A Brief Overview on the Obstacle Problem

R. Monneau

Ecole Nationale des Ponts et Chaussées, CERMICS, 6 et 8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France

Abstract

We present a short survey on the obstacle problem including the theory developped by L.A. Caffarelli and the theory developped independently by G.S. Weiss. We also present some other recent results on the regularity of the free boundary.

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1 Introduction

In this note we will present a short survey on the obstacle problem. This presentation is obviously far to be exhaustive. The author has decided to present some new and old important results chosen in a large literature. In particular the references of this note are voluntary limited.

Some books deal with the obstacle problem like the book [17] of D. Kinderlehrer and G. Stampacchia on variational inequalities in 1980, the book [12] of A. Friedman recalls some results on the obstacle problem until 1982 and also many other free boundary problems. To have examples of applications of the obstacle problem to Mathematical Physics and many references until 1987 look at the book [20] of J.F. Rodrigues. Its chapter 6 gives in particular a good overview and detailed results on the obstacle problem.

1.1 A physical example

Let us consider a horizontal wire and a membrane hanging on this wire. We assume that this membrane is above a plate. We push up the plate on the membrane. We get a contact area between the membrane and the obstacle which is the plate. This contact

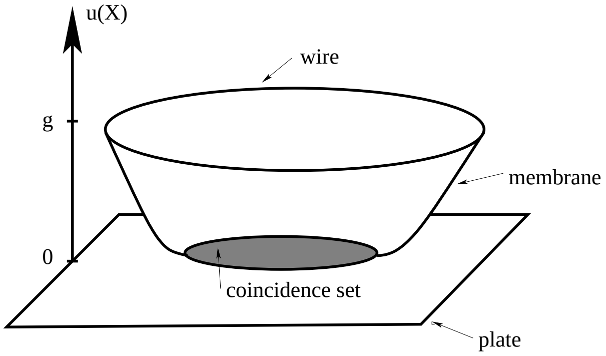

Figure 1: Membrane over a plate obstacle

area is called the coincidence set (see fig. 1). The boundary of the coincidence set is called the free boundary for the obstacle problem. The goal of this note is to give some light on the properties of this free boundary.

1.2 Mathematical formulation

We now give a simple mathematical model for this problem. We assume that the membrane is given by the graph of a function

$$u : \Omega \subset \mathbf{R}^n \longrightarrow \mathbf{R}$$

where Ω is a smooth bounded domain.

We assume that the wire is given by the value of u on the fixed boundary $\partial\Omega$ which is assumed to be a constant

$$u = constant = g \quad \text{on} \quad \partial\Omega \tag{1.1}$$

We assume that the plate is at the 0-level, and we assume that the membrane stays above the plate, i.e. the obstacle

$$u \geq 0 \quad \text{on} \quad \Omega$$

We assume that u minimizes an energy on Ω which is given by

$$\int_\Omega |\nabla u|^2 + 2u \tag{1.2}$$

where the minimization is mathematically taken on the Sobolev space of functions

$$K = \left\{ u \in H^1(\Omega), \quad u = g \text{ on } \partial\Omega, \quad u \geq 0 \text{ on } \Omega \right\} \tag{1.3}$$

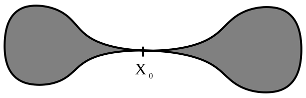

Figure 2: Contact point

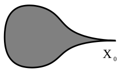

Figure 3: Cusp

The first term in the energy (1.2) is a kind of elastic energy and the second term is a kind of gravity energy for a heavy membrane.

