On the Number of Singularities for the Obstacle Problem in Two Dimensions.

R. Monneau*

– dedicated to Henri Berestycki and Alexis Bonnet

Abstract

We study the obstacle problem in two dimensions. On the one hand we improve a result of L.A. Caffarelli and N.M. Rivière: we state that every connected component of the interior of the coincidence set has at most N_0 singular points, where N_0 is only dependent on some geometric constants. Moreover if the component is small enough, then this component has at most two singular points. On the other hand, we prove in a simple case a conjecture of D.G. Schaeffer on the generic regularity of the free boundary: for a familly of obstacle problems in two dimensions continuously indexed by a parameter λ, the free boundary of the solution u_λ is analytic for almost every λ. Finally we present a new monotonicity formula for singular points.

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1 Introduction

In this article we are interested in the number of singularities on the free boundary for the obstacle problem. The obstacle problem appears naturally in several physical situations and an important literature exists on this subject. See for instance the monograph [24] (1980) by D. Kinderlehrer and G. Stampacchia on variational inequalities, the monograph [19] (1982) by A. Friedman which shows links with many other free boundary problems, and the monograph [28] (1987) by J.F. Rodrigues which gives in particular many applications to Mathematical Physics. These monographs also provide many references on

*Ecole Nationale des Ponts et Chaussées, CERMICS, 6 et 8 avenue Blaise Pascal, Cité Descartes Champs-sur-Marne, 77455 Marne-la-Vallée Cedex 2, France

the subject.
We first give a simple physical example and the mathematical formulation of this obstacle problem. We then present our main results, some natural generalizations of these results, and a new monotonicity formula for singular points.

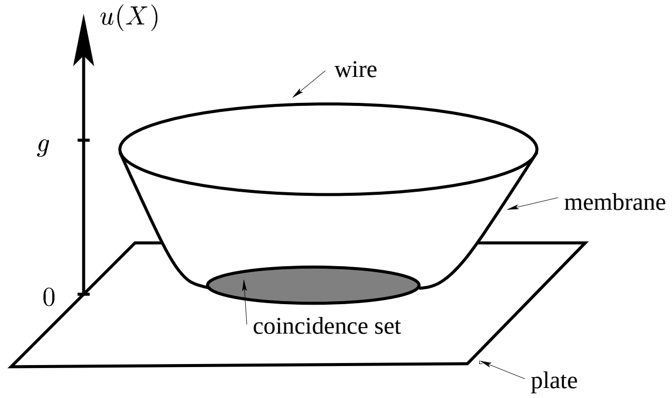

Figure 1: Membrane over a plate obstacle

1.1 A physical example

Let us consider a horizontal wire and an elastic membrane hanging on this wire. We assume that this membrane is submitted to a uniform density of forces, for example gravity. So the membrane is not flat, and its shape is like a bowl. We then introduce a plate below the membrane as an obstacle. We push up the plate and we get a contact area between the plate and the membrane. This contact area is called the coincidence set in the literature (see fig. 1). The boundary of the coincidence set is called the free boundary. The goal of this article is to give some new results on the number of singularities on this free boundary.

1.2 Mathematical formulation

We now give a simple mathematical model for this particular obstacle problem. We assume that the membrane is given by the graph of a function

$$u : \Omega \subset \mathbf{R}^n \longrightarrow \mathbf{R}$$

where Ω is a smooth bounded domain. We assume that the wire is given by the value of u on the fixed boundary $\partial\Omega$ which is assumed to be a constant

$$u = constant = \lambda \quad \text{on} \quad \partial\Omega \tag{1.1}$$

We assume that the plate is at the 0-level, and we assume that the membrane stays above the plate obstacle

$$u \geq 0 \quad \text{on} \quad \Omega$$

We assume that u minimizes an energy on Ω which is given by

$$\int_\Omega |\nabla u|^2 + 2u \tag{1.2}$$

where the minimization is mathematically taken on the convex set of functions

$$K = \left\{ u \in H^1(\Omega), \quad u = \lambda \text{ on } \partial\Omega, \quad u \geq 0 \text{ on } \Omega \right\} \tag{1.3}$$

where $H^1(\Omega)$ is the Sobolev space of functions in $L^2(\Omega)$ with gradient in $L^2(\Omega)$. The first term in the energy (1.2) is a kind of elastic energy and the second term is a kind of gravity energy for a heavy membrane.

The well-posedness and good properties of the obstacle problem are guaranteed by the following result for $\lambda > 0$:

Theorem 1.1 (J. Frehse [18], $C^{1,1}$ Regularity of the Minimizer)
There exists a unique solution u minimizing energy (1.2) on the convex set K defined in (1.3). Moreover this solution satisfies

$$\left\{ \begin{array}{l} \Delta u = 1 \quad on \quad \{u > 0\} \cap \Omega \\ u \geq 0 \quad on \quad \Omega \\ u \in C^{1,1}(\overline{\Omega}) \end{array} \right. \tag{1.4}$$

The coincidence set is

$$\{u = 0\}$$

and the free boundary is $\partial\{u = 0\}$.

1.3 Main results

Let us recall the

Conjecture 1.2 (D.G. Schaeffer [29] (1974))
We conjecture that generically the weak solution of the obstacle problem that one obtains variationally is also a strong solution, by which we mean that the free boundary is a C^∞ manifold.

Our first main theorem proves the conjecture for a simple case in two dimensions.

Theorem 1.3 (Generic Smoothness of the Free Boundary)
For $n = 2$, the solutions u_λ to (1.4),(1.1) satisfy that for almost every constant $\lambda > 0$, the free boundary $\partial\{u_\lambda = 0\}$ is an analytic one-dimensional submanifold.

Although some partial results are given in the present paper, D.G. Schaeffer's conjecture remains open in higher dimensions.

We will now discuss the number of singularities of the free boundary. Let us recall the

Theorem 1.4 (L.A. Caffarelli, N.M. Rivière [15])
The boundary of a connected component of the interior of the coincidence set is analytic except at finitely many singular points.

In his article [31] D.G. Schaeffer wrote:
"*In our examples, if $n = 2$, the exceptional points all lie along the real axis with at most two exceptional points per component. One's appreciation of the delicacy of the arguments of [15] [results of L.A. Caffarelli and N.M. Rivière] may be enhanced by the observation that in our examples the exceptional points from different components may have limit points, and that these limit points may themselves have limit points, etc.*"
This introduces naturally our second main theorem:

Theorem 1.5 (Small Components have at most Two Singular Points)
Let us consider a connected component $\mathcal{C}$ of the interior of the coincidence set for a solution u to (1.4) in $\mathbf{R}^2$. Then there exists a constant $\rho = \rho\left(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega)\right) > 0$ such that
i) if diam $(\mathcal{C}) < \rho$, *then the free boundary $\partial\mathcal{C}$ is analytic except in at most two singular points.*
ii) if diam $(\mathcal{C}) \geq \rho$, *and $X_1, X_2 \in \partial\mathcal{C}$ are two singular points, then $|X_1 - X_2| \geq \frac{\rho}{2}$.*

In fig. 2 is pictured a coincidence set with two singular points.

Theorem 1.4 was remarkably proved in 1975 using H. Lewy reflections and holomorphic functions. In fact our theorem 1.5 only requires the geometric approach developed later by L.A. Caffarelli in [8], and provides an upper bound N_0 on the number of singularities on the free boundary $\partial\mathcal{C}$ only as a function of some geometric quantities. For instance, as a corollary of theorem 1.5 we get that $N_0 = N_0(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega), |\Omega|)$, where $|\Omega|$ denotes the 2-volume of Ω.

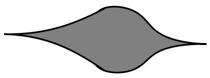

Figure 2: A free boundary with two singular points

1.4 Generalizations

As an application, we can adapt the results of this article to some non-linear obstacle problem like the minimization of

$$\int_\Omega F(|\nabla u|^2) + G(u)$$

over the convex set of functions

$$K = \left\{ u \in H^1(\Omega), \quad u = \phi + \lambda \text{ on } \partial\Omega, \quad u \geq 0 \text{ on } \Omega \right\}$$

where Ω is a bounded domain in $\mathbf{R}^2$ and for instance $\partial\Omega, \phi \in C^{2,\alpha}$, $\phi > 0$ on $\partial\Omega$, and $F, G \in C^2$ are nondecreasing convex functions satisying $F'(0) > 0, G'(0) > 0$. The various constants appearing in this article will then depend on the functions F, G, ϕ.

For instance, we can also consider a general (linear) obstacle problem on a smooth bounded domain $\Omega \subset \mathbf{R}^n$:

$$\left\{ \begin{array}{l} \mathcal{L}u = f \geq 1 > 0 \text{ on } \{u > 0\} \cap \Omega \\ u \geq 0 \text{ on } \Omega \\ u \in C^{1,1}(\overline{\Omega}) \end{array} \right. \tag{1.5}$$

where the linear operator

$$\mathcal{L}u = a_{ij}(X)u_{ij} + b_i(X)u_i + c(X)u$$

is elliptic, i.e.

$$\exists c_0 > 0, \quad \forall \xi \in \mathbf{R}^N, \quad \forall X \in \Omega, \quad a_{ij}(X)\xi_i\xi_j \geq c_0|\xi|^2$$

We assume for example that there exists a constant C_0 such that

$$|a_{ij}|_\alpha, |b_i|_\alpha, |c|_\alpha, |f|_\alpha \leq C_0$$

In particular a variant of theorem 1.5 is true for solutions to (1.5) in two dimensions. Then the constant ρ in theorem 1.5 depends here on $(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega), c_0, C_0)$, and the conclusion of theorem 1.5 holds with the free boundary only $C^{2,\alpha}$ (and not analytic) outside the singularities.

Remark 1.6 *From the results of D. Kinderlehrer, L. Nirenberg [23] and V. Isakov [21], on the regularity of the free boundary, it is clear that the more regular the data* (a_{ij}, b_i, c, f) *are, the more regular the free boundary is outside its singularities. In particular if the data* (a_{ij}, b_i, c, f) *are analytic, then the free boundary is analytic outside its singularities.*

1.5 A new monotonicity formula for singular points

A key argument to prove theorem 1.3 is a new monotonicity formula for singular points. This argument will be in particular used in subsection 4.3.

