call

$$B^\varepsilon_{ij}(u) = \varepsilon A^\varepsilon_{ij}(u)$$

we have

$$B^\varepsilon_{ij}(u) = B^\varepsilon_{ij,I}U_I + B^\varepsilon_{ij,IJ}U_IU_J + B^\varepsilon_{ij,IJK}U_IU_JU_K$$

where the coefficients $B^\varepsilon_{ij} = (B^\varepsilon_{ij,I}, B^\varepsilon_{ij,IJ}, B^\varepsilon_{ij,IJK})$ are rational fractions in ε such that

$B^{\varepsilon}_{ij} = B^{-,\varepsilon}_{ij} + B^{0,\varepsilon}_{ij}$ where $B^{0,\varepsilon}_{ij}$ are polynomials in ε and

$$\begin{aligned}
B^{-,\varepsilon}_{33}(u) &= \frac{1}{\varepsilon}\left\{(\lambda+2\mu)\frac{e_{33}(u)}{\varepsilon^2} + (\lambda+2\mu)E^2_{33}(u) + \lambda E^0_{\gamma\gamma}(u) + 2\mu(\partial_\gamma u_3)e_{3\gamma}(u)\right\} \\
&= \frac{1}{\varepsilon}\left\{\begin{array}{l} (\lambda+2\mu)\dfrac{e_{33}(u)}{\varepsilon^2} + \lambda\partial_\gamma u_\gamma + (\lambda+\mu)(\partial_\gamma u_3)(\partial_\gamma u_3) \\ \\ +2(\lambda+2\mu)(e_{3\gamma}(u))(e_{3\gamma}(u)) - 2(\lambda+\mu)e_{3\gamma}(u)\partial_\gamma u_3 \end{array}\right\} \\
B^{-,\varepsilon}_{\rho 3}(u) &= 2\mu\frac{e_{\rho 3}(u)}{\varepsilon} \\
B^{-,\varepsilon}_{3\alpha}(u) &= 2\mu\frac{e_{3\alpha}(u)}{\varepsilon} \\
B^{-,\varepsilon}_{\rho\alpha}(u) &= 0
\end{aligned}$$

Because the term $U = \nabla^\varepsilon u$ contains the term $\frac{e_{\rho 3}(u)}{\varepsilon}$, up to redefine $B^{-,\varepsilon}_{ij}, B^{0,\varepsilon}_{ij}$ in $\tilde{B}^{-,\varepsilon}_{ij}, \tilde{B}^{0,\varepsilon}_{ij}$ we can always put the term $2\mu\frac{e_{\rho 3}(u)}{\varepsilon}$ in $\tilde{B}^{0,\varepsilon}_{\rho 3}$, the term $2\mu\frac{e_{3\alpha}(u)}{\varepsilon}$ in $\tilde{B}^{0,\varepsilon}_{3\alpha}$ and the terms $2(\lambda+2\mu)(e_{3\gamma}(u))(e_{3\gamma}(u)) - 2(\lambda+\mu)e_{3\gamma}(u)\partial_\gamma u_3$ in $\tilde{B}^{0,\varepsilon}_{33}$. Therefore we get

$$B^{\varepsilon}_{ij} = \tilde{B}^{-,\varepsilon}_{ij} + \tilde{B}^{0,\varepsilon}_{ij}$$

where $\tilde{B}^{0,\varepsilon}_{ij}$ are polynomials in ε and

$$\tilde{B}^{-,\varepsilon}_{33}(u) = \frac{1}{\varepsilon}\left\{(\lambda+2\mu)\frac{e_{33}(u)}{\varepsilon^2} + \lambda\partial_\gamma u_\gamma + (\lambda+\mu)(\partial_\gamma u_3)(\partial_\gamma u_3)\right\}$$

$$\tilde{B}^{-,\varepsilon}_{\rho 3}(u) = \tilde{B}^{-,\varepsilon}_{3\alpha}(u) = 2\mu\frac{e_{3\alpha}(u)}{\varepsilon} = \tilde{B}^{-,\varepsilon}_{\rho\alpha}(u) = 0$$

Similarly to (4.8), we have

$$\partial_3(\tilde{B}^{-,\varepsilon}_{3j}(u) - \tilde{B}^{-,\varepsilon}_{3j}(v)) + \partial_\beta(\tilde{B}^{-,\varepsilon}_{\beta j}(u) - \tilde{B}^{-,\varepsilon}_{\beta j}(v)) = -\varepsilon\tilde{f}_j - \partial_i(\tilde{B}^{0,\varepsilon}_{ij}(u) - \tilde{B}^{0,\varepsilon}_{ij}(v)) \quad (4.9)$$

Now because $\partial_3\left((\partial_\gamma u_3)(\partial_\gamma u_3)\right) = 2\varepsilon^2\left[\partial_\gamma\left(\frac{e_{33}(u)}{\varepsilon^2}\right)\right]\partial_\gamma u_3$, we get:

$$\left|\partial_3\left(\frac{e_{33}(w)}{\varepsilon^2} + \frac{\lambda}{\lambda+2\mu}\partial_\gamma w_\gamma\right)\right|_2 \le C\{\varepsilon^2 + \varepsilon|D(U^3 - V^3)|_2 + \varepsilon^2|U\otimes D_T U - V\otimes D_T V|_2\}$$

$$\left|\partial_{j3}\left(\frac{e_{33}(w)}{\varepsilon^2} + \frac{\lambda}{\lambda+2\mu}\partial_\gamma w_\gamma\right)\right|_2 \le C\{\varepsilon^2 + \varepsilon|D^2(U^3 - V^3)|_2 + \varepsilon^2|D(U\otimes D_T U - V\otimes D_T V)|_2\}$$

which implies

$$\left|\partial_3\left(\frac{e_{33}(w)}{\varepsilon^2}\right)\right|_2 \leq C\{|D_TW|_2+\varepsilon^2+\varepsilon|D(U^3-V^3)|_2+\varepsilon^2|U\otimes D_TU-V\otimes D_TV|_2\}$$

and then

$$|DW|_2 \leq C\left\{\varepsilon+|U^2-V^2|_2+\varepsilon|D(U^3-V^3)|_2+\varepsilon^2|U\otimes D_TU-V\otimes D_TV|_2\right\}$$

Moreover for $j=\alpha$, we have $\partial_{\alpha 3}(\partial_\gamma w_\gamma)=\partial_{\alpha\gamma}(\partial_3 w_\gamma)$, and then $|\partial_{\alpha 3}(\partial_\gamma w_\gamma)|_2 \leq C|D_T^2 W|_2$. For $j=3$, we have $\partial_{33}(\partial_\gamma w_\gamma)=\partial_{\gamma 3}(2e_{3\gamma}(w)-\partial_\gamma w_3)=-\partial_{\gamma\gamma}(e_{33}(w))+2\partial_{\gamma 3}e_{3\gamma}(w)$, and then

$$|\partial_{33}(\partial_\gamma w_\gamma)|_2 \leq C\left(|D_T^2W|_2+\varepsilon\left|\partial_{\gamma 3}\left(\frac{e_{3\gamma}(w)}{\varepsilon}\right)\right|_2\right)$$

