Non-convex coercive Hamilton-Jacobi equations: Guerand's relaxation revisited

Nicolas Forcadel, Cyril Imbert et Régis Monneau

September 14, 2023

Abstract

This work is concerned with Hamilton-Jacobi equations of evolution type posed in domains and supplemented with boundary conditions. Hamiltonians are coercive but are neither convex nor quasi-convex. We analyse boundary conditions when understood in the sense of viscosity solutions. This analysis is based on the study of boundary conditions of evolution type. More precisely, we give a new formula for the relaxed boundary conditions derived by J. Guerand (*J. Differ. Equations*, 2017). This new point of view unveils a connection between the relaxation operator and the classical Godunov flux from the theory of conservation laws. We apply our methods to two classical boundary value problems. It is shown that the relaxed Neumann boundary condition is expressed in terms of Godunov's flux while the relaxed Dirichlet boundary condition reduces to an obstacle problem at the boundary associated with the lower non-increasing envelope of the Hamiltonian.

1 Introduction

When a partial differential equation is posed in a domain, the boundary condition may be in conflict with the equation. This typically happens when characteristics reach the boundary. More specifically, such a phenomenon is observed for evolutive Hamilton-Jacobi (HJ) equations. A classical way to handle this discrepancy is to impose either the boundary condition or the equation at the boundary, both in terms of viscosity solutions. Such viscosity solutions are called *weak*.

The second and third authors studied HJ equations on networks [17] for coercive and convex Hamiltonians. The equations are supplemented with conditions at junctions (vertices). When these conditions are compatible with the maximum principle, it is easy to construct weak viscosity solutions by Perron's method. In this previous work, the authors proved that these weak solutions satisfy *other* boundary (junction) conditions in a strong sense. The family of these *relaxed* boundary conditions is completely characterized by a real parameter, the *flux limiter*.

When Hamiltonians are coercive but not necessarily convex, J. Guerand has shown in the mono-dimensional setting [15] that it is also possible to characterize relaxed boundary conditions associated with general boundary conditions compatible with the maximum principle. In this case, the family of relaxed boundary conditions is much richer and is characterized by a family of *limiter* points. With this tool at hand, she established a comparison principle for general boundary conditions in this framework.

In this work, we are interested in the multi-dimensional case and we treat both dynamic, Neumann and Dirichlet boundary conditions. As far as dynamic boundary conditions are concerned, we give a new formula for the relaxed boundary conditions obtained by J. Guerand. It is easily derived from the definition of weak viscosity solutions. We also exhibit a deeply rooted connection between the relaxed dynamic boundary condition and Godunov's flux for conservation laws. This classical numerical flux also appears in the formula for the relaxed Neumann boundary condition. As far as the Dirichlet boundary condition is concerned, relaxation yields an obstacle problem at the boundary.

1.1 Coercive Hamilton-Jacobi equations posed on domains

In this article, we are interested in the study of Hamilton-Jacobi (HJ) equations of evolution type posed in a C^1 domain Ω of $\mathbb{R}^d$ and supplemented with boundary conditions. We shall see that the study of boundary

conditions of evolution type

$$\begin{cases} u_t + H(t,x,Du) = 0, & t>0, x\in\Omega, \\ u_t + F_0(t,x,Du) = 0, & t>0, x\in\partial\Omega \end{cases} \tag{1.1}$$

is suprisingly fruitful in the understanding of general boundary conditions that are compatible with the maximum principle. In particular, it gives a new insight on the classical inhomogeneous Neumann problem,

$$\begin{cases} u_t + H(t,x,Du) = 0, & t>0, x\in\Omega, \\ \frac{\partial u}{\partial n} + h(t,x) = 0, & t>0, x\in\partial\Omega \end{cases} \tag{1.2}$$

and on the Dirichlet problem,

$$\begin{cases} u_t + H(t,x,Du) = 0, & t>0, x\in\Omega, \\ u = g(t,x), & t>0, x\in\partial\Omega. \end{cases} \tag{1.3}$$

In (1.2) and (1.3), the functions $h, g\colon (0,+\infty)\times\partial\Omega\to\mathbb{R}$ are continuous and $\frac{\partial u}{\partial n}$ denotes the normal derivative associated with the outward unit normal vector field $n\colon \partial\Omega\to\mathbb{R}^d$. Throughout this work, we make the following assumption,

(1.4) $H, F_0\colon (0+\infty)\times\Omega\times\mathbb{R}^d\to\mathbb{R}$ are continuous, $\partial\Omega\in C^1$, F_0 is non-decreasing in $\dfrac{\partial u}{\partial n}$ and H is coercive.