We first need to give a definition for singular points. For sake of simplicity we choose the

Definition 1.7 (Regular Points, Singular Points)
Let u *a solution to the obstacle problem (1.4). A point* X_0 *of the free boundary* $\partial\{u=0\}$ *is said to be* **regular** *if locally in some appropriate coordinates the coincidence set* $\{u=0\}$ *is the subgraph* $\{x_n \leq g(x_1,...,x_{n-1})\}$ *of an analytic function* g*. And a point* X_0 *is said to be* **singular** *if it is not regular.*

With this definition, outside the singular points, the free boundary is in particular an analytic $(n-1)$-dimensional submaniflod in $\Omega \subset \mathbf{R}^n$. On the other hand, in a singular point we have the following monotonicity formula:

Theorem 1.8 (Monotonicity Formula for Singular Points)
Let $v(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X \quad \geq 0$ *where* Q *is a symmetric matrix* $n \times n$ *such that tr* $Q = 1$*. If* u *is a solution to (1.4) and the origin* 0 *is a singular point, then*

$$\frac{1}{r^{n+3}} \int_{\partial B_r(0)} (u-v)^2 \quad \textit{is non-decreasing in} \quad r$$

Theorem 1.8 is a simple corollary of a monotonicity formula of G.S. Weiss in [33]. As a simple consequence of theorem 1.8, we obtain that the solution to the obstacle problem (1.4) has a Taylor expansion of order 2 in a singular point. More generally with an adapted definition of singular points for the obstacle problem (1.5) and an adapted version of theorem 1.8, we prove the

Theorem 1.9 (Taylor Expansion at a Singular Point)
If u *is a solution to (1.5) and the origin* 0 *is a singular point, then*

$$u(X) = \frac{1}{2} \quad {}^tX \cdot D^2u(0) \cdot X + o(X^2)$$

Theorem 1.9 was first proved in dimension two by L.A. Caffarelli and N.M. Rivière in [16] establishing a strong estimate on the growth of the second derivatives of the solution near the free boundary in two dimensions. G.S. Weiss gave a different proof using an epiperimetric inequality (see [33]). Finally theorem 1.9 was first proved for the Laplace operator in every dimension only recently by L.A. Caffarelli in [13]. His proof uses the following monotonicity formula applied to the first derivatives of the solution u ($v = \nabla u$):

Theorem 1.10 (H.W. Alt, L.A. Caffarelli, A. Friedman [1], Monotonicity Formula for the Two Phases Problem)
If v is a continuous function such that $\Delta v = 0$ in $\{v \neq 0\}$, and $v(0) = 0$, then

$$\Phi(r) := \frac{1}{r^4}\left(\int_{B_r(0)\cap\{v>0\}} \frac{|\nabla v|^2}{|X|^{n-2}}\right)\left(\int_{B_r(0)\cap\{v<0\}} \frac{|\nabla v|^2}{|X|^{n-2}}\right) \quad \textit{is nondecreasing in} \quad r$$

This argument of L.A. Caffarelli remained unknown to the present author during his Ph.D. and was independently rediscovered by the author and used to prove theorem 1.3 in the first version of this article which can be found in his Doctoral Dissertation [25]. Let us remark that some generalizations of the monotonicity formula theorem 1.10 exist: see the work [12] of L.A. Caffarelli, and a recent general monotonicity formula [14] of L.A. Caffarelli, C.E. Kenig and D. Jerison. On the contrary, in this final version of the article we do not use theorem 1.10, and prefer to use our monotonicity formula for singular points, namely theorem 1.8. In particular theorem 1.8 avoids having to differentiate the solution and then seems to have a more important potential of applications in the special context of the obstacle problem (see the appendix).

1.6 Organization of the article

In section 2 we have put forth some short preliminary results which are essentially used at the end of each section 3, 4. Sections 3, 4 and 5 are independent. In section 3 we give the proof of theorem 1.5 on the finite number of singularities. And in section 4 we give the proof of theorem 1.3 on the generic smoothness of the free boundary. In section 5 we present the short proof of the monotonicity formula for singular points (theorem 1.8) and we explain in an appendix (section 6) how this monotonicity formula can be adapted to more general operators than the Laplace operator.

2 Some well-known results on blow-up limits

2.1 The simple blow-up limit

To prove regularity results on the free boundary, the main tool (first introduced for the obstacle problem by L.A. Caffarelli in [7]) is the notion of blow-up.

Let us consider a solution u to (1.4) and assume that X_0 is a point on the free boundary $\partial\{u=0\}$. Let us consider the following blow-up sequence of functions

$$u^\epsilon(X) = \frac{u(X_0+\epsilon X)}{\epsilon^2}$$

By theorem 1.1, $u^\epsilon(0) = \nabla u^\epsilon(0) = 0$ and the second derivatives $|D^2u^\epsilon|$ are bounded by a constant independent of $\epsilon > 0$. By Ascoli-Arzela theorem, up to extraction of a convergent subsequence (ϵ'), we get

$$u^{\epsilon'} \longrightarrow u^0 \quad \text{uniformly on compact sets of} \quad \mathbf{R}^n$$

This function u^0 is called a blow-up limit of the function u at the point X_0. In fig. 3 we have represented some possible blow-up limits in two dimensions.

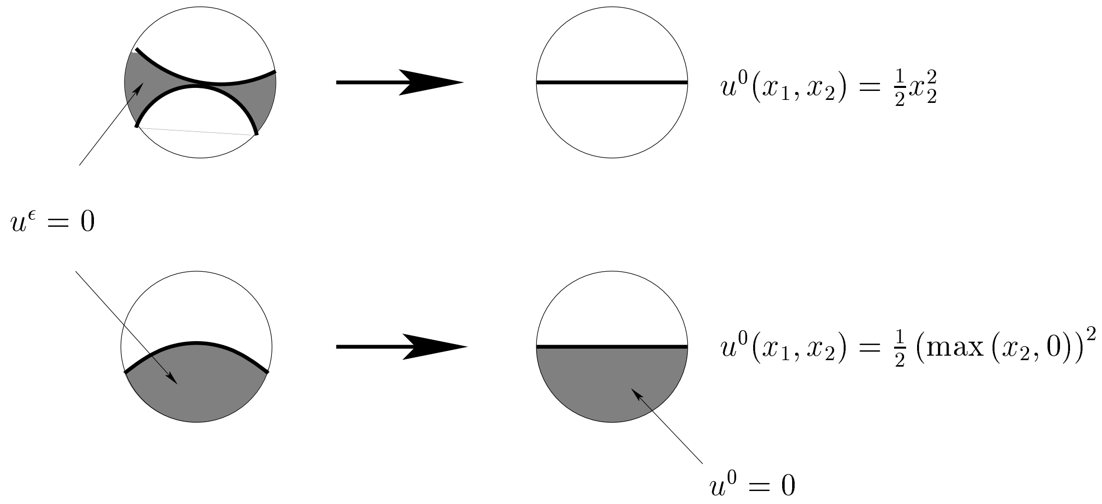

Figure 3: Some blow-up limits in two dimensions

In any dimensions, the main result for blow-up limits is the following

Theorem 2.1 (L.A. Caffarelli [7, 8, 13], Characterization of a Simple Blow-up Limit)
The blow-up limit u^0 is unique and only depends on the point X_0 on the free boundary. Moreover either X_0 is a **singular** *point and then u^0 is a quadratic form, i.e.*

$$u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q_{X_0} \cdot X \quad \geq 0$$

where Q_{X_0} is a symmetric matrix $n \times n$ such that tr $Q_{X_0} = 1$.
Or X_0 is a **regular** *point and there exists a unit vector $\nu_{X_0} \in \mathbf{S}^{n-1}$ such that*

$$u^0(X) = \frac{1}{2}\left(\max\left(< X, \nu_{X_0} >, 0\right)\right)^2$$

A partial result in this direction was obtained by G.S. Weiss in [33] who proved that the blow-up limits on $\mathbf{R}^n$ are invariant by certain dilations.

2.2 More general blow-up limits

We now recall a result which characterizes the limit of some more general blow-up sequences where the origin moves with the scaling.

Lemma 2.2 (Characterization of General Blow-up Limits on $\mathbf{R}^n$, [8])
Let

$$u^\epsilon(X) = \frac{u_\epsilon(X_\epsilon + \epsilon X)}{\epsilon^2}$$

where u_ϵ is a sequence of solutions to (1.4) satisfying $u_\epsilon(X_\epsilon) = 0$, and

$$|D^2 u_\epsilon|_{L^\infty(\Omega)} \leq M$$

Let us assume that $\frac{1}{\epsilon} d(X_\epsilon, \partial\Omega) \to +\infty$ as $\epsilon \to 0$. Then up to extraction of a convergent subsequence (ϵ'), we get

$$u^{\epsilon'} \longrightarrow u^0 \quad \textit{uniformly on compact sets of} \quad \mathbf{R}^n$$

where u^0 is defined on $\mathbf{R}^n$ and is convex. Moreover either
i) the blow-up limit can be written

$$u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X \quad \geq 0$$

where Q is a symmetric matrix $n \times n$ such that tr $Q = 1$.
Or
ii) the interior of the coincidence set of the blow-up limit satisfies

$$\{u^0 = 0\}^0 \neq \emptyset$$

and 0 is a regular point for u^0 and also for $u^{\epsilon'}$ with ϵ' small enough.

As a useful complement let us recall the

Lemma 2.3 (Blow-down Limit, [8])
Let u^0 a solution to the obstacle problem (1.4) with bounded second derivatives on $\Omega = \mathbf{R}^n$. Let the blow-down sequence with $\mu \to +\infty$:

$$u^{0\mu} = \frac{u^0(\mu X)}{\mu^2}$$

Then up to extraction of a convergent subsequence (μ')*, we have*

$$u^{0\mu'} \longrightarrow u^{0\infty} \quad \textit{uniformly on compact sets of} \quad \mathbf{R}^n$$

where

$$u^{0\infty}(X) = \begin{cases} \frac{1}{2}{}^tX\cdot Q\cdot X \geq 0 \quad \textit{with} \quad trQ = 1 \\ \textit{or} \\ \frac{1}{2}\max(< X,\nu >, 0)^2 \end{cases}$$

Another useful result is the following nondegeneracy property of the solution:

Lemma 2.4 (Nondegeneracy, [8])
Let u *a solution to problem (1.4) and* $0 \in \overline{\{u > 0\}}$*. If* $B_r(0) \subset \Omega$*, then*

$$\sup_{B_r(0)} (u(X) - u(0)) \geq \frac{r^2}{2n}$$

Proof of lemma 2.4. Apply the maximum principle to $w(X) = u(X) - u(0) - \frac{1}{2n}|X|^2$ in $B_r(0) \cap \{u > 0\}$.