The well-posedness and good properties of the obstacle problem are garanteed by the following result for $g > 0$:

Theorem 1.1 (J. Frehse [11], $C^{1,1}$ Regularity of the Minimizer)
There exists a unique solution u minimizing energy (1.2) on the convex set K defined in (1.3). Moreover this solution satisfies

$$\left\{ \begin{array}{l} \Delta u = 1 \quad on \quad \{u > 0\} \cap \Omega \\ u \geq 0 \quad on \quad \Omega \\ u \in C^{1,1}(\overline{\Omega}) \end{array} \right. \tag{1.4}$$

The coincidence set is

$$\{u = 0\}$$

and the free boundary is $\partial\{u = 0\}$. Moreover it is proved that the free boundary has a finite (hyper)area:

Theorem 1.2 (H. Brezis, D. Kinderlehrer [3], L.A. Caffarelli [6], Finite Hausdorff Measure)

$$\mathcal{H}^{n-1}(\partial\{u = 0\}) \leq C$$

1.3 Examples of singularities

Let us now give some examples of singularities presented by D.G. Schaeffer in [23]. In two dimensions, the free boundary can have a contact point (see fig. 2) or a cusp point

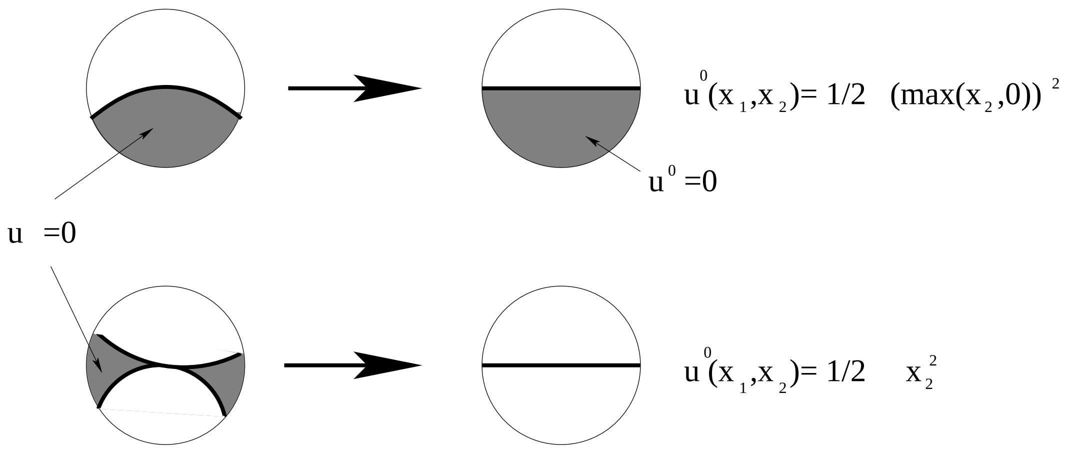

Figure 4: Blow-up limits

(see fig. 3). Following Schaeffer [23], the intuitive origin of these singularities can be obtained by "continuous deformation"with varying obstacles with different components, being the case of fig. 2 the "product" of the joining of two components, and the case of fig. 3 a further consequence of the vanishing of the diameter in the right part of fig. 2. Notice that the double point of $\partial\{u=0\}$ consists of two tangent curves and not two curves intersecting at a non-zero angle.

2 The blow-up Method

To prove regularity results on the free boundary, the main tool (first introduced for the obstacle problem by L.A. Caffarelli in [4]) is the notion of blow-up.

Let us consier a solution u of (1.4) and assume that X_0 is a point on the free boundary $\partial\{u=0\}$. Let us consider the sequence of functions

$$u^\epsilon(X) = \frac{u(X_0+\epsilon X)}{\epsilon^2}$$

By theorem 1.1, $u^\epsilon(0) = \nabla u^\epsilon(0) = 0$ and the second derivatives $|D^2 u^\epsilon|$ are bounded by a constant independent on $\epsilon > 0$. Up to extraction of a subsequence we get a limit as $\epsilon \to 0$

$$u^0(X) \quad \text{defined on} \quad \mathbf{R}^n$$

In fig. 4 we have represented some possible limits in two dimensions. In any dimensions, the main result for the regularity of the free boundary is the following