Therefore

$$\left|\partial_{j3}\left(\frac{e_{33}(w)}{\varepsilon^2}\right)\right|_2 \leq C\left\{\begin{array}{l}\varepsilon\left|\partial_{\gamma 3}\left(\frac{e_{3\gamma}(w)}{\varepsilon}\right)\right|_2+|D_T^2W|_2 \\ \\ +\varepsilon^2+\varepsilon|D^2(U^3-V^3)|_2+\varepsilon^2|D(U\otimes D_TU-V\otimes D_TV)|_2\end{array}\right\}$$

and then

$$|D^2W|_2 \leq C\left\{\begin{array}{l}\varepsilon+|U^2-V^2|_2+|U|_\infty|\left(D_TU\right)^2-\left(D_TV\right)^2|_2 \\ \\ +\varepsilon|D^2(U^3-V^3)|_2+\varepsilon^2|D(U\otimes D_TU-V\otimes D_TV)|_2\end{array}\right\}$$

We will use the fact that formally for $U^{\otimes n}=U\otimes U\otimes\ldots\otimes U$:

$$D(U^{\otimes n}-V^{\otimes n})=n(U^{\otimes(n-1)}DU-V^{\otimes(n-1)}DV)=n(U^{\otimes(n-1)}DW+(U^{\otimes(n-1)}-V^{\otimes(n-1)})DV)$$

$$D^2(U^{\otimes n}-V^{\otimes n})=n\left\{\begin{array}{l}U^{\otimes(n-1)}D^2W+(U^{\otimes(n-1)}-V^{\otimes(n-1)})D^2V \\ \\ +(n-1)\{U^{\otimes(n-2)}((DU)^{\otimes 2}-(DV)^{\otimes 2})+(U^{\otimes(n-2)}-V^{\otimes(n-2)})(DV)^{\otimes 2}\}\end{array}\right\}$$

Of course, rigorously we should write $D(U\otimes U)=DU\otimes U+U\otimes DU$, but for the computation of an upper bound it is sufficient to write formally $D(U\otimes U)=2U\otimes DU$ like in the previous formal computations of the derivatives.

On the other hand we have

$$|U^2-V^2|_2 \leq C(1+|W|_\infty)|W|_2$$

$$|\left(DU\right)^2-\left(DV\right)^2|_2 \leq C\left(|DW|_2+|DW|_4|DW|_4\right)$$

So we deduce the

Proposition 4.9 (A priori estimate on W)
If $\varepsilon\,(1+|W|_\infty)^2 \leq C$, then

$$|W|_2 \leq C\varepsilon$$

$$|DW|_2 \leq C\varepsilon\,(1+|W|_\infty)$$

$$|D^2W|_2 \leq C\varepsilon\,(1+|W|_\infty)^2\,(1+|DW|_4^2)$$

Using Gagliardo-Nirenberg inequality [23]: $|W|_\infty, |DW|_4 \leq C|W|_{2,2}$, it is easy to check that proposition 3.3 is a corollary of proposition 4.9.

5 Proof of the coercivity of the hessian of the energy

We will prove the coercivity of the hessian:

$$\exists c_0(R_0)>0, \quad \forall |F|_{4,2} \leq R_0, \quad \forall M>0, \quad \exists \varepsilon'_0(M,R_0)>0,$$

$$\forall \varepsilon \in (0,\varepsilon'_0], \quad \forall u \in K^\varepsilon_M(F), \quad \forall v \in H^1(\Omega), \quad \mathcal{E}^{\varepsilon\prime\prime}(u)\cdot(v,v) \geq c_0|\nabla^\varepsilon v|_2^2$$

Let us recall that the radius R_0 is every radius which guarantees the existence of solutions u^0 by the application of the inverse function theorem in the case $\varepsilon=0$. This means that the radius R_0 guarantees the inversibility of $D_{u^0}\mathcal{A}^0(u^0)$ which has to be understood as the linearization of system (2.5). We will show that the coercivity of $\mathcal{E}_0^{\varepsilon\prime\prime}$ is a consequence of the coercivity of the hessian $\mathcal{E}_0^{0\prime\prime}$ for $\varepsilon=0$:

Lemma 5.1 (Coercivity of the hessian for $\varepsilon=0$)
Let the space of Kirchhoff-Love displacements

$$V^{KL}=\left\{\overline{v}^0 \in H^1(\Omega), \quad \exists(\eta_\alpha,\eta_3), \quad \overline{v}^0 = \left|\begin{array}{l} \overline{v}^0_\alpha = \eta_\alpha(x') - x_3\partial_\alpha\eta_3(x') \\ \overline{v}^0_3 = \eta_3(x') \end{array}\right.\right\}$$

Let

$$\mathcal{E}^0_F(u^0) := \int_\Omega \frac{\lambda\mu}{\lambda+2\mu}(E^0_{\beta\beta})^2 + \mu E^0_{\alpha\beta}E^0_{\beta\alpha} - \int_\omega I(F_i u^0_i)$$

where $E^0_{ij}=E^0_{ij}(u^0)$. Equation (2.5) (i.e. $\mathcal{A}^0(u^0)=F$) is the Euler-Lagrange equation of $\mathcal{E}^0_F$ and moreover there exists $c'_0>0$ such that

$$\forall \overline{v}^0 \in V^{KL}, \quad \mathcal{E}_0^{0\prime\prime}(u^0)\cdot(\overline{v}^0,\overline{v}^0) \geq c'_0|\nabla\overline{v}^0|_2^2 \tag{5.1}$$

for every solution u^0 found for $|F|_{4,2}\leq R_0$.

Proof of lemma 5.1

The only thing to prove is to check the coercivity of $\mathcal{E}_0^{0\prime\prime}(u^0)$ on the space V^{KL}. But this is a straightforward consequence of the coercivity of $\mathcal{E}_0^{0\prime\prime}(0)$ and of the inversibility of $D_{u^0}\mathcal{A}^0(u^0)$ for each u^0 given for a $|F|_{4,2} \leq R_0$. This ends the proof of the lemma.