3 Proof of theorem 1.5

3.1 Bounds on the number of singularities on the free boundary

In this section we focus on the singularities of the free boundary for the obstacle problem in two dimensions. Let us recall that the free boundary is the boundary of the coincidence set which can have many connected components and even an infinite number of connected components with accumulation points (see the examples of D.G. Schaeffer [31]). It is then necessary to restrict our analysis to connected components. As in [15] we will study connected components $\mathcal{C}$ of the interior of the coincidence set $\{u = 0\}$ where u is a solution to (1.4) for $n = 2$.

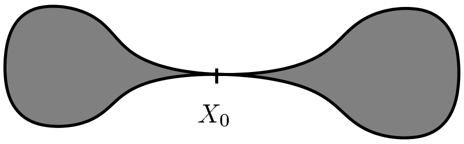

Figure 4: Contact point

In two dimensions, the free boundary can have a contact point (see fig. 4) or a cusp point (see fig. 5). Following D.G. Schaeffer [31], the intuitive origin of these singularities

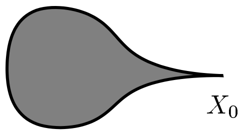

Figure 5: Cusp

can be obtained by "continuous deformation" with varying obstacles with different components, being the case of fig. 4 the "product" of the joining of two components, and the case of fig. 5 a further consequence of the vanishing of the diameter in the right part of fig. 4. Notice that the double point of $\partial\{u = 0\}$ consists of two tangent curves and not two curves intersecting at a non-zero angle.

This study of the singularities on the free boundary is based on a "half strip" lemma (lemma 3.3 below). Before to present this lemma, we want to recall a related geometric criterion found by L.A. Caffarelli. First, we need the following

Definition 3.1 (Thickness of the Coincidence Set)
We define the thickness of the coincidence set $\{u = 0\}$ in a ball $B_r(X_0)$ by

$$\delta_r(X_0) = \frac{1}{r} m.d.\,(\{u = 0\} \cap B_r(X_0))$$

where the minimum diameter $(m.d.)$ of $\{u = 0\} \cap B_r(X_0)$ is the infimum of the distances between pairs of parallel hyperplanes whose strip determined by them contains it.

Theorem 3.2 (L.A. Caffarelli [13], Geometric Criterion)
For each $r > 0$, there exists a critical thickness $\sigma_0(r)$ with $\sigma_0(r) \to 0$ as $r \to 0$, such that if

$$\delta_r(X_0) > \sigma_0(r)$$

for some point X_0 of the free boundary and for one radius $r > 0$, then the point X_0 is regular. Moreover this function $\sigma_0(r)$ only depends on bounds on $|D^2 u|_{L^\infty(\Omega)}$ and $d(X_0, \partial\Omega)$.

Our "half strip" lemma is an improved version for singularities in dimension two of the geometric criterion of L.A. Caffarelli:

Lemma 3.3 (Half Strip Near a Singular Point)
Let $\mathcal{C}$ a connected component of the interior of the coincidence set $\{u=0\}$, where u is a solution to the obstacle problem (1.4) in dimension $n=2$. Then there exists a continuous nondecreasing function $\sigma(r)$ with $\sigma(0)=0$, such that if $X_0 \in \partial\mathcal{C}$ is a singular point of the free boundary, then the intersection of $\mathcal{C}$ with the ball $B_r(X_0)$ is included in a half strip of thickness equal to $r\sigma(r)$.
More precisely, there exists a half straight line $A_\tau = \{X_0 + s\tau, s \in (0,+\infty)\}$ with $\tau \in \mathbf{S}^1$, such that

$$\mathcal{C} \cap B_r(X_0) \quad \subset \quad \{X, d(X, A_\tau) < r\sigma(r)\} \cap B_r(X_0) \quad =: \quad \Sigma_{\sigma(r)}(X_0) \tag{3.1}$$

Moreover this function $\sigma(r)$ only depends on bounds on $|D^2u|_{L^\infty(\Omega)}$ and $d(\mathcal{C}, \partial\Omega)$.

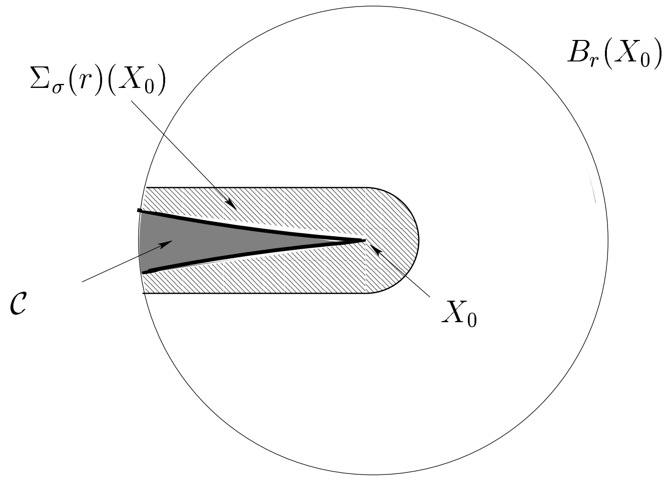

Figure 6: Near a singular point, a connected component is included in a half strip of small thickness

In particular estimation (3.1) is uniform over all connected components $\mathcal{C}$ of the interior of the coincidence set at a distance bounded from below to the fixed boundary $\partial\Omega$. Roughly speaking, this means that in a small enough ball of center a singular point, the connected component $\mathcal{C}$ is included in a half strip of much smaller thickness as the radius of the ball tends to zero. In fig. 6 is pictured the situation (3.1).

We then apply this lemma to prove theorem 1.5.

Proof of theorem 1.5 i).
Let us fix a constant ρ small enough, for instance given by

$$2\rho = \sigma^{-1}(10^{-2})$$

Let us assume that a connected component $\mathcal{C}$ of the interior of the coincidence set satisfies

$$\text{diam } (\mathcal{C}) \quad < \rho$$

and has at least three singular points X_1, X_2, X_3. We can always assume that

$$|X_3 - X_2| \geq |X_3 - X_1| \geq |X_2 - X_1| \tag{3.2}$$

Let

$$\rho_1 = \frac{3}{2}|X_2 - X_1|$$

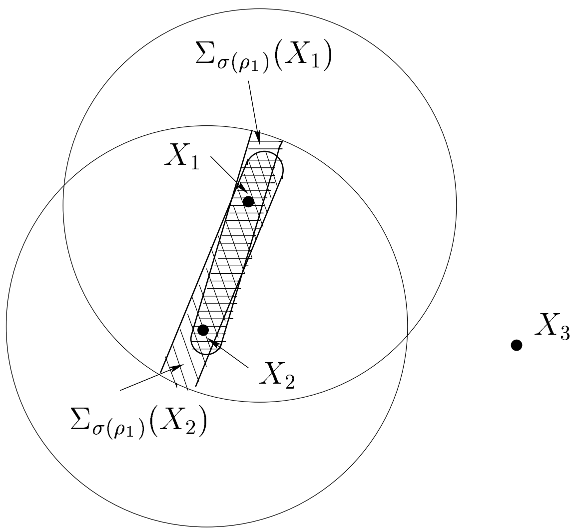

Figure 7: The component $\mathcal{C}$ is in the intersection of two half strips $\Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2)$

Then in particular X_2 has to belong to the half strip $\Sigma_{\sigma(\rho_1)}(X_1)$ given by lemma 3.3. Similarly we get $X_1 \in \Sigma_{\sigma(\rho_1)}(X_2)$. By construction we have

$$\Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2) \quad \subset \quad B_{\rho_1}(X_1) \cap B_{\rho_1}(X_2)$$

Now because

$$\rho_1 < 2\rho \tag{3.3}$$

we deduce that

$$\sigma(\rho_1) \leq \sigma(2\rho)$$

which is small enough such that we have geometrically (see fig. 7):

$$\Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2) \quad \subset\subset \quad B_{\rho_1}(X_1) \cap B_{\rho_1}(X_2)$$

and consequently by connectedness of $\mathcal{C}$ we get

$$\mathcal{C} \quad \subset \quad \Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2)$$

The contradiction comes from the fact that by construction (because of (3.2) and $\sigma(2\rho) = 10^{-2}$ was chosen sufficiently small)

$$X_3 \quad \notin \quad \Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2)$$

Proof of theorem 1.5 ii).
Let us assume that a connected component $\mathcal{C}$ of the interior of the coincidence set satisfies

$$\text{diam }(\mathcal{C}) \quad \geq \rho$$

Let us consider two singular points $X_1, X_2 \in \partial\mathcal{C}$. Let us assume that $|X_2 - X_1| < \frac{\rho}{2}$. Then the same argument as in i) gives

$$\mathcal{C} \quad \subset \quad \Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2)$$

where $\rho_1 = 1,5|X_2 - X_1|$. Moreover

$$\text{diam } \left(\Sigma_{\sigma(\rho_1)}(X_1) \cap \Sigma_{\sigma(\rho_1)}(X_2)\right) \quad < 2|X_2 - X_1| < \rho$$

which gives the contradiction. This ends the proof of theorem 1.5.

As a straightforward corollary of theorem 1.5, we get by a covering argument of Ω by small balls:

Corollary 3.4 *Let $\mathcal{C}$ a connected component of the interior of the coincidence set. Then there exists a constant $N_0 = N_0(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega), |\Omega|)$ where $|\Omega|$ denotes the 2-volume of Ω, such that $\partial\mathcal{C}$ has at most N_0 singular points.*

Let us point out that the bound on the number of singular points given by L.A. Caffarelli and N.M. Rivière in theorem 1.4 depends on the solution, and not just on the geometry as in corollary 3.4.