The coercivity of the Hamiltonian H means that $H(t,x,p)$ tends to $+\infty$ as $|p|\to+\infty$. We assume very often that F_0 is *semi-coercive*, that is to say it satisfies the following condition,

$$F_0(t,x,p)\to+\infty \quad \text{as} \quad p\cdot n(x)\to+\infty. \tag{1.5}$$

It is also useful to deal with cases in which this condition on the function F_0 is not satisfied. It is for instance interesting to consider constant F_0 functions.

Weak and strong viscosity solutions for HJ equations posed in domains. It is known that classical solutions to Hamilton-Jacobi equations do not exist in general while viscosity solutions are easily constructed by Perron's method [22]. As far as boundary conditions are concerned, because characteristics can exit the domain, boundary conditions are generally "lost" for Hamilton-Jacobi equations. As first observed by H. Ishii [21], it is useful to consider viscosity solutions that satisfy either the equation or the boundary condition on $\partial\Omega$. Such viscosity solutions are called weak. They are easily constructed thanks to Perron's method [22]. On the contrary, if the boundary condition is always satisfied on $\partial\Omega$, we say that viscosity solutions are strong. It is usually easier to prove uniqueness of strong viscosity solutions than to prove uniqueness of weak ones.

In this article, it is proved that weak viscosity solutions associated with (1.1) or (1.2) or (1.3) are strong viscosity solutions for *other* boundary conditions that we identify. We start with (1.1).

Theorem 1.1 (Relaxed boundary condition – [17, 18, 15])**.** *Assume that $H, F_0\colon \mathbb{R}^d\to\mathbb{R}$ are continuous, $p\mapsto F_0(p)$ is non-decreasing with respect to $p\cdot n$, H is coercive and F_0 is semi-coercive (in the sense of (1.5)).*

Then, there exists a continuous semi-coercive function $\mathfrak{R}F_0\colon\mathbb{R}^d\to\mathbb{R}$ such that a function $u\colon (0,+\infty)\times\Omega$ is a weak viscosity solution of (1.1) if and only if it is a strong viscosity solution of

$$\begin{cases} u_t + H(t,x,Du) = 0, & t>0, x\in\Omega, \\ u_t + \mathfrak{R}F_0(t,x,Du) = 0, & t>0, x\in\partial\Omega. \end{cases}$$

If F_0 is not semi-coercive, the result still holds true if u satisfies a weak continuity assumption at the boundary of $\partial\Omega$: for all $x\in\partial\Omega$ and $t>0$,

$$u^*(t,x) = \limsup_{(s,y)\to(t,x), y\in\Omega} u(s,y),$$

see Theorem 3.14 in Section 3.

The application mapping F_0 to $\mathfrak{R}F_0$ is referred to as the *relaxation operator*. Theorem 1.1 was proved by the second and the third authors [17, 18] under the additional assumption that the Hamiltonian H is convex and Ω is a half-space. In this case, the relaxation operator takes a very simple form since $\mathfrak{R}F_0$ is the maximum of a constant A_0 (depending on H and F_0) and the the lower non-increasing envelope of H given by the formula $H_-(t,x,p) := \inf_{\rho\le 0} H(t,x,p-\rho n(x))$. When Hamiltonians are coercive but are not convex, J. Guerand [15] identified the relaxation operator in the monodimensional setting. The formula she obtained for $\mathfrak{R}F_0$ is referred in this article as Guerand's operator and is denoted by $\mathfrak{J}F_0$; it is given in Definition 4.3.

1.2 A new formula for the relaxation operator

The first main result of this article is a new formula for the relaxation operator. We first present it in the mono-dimensional setting for the sake of clarity.

1.2.1 The homogeneous mono-dimensional case

To simplify the presentation, we assume here that $\Omega = (0,+\infty)$ and H and F_0 don't depend on (t,x). Let u be a (continuous) weak viscosity solution to (1.1). As explained above, this means that either the equation or the boundary condition is satisfied in the sense of viscosity solutions; see Definition 3.1 for a precise definition. Consequently, if φ is a test function touching u from above at $P_0 = (t_0, 0)$, then

$$\varphi_t + H(\varphi_x) \le 0 \quad \text{or} \quad \varphi_t + F_0(\varphi_x) \le 0 \quad \text{at } P_0$$

or equivalently $\varphi_t + (F_0 \wedge H)(\varphi_x) \le 0$ at P_0 where $F_0 \wedge H$ denotes the minimum of F_0 and H. Keeping in mind the discussion above about weak and strong viscosity solutions, we obtained a first boundary condition that is satisfied in a strong sense.