We will write

$$\mathcal{E}_0^{\varepsilon\prime\prime}(u)\cdot(\overline{v},\overline{v}) = \mathcal{G}^{\varepsilon}_{u^0}(\overline{v}) + \mathcal{O}^{\varepsilon}$$

and we will prove later the following coercivity property for $\mathcal{G}^{\varepsilon}_{u^0}(\overline{v})$:

Proposition 5.2 (Coercivity of the hessian for $\varepsilon > 0$ at u^0)
For a given function $u^0 \in V^{KL}$ smooth enough, let

$$\begin{aligned}
\mathcal{G}^{\varepsilon}_{u^0}(\overline{v}) = \; & \int_\Omega \lambda\left(\frac{\overline{e}_{33}}{\varepsilon^2} + \overline{e}_{\alpha\alpha} + 2e^{-}_{\alpha 3}\partial_\alpha \overline{v}_3\right)^2 \\
& +2\mu\left\{\left(\frac{\overline{e}_{33}}{\varepsilon^2} + e^{-}_{\alpha 3}\partial_\alpha\overline{v}_3\right)^2 + 2\Sigma\left(\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right)^2 + \Sigma\left(\overline{e}_{\alpha\beta} + \frac{1}{2}(e^{-}_{\beta 3}\partial_\alpha\overline{v}_3 + e^{-}_{\alpha 3}\partial_\beta\overline{v}_3)\right)^2\right\} \\
& +2\mu E^0_{\alpha\beta}\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3
\end{aligned}$$

where $e^{-}_{\alpha 3} = e^{-}_{\alpha 3}(u^0)$, $E^0_{\alpha\beta} = E^0_{\alpha\beta}(u^0)$. If u^0 satisfies property (5.1), then there exists $c_0'' > 0$ and $\varepsilon_0'' > 0$ (which only depend on c_0' and a bound on u^0) such that for every $\varepsilon \in (0, \varepsilon_0'']$:

$$\forall \overline{v} \in H^1(\Omega), \quad \mathcal{G}^{\varepsilon}_{u^0}(\overline{v}) \quad \geq \quad c_0''|\nabla^{\varepsilon}\overline{v}|_2^2$$

Remark 5.3 (Formal asymptotics)
We have formally $\mathcal{E}^{\varepsilon}_F(u^0 + \varepsilon^2 u^2) \to \mathcal{E}^0_F(u^0, e_{33}(u^2))$, where the new energy $\mathcal{E}^0_F(u^0, z)$ is defined by

$$\mathcal{E}^0_F(u^0, z) = \int_\Omega \lambda/2\left\{z + E^2_{33} + E^0_{\beta\beta}\right\}^2 + \mu\left\{(z + E^2_{33})^2 + E^0_{\alpha\beta}E^0_{\alpha\beta}\right\} - \int_\omega I(F_i u_i^0)$$

where $E^k_{ij} = E^k_{ij}(u^0)$. This energy is minimum for the particular choice

$$z^* = -\left(E^2_{33} + \lambda/(\lambda + 2\mu)E^0_{\beta\beta}\right)$$

which corresponds to the choice $u^2 = u^2[u^0]$. Then it is possible to compute and see that

$$\mathcal{E}^0_F(u^0, e_{33}(u^2[u^0])) = \mathcal{E}^0_F(u^0)$$

Similarly $\mathcal{G}^{\varepsilon}_{u^0}(\overline{v}^0 + \varepsilon^2\overline{v}^2\left\{\overline{v}^0\right\}) \to \mathcal{E}^{0\prime\prime}_0(u^0)\cdot(\overline{v}^0,\overline{v}^0)$. *Here* $\overline{v}^2\left\{\overline{v}^0\right\}$ *is defined by* $\overline{v}^2 = \begin{cases}\overline{v}_\alpha = 0\\ \overline{v}_3 = T(e_{33}(\overline{v}^2))\end{cases}$ *and* $e_{33}(\overline{v}^2)$ *satisfies the linearized of equation (2.9) defining* $e_{33}(u^2)$*:*

$$e_{33}(\overline{v}^2) = -\left(e^{-}_{\alpha 3}\partial_\alpha \overline{v}^0_3 + \frac{\lambda}{\lambda+2\mu}e_{\alpha\alpha}(\overline{v}^0)\right)$$

Proof of proposition 3.1

We show how we can bound the term $\mathcal{O}^\varepsilon = \mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(\overline{v},\overline{v}) - \mathcal{G}^{\varepsilon}_{u^0}(\overline{v})$, and then get an estimate on $\mathcal{E}^{\varepsilon\prime\prime}_0(u)$ using proposition 5.2.

Let us remark that with $U = \nabla^\varepsilon u$

$$\mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(\overline{v},\overline{v}) = \int_\Omega (\nabla^\varepsilon\overline{v})_I(\nabla^\varepsilon\overline{v})_J G^\varepsilon_{IJ}(U)$$

where we can split $G^\varepsilon_{IJ}(U)$ in

$$G^\varepsilon_{IJ}(U) = G^{0,\varepsilon}_{IJ}(U) + \varepsilon G^{1,\varepsilon}_{IJ}(U)$$

such that

$$\mathcal{G}^\varepsilon_{u^0}(\overline{v}) = \int_\Omega (\nabla^\varepsilon\overline{v})_I(\nabla^\varepsilon\overline{v})_J G^{0,\varepsilon}_{IJ}(U^0)$$

with $U^0 = (e_{33}(u^2), 0, e_{\alpha\beta}(u^0), e^{-}_{\alpha 3}(u^0))$. Moreover for each $i = 0,1$ we have

$$G^{i,\varepsilon}_{IJ}(U) = G^{i,\varepsilon}_{IJ,0} + G^{i,\varepsilon}_{IJ,K}U_K + G^{i,\varepsilon}_{IJ,KL}U_K U_L$$

and $G^{i,\varepsilon}_{IJ} = (G^{i,\varepsilon}_{IJ,0}, G^{i,\varepsilon}_{IJ,K}, G^{i,\varepsilon}_{IJ,KL})$ are polynomials in ε with constant coefficients. In particular we get

$$\mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(\overline{v},\overline{v}) = \mathcal{G}^\varepsilon_{u^0}(\overline{v}) + \int_\Omega (\nabla^\varepsilon\overline{v})_I(\nabla^\varepsilon\overline{v})_J\left\{\varepsilon G^{1,\varepsilon}_{IJ}(U) + (G^{0,\varepsilon}_{IJ}(U) - G^{0,\varepsilon}_{IJ}(U^0))\right\}$$

Using the fact that $|G^{1,\varepsilon}_{IJ}(A)|_\infty \le C(1+|A|_\infty)^2$, $|G^{0,\varepsilon}_{IJ}(A) - G^{0,\varepsilon}_{IJ}(B)|_\infty \le C|A-B|_\infty(1 + |B|_\infty + |A-B|_\infty)$, and $U = U^0 + \varepsilon V^1$ with $|V^1|_\infty \le C\left(1+\frac{|W|_\infty}{\varepsilon}\right)$, we deduce that

$$|\mathcal{O}^\varepsilon| \le C\varepsilon|\nabla^\varepsilon\overline{v}|_2^2\left(1+\frac{|W|_\infty}{\varepsilon}\right)^2$$

Because $u \in K^\varepsilon_M(F)$ and $|W|_\infty \le C|W|_{2,2}$ (by Gagliardo-Nirenberg inequality), this proves that $\mathcal{E}^{\varepsilon\prime\prime}_0(u)\cdot(\overline{v},\overline{v}) \ge c_0|\nabla^\varepsilon\overline{v}|_2^2$ for ε small enough. This ends the proof of proposition 3.1.