As an other corollary we get a different kind of estimate:

Corollary 3.5 *Let $\mathcal{C}$ a connected component of the interior of the coincidence set. Then there exists a constant $N_1 = N_1(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega), \mathcal{H}^1(\partial\mathcal{C}))$ such that $\partial\mathcal{C}$ has at most N_1 singular points. More precisely $N_1 \leq 2\max\left(\frac{\mathcal{H}^1(\partial\mathcal{C})}{\rho}, 1\right)$ where ρ was given in theorem 1.5.*

Here $\mathcal{H}^1$ is the onedimensional Hausdorff measure. As an estimate of the length $\mathcal{H}^1(\partial\{u=0\})$ of the free boundary we have more generally:

Theorem 3.6 (H. Brezis, D. Kinderlehrer [6], L.A. Caffarelli [9])
Let u a solution to (1.4) in dimension $n \geq 1$. Then there exists a constant $C_1 = C_1\left(|D^2u|_{L^\infty(\Omega)}, \mathcal{H}^{n-1}(\partial\Omega), |\Omega|, n\right)$ such that

$$\mathcal{H}^{n-1}(\Gamma) \leq C_1$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

Let us mention that with a similar proof we can get an adapted version of theorem 1.5 for connected components $\mathcal{C}$ of the coincidence set (and not of the interior of the coincidence set as previously). We have

Theorem 3.7 *Let us consider a connected component $\mathcal{C}$ of the coincidence set. Then there exists a constant $\rho = \rho\left(|D^2u|_{L^\infty(\Omega)}, d(\mathcal{C}, \partial\Omega)\right) > 0$ such that $\partial\mathcal{C}$ is the union of at most N_2 analytic curves, possibly with tangent intersections. Moreover*
i) if diam $(\mathcal{C}) < \rho$, *then* $N_2 \leq 2$.
ii) if diam $(\mathcal{C}) \geq \rho$, *and $X_1, X_2 \in \partial\mathcal{C}$ are two distinct endpoints of these analytic curves, then $|X_1 - X_2| \geq \frac{\rho}{2}$.*

Remark 3.8 *These endpoints are necessarily cusps. For example in fig. 4 there is one closed curve with a tangent self-intersection (but without endpoints), in fig. 5 there is one curve with two ends on the same point X_0 which is a cusp. And in fig. 2, there are two curves with two endpoints (two cusps).*

3.2 Proof of lemma 3.3

Let us assume that the lemma is false. Then there exists a constant $\epsilon_0 \in (0,1)$ and a sequence

$$r \longrightarrow 0$$

(here we drop the index k of the sequences to simplifiy the notations) and there exist sequences

$$\left\{\begin{array}{l} u \quad \text{solution to (1.4) with} \quad |D^2u| \leq M \\ \mathcal{C} \quad \text{a connected component of} \quad \{u=0\}^0 \quad \text{with} \quad d(\mathcal{C}, \partial\Omega) \geq \delta > 0 \\ X_0 \quad \text{a singular point on} \quad \partial\mathcal{C} \end{array}\right.$$

such that for every $\nu \in \mathbf{S}^1$:

$$\mathcal{C} \cap B_r(X_0) \quad \not\subset \quad \Sigma_{\epsilon_0}(X_0) \tag{3.4}$$

where

$$\begin{cases} \Sigma_{\epsilon_0}(X_0) = \{X, d(X, A_{\nu^\perp}) < \epsilon_0 r\} \cap B_r(X_0) \\ \\ A_{\nu^\perp} = \left\{X_0 + s\nu^\perp, s \in (0, +\infty)\right\} \end{cases}$$

From the geometric criterion (theorem 3.2) there exists $\nu \in \mathbf{S}^1$ such that

$$\mathcal{C} \cap B_r(X_0) \quad \subset \quad \{X, | < X - X_0, \nu > | < r\sigma_0(r)\} \cap B_r(X_0)$$

We now define a new sequence by

$$u^r(X) = \frac{u(X_0 + rX)}{r^2}$$

and

$$\mathcal{C}^r = \frac{\mathcal{C} - X_0}{r}$$

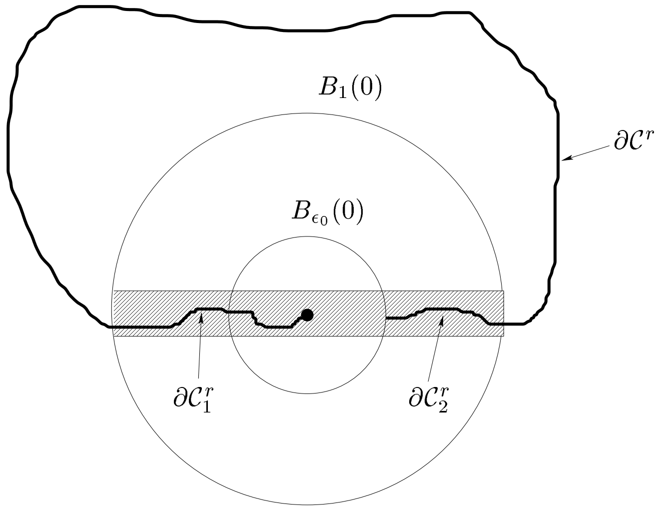

Figure 8: $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ are connected outside $B_1(0)$

Then up to introduce moreover a rotation of the coordinates, we can assume that the direction ν is fixed and does not depend on the sequence r. With these new coordinates we have (from the geometric criterion theorem 3.2)

$$\mathcal{C}^r \cap B_1(0) \quad \subset \quad \{X, | < X, \nu > | < \sigma_0(r)\} \cap B_1(0) \quad =: \quad S_{\sigma_0(r)}$$

and from (3.4), we deduce that each connected component of

$$S_{\sigma_0(r)} \backslash B_{\epsilon_0}(0)$$

has a non-empty intersection with $\mathcal{C}^r$. Let us call $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ the parts of $\mathcal{C}^r$ included in the two components of $S_{\sigma_0(r)} \backslash B_{\epsilon_0}(0)$. Then by connectedness of $\mathcal{C}$ we have two cases (see fig. 8 and 9):

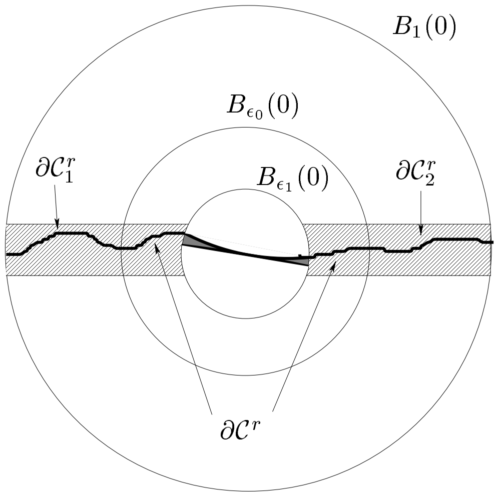

Figure 9: $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ are only connected inside $S_{\sigma_0(r)} \cap B_{\epsilon_0}(0)$

Case 1 : $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ are connected outside $B_1(0)$

Then the maximum principle applied to u^r on the open set which is bordered by

$$\mathcal{C}^r \cup \left(S_{\sigma_0(r)} \cap B_{\epsilon_0}(0)\right)$$

implies

$$\sup_{B_1^+(0)\backslash S_{\sigma_0(r)}} u^r \leq \sup_{S_{\sigma_0(r)} \cap B_{\epsilon_0}(0)} u^r \tag{3.5}$$

where up to change ν in $-\nu$, the "half of the ball" $B_1^+(0)$ is defined by

$$B_1^+(0) = \{X, < X, \nu > \geq 0\} \cap B_1(0)$$

Moreover, u^r tends to u^0 uniformly. The set $\left\{u^0 = 0\right\}$ contains $\pm\nu^\perp$ and hence because it is convex, it contains the segment between them. Consequently the limit in (3.5) gives

$$\sup_{B_1^+(0)} u^0 \leq 0$$

Then lemma 2.2 implies that 0 is a regular point for u^r with r small enough. Contradiction.

Case 2 : $\mathcal{C}_1^r$ and $\mathcal{C}_2^r$ are only connected inside $S_{\sigma_0(r)} \cap B_{\epsilon_0}(0)$

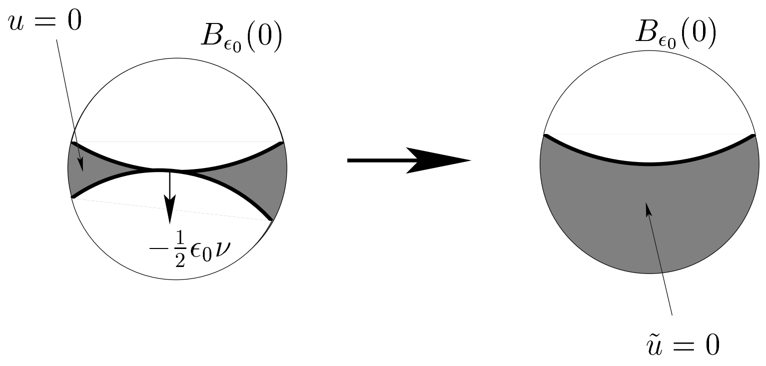

Figure 10: Construction of the function $\tilde{u}$

We introduce (see fig. 10)

$$\tilde{u}^r = \begin{cases} 0 \quad \text{on the connected component of} \quad \{u^r > 0\} \cap B_{\epsilon_0}(0) \quad \text{which contains} \quad -\frac{1}{2}\epsilon_0\nu \\ u^r \quad \text{elsewhere on} \quad B_{\epsilon_0}(0) \end{cases}$$

Then $\tilde{u}^r$ satisfies the obstacle problem $B_{\epsilon_0}(0)$ and from the geometric criterion (theorem 3.2), there exists $\epsilon_1 \in (0, \epsilon_0)$ such that up to define new coordinates, we have

$$\{\tilde{u}^r = 0\} \cap B_{\epsilon_1}(0) \quad \text{is a subgraph} \quad \{x_2 \le g_+(x_1)\}$$

We can also get a similar result with the construction of $\tilde{u}^r$ using the connected component of $\{u^r > 0\} \cap B_{\epsilon_0}(0)$ which contains $\frac{1}{2}\epsilon_0\nu$. Therefore

$$\mathcal{C}^r \cap B_{\epsilon_1}(0) \subset \{g_-(x_1) \le x_2 \le g_+(x_1)\}$$

with $g_-, g_+ \in C^1$ and $g_-(0) = g_+(0) = 0$. Moreover we have

$$\mathcal{C}^r \cap B_{\epsilon_0}(0) \quad \subset \bigcap_{\epsilon_1 < \epsilon < \epsilon_0} B_\epsilon(0) \cap \{X, | < X, \nu_\epsilon > | < \sigma_0(\epsilon r)\}$$

where ν_ϵ is a unit vector depending on ϵ. And for $\sigma_0(r)$ small enough (and then r small enough), the only way to connect $\mathcal{C}_1^r$ to $\mathcal{C}_2^r$ is first (let say)

$$\text{to connect} \quad \mathcal{C}_1^r \quad \text{to} \quad \{g_-(x_1) \le x_2 \le g_+(x_1), \quad x_1 < 0\}$$

$$\text{to connect} \quad \mathcal{C}_2^r \quad \text{to} \quad \{g_-(x_1) \le x_2 \le g_+(x_1), \quad x_1 > 0\}$$

and

$$\begin{array}{ll} \text{to connect} & \{g_-(x_1) \le x_2 \le g_+(x_1), \quad x_1 < 0\} \\ \text{to} & \{g_-(x_1) \le x_2 \le g_+(x_1), \quad x_1 > 0\} \quad \text{via the origin} \quad 0 \end{array}$$

In particular the interior of $\mathcal{C}^r$ is broken at the origin, which implies that the interior of $\mathcal{C}^r$ has at least two connected components. This is in contradiction with the fact that $\mathcal{C}$ is a connected component of the interior of the coincidence set.