Theorem 2.1 (L.A. Caffarelli [5, 7], Caracterization of the blow-up limit) *The blow-up limit is unique and only depends on the point X_0 on the free boundary. Moreover either there there exists a unit vector $\nu_{X_0} \in \mathbf{S}^{n-1}$ such that*

$$u^0(X) = \frac{1}{2}(\max(< X, \nu_{X_0} >, 0)^2$$

and the point X_0 is called **regular**.
Or u^0 is a quadratic form, i.e.

$$u^0(X) = \frac{1}{2} \quad {}^t X \cdot Q_{X_0} \cdot X \qquad \geq 0$$

where Q_{X_0} is a symmetric matrix $n \times n$ such that tr $Q_{X_0} = 1$. Such a point X_0 is called **singular**.

This theorem says two things:
i) a Liouville result which classifies all possible blow-up limits on $\mathbf{R}^n$.
ii) the uniqueness of the blow-up limit for each point X_0.
The Liouville result was obtained by L.A. Caffarelli in [5] and essentially uses the maximum principle. We can also cite the work of G.S. Weiss [25] which helps to classify all solutions on $\mathbf{R}^n$ invariant by some dilatations.
The uniqueness of the blow-up limit is difficult to prove in the case of regular points and will be discussed in section 3. On the contrary, in the case of singular points, this uniqueness is quite easy to prove . In [7] L.A. Caffarelli uses the monotonicity formula of H.W. Alt, L.A. Caffarelli and A. Friedman [1] applied to the first derivatives of the solution to prove it. A more elementary fact which avoids to derive the solution u is the following:

Theorem 2.2 (R. Monneau [19], Monotonicity Formula for Singular Points) *Let $v(X) = \frac{1}{2} \quad {}^t X \cdot Q \cdot X \qquad \geq 0$ where Q is a symmetric matrix $n \times n$ such that tr $Q = 1$. If u is a solution to (1.4) and the origine 0 is a singular point, then*

$$\frac{1}{r^{n+3}} \int_{\partial B_r(0)} (u-v)^2 \quad \textit{is nondecreasing in} \quad r$$

This result is a simple corollary of the following monotonicity formula discovered by G.S. Weiss:

Theorem 2.3 (G.S. Weiss [25], Monotonicity Formula for the Obstacle Problem)
For each point $X_0 \in \Omega$ such that the ball $B_r(X_0) \subset \Omega$ we have

$$\Phi_{X_0,u}(r) := \left(\frac{1}{r^{n+2}} \int_{B_r(X_0)} |\nabla u|^2 + 2u \quad - \frac{1}{r^{n+3}} \int_{\partial B_r(X_0)} 2u^2 \right) \quad \textit{is nondecreasing in} \quad r$$

In particular with these monotonicity formulas it is possible to prove

Theorem 2.4 (L.A. Caffarelli [7], Continuity on the Set of Singular Points) *The map*

$$X_0 \longmapsto Q_{X_0}$$

is continuous on the set of singular points of the free boundary. Moreover the set of singular points is a closed set included in a C^1 $(n-1)$-dimensional submanifold.

More precise results can be found in [7].

Remark 2.5 *We have*

$$\Phi_{0,u^\epsilon}(r) = \Phi_{X_0,u}(\epsilon r)$$

and taking the limit we get

$$\Phi_{0,u^0}(r) = constant = \Phi_{X_0,u}(0^+)$$

Using theorem 2.1, it is possible to compute for a point X_0 of the free boundary

$$\Phi_{X_0,u}(0^+) = \begin{cases} \alpha_n & \text{for a regular point} \quad X_0 \\ 2\alpha_n & \text{for a singular point} \quad X_0 \end{cases} \tag{2.1}$$

where α_n is a constant which only depends on the dimension n.