Proof of proposition 5.2

To prove the coercivity of $\mathcal{G}^\varepsilon_{u^0}$, we split it in two terms:

$$\mathcal{G}^\varepsilon_{u^0} = \eta\mathcal{G}^\varepsilon_c + (1-\eta)\mathcal{G}^{\varepsilon,\eta} \tag{5.2}$$

where $\mathcal{G}^\varepsilon_c$ is coercive, $\mathcal{G}^{\varepsilon,\eta}$ will be bounded from below and $\eta \in (0, \frac{1}{2})$ will be fixed later. We choose:

$$\begin{aligned}\mathcal{G}^\varepsilon_c(\overline{v}) = & \int_\Omega \lambda\left(\frac{\overline{e}_{33}}{\varepsilon^2} + \overline{e}_{\alpha\alpha} + 2e^-_{\alpha 3}\partial_\alpha\overline{v}_3\right)^2 \\ & +2\mu\left\{\left(\frac{\overline{e}_{33}}{\varepsilon^2} + e^-_{\alpha 3}\partial_\alpha\overline{v}_3\right)^2 + 2\Sigma\left(\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right)^2 + \Sigma\left(\overline{e}_{\alpha\beta} + \frac{1}{2}(e^-_{\beta 3}\partial_\alpha\overline{v}_3 + e^-_{\alpha 3}\partial_\beta\overline{v}_3)\right)^2\right\} \\ & +2\mu B_{\alpha\beta}\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3\end{aligned}$$

where $B_{\alpha\beta}\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3 = (e^-_{\beta 3}\partial_\alpha\overline{v}_3 + e^-_{\alpha 3}\partial_\beta\overline{v}_3)^2$ and then $B_{\alpha\beta} = B_{\alpha\beta\rho\sigma}e^-_{\rho 3}e^-_{\sigma 3}$.
It is then easy to prove by contradiction that there exists a constant $C > 0$ such that $\mathcal{G}^\varepsilon_c(\overline{v}) \geq C\left\{\left|\frac{\overline{e}_{33}}{\varepsilon^2}\right|_2^2 + \Sigma_\alpha\left|\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right|_2^2 + \Sigma_{\alpha,\beta}\left|\overline{e}_{\alpha\beta}\right|_2^2\right\}$ which implies from lemma 4.1 that there exists a constant $C > 0$ such that

$$\mathcal{G}^\varepsilon_c(\overline{v}) \quad \geq \quad C|\nabla^\varepsilon\overline{v}|_2^2$$

From our choice of $\mathcal{G}^\varepsilon_c$, we deduce that

$$\begin{aligned}\mathcal{G}^{\varepsilon,\eta}(\overline{v}) = & \int_\Omega \lambda\left(\frac{\overline{e}_{33}}{\varepsilon^2} + \overline{e}_{\alpha\alpha} + 2e^-_{\alpha 3}\partial_\alpha\overline{v}_3\right)^2 \\ & +2\mu\left\{\left(\frac{\overline{e}_{33}}{\varepsilon^2} + e^-_{\alpha 3}\partial_\alpha\overline{v}_3\right)^2 + 2\Sigma\left(\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right)^2 + \Sigma\left(\overline{e}_{\alpha\beta} + \frac{1}{2}(e^-_{\beta 3}\partial_\alpha\overline{v}_3 + e^-_{\alpha 3}\partial_\beta\overline{v}_3)\right)^2\right\} \\ & +2\mu E^{0,\eta}_{\alpha\beta}\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3\end{aligned}$$

where $E^{0,\eta}_{\alpha\beta} = \frac{1}{1-\eta}(E^0_{\alpha\beta} - \eta B_{\alpha\beta})$. To bound $\mathcal{G}^{\varepsilon,\eta}$ from below, we consider the minimization

$$\theta^{\varepsilon,\eta} \quad = \quad \inf_{|\overline{v}|_2^2=1, \overline{v}\in H^1(\Omega)} \mathcal{G}^{\varepsilon,\eta}(\overline{v}) \tag{5.3}$$

and we prove the following

Lemma 5.4 (Lower bound on the second term $\mathcal{G}^{\varepsilon,\eta}$)
The infimum in (5.3) is reached in a function $\overline{v}^{\varepsilon,\eta} \in H^1(\Omega)$ which satisfies

$$\mathcal{G}^{\varepsilon,\eta}(\overline{v}^{\varepsilon,\eta}) \quad \geq \quad \mathcal{G}^{0,\eta}(\overline{v}^{0,\varepsilon,\eta}) - C\varepsilon$$

where $\mathcal{G}^{0,\eta}$ is defined by

$$
\begin{aligned}
\mathcal{G}^{0,\eta}(\overline{v}) \;&= \int_\Omega \frac{2\lambda\mu}{\lambda+2\mu}\left(\overline{e}_{\alpha\alpha} + e^-_{\alpha 3}\partial_\alpha \overline{v}_3\right)^2 \\
&+2\mu\left\{\Sigma\left(\overline{e}_{\alpha\beta} + \frac{1}{2}(e^-_{\beta 3}\partial_\alpha\overline{v}_3 + e^-_{\alpha 3}\partial_\beta\overline{v}_3)\right)^2\right\} \\
&+2\mu E^{0,\eta}_{\alpha\beta}\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3
\end{aligned}
$$

for $e^-_{\alpha 3} = e^-_{\alpha 3}(u^0)$ and $E^{0,\eta}_{\alpha\beta} = E^{0,\eta}_{\alpha\beta}(u^0)$ and

$$\overline{v}^{0,\varepsilon,\eta} = P^{KL}(\overline{v}^{\varepsilon,\eta})$$

with $P^{KL} : H^1(\Omega) \to V^{KL}$ is the projector defined by

$$P^{KL}(\overline{v}) = \begin{pmatrix} \xi_\alpha - x_3\partial_\alpha\xi_3 \\ \xi_3 \end{pmatrix} \quad \textit{where} \quad \xi(x') = \frac{1}{2}\int_{-1}^{1} \overline{v}(x', x_3)dx_3$$

Moreover

$$\left|\nabla\left(\overline{v}^{\varepsilon,\eta} - \overline{v}^{0,\varepsilon,\eta}\right)\right|_2^2 \leq C\varepsilon \tag{5.4}$$

Because $\mathcal{G}^{0,\eta}(\overline{v}) = \mathcal{G}^{0,0}(\overline{v}) + \dfrac{\eta}{1-\eta}\displaystyle\int_\Omega 2\mu(E^0_{\alpha\beta} - B_{\alpha\beta})\partial_\alpha\overline{v}_3\partial_\beta\overline{v}_3$ with $\mathcal{G}^{0,0}(\overline{v}) = \mathcal{E}^{0\prime\prime}_0(u^0)\cdot$ $(\overline{v}, \overline{v})$, we deduce from the coercivity (5.1) that for every $\overline{v} \in V^{KL}$:

$$
\begin{aligned}
\mathcal{G}^{0,\eta}(\overline{v}) \;&\geq\; c_0'|\nabla\overline{v}|_2^2 - C\eta|\nabla\overline{v}|_2^2 \\
&\geq\; C|\nabla\overline{v}|_2^2 \quad \text{for} \quad \eta \leq C
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
\theta^{\varepsilon,\eta} = \mathcal{G}^{\varepsilon,\eta}(\overline{v}^{\varepsilon,\eta}) \;&\geq\; C|\nabla\overline{v}^{0,\varepsilon,\eta}|_2^2 - C\varepsilon \\
&\geq\; C|\nabla\overline{v}^{\varepsilon,\eta}|_2^2 - C\varepsilon \\
&\geq\; C \quad \text{for} \quad \varepsilon \leq C
\end{aligned}
$$