4 Proof of theorem 1.3

4.1 Generic smoothness of the free boundary

Let us recall the following result

Theorem 4.1 (L.A. Caffarelli [13], Structure of the Singular Set)
For each $\lambda > 0$, let

$$\mathcal{S}_\lambda = \quad \text{the set of singular points of the coincidence set of} \quad u_\lambda$$

Then $\mathcal{S}_\lambda \backslash \mathcal{E}_\lambda$ is included in a C^1 $(n-1)$-dimensional submanifold of finite $(n-1)$-volume and $\mathcal{H}^{n-1}(\mathcal{E}_\lambda) = 0$.

Using the blow-up techniques developed in section 4.3, we improve here this result in the following proposition

Proposition 4.2 (A C^1-Submanifold Contains Almost All the Singularities)
Let

$$\mathcal{S} = \bigcup_{\lambda > 0} \mathcal{S}_\lambda$$

Then $\mathcal{S} \backslash \mathcal{E}$ is included in a C^1 $(n-1)$-dimensional submanifold of finite $(n-1)$-volume and $\mathcal{H}^{n-1}(\mathcal{E}) = 0$.

Now let us introduce the function h defined on $\overline{\Omega}$ by

$$h(X) = \sup\{\lambda \in [0, +\infty), \quad u_\lambda(X) = 0\}$$

This function is not C^1 in general as shows the following example:

Example: In dimension $n = 1$, for $\Omega = (-1, 1)$ we can compute (see fig. 11)

$$h(x) = \frac{1}{2}(1 - |x|)^2$$

Nevertheless we can prove the

Proposition 4.3 *The function h is Lipschitz-continuous on $\overline{\Omega}$.*

and the

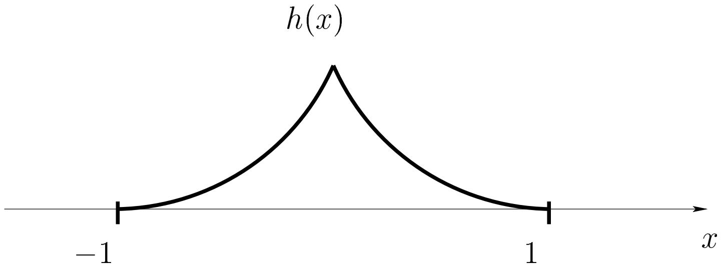

Figure 11: The function h is not C^1

Proposition 4.4

$$\{h \geq \lambda\} = \{u_\lambda = 0\} \tag{4.1}$$

$$\{h = \lambda\} = \partial\{u_\lambda = 0\}$$

Let us recall the following coarea formula:

Theorem 4.5 (H. Federer [17], Coarea Formula)
Let $A \subset \mathbf{R}^m$ a set such that $A \backslash B$ is included in a C^1 k-dimensional submanifold of finite k-volume and $\mathcal{H}^k(B) = 0$. Let a function

$$f : A \longrightarrow \mathbf{R}$$

which is Lipschitz on A, i.e.

$$\exists C > 0, \quad |f(X') - f(X)| \leq C|X' - X| \quad \textit{for all} \quad X, X' \in A$$

Then the gradient ∇f is defined $\mathcal{H}^k$ almost everywhere on A and

$$\int_A |\nabla f| \quad d\mathcal{H}^k = \int_{\mathbf{R}} \mathcal{H}^{k-1}(f^{-1}(y)) \quad dy$$

Using the fact that in any dimensions we have

Proposition 4.6 (The Function h Has a Null Gradient on the Singular Set)

$$\limsup_{\substack{X, X' \in \mathcal{S} \\ |X - X'| \to 0}} \frac{|h(X') - h(X)|}{|X' - X|} = 0$$

We deduce that

$$\int_0^{+\infty} \mathcal{H}^{n-2}\left(\left(h_{|\mathcal{S}}\right)^{-1}(\lambda)\right) \quad d\lambda = 0$$

with

$$\left(h_{|\mathcal{S}}\right)^{-1}(\lambda) = \mathcal{S}_\lambda$$

We have proved the

Theorem 4.7 *For almost every $\lambda > 0$, the free boundary $\partial\{u_\lambda = 0\}$ is analytic except on a set of zero $\mathcal{H}^{n-2}$ measure.*

In particular in dimension $n = 2$, this proves theorem 1.3.

Remark 4.8 *With the same ideas, we can prove in any dimension $n \geq 1$, that the free boundary is generically free of singularities X_0 which satisfy*

$$\dim Ker\ D^2u(X_0) \quad \leq 1$$

Proof of proposition 4.6
If the proposition is false, then there exists $\delta > 0$, and sequences of singular points $X_k, X_k' \in \mathcal{S}$ such that

$$\frac{h(X_k') - h(X_k)}{|X_k' - X_k|} \geq \delta > 0 \quad \text{and} \quad |X_k' - X_k| \to 0$$

By equality (4.1) we have in particular

$$\{u_{h(X_k)} = 0\} \ni X_k'$$

which (since h is Lipschitz) is improved in

$$\{u_{h(X_k)} = 0\} \supset B_{r_k}(X_k') \quad \text{where} \quad r_k = \frac{\delta|X_k' - X_k|}{Lip(h)}$$

Let us introduce the following blow-up sequence:

$$u^k(X) = \frac{u_{h(X_k)}(X_k + r_k X)}{r_k^2}$$

In particular

$$u^k = 0 \quad \text{on the ball} \quad B_1(Y_k) \quad \text{with} \quad |Y_k| = \frac{Lip(h)}{\delta}$$

From the geometric criterion theorem 3.2, this gives a contradiction with the fact that 0 is a singular point for u^k.

We will now prove proposition 4.4. We first need the following two lemmata:

Lemma 4.9

$$\{u_\lambda = 0\} \subset \{u_{\lambda'} = 0\} \quad \text{if} \quad \lambda \geq \lambda'$$

Proof of lemma 4.9. Apply the maximum principle to $u_\lambda - u_{\lambda'}$.

Lemma 4.10 *For every point $X_0 \in \Omega$, $X_0 \in \partial\{u_{h(X_0)} = 0\}$.*

Proof of lemma 4.10.
This lemma is a consequence of the fact that for $\lambda_1 = h(X_0)$ and for every $s > 0$:

$$u_{\lambda_1+s}(X_0) > 0$$

and from the nondegeneracy lemma:

$$\sup_{B_r(X_0)} (u_{\lambda_1+s} - u_{\lambda_1+s}(X_0)) \geq \frac{r^2}{2n}$$

By continuity of the map $(X, \lambda) \to u_\lambda(X)$ we get in $s = 0$

$$\sup_{B_r(X_0)} u_{\lambda_1} \geq \frac{r^2}{2n}$$

which proves the lemma.

Proof of proposition 4.4.
From lemma 4.9 and the continuity of the map $(X, \lambda) \mapsto u_\lambda(X)$, we get

$$\{u_\lambda = 0\} = \{h \geq \lambda\}$$

To prove that

$$\partial\{u_\lambda = 0\} = \{h = \lambda\}$$

we only need (from lemma 4.10) to prove that

$$\partial\{u_\lambda = 0\} \subset \{h = \lambda\}$$

which is a consequence of

$$\partial\{u_\lambda = 0\} \subset \{h \geq \lambda\}$$

and the fact that h is Lipschitz (proposition 4.3) which avoids the values $h > \lambda$. This ends the proof of proposition 4.4.

4.2 The function h is Lipschitz-continuous : proof of proposition 4.3

We first assume the following result which will be proved later:

Proposition 4.11 *The function h is continuous.*

To prove that h is Lipschitz, we need (in the spirit of the work of L.A. Caffarelli [10, 11]) to introduce the following familly of functions for $\delta > 0$ and $\epsilon > 0$:

$$v^\epsilon(X) = \sup_{Y \in B_{\epsilon\delta}(X)} u_{\lambda-\delta}(Y)$$

which are subsolutions (as it will be proved below) to the obstacle problem (1.4) on

$$\Omega_{(-\epsilon\delta)} = \{X \in \Omega,\ d(X, \partial\Omega) > \epsilon\delta\}$$

For $\delta > 0$ fixed, by a continuity method varying the parameter $\epsilon > 0$, we will prove that these subsolutions stay under the solution u_λ until some critical value $\epsilon_c > 0$:

$$v^\epsilon \leq u_\lambda \quad \text{on} \quad \Omega_{(-\epsilon\delta)} \quad \text{for} \quad 0 \leq \epsilon \leq \epsilon_c := 1/|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\epsilon\delta)})} \tag{4.2}$$

For $\epsilon = \epsilon_c$ this implies

$$d(\{u_\lambda = 0\}, \{u_{\lambda-\delta} > 0\}) \geq \epsilon_c \delta$$

In particular if $X' \in \partial\{u_\lambda = 0\}$ and $X \in \partial\{u_{\lambda-\delta} = 0\}$ then

$$0 \leq \frac{h(X') - h(X)}{|X' - X|} \leq \frac{1}{\epsilon_c}$$

Now as $\delta \to 0$, we get

$$0 \leq \limsup_{\substack{h(X') > h(X) \\ h(X') \to \lambda \\ h(X) \to \lambda}} \frac{h(X') - h(X)}{|X' - X|} \leq |\nabla u_\lambda|_{L^\infty(\partial\Omega)}$$

where we have used the continuity of the map $(X, \lambda) \longmapsto \nabla u_\lambda(X)$. In particular we conclude that

$$Lip(h) \leq \sup_{\lambda \in [0, +\infty)} |\nabla u_\lambda|_{L^\infty(\partial\Omega)} = |\nabla u_\infty|_{L^\infty(\partial\Omega)} < +\infty$$

where the function u_∞ satisfies

$$\begin{cases} \Delta u_\infty = 1 \quad \text{on} \quad \Omega \\ u_\infty = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