3 Regular points

The expected property for regular points is the following

Theorem 3.1 (L.A. Caffarelli [4, 5], G.S. Weiss [25], Smoothness of the Free Boundary near a Regular Point)
If X_0 is a regular point, then in some appropriate coordinates, the coincidence set is a subgraph $\{x_n \le h(x_1, ..., x_{n-1})\}$ in a neighborhood of X_0, where the function h is of class C^1.

This result is completed by the

Theorem 3.2 (D. Kinderlehrer, L. Nirenberg [16], V. Isakov [14], Higher Regularity of the Free Boundary)
If for a solution of (1.4), the free boundary is locally C^1, this free boundary is analytic.

3.1 Criterions for regular points

We will present an energetic criterion first and then a geometric criterion.
Using the monotonicity formula (theorem 2.3) and the property (2.1), we get

Theorem 3.3 (G.S. Weiss [25], Energetic Criterion)
If

$$\Phi_{X_0,u}(r) < 2\alpha_n$$

for some r, then the point X_0 is a regular point.

To introduce the geometric criterion found by L.A. Caffarelli, we need the following

Definition 3.4 (Thickness of the Coincidence Set)
We define the thickness of the coincidence set $\{u = 0\}$ in a ball $B_r(X_0)$ by

$$\delta_r(X_0) = \frac{1}{r} m.d. \left(\{u = 0\} \cap B_r(X_0)\right)$$

where the minimum diameter (m.d.) of $\{u = 0\} \cap B_r(X_0)$ is the infimum of the distances between pairs of parallel hyperplanes whose strip determined by them countains it.

Theorem 3.5 (L.A. Caffarelli [7], Geometric Criterion)
For each $r > 0$, there exists a critical thickness $\sigma(r)$ with $\sigma(r) \to 0$ as $r \to 0$, such that if

$$\delta_r(X_0) > \sigma(r)$$

for some point X_0 of the free boundary and for one radius $r > 0$, then the point X_0 is regular.

In fact the uniqueness of the blow-up limit for regular points is a consequence of theorem 3.1. The main difficulty is to avoid the free boundary to rotate (slowly and slowly) around X_0. In the approach of G.S. Weiss, the proof of theorem 3.1 is based on an energy power decay estimate on $\Phi_{X_0,u}(r) - \Phi_{X_0,u}(0^+) \geq 0$ which is enough to control the axis of the blow-up limit.
The approach of L.A. Caffarelli to prove theorem 3.1 is based on classical tools like the Maximum Principle and Harnack Inequality, used very cleverly to control the rotation of the axis.

3.2 A counter-example

Let us consider the slightly more general obstacle problem

$$\left\{ \begin{array}{l} \Delta u = f \geq 1 \quad \text{on} \quad \{u > 0\} \cap \Omega \\ u \geq 0 \quad \text{on} \quad \Omega \\ u \in C^{1,1}(\overline{\Omega}) \end{array} \right. \tag{3.1}$$

where $f \in C^\alpha(\overline{\Omega})$, then theorems 2.1 and 3.1 are still true. But we can not go up to the regularity $f \in C^0$ as shows the following counter-example

Theorem 3.6 (I. Blank [2], Example of Non-Uniqueness of the Blow-up Limit)
In $\mathbf{R}^2$, there exists a continuous function f, a solution u to (3.1) and a point X_0 of the free boundary such that the blow-up limit in X_0 is not unique. More precisely for each $\nu \in \mathbf{S}^1$, there exists a sequence $\epsilon_k \to 0$ such that

$$u^{\epsilon_k}(X) \longrightarrow \frac{1}{2}(\max(< X, \nu >, 0))^2$$

4 More results in two dimensions

Let us recall the

Theorem 4.1 (L.A. Caffarelli, N.M. Rivière [8], Regularity of the Free Boundary in Two Dimensions)
If $\mathcal{C}$ is a connected component of the interior of $\{u = 0\}$ in $\mathbf{R}^2$, then $\partial\mathcal{C}$ is analytic except in a finite number of points.