The second line is a consequence of (5.4), the third line comes from Poincaré inequality $|\nabla\overline{v}^{\varepsilon,\eta}|_2^2 \geq C|\overline{v}^{\varepsilon,\eta}|_2^2$ with $|\overline{v}^{\varepsilon,\eta}|_2^2 = 1$. Then

$$
\begin{aligned}
\mathcal{G}^\varepsilon_{u^0}(\overline{v}) \;&\geq\; C\eta|\nabla^\varepsilon\overline{v}|_2^2 + (1-\eta)|\overline{v}|_2^2\theta^{\varepsilon,\eta} \\
&\geq\; C|\nabla^\varepsilon\overline{v}|_2^2
\end{aligned}
$$

which ends the proof of proposition 5.2.

Proof of lemma 5.4

We prove here the lemma in the particular case $\eta = 0$. For $\eta > 0$ the proof is similar. In case $\eta = 0$, we simply drop the index η and use the notations $\mathcal{G}^\varepsilon = \mathcal{G}^{\varepsilon,0}$, $\mathcal{G}^0 = \mathcal{G}^{0,0}$ $\theta^\varepsilon = \theta^{\varepsilon,0}$, $\overline{v}^\varepsilon = \overline{v}^{\varepsilon,0}$, $\overline{v}^{0,\varepsilon} = \overline{v}^{0,\varepsilon,0}$. We perform the proof in two steps:

Step 1: $|\theta^\varepsilon| \le C$

Using the general inequality $2ab \le \kappa^2 a^2 + \frac{b^2}{\kappa^2}$, we get

$$\mathcal{G}^\varepsilon(\overline{v}) \ge C\left\{\left|\frac{\overline{e}_{33}}{\varepsilon^2}\right|_2^2 + \Sigma_\alpha \left|\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right|_2^2 + \Sigma_{\alpha,\beta}\left|\overline{e}_{\alpha\beta}\right|_2^2\right\} - C(\Sigma_{\rho\sigma}(|e^-_{\rho 3}e^-_{\sigma 3}|_\infty + |E^0_{\rho\sigma}|_\infty)(\Sigma_\alpha|\partial_\alpha \overline{v}_3|_2^2)$$

Because of the following result (whose proof is rejected in the Appendix):

Lemma 5.5 (An adapted Korn's inequality)

There exists $C > 0$, such that for every $\kappa > 0$ we have:

$$\Sigma|\partial_\alpha \overline{v}_3|_2^2 \le C\left\{\sup\left(\varepsilon^2, \kappa^2, \frac{\varepsilon^4}{\kappa^2}\right)\left\{\left|\frac{\overline{e}_{33}}{\varepsilon^2}\right|_2^2 + \Sigma_\alpha\left|\frac{\overline{e}_{\alpha 3}}{\varepsilon}\right|_2^2 + \Sigma_{\alpha,\beta}\left|\overline{e}_{\alpha\beta}\right|_2^2\right\} + \sup\left(1, \frac{1}{\kappa^2}\right)|\overline{v}|_2^2\right\}$$

we deduce for a choice of κ small enough and $\varepsilon^2 \le C$ that (using as previously lemma 4.1):

$$\mathcal{G}^\varepsilon(\overline{v}) \quad \ge \quad C|\nabla^\varepsilon \overline{v}|_2^2 - C|\overline{v}|_2^2 \tag{5.5}$$

We can now prove that the infimum in the definition of $\theta^\varepsilon = \inf_{\overline{v}\in H^1(\Omega), |\overline{v}|_2^2=1} \mathcal{G}^\varepsilon(\overline{v})$ is reached in some function $\overline{v}^\varepsilon$. It is sufficient to minimize the convex energy $\mathcal{G}^\varepsilon(\overline{v}) + C|\overline{v}|_2^2$ under the constraint $|\overline{v}|_2^2 = 1$ (whose minimum is then $\theta^\varepsilon + C$). We get a lower bound on θ^ε from (5.5), and an upper bound taking the particular function $\overline{v} = (0, 0, C\sin(2\pi x_1))$. This proves

$$|\theta^\varepsilon| \le C \tag{5.6}$$

Step 2

Minimizing the quadratic functional $\mathcal{G}^\varepsilon(\overline{v})$ under the constraint $\int_\Omega (\overline{v})^2 = 1$, makes appear a Lagrange multiplier in the Euler-Lagrange equations:

$$\forall \overline{w} \in H^1(\Omega), \quad \mathcal{G}^{\varepsilon\prime}(\overline{v}^\varepsilon)\cdot \overline{w} = \theta^\varepsilon \int_\Omega 2\overline{v}^\varepsilon \cdot \overline{w} \tag{5.7}$$

Taking $\overline{w} = \overline{v}^\varepsilon$, it is easy to check that this Lagrange multiplier is nothing else than θ^ε. The variational equation (5.7) is equivalent to the following PDE

$$\mathcal{L}^\varepsilon \overline{v}^\varepsilon = \begin{pmatrix} -\theta^\varepsilon \overline{v}^\varepsilon \\ 0 \end{pmatrix}$$

where $\mathcal{L}^\varepsilon$ is a linear differential operator with coefficients which only depend on u^0. Using the Nirenberg's translations method, we prove in the Appendix the

Lemma 5.6 (Bounds on the eigenfunction)
Let $\overline{v}^{0,\varepsilon} = P^{KL}(\overline{v}^\varepsilon)$ *and* $\overline{v}^{2,\varepsilon} = \overline{v}^{2,\varepsilon}\left\{\overline{v}^{0,\varepsilon}\right\}$ *where* $\overline{v}^{2,\varepsilon} = \left\{ \begin{array}{l} \overline{v}_\alpha = 0 \\ \overline{v}_3 = T(e_{33}(\overline{v}^{2,\varepsilon})) \end{array} \right.$ *and* $e_{33}(\overline{v}^{2,\varepsilon})$ *satisfies the linearized of equation (2.9) defining* $e_{33}(u^0)$*:*

$$e_{33}(\overline{v}^{2,\varepsilon}) = -\left(e^-_{\alpha 3}\partial_\alpha \overline{v}^{0,\varepsilon}_3 + \frac{\lambda}{\lambda+2\mu} e_{\alpha\alpha}(\overline{v}^{0,\varepsilon})\right)$$

Then for $\overline{w}^\varepsilon = \overline{v}^\varepsilon - (\overline{v}^{0,\varepsilon} + \varepsilon^2 \overline{v}^{2,\varepsilon})$ *we have:*

$$|\nabla^\varepsilon \overline{w}^\varepsilon|_2^2 \leq C\varepsilon^2 \tag{5.8}$$

$$|\nabla^\varepsilon \overline{v}^\varepsilon|_2^2 + |\nabla^\varepsilon \partial_\alpha \overline{v}^\varepsilon|_2^2 + |\nabla^\varepsilon \partial_{\alpha\beta} \overline{v}^\varepsilon|_2^2 \leq C \tag{5.9}$$