End of the proof of proposition 4.3.
By definition of v^ϵ, if $X_1 \in \partial\Omega_{(-\epsilon\delta)}$, there exists $X_2 \in \partial\Omega$ such that (see fig. 12)

$$|X_2 - X_1| = \epsilon\delta \quad \text{and} \quad [X_1, X_2] \subset \overline{\Omega}\setminus\Omega_{(-\epsilon\delta)}$$

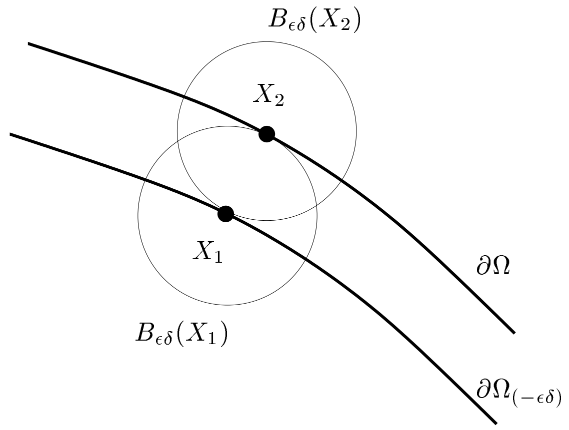

Figure 12: The situation close to the fixed boundary

Moreover

$$\begin{aligned} v^\epsilon(X_1) &= \sup_{Y \in B_{\epsilon\delta}(X_1)} u_{\lambda-\delta}(Y) \\ &\leq \lambda - \delta \\ &= u_\lambda(X_2) - \delta \\ &= u_\lambda(X_1) - \delta + (u_\lambda(X_2) - u_\lambda(X_1)) \\ &\leq u_\lambda(X_1) - \delta(1 - \epsilon|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\epsilon\delta)})}) \end{aligned}$$

Consequently

$$v^\epsilon < u_\lambda \quad \text{on} \quad \partial\Omega_{(-\epsilon\delta)} \quad \text{while} \quad \epsilon < \epsilon_c = 1/|\nabla u_\lambda|_{L^\infty(\overline{\Omega}\setminus\Omega_{(-\epsilon\delta)})} \tag{4.3}$$

Here the fact that v^ϵ is a subsolution for the obstacle problem means

Lemma 4.12 *The function v^ϵ satisfies*

$$\begin{cases} \Delta v^\epsilon \geq 1 \quad \textit{on} \quad \{v^\epsilon > 0\} \cap \Omega_{(-\epsilon\delta)} \\ v^\epsilon \geq 0 \\ \textit{if} \quad v^\epsilon(X_0) = 0, \quad \textit{then} \quad v^\epsilon(X_0 + X) \leq CX^2 \quad \textit{with} \quad C = \frac{1}{2}|D^2 u|_{L^\infty(\Omega)} \end{cases}$$

Now let us consider the open set where v^ϵ is bigger than u_λ:

$$\omega := \{v^\epsilon > u_\lambda\} \cap \Omega_{(-\epsilon\delta)}$$

We want to prove that $\omega = \emptyset$ while $\epsilon < \epsilon_c$. Let us assume on the contrary that $\omega \neq \emptyset$. Then from lemma 4.12 and the maximum principle we deduce that the maximum

$$\max_\omega (v^\epsilon - u_\lambda) > 0$$

is reached on

$$\partial\omega \cup ((\partial\{u_\lambda = 0\} \cap \omega)$$

But

$$\partial\omega \quad \subset \quad (\partial\Omega_{(-\epsilon\delta)}) \cup \{v^\epsilon = u_\lambda\}$$

and from (4.3) we get

$$v^\epsilon - u_\lambda \leq 0 \quad \text{on} \quad \partial\omega \quad \text{while} \quad \epsilon < \epsilon_c$$

We can resume what we have proved in the following

Lemma 4.13

$$\left\{ \begin{array}{l} v^\epsilon < u_\lambda \quad on \quad \partial\Omega_{(-\epsilon\delta)} \\ \\ \max\limits_{\Omega_{(-\epsilon\delta)}} (v^\epsilon - u_\lambda) \leq \max\limits_{\partial\{u_\lambda=0\}} v^\epsilon \end{array} \right\} \quad while \quad \epsilon < \epsilon_c$$

We now apply a continuity method in three steps (in the spirit of the sliding method [3]):

Step 1: Initialization

By the continuity of h (proposition 4.11) and equality (4.1), there exists an $\epsilon_1 > 0$ small enough (and in particular smaller that ϵ_c) such that

$$d(\{u_\lambda = 0\}, \{u_{\lambda-\delta} > 0\}) \geq \epsilon_1\delta$$

In particular we get that

$$v^\epsilon = 0 \quad \text{on} \quad \partial\{u_\lambda = 0\} \quad \text{for} \quad \epsilon \leq \epsilon_1$$

and from lemma 4.13, we have

$$v^\epsilon \leq u_\lambda \quad \text{on} \quad \Omega_{(-\epsilon\delta)} \quad \text{for} \quad \epsilon \leq \epsilon_1$$

Step 2: Continuation

Let

$$\epsilon^* = \sup\{\epsilon' \in [0, \epsilon_c], \quad v^\epsilon \leq u_\lambda \quad \text{on} \quad \Omega_{(-\epsilon\delta)} \quad \text{for all} \quad \epsilon \leq \epsilon'\}$$

In particular

$$v^{\epsilon^*} \leq u_\lambda \quad \text{on} \quad \Omega_{(-\epsilon^*\delta)}$$

If $\epsilon^* < \epsilon_c$, then from lemma 4.13 and the maximum principle, we get

$$v^{\epsilon^*} < u_\lambda \quad \text{on} \quad \{v^{\epsilon^*} > 0\} \cap \Omega_{(-\epsilon^*\delta)} \tag{4.4}$$

On the other hand we have

$$\{v^{\epsilon^*} = 0\} \supset \{u_\lambda = 0\}$$

We will prove that

$$\partial\{u_\lambda = 0\} \cap \partial\{v^{\epsilon^*} > 0\} \neq \emptyset \tag{4.5}$$

If not, we have

$$d(\partial\{u_\lambda = 0\}, \{v^{\epsilon^*} > 0\}) = \eta > 0$$

and then

$$\begin{aligned} v^{\epsilon^*+s}(X) &= \sup_{Y \in B_{(\epsilon^*+s)\delta}(X)} u_{\lambda-\delta}(Y) \\ &= \sup_{Y \in B_{s\delta}(X)} \sup_{Y' \in B_{\epsilon^*\delta}(Y)} u_{\lambda-\delta}(Y') \\ &= \sup_{Y \in B_{s\delta}(X)} v^{\epsilon^*}(Y) \end{aligned}$$

Then

$$v^{\epsilon^*+s} = 0 \quad \text{on} \quad \partial\{u_\lambda = 0\} \quad \text{if} \quad s < \frac{\eta}{\delta} \quad \text{and} \quad \epsilon^* + s < \epsilon_c$$

which by lemma 4.13 would give a contradiction to the definition of ϵ^*.
Then (4.5) is true and there exists

$$X_3 \in \partial\{u_\lambda = 0\} \cap \partial\{v^{\epsilon^*} > 0\}$$

Moreover there exists (see fig. 13)

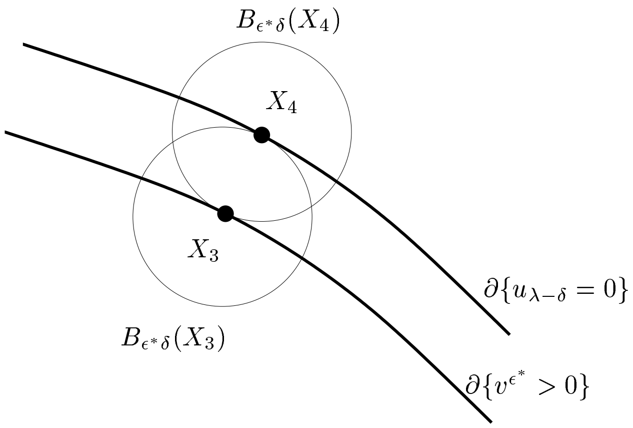

Figure 13: The situation near the free boundary

$$X_4 \in \partial\{u_{\lambda-\delta} = 0\} \quad \text{such that} \quad |X_4 - X_3| = \epsilon^*\delta$$

As a consequence

$$v^{\epsilon^*} > 0 \quad \text{on} \quad B_{\epsilon^*\delta}(X_4)$$

Using lemma 4.13, we can resume the properties of $u_\lambda - v^{\epsilon^*}$ on the ball $B_{\epsilon^*\delta}(X_4)$ by

$$\begin{cases} \Delta(u_\lambda - v^{\epsilon^*}) \leq 0 \quad \text{on} \quad B_{\epsilon^*\delta}(X_4) \\ u_\lambda - v^{\epsilon^*} > 0 \quad \text{on} \quad B_{\epsilon^*\delta}(X_4) \\ u_\lambda(X_3) = v^{\epsilon^*}(X_3) = 0 \quad \text{with} \quad X_3 \in \partial B_{\epsilon^*\delta}(X_4) \end{cases}$$

The Hopf lemma implies

$$\frac{d}{dn}\left(u_\lambda - v^{\epsilon^*}\right)(X_3) > 0 \quad \text{with} \quad n = \frac{X_3 - X_4}{|X_3 - X_4|}$$

This is in contradiction with the fact that the nonnegative functions u_λ and v^{ϵ^*} satisfy

$$\begin{cases} \nabla u_\lambda(X_3) = 0 \quad \text{because} \quad u_\lambda \in C^{1,1} \\ \\ \nabla v^{\epsilon^*}(X_3) = 0 \quad \text{because} \quad v^{\epsilon^*}(X_3 + X) \leq CX^2 \end{cases}$$

Step 3: Conclusion

As a consequence we get

$$\epsilon^* = \epsilon_c$$

which proves (4.2).

Proof of lemma 4.12

This lemma is a straightforward consequence of the

Lemma 4.14 (T. Kato [22], Variant of Kato's Inequality)
Let two Lipschitz functions u_i for $i = 1, 2$, which satisfy $u_i \geq 0$ on ω, $\Delta u_i \geq 0$ on ω, and

$$\Delta u_i \geq 1 \text{ on } \{u_i > 0\} \cap \omega \tag{4.6}$$

Then $v = sup(u_1, u_2)$ satisfies

$$\Delta v \geq 1 \text{ on } \{v > 0\} \cap \omega$$

And lemma 4.14 can be proved using the original mollification argument of T. Kato in [22].