Moreover these authors give a precise behaviour of the free boundary near a singularity in $\mathbf{R}^2$ in [9]. See also [24]. We now give more informations on the number of singularities:

Theorem 4.2 (R. Monneau [18], Small Components Have at Most Two Singular Points)
If $\mathcal{C}$ is connected component of the interior of $\{u = 0\}$ in $\mathbf{R}^2$, then there exists

$$\rho = \rho(|D^2 u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega)) > 0$$

such that
i) if $\mathrm{diam}(\mathcal{C}) < \rho$, then $\partial\mathcal{C}$ has at most two singular points.
ii) if $\mathrm{diam}(\mathcal{C}) \geq \rho$ and $X_1, X_2 \in \partial\mathcal{C}$ are singular points, then $|X_2 - X_1| > \frac{\rho}{2}$.

5 Stability and genericity

In this section we consider solutions u to the obstacle problem (1.4) such that $u =$ *constant* $= g$ on $\partial\Omega$. We will make perturbations varying the parameter $g > 0$. To this end we denote u by u_g to explicit the dependence of the solution on g.

Theorem 5.1 (Stability of the Free Boundary)
If the free boundary $\partial\{u_{g_0} = 0\}$ is analytic for a particular value g_0, then $\partial\{u_g = 0\}$ is analytic for g is a neighborhood of g_0.

This result was first proved by D.G. Schaeffer in [22] when $\partial\Omega \in C^\infty$, using the Nash-Moser inverse function theorem. Using the geometric criterion (theorem 3.5) of L.A. Caffarelli, it is possible to see that theorem 5.1 is a consequence of the

Theorem 5.2 (L.A. Caffarelli [6], Measure Stability)
If u_1, u_2 satisfy the same obstacle problem (1.4), then we can estimate the measure of the symmetric difference of the coincidence sets by

$$|\{u_2 = 0\}\Delta\{u_1 = 0\}| \le C|u_2 - u_1|^{\frac{1}{2}}_{L^\infty(\Omega)}$$

where $C = C(|D^2u_1|_{L^\infty(\Omega)}, |D^2u_2|_{L^\infty(\Omega)}, \Omega, n)$.

See also the book [20] of J.F Rodrigues for detailed results on more general operators.

Let us recall the

Conjecture 5.3 (D.G. Schaeffer [21] (1974))
We conjecture that generically the weak solution of the obstacle problem that one obtains variationally is also a strong solution, by which we mean that the free boundary is a C^∞ manifold.

The following result is a first answer in a simple case:

Theorem 5.4 (R. Monneau [18], Generic regularity of the Free Boundary)
In $\mathbf{R}^2$, for almost every constant $g > 0$, the free boundary $\partial\{u_g = 0\}$ is analytic.

The question stays open in higher dimensions.

6 Convexity property of the coincidence set

Theorem 6.1 (B. Kawohl [15], Convexity of the Coincidence Set)
Let $\Omega \subset \mathbf{R}^n$ a smooth bounded domain. If Ω is convex, then for each constant $g > 0$, the coincidence set $\{u_g = 0\}$ is convex (and analytic).

This result was first proved in the particular case $n = 2$ by A. Friedman and D. Phillips in [13]. Let us give an example of a generalization of this result in two dimensions

Theorem 6.2 (J. Dolbeault, R. Monneau [10], Convexity for Nonlinear Obstacle Problems)
Let $\Omega \subset \mathbf{R}^2$ a smooth bounded domain. Let $u_g \in H^1(\Omega)$ minimizing

$$\int_\Omega F(|\nabla u|^2) + G(u)$$

under the constraints $u \geq 0$ on Ω and $u = constant = g$ on $\partial\Omega$.
We assume that $F, G \in C^2$ are nondecreasing convex functions, with $F'(0) > 0, G'(0) > 0$. If Ω is convex, then the coincidence set $\{u_g = 0\}$ is convex (and C^1).

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