Let us remark that

$$\mathcal{G}^\varepsilon(\overline{v}) = \int_\Omega 4\mu\Sigma \left(\frac{e_{\alpha 3}(\overline{v})}{\varepsilon}\right)^2 + G'_{IJ}(\nabla'^\varepsilon \overline{v})_I (\nabla'^\varepsilon \overline{v})_J$$

where $|G'_{IJ}|_\infty \leq C$ and $\nabla'^\varepsilon \overline{v} = \left(\dfrac{e_{33}(\overline{v})}{\varepsilon^2}, \partial_\alpha \overline{v}\right)$.
Using $\overline{v}^\varepsilon = \overline{v}^{0,\varepsilon} + \varepsilon^2 \overline{v}^{2,\varepsilon} + \overline{w}^\varepsilon$ with $\overline{v}^{0,\varepsilon} \in V^{KL}$ and $\overline{v}^{2,\varepsilon} = \overline{v}^{2,\varepsilon}\left\{\overline{v}^{0,\varepsilon}\right\}$, we compute

$$\nabla'^\varepsilon \overline{v}^\varepsilon = W'^{0,\varepsilon} + \varepsilon W'^{1,\varepsilon}$$

with $W'^{0,\varepsilon} = (e_{33}(\overline{v}^{2,\varepsilon}), \partial_\alpha \overline{v}^{0,\varepsilon})$. Because

$$\int_\Omega G'_{IJ}(\nabla'^\varepsilon \overline{v}^\varepsilon)_I (\nabla'^\varepsilon \overline{v}^\varepsilon)_J = \int_\Omega G'_{IJ}(W'^{0,\varepsilon}_I W'^{0,\varepsilon}_J + 2\varepsilon W'^{0,\varepsilon}_I W'^{1,\varepsilon}_J + \varepsilon^2 W'^{1,\varepsilon}_I W'^{1,\varepsilon}_J)$$

and $\mathcal{G}^0(\overline{v}^{0,\varepsilon}) = \int_\Omega G'_{IJ} W'^{0,\varepsilon}_I W'^{0,\varepsilon}_J$ we see that

$$\mathcal{G}^\varepsilon(\overline{v}^\varepsilon) \quad \geq \quad \mathcal{G}^0(\overline{v}^{0,\varepsilon}) - C\varepsilon \tag{5.10}$$

is a consequence of $|W'^{1,\varepsilon}|_2^2 \leq C$. The following inequalities prove it

$$\begin{aligned} |\varepsilon W'^{1,\varepsilon}|_2^2 &\leq |\frac{e_{33}(\overline{w}^\varepsilon)}{\varepsilon^2}|_2^2 + \Sigma|\varepsilon^2 \partial_\alpha \overline{v}^{2,\varepsilon} + \partial_\alpha \overline{w}^\varepsilon)|_2^2 \\ &\leq C(\varepsilon^4 \Sigma |\partial_\alpha \overline{v}^{2,\varepsilon}|_2^2 + |\nabla^\varepsilon \overline{w}^\varepsilon|_2^2) \\ &\leq C\varepsilon^2 \end{aligned}$$

where we have used the bounds given in lemma 5.6. This ends the proof of lemma 5.4.

6 Appendix: proofs of two lemmata

6.1 Proof of lemma 5.5

We use a function $\psi_+ \in C^\infty(]-\infty,1])$, supp $(\psi_+) \subset [-2,1]$, $\psi_+ > 0$ on $(-1,1)$, and $\psi_+^2 + \psi_-^2 = 1$ on $[-1,1]$ for $\psi_-(x_3) = \psi_+(-x_3)$. We set $\overline{w} = \psi\overline{v}$ with $\psi(x) = \psi_+(x_3)$ or $\psi(x) = \psi_-(x_3)$. Then we have (where the extension $\tilde{w}$ of $\overline{w}$ is taken as the linear combination of two dilated reflexions):

Lemma 6.1 (Nitsche [24], p 240; An extension result)
For the previous ψ, there exists an extension $\tilde{w}$ of $\overline{w} = \psi\overline{v}$ on $D = \omega \times \mathbf{R}$ such that $\tilde{w} = \overline{w}$ on $\Omega = \omega \times (-1,1)$, and for all $i,j \in \{1,2,3\}$ we have

$$\frac{1}{C}|e_{ij}(\overline{w})|^2_{L^2(\Omega)} \le |e_{ij}(\tilde{w})|^2_{L^2(D)} \le C|e_{ij}(\overline{w})|^2_{L^2(\Omega)}$$

Then we have:

$$\begin{aligned}
\int_\Omega (\partial_\alpha \overline{w}_3)^2 &\le \int_D (\partial_\alpha \tilde{w}_3)^2 \\
&\le \int_D (2e_{\alpha 3}(\tilde{w}))^2 - 2\partial_\alpha \tilde{w}_3 \partial_3 \tilde{w}_\alpha \\
&= \int_D (2e_{\alpha 3}(\tilde{w}))^2 - 2\partial_\alpha \tilde{w}_\alpha \partial_3 \tilde{w}_3
\end{aligned}$$

where the second line comes from $(2e_{\alpha 3}(\tilde{w}))^2 = (\partial_\alpha \tilde{w}_3)^2 + (\partial_3 \tilde{w}_\alpha)^2 + 2\partial_\alpha \tilde{w}_3 \partial_3 \tilde{w}_\alpha$, and the third is obtained by an integration by part. Using the general fact that $2ab \le \kappa^2 a^2 + \frac{b^2}{\kappa^2}$, we get $\int_D |\partial_\alpha \tilde{w}_\alpha \partial_3 \tilde{w}_3| \le \int_D C(\kappa^2(\partial_\alpha \tilde{w}_\alpha)^2 + \frac{1}{\kappa^2}(\partial_3 \tilde{w}_3)^2$. From lemma 6.1 we deduce that

$$\begin{aligned}
\int_\Omega (\partial_\alpha \overline{w}_3)^2 &\le C\int_\Omega \left(2e_{\alpha 3}(\psi\overline{v})\right)^2 + \kappa^2\left(\partial_\alpha(\psi\overline{v}_\alpha)\right)^2 + \frac{1}{\kappa^2}\left(\partial_3(\psi\overline{v}_3)\right)^2 \\
&\le C\int_\Omega \psi^2 \left\{4\varepsilon^2 \left(\frac{e_{\alpha 3}(\overline{v})}{\varepsilon}\right)^2 + \kappa^2(\partial_\alpha \overline{v}_\alpha)^2 + \frac{\varepsilon^4}{\kappa^2}\left(\frac{\partial_3 \overline{v}_3}{\varepsilon^2}\right)^2\right\} \\
&\quad + (\partial_3\psi)^2(\overline{v}_\alpha)^2 + \frac{1}{\kappa^2}(\partial_3\psi)^2(\overline{v}_3)^2
\end{aligned}$$

Then

$$\begin{aligned}\int_\Omega (\partial_\alpha \overline{v}_3)^2 &= \int_\Omega \Sigma_\pm \psi_\pm^2 (\partial_\alpha \overline{v}_3)^2 \\ &= \int_\Omega \Sigma_\pm (\partial_\alpha(\psi_\pm \overline{v}_3))^2 \\ &\leq C\left(\sup\left(\varepsilon^2, \kappa^2, \frac{\varepsilon^4}{\kappa^2}\right) |\nabla^\varepsilon \overline{v}|_2^2 + C \sup\left(1, \frac{1}{\kappa^2}\right) |\overline{v}|_2^2\right)\end{aligned}$$

This ends the proof of lemma 5.5.