Proof of proposition 4.11.

Let us recall that we have proved that if h is continuous, then h is much better than continuous: h is Lipschitz. Hence we will here introduce a perturbation argument which will insure easily the continuity of h. We will denote by $\eta > 0$ the parameter of the perturbation:

$$u_\lambda^\eta = (1 - \eta\lambda)u_\lambda$$

For $\eta > 0$ small enough, the maximum principle implies

$$u^\eta_\lambda \le u^\eta_{\lambda'} \quad \text{on} \quad \Omega \quad \text{if} \quad \lambda \le \lambda'$$

Now let us assume that h is not continuous in $X_0 \in \Omega$. Then there exist $\delta > 0$ and a sequence of points

$$X_k \to X_0 \quad \text{with} \quad |h(X_k) - h(X_0)| \ge \delta > 0$$

If $h(X_k) - h(X_0) \le -\delta$, then let

$$v_2^k(X) := (1 - \eta h(X_k)) \frac{u_{h(X_k)}(X_0 + |X_k - X_0| X)}{|X_k - X_0|^2}$$

$$\le \quad v_1^k(X) := (1 - \eta h(X_0)) \frac{u_{h(X_0)}(X_0 + |X_k - X_0| X)}{|X_k - X_0|^2}$$

Because by lemma 4.10 X_0 is a point of the free boundary $\partial\{u_{h(X_0)} = 0\}$, theorem 2.1 states that the blow-up limit v_1^0 satisfies with $\lambda_1 = h(X_0)$:

$$\frac{1}{1 - \eta\lambda_1} v_1^0(X) = \begin{cases} \frac{1}{2} \quad {}^t X \cdot Q_1 \cdot X \ge 0 \quad \text{with} \quad trQ_1 = 1 \\ \text{or} \\ \frac{1}{2} \quad (\max(<X, \nu_1>, 0))^2 \end{cases}$$

The sequence v_2^0 has also a blow-up limit:

$$v_2^0 \quad \text{defined on} \quad \mathbf{R}^n$$

Then if we introduce a blow-down sequence with $\mu \to +\infty$:

$$v_2^{0\mu}(X) := \frac{v_2^0(\mu X)}{\mu^2} \le v_1^{0\mu}(X) := v_1^0(X)$$

Lemma 2.3 insures that the blow-down limit $v_2^{0\infty}$ satisfies with $\lambda_2 = \lim h(X_k) \le \lambda_1 - \delta$:

$$\frac{1}{1 - \eta\lambda_2} v_2^{0\infty}(X) = \begin{cases} \frac{1}{2} {}^t X \cdot Q_2 \cdot X \ge 0 \quad \text{with} \quad trQ_2 = 1 \\ \text{or} \\ \frac{1}{2} (\max(<X, \nu_2>, 0))^2 \end{cases}$$

The fact that

$$1 - \eta\lambda_2 > 1 - \eta\lambda_1$$

gives a contradiction with the inequality

$$v_2^{0\infty} \le v_1^{0\infty} := v_1^0 \quad \text{on} \quad \mathbf{R}^n$$

We get a similar contradiction with $h(X_k) - h(X_0) \ge \delta$.

4.3 Structure of the singular set: proof of proposition 4.2

Let us recall that we denote by $\mathcal{S}$ the set of all singular points for all values of the parameter λ:

$$\mathcal{S} = \left\{ X \in \overline{\Omega}, \quad \exists \lambda \in [0, +\infty), \quad X \quad \text{is a singular point of} \quad \partial \{u_\lambda = 0\} \right\}$$

To prove proposition 4.2, we first need the following result

Lemma 4.15 *The set $\mathcal{S}$ is a closed set and there is a modulus of continuity $\sigma(r)$ such that*

$$|Q_{X_1} - Q_{X_0}| \leq \sigma(|X_1 - X_0|) \quad \textit{for every points} \quad X_0, X_1 \in \mathcal{S}$$

Here $Q_X \geq 0$ is a $n \times n$ matrix such that $trQ_X = 1$, which characterizes the (unique) blow-up limit in a singular point X (see theorem 2.1).

Moreover let us recall the

Theorem 4.16 (Whitney's Extension, see [17])
Let A a compact set in $\mathbf{R}^m$, and two functions

$$f : A \longrightarrow \mathbf{R}^p, \quad Tf : A \longrightarrow \mathbf{R}^{m \times p}$$

such that

$$\limsup_{\substack{|a-b| \to 0 \\ a,b \in A}} \quad |Tf(b) - Tf(a)| = 0$$

and

$$\limsup_{\substack{|a-b| \to 0 \\ a,b \in A}} \frac{|f(b) - f(a) - Tf(a) \cdot (b-a)|}{|b-a|} = 0$$

Then there exists a C^1 map

$$g : \mathbf{R}^m \longrightarrow \mathbf{R}^p$$

such that

$$g_{|A} = f \quad \textit{and} \quad (\nabla g)_{|A} = Tf$$

Applying the Whitney's extension theorem we get that locally near a singular point X_0 such that

$$\dim \mathrm{Ker} \quad Q_{X_0} = k$$

the singular set $\mathcal{S}$ is included in a C^1 k-dimensional submanifold in Ω. In particular if we define the set

$$\mathcal{E} = \{X \in \mathcal{S}, \quad \dim \mathrm{Ker} \quad Q_X < n-1\}$$

we get by a denombrable covering argument that

$$\mathcal{H}^{n-1}(\mathcal{E}) = 0$$

And by another covering argument, we get that the closed set $\mathcal{S}\backslash\mathcal{E}$ is included in a C^1 $(n-1)$-dimensional submanifold of finite $(n-1)$-volume. This ends the proof of proposition 4.2.

Proof of lemma 4.15.
Let us assume that the lemma is false. Then there exists a constant $\delta > 0$, and two sequences of points $X_k, X'_k \in \mathcal{S}$ such that

$$|Q_{X_k} - Q_{X'_k}| \geq \delta > 0 \quad \text{and} \quad |X_k - X'_k| \longrightarrow 0$$

We can always assume that $h(X_k) \leq h(X'_k)$. Then let

$$\begin{aligned} v_1^k(X) &:= \frac{u_{h(X_k)}(X_k + |X_k - X'_k|X)}{|X_k - X'_k|^2} \\ \leq \quad v_2^k(X) &:= \frac{u_{h(X'_k)}(X_k + |X_k - X'_k|X)}{|X_k - X'_k|^2} \end{aligned}$$

Up to extract a convergent subsequence (k'), we have

$$v_1^{k'}(X) \longrightarrow v_1^0(X) = \frac{1}{2} \quad {}^tX \cdot Q_1 \cdot X \geq 0 \quad \text{with} \quad trQ_1 = 1$$

and

$$v_2^{k'} \longrightarrow v_2^0$$

where for some $|Y_0| = 1$:

$$v_2^0(Y_0 + X) = \frac{1}{2} \quad {}^tX \cdot Q_2 \cdot X \geq 0 \quad \text{with} \quad trQ_2 = 1$$

The following inequality

$$v_1^0 \leq v_2^0 \quad \text{on} \quad \mathbf{R}^n$$

implies $Q_1 = Q_2$ and

$$v_1^0 = v_2^0$$

Now by the monotonicity formula theorem 1.8 we have

$$\begin{aligned} \int_{\partial B_1(0)} \left(v_1^k(X) - v_1^0(X)\right)^2 &\geq \lim_{r\to 0} \int_{\partial B_1(0)} \left(\frac{v_1^k(rX)}{r^2} - \frac{v_1^0(rX)}{r^2}\right)^2 \\ &= \int_{\partial B_1(0)} \left(\frac{1}{2} \quad {}^tX \cdot Q_{X_k} \cdot X - v_1^0(X)\right)^2 \end{aligned}$$

Similarly we have for v_2^k:

$$\int_{\partial B_1(0)} \left(v_2^k(X) - v_2^0(X)\right)^2 \geq \int_{\partial B_1(0)} \left(\frac{1}{2} \quad {}^tX \cdot Q_{X'_k} \cdot X - v_2^0(X)\right)^2$$

From the fact that $v_i^{k'} \to v_1^0 = v_2^0$ for $i = 1, 2$ we deduce that

$$|Q_{X_{k'}} - Q_{X'_{k'}}| \longrightarrow 0$$

Contradiction. This ends the proof of lemma 4.15.

5 The monotonicity formula for singular points

5.1 Preliminaries

Let us recall the following fundamental result for the obstacle problem:

Theorem 5.1 (G.S. Weiss [33], Monotonicity Formula for the Obstacle Problem)
If u is a solution to (1.4), then for any origin $0 \in \Omega$, we have

$$\Phi_u(r) := \left(\frac{1}{r^{n+2}} \int_{B_r(0)} |\nabla u|^2 + 2u \quad - \frac{1}{r^{n+3}} \int_{\partial B_r(0)} 2u^2\right) \quad \textit{is nondecreasing in} \quad r$$

This quantity has in particular the following invariance by scaling:

$$\Phi_{u^\epsilon}(r) = \Phi_u(\epsilon r)$$

where

$$u^\epsilon(X) = \frac{u(\epsilon X)}{\epsilon^2}$$

Then taking the limit in a point 0 of the free boundary, we get

$$\Phi_{u^0}(r) = constant = \Phi_u(0^+)$$

Using theorem 2.1, it is possible to compute

$$\Phi_u(0^+) = \begin{cases} \alpha_n & \text{if} \quad 0 \quad \text{is a regular point} \\ 2\alpha_n & \text{if} \quad 0 \quad \text{is a singular point} \end{cases} \tag{5.1}$$

where α_n is a constant which only depends on the dimension n.

Using the monotonicity formula (theorem 5.1) and the property (5.1), we get in particular

Theorem 5.2 (G.S. Weiss [33], Energetic Criterion)
If

$$\Phi_u(r) < 2\alpha_n$$

for some r, then the point 0 is a regular point.