6.2 Proof of lemma 5.6

Let us remark that from (5.5)-(5.6) we have:

$$|\nabla^\varepsilon \overline{v}^\varepsilon|_2^2 \leq C \tag{6.1}$$

Let us first assume that we have more, i.e.

$$|\nabla^\varepsilon \overline{v}^\varepsilon|_2^2 + |\nabla^\varepsilon \partial_\rho \overline{v}^\varepsilon|_2^2 + |\nabla^\varepsilon \partial_{\rho\sigma} \overline{v}^\varepsilon|_2^2 \leq C \tag{6.2}$$

Asumming (6.2) (which will be proved later) we now prove that $|\nabla^\varepsilon \overline{w}^\varepsilon|_2^2 \leq C\varepsilon^2$.

Estimates on $\frac{\overline{e}_{\alpha 3}}{\varepsilon}$ and $\frac{\overline{e}_{33}}{\varepsilon^2}$.

To simplify the notations let us note $\overline{v} = \overline{v}^\varepsilon$. To estimate $\overline{e}_{\alpha 3}$ and $\overline{e}_{33}$ we use equations

$\mathcal{L}^\varepsilon \overline{v} = \begin{pmatrix} -\theta^\varepsilon \overline{v} \\ 0 \end{pmatrix}$ which we write explicitly:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \left.\begin{array}{l} 2\mu\partial_3\left\{\dfrac{\overline{e}_{\alpha 3}}{\varepsilon^2}\right\} \\ +\partial_\beta\left\{\delta_{\alpha\beta}\lambda\left(\dfrac{\overline{e}_{33}}{\varepsilon^2}+\overline{e}_{\gamma\gamma}+2\partial_\alpha\overline{v}_3 e^-_{\alpha 3}\right)+2\mu\left(\overline{e}_{\alpha\beta}+\dfrac{1}{2}(\partial_\alpha\overline{v}_3 e^-_{\beta 3}+\partial_\beta\overline{v}_3 e^-_{\alpha 3})\right)\right\} \end{array}\right\} = -\theta^\varepsilon\overline{v}_\alpha \\ \left.\begin{array}{l} \dfrac{1}{\varepsilon^2}\partial_3\left\{(\lambda+2\mu)\dfrac{\overline{e}_{33}}{\varepsilon^2}+\lambda\overline{e}_{\gamma\gamma}+2(\lambda+\mu)\partial_\gamma\overline{v}_3 e^-_{\gamma 3}\right\} \\ +\partial_\alpha\left\{\begin{array}{l} \delta_{\alpha\beta}\left(2\mu\dfrac{\overline{e}_{\beta 3}}{\varepsilon^2}+2e^-_{\beta 3}((\lambda+\mu)\dfrac{\overline{e}_{33}}{\varepsilon^2}+\lambda\overline{e}_{\gamma\gamma}+(2\lambda+\mu)\partial_\gamma\overline{v}_3 e^-_{\gamma 3})\right) \\ +2\mu\left(e^-_{\beta 3}(\overline{e}_{\alpha\beta}+\frac{1}{2}(\partial_\alpha\overline{v}_3 e^-_{\beta 3}+\partial_\beta\overline{v}_3 e^-_{\alpha 3}))+E^0_{\alpha\beta}\partial_\beta\overline{v}_3\right) \end{array}\right\} \end{array}\right\} = -\theta^\varepsilon\overline{v}_3 \end{array}\right| \text{ on } \Omega \\ \left.\begin{array}{l} 2\mu\dfrac{\overline{e}_{\alpha 3}}{\varepsilon^2}=0 \\ \dfrac{1}{\varepsilon^2}\left\{(\lambda+2\mu)\dfrac{\overline{e}_{33}}{\varepsilon^2}+\lambda\overline{e}_{\gamma\gamma}+2(\lambda+\mu)\partial_\gamma\overline{v}_3 e^-_{\gamma 3}\right\}=0 \end{array}\right| \text{ on } \partial\Omega \end{array}\right. \tag{6.3}$$

We have $\dfrac{\overline{e}_{\alpha 3}}{\varepsilon^2} = \displaystyle\int_{-1}^{x_3} \partial_3(\frac{\overline{e}_{\alpha 3}}{\varepsilon^2})$ because $\dfrac{\overline{e}_{\alpha 3}}{\varepsilon^2}_{|x_3=-1} = 0$. And from (6.3), we can estimate $|\partial_3(\frac{\overline{e}_{\alpha 3}}{\varepsilon^2})|_2^2 \le C(|\nabla^\varepsilon\partial_\rho\overline{v}|_2^2 + |\nabla^\varepsilon\overline{v}|_2^2)$, so we conclude that:

$$\left|\frac{\overline{e}_{\alpha 3}}{\varepsilon^2}\right|_2^2 \quad\le\quad C \tag{6.4}$$

Deriving the first and the third equation of (6.3), we get similarly

$$\left|\partial_\alpha\left(\frac{\overline{e}_{\beta 3}}{\varepsilon^2}\right)\right|_2^2 \quad\le\quad C\left(|\nabla^\varepsilon\partial_{\alpha\beta}\overline{v}|_2^2+|\nabla^\varepsilon\partial_\alpha\overline{v}|_2^2+|\nabla^\varepsilon\overline{v}|_2^2\right) \quad\le\quad C$$

From equations (6.3) we can also estimate:

$$\left|\frac{1}{\varepsilon^2}\left((\lambda+2\mu)\frac{\overline{e}_{33}}{\varepsilon^2}+\lambda\overline{e}_{\gamma\gamma}+2(\lambda+\mu)\partial_\gamma\overline{v}_3 e^-_{\gamma 3}\right)\right|_2^2 \le C\left(|\nabla^\varepsilon\partial_\alpha\overline{v}|_2^2+\Sigma\left|\partial_\alpha\left(\frac{\overline{e}_{\beta 3}}{\varepsilon^2}\right)\right|_2^2\right)\le C \tag{6.5}$$

Estimates on $\partial_\alpha\overline{w}^\varepsilon$.