5.2 Proof of theorem 1.8

Let

$$v(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X \quad \geq 0$$

where Q is a symmetric matrix $n \times n$ such that $tr\ Q = 1$, which satisfies

$$\Phi_v(r) = 2\alpha_n = \Phi_u(0^+)$$

And let

$$w = u - v$$

Then by integration by parts we compute

$$\begin{aligned}
\Phi_u(r) - \Phi_u(0^+) &= \Phi_u(r) - \Phi_v(r) \\
&= \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 + 2\nabla v \cdot \nabla w + 2w - \frac{2}{r^{n+3}} \int_{\partial B_r} w^2 + 2vw \\
&= \frac{1}{r^{n+2}} \int_{B_r} |\nabla w|^2 - \frac{2}{r^{n+3}} \int_{\partial B_r} w^2 + \frac{2}{r^{n+3}} \int_{\partial B_r} w(r\partial_r v - 2v) \\
&= \frac{1}{r^{n+2}} \int_{B_r} -w\Delta w + \frac{1}{r^{n+3}} \int_{\partial B_r} w(r\partial_r w - 2w)
\end{aligned}$$

because $r\partial_r v - 2v = 0$. But the second term of the last line is

$$\begin{aligned}
\frac{1}{r^{n+3}} \int_{\partial B_r} w(r\partial_r w - 2w) &= \frac{1}{2r^{n+3}} \int_{\partial B_r} \left(\frac{1}{r^4} \partial_r (w^2) - 4\frac{w^2}{r^5} \right) \\
&= \frac{r}{2} \frac{d}{dr} \left(\frac{1}{r^{n+3}} \int_{\partial B_r} w^2 \right)
\end{aligned}$$

which gives

$$\frac{d}{dr} \left(\frac{1}{r^{n+3}} \int_{\partial B_r} (u-v)^2 \right) = \frac{2}{r} \left(\Phi_u(r) - \Phi_u(0^+) \right) + \frac{2}{r^{n+3}} \int_{B_r} (u-v)\Delta(u-v) \qquad (5.2)$$

Now

$$(u-v)\Delta(u-v) = \left\{ \begin{array}{ll} 0 & \text{on} \quad u > 0 \\ v & \text{on} \quad \{u=0\}^0 \end{array} \right\} \quad \geq \quad 0$$

Then we get

$$\frac{d}{dr} \left(\frac{1}{r^{n+3}} \int_{\partial B_r} (u-v)^2 \right) \geq \frac{2}{r} \left(\Phi_u(r) - \Phi_u(0^+) \right) \geq 0$$

This ends the proof of the theorem.

6 Appendix: further results on the uniqueness of the blow-up limit in singular points

Proof of theorem 1.9
Write the equation

$$a_{ij}(X)u_{ij} + b_i(X)u_i + c(X)u = f(X)$$

in the form

$$a_{ij}(0)u_{ij} = f(0)+(f(X)-f(0))-(a_{ij}(X)-a_{ij}(0))u_{ij}-(b_i(X)-b_i(0))u_i-(c(X)-c(0))u$$

After a linear change of coordinates, the problem reduces to

$$\begin{cases} \Delta u = 1 + f_1(X) \quad \text{on} \quad \{u>0\}\cap\Omega \\ u \geq 0 \text{ on } \Omega \\ u \in C^{1,1}(\overline{\Omega}) \end{cases}$$

Let

$$\Psi_u(r) = \frac{1}{r^{n+2}}\int_{B_r} |\nabla u|^2 + 2u$$

and

$$u^\epsilon(X) = \frac{u(\epsilon X)}{\epsilon^2}$$

Then

$$\Psi_{u^\epsilon}(\frac{r}{\epsilon}) = \Psi_u(r)$$

and by differentiating with respect to ϵ at $\epsilon = 1$, we get

$$-r\left(\Psi_u\right)'(r) + \frac{1}{r^{n+2}}\int_{B_r} 2\nabla u\cdot\nabla U + 2U = 0$$

where

$$U = \frac{d}{d\epsilon}(u^\epsilon)_{|\epsilon=1} = X\cdot\nabla u - 2u$$

i.e.

$$\left(\Psi_u\right)'(r) = \frac{1}{r^{n+3}}\left(\int_{B_r} -2Uf_1 + \int_{\partial B_r} 2U\nabla u\cdot n\right)$$

which can be rewritten

$$\left(\Phi_u\right)'(r) = \frac{2}{r^{n-1}}\int_{\partial B_r} r\left|\frac{d}{dr}\left(\frac{u}{r^2}\right)\right|^2 - \frac{2}{r^{n+3}}\int_{B_r} Uf_1$$

This proves that

$$\Phi_u(r) - \Phi_u(0^+) \geq -2\int_0^r \frac{ds}{s^{n+3}}\int_{B_s} Uf_1$$

Because 0 is assumed to be a singular point, $\Phi_u(0^+) = 2\alpha_n$ and (5.2) can be rewritten here

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r}(u-v)^2\right) = \frac{2}{r}\left(\Phi_u(r) - \Phi_u(0^+)\right) + \frac{2}{r^{n+3}}\int_{B_r\cap\{u>0\}}(u-v)f_1 + \frac{2}{r^{n+3}}\int_{B_r\cap\{u=0\}} v$$

By the nonnegativity of v this implies

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r}(u-v)^2\right) \geq g(r)$$

where

$$g(r) := -\frac{4}{r}\int_0^r \frac{ds}{s^{n+3}}\int_{B_s} U f_1 + \frac{2}{r^{n+3}}\int_{B_r\cap\{u>0\}}(u-v)f_1$$

Let us define the quantity

$$\Xi_u^v(r) := \frac{1}{r^{n+3}}\int_{\partial B_r}(u-v)^2$$

We want to prove that the limit $\Xi_u^v(0^+)$ in $r=0$ exists. To this end, we will use the following result which will be proved later:

Lemma 6.1 *Let us consider a function $h \in C^1\left((0,1);\mathbf{R}\right)$. We assume the existence of a constant $C>0$ and a function $g_0 : (0,1) \to [0,+\infty)$ such that*

$$h' \geq -g_0 \quad \textit{and} \quad h \geq -C$$

If $\int_0^1 g_0(r)dr < \infty$, then the limit $h(0^+) := \lim_{r\to 0^+} h(r)$ exists and is finite.

Because of the $C^{1,1}$ bound on the solution u of (1.5), we see that there exists a constant $C>0$ such that $\Xi_u^v(r) \geq -C$ for all $r \in (0,1)$. As a consequence of lemma 6.1, we see that the limit $\Xi_u^v(0^+)$ exists if the following sufficient condition is satisfied:

$$\int_0^1 |g(r)|\, dr \quad \text{converges} \tag{6.1}$$

Let a modulus of continuity σ defined by

$$|f_1(X)| \leq \sigma(|X|)$$

Then an integration by parts in the variable s allows us to see that condition (6.1) is satisfied if

$$\int_0^1 \frac{dr}{r}\int_0^r \sigma(s)\frac{ds}{s} < +\infty$$

which is in particular true for problem (1.5) because all the coefficients are $C^{0,\alpha}$.
To prove the uniqueness of the blow-up limit, let us remark that for $u^\epsilon(X) = \dfrac{u(\epsilon X)}{\epsilon^2}$, we have

$$\Xi_{u^\epsilon}^v(r) = \Xi_u^v(\epsilon r)$$

Now, up to extraction of a convergent subsequence (ϵ'), we can assume that $u^{\epsilon'}$ tends, uniformly on compact sets of $\mathbf{R}^n$, to a limit u^0 which satisfies

$$\Xi^v_{u^0}(r) = \Xi^v_u(0^+)$$

Using the results of section 2, which are also true for solutions of (1.5) (see for instance [8]), we deduce that

$$u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X$$

where Q is a symmetric matrix $n \times n$ such that $tr\ Q = 1$. Then choosing $v = u^0$, we see that

$$\Xi^v_u(0^+) = \Xi^{u^0}_{u^0}(r) = 0$$

and then every other blow-up limit $\tilde{u}^0$ has to satisfy

$$\Xi^v_{\tilde{u}^0}(r) = \Xi^v_u(0^+) = 0$$

Consequently $\tilde{u}^0 = u^0$, which proves the uniqueness of the blow-up limit. Finally the whole sequence u^ϵ converges to the unique blow-up limit $u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X$. Moreover, because this convergence is uniform on compact sets of $\mathbf{R}^n$, we deduce that

$$u(X) = \frac{1}{2} \quad {}^tX \cdot Q \cdot X + o(X^2)$$

To finish the proof, we only have to state that $D^2u(0)$ exists and is equal to Q. To this end, we remark that ∇u is Lipschitz-continuous because of the $C^{1,1}$ bound on u, and then $D^2u(0)$ is defined by

$$D^2u(0) \cdot X = \lim_{\epsilon \to 0} \left(\frac{\nabla u(\epsilon X) - \nabla u(0)}{\epsilon} \right) \tag{6.2}$$

where $\nabla u(0) = 0$. Because $\dfrac{(\nabla u)(\epsilon X)}{\epsilon} = \nabla (u^\epsilon)(X)$ converges to $\nabla u^0(X) = Q \cdot X$, uniformly on compact sets of $\mathbf{R}^n$, we deduce that the limit in (6.2) exists as ϵ tends to zero, and that $D^2u(0) = Q$. This ends the proof of the theorem.

Remark 6.2 *When condition*

$$\int_0^1 \sigma(s) \frac{ds}{s} < +\infty \tag{6.3}$$

is not satisfied, I. Blank found in [4] an example of non uniqueness of the blow-up limit in a "regular point". Using the symmetry $X \mapsto -X$, it is easy to transform his result in an example of non uniqueness of the blow-up limit in a singular point when condition (6.3) is not satisfied.
We do not know if (6.3) is a sufficient condition to insure the uniqueness of the blow-up limit in singular points.

Proof of lemma 6.1
Let us consider $0 < a < b < 1$. Then by integration we get

$$h(a) \leq h(b) + \int_a^b g_0(r)dr$$

We first take the lim sup in a, on both sides:

$$\limsup_{a \to 0^+} h(a) \leq h(b) + \int_0^b g_0(r)dr \tag{6.4}$$

and then we take the lim inf in b

$$\limsup_{a \to 0^+} h(a) \leq \liminf_{b \to 0^+} h(b) + 0$$

This implies the equality $\liminf_{0^+} h = \limsup_{0^+} h < +\infty$ and then the existence of the limit $h(0^+)$ in zero. We will now prove that this limit is finite. On the one hand, the bound from below comes from the fact that $h \geq -C$, and on the other hand the bound from above is given by (6.4) for a fixed $b \in (0, 1)$. This ends the proof of the lemma.

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