Let $\xi = \frac{1}{2}\int_{-1}^1 \overline{v}(x', x_3)dx_3$. Then $\overline{v}(x) = \overline{v}(x',0) + \int_0^{x_3}\partial_3\overline{v}$ and $\xi = \overline{v}(x',0) + \frac{1}{2}\int_{-1}^1\int_0^{x_3}\partial_3\overline{v}$, which gives:

$$\overline{v} = \xi + \int_0^{x_3} \partial_3 \overline{v} - \frac{1}{2}\int_{-1}^{1}\int_0^{x_3} \partial_3 \overline{v}$$

In particular using the fact that $\partial_\alpha \partial_3 \overline{v}_3 = \varepsilon^2 \left(\dfrac{e_{33}(\partial_\alpha \overline{v})}{\varepsilon^2} \right)$ we see that

$$|\partial_\alpha \left(\overline{v}_3 - \xi_3\right)|_2^2 \leq C\varepsilon^4 |\nabla^\varepsilon \partial_\alpha \overline{v}|_2^2 \leq C\varepsilon^4 \tag{6.6}$$

and

$$|\partial_{\beta\alpha} \left(\overline{v}_3 - \xi_3\right)|_2^2 \leq C\varepsilon^4 |\nabla^\varepsilon \partial_{\beta\alpha} \overline{v}|_2^2 \leq C\varepsilon^4$$

Now $\partial_3 \overline{v}_\alpha = (\partial_3 \overline{v}_\alpha + \partial_\alpha \xi_3) - \partial_\alpha \xi_3$, and we get

$$\overline{v}_\alpha = \xi_\alpha - x_3 \partial_\alpha \xi_3 + \int_0^{x_3} (\partial_3 \overline{v}_\alpha + \partial_\alpha \xi_3) - \frac{1}{2}\int_{-1}^{1}\int_0^{x_3} (\partial_3 \overline{v}_\alpha + \partial_\alpha \xi_3)$$

Using $\partial_3 \overline{v}_\alpha + \partial_\alpha \xi_3 = 2\overline{e}_{\alpha 3} - \partial_\alpha \left(\overline{v}_3 - \xi_3\right)$, we get

$$|\partial_\beta (\overline{v}_\alpha - (\xi_\alpha - x_3 \partial_\alpha \xi_3))|_2^2 \leq C\varepsilon^4 \left(\Sigma \left| \partial_\beta \left(\frac{\overline{e}_{\alpha 3}}{\varepsilon^2} \right) \right|_2^2 + \left| \frac{\partial_{\beta\alpha} \left(\overline{v}_3 - \xi_3\right)}{\varepsilon^2} \right|_2^2 \right) \leq C\varepsilon^4 \tag{6.7}$$

Then (6.4)-(6.5)-(6.6)-(6.7) give $|\nabla^\varepsilon \overline{w}^\varepsilon|_2^2 \leq C\varepsilon^2$.

Proof of (6.2)

We apply the method of Nirenberg's translations. Let us derive equation $\mathcal{L}^\varepsilon \overline{v} = \begin{pmatrix} -\theta^\varepsilon \overline{v} \\ 0 \end{pmatrix}$, we get

$$\mathcal{L}^\varepsilon (\partial_\rho \overline{v}) + \mathcal{R}_\rho^\varepsilon (\overline{v}) = \begin{pmatrix} -\theta^\varepsilon \partial_\rho \overline{v} \\ 0 \end{pmatrix}$$

where $\mathcal{R}_\rho^\varepsilon$ is a linear differential operator. Multiplying by $\partial_\rho \overline{v}$ and integrating by part we get:

$$\mathcal{G}^\varepsilon (\partial_\rho \overline{v}) = \theta^\varepsilon |\partial_\rho \overline{v}|_2^2 + \int_\Omega \mathcal{R}_\rho^\varepsilon (\overline{v}) \partial_\rho \overline{v}$$

where

$$
-\int_\Omega \mathcal{R}^\varepsilon_\rho(\overline{v})\partial_\rho\overline{v} = \int_\Omega \left\{ \begin{array}{l}
(\partial_{\beta\rho}\overline{v}_\alpha)\left(\delta_{\alpha\beta}2\lambda(\partial_\rho e^{-}_{\gamma 3})\partial_\gamma\overline{v}_3 + \mu((\partial_\rho e^{-}_{\beta 3})\partial_\alpha\overline{v}_3 + (\partial_\rho e^{-}_{\alpha 3})\partial_\beta\overline{v}_3)\right) \\[2ex]
+\dfrac{1}{\varepsilon^2}(\partial_{3\rho}\overline{v}_3)2(\lambda+\mu)(\partial_\rho e^{-}_{\gamma 3})\partial_\gamma\overline{v}_3 \\[2ex]
+(\partial_{\alpha\rho}\overline{v}_3)\left[\begin{array}{l}
\delta_{\alpha\beta}\left\{ \begin{array}{l}
2(\partial_\rho e^{-}_{\beta 3})((\lambda+\mu)\frac{\overline{e}_{33}}{\varepsilon^2} + \lambda\overline{e}_{\gamma\gamma}) \\[1ex]
+2(2\lambda+\mu)(\partial_\rho(e^{-}_{\beta 3}e^{-}_{\gamma 3}))\partial_\gamma\overline{v}_3
\end{array}\right\} \\[3ex]
+2\mu\left\{ \begin{array}{l}
(\partial_\rho e^{-}_{\beta 3})\overline{e}_{\alpha\beta} + (\partial_\rho E^0_{\alpha\beta})\partial_\beta\overline{v}_3 \\[1ex]
+\frac{1}{2}\left((\partial_\rho(e^{-}_{\beta 3}e^{-}_{\beta 3}))\partial_\alpha\overline{v}_3 + (\partial_\rho(e^{-}_{\beta 3}e^{-}_{\alpha 3}))\partial_\beta\overline{v}_3\right)
\end{array}\right\}
\end{array}\right]
\end{array}\right\}
$$

Using the fact that $2ab \leq \kappa^2 a^2 + \frac{b^2}{\kappa^2}$ we get $\left|\int_\Omega \mathcal{R}^\varepsilon_\rho(\overline{v})\partial_\rho\overline{v}\right| \leq C(\kappa^2|\nabla^\varepsilon\partial_\rho\overline{v}|_2^2 + \dfrac{1}{\kappa^2}|\nabla^\varepsilon\overline{v}|_2^2)$, and then for κ small enough, we get from (5.5)-(5.6) and (6.1):

$$
|\nabla^\varepsilon\partial_\rho\overline{v}|_2^2 \leq C \tag{6.8}
$$

Deriving once more, we get with obvious notations $\mathcal{L}^\varepsilon(\partial_{\rho\sigma}\overline{v}) + \mathcal{R}^\varepsilon_\sigma(\partial_\rho\overline{v}) + \mathcal{R}^\varepsilon_\rho(\partial_\sigma\overline{v}) + \mathcal{R}^\varepsilon_{\rho\sigma}(\overline{v}) = \begin{pmatrix} -\theta^\varepsilon\partial_{\rho\sigma}\overline{v} \\ 0 \end{pmatrix}$, which gives as previously

$$
|\nabla^\varepsilon\partial_{\rho\sigma}\overline{v}|_2^2 \leq C \tag{6.9}
$$

This ends the proof of (6.2) and of lemma 5.6.

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