We next derive a more precise one. For any $q \ge \varphi_x(P_0) =: p$, the test function $\tilde\varphi(t,x) = \varphi(t,x) + (q-p)\cdot x \ge u(t,x)$ also touches u at P_0 from above. In particular, we also have $\varphi_t + (F_0 \wedge H)(q) \le 0$ at P_0. We conclude that

$$\varphi_t + \underline{R}F_0(\varphi_x) \le 0 \quad \text{at } P_0$$

where the operator $\underline{R}$ is defined by

$$\underline{R}F_0(p) := \sup_{q\ge p} (F_0 \wedge H)(q). \tag{1.6}$$

Similarly if a test function φ touches a weak solution u of (1.1) from below at P_0, we get

$$\varphi_t + \overline{R}F_0(\varphi_x) \ge 0 \quad \text{at } P_0$$

where the operator $\overline{R}$ is defined by

$$\overline{R}F_0(p) := \inf_{q\le p} (F_0 \vee H)(q). \tag{1.7}$$

We refer the reader to Figure 1 for a representation of the effects of $\underline{R}$ and $\overline{R}$ on F_0.

We next remark that $\overline{R}F_0 = \underline{R}F_0 = F_0$ in $\{F_0 = H\}$ (see Remark 2.2 below). We define the relaxation operator $\mathfrak{R}F_0$ as follows,

$$\mathfrak{R}F_0 = \begin{cases} \underline{R}F_0 & \text{in } \{F_0 \ge H\}, \\ \overline{R}F_0 & \text{in } \{F_0 \le H\}. \end{cases} \tag{1.8}$$

We refer the reader to Figure 1 for a representation of the effects of $\mathfrak{R}$ on F_0.

Example 1.2. In the totally degenerate case, *i.e.* in the case where F_0 is constant, the relaxed boundary function $\mathfrak{R}F_0$ is given by,

$$\mathfrak{R}F_0 = \max(A, H_-) \quad \text{when} \quad F_0 \equiv const = A \quad \text{with} \quad H_-(p) := \inf_{(-\infty,p]} H.$$

This computation is used in the derivation of the relaxed Dirichlet condition, see the proof of Theorem 1.6.

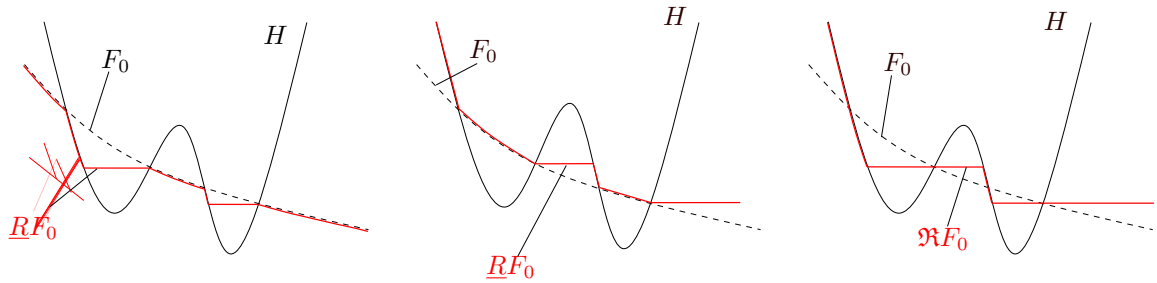

Figure 1: Effects of $\underline{R}$, $\overline{R}$ and $\mathfrak{R}$ on F_0. The Hamiltonian H is represented with a plain line, while a dashed line is used for the function F_0. The relaxation operators appear in red. We see that $\underline{R}F_0 \leq F_0$ while $\overline{R}F_0 \geq F_0$. We can also observe that $\mathfrak{R}F_0 = \underline{R}F_0$ in $\{F_0 \geq H\}$ and $\mathfrak{R}F_0 = \overline{R}F_0$ in $\{F_0 \leq H\}$.

The first main result of this work states that Guerand's relaxation operator coincides with the one defined by (1.8).

Theorem 1.3 (Guerand's operator and the relaxation operator coincide)**.** *Assume $H, F_0 \colon \mathbb{R} \to \mathbb{R}$ are continuous, H is coercive and F_0 is non-increasing and semi-coercive (in the sense of (1.5)). Then we have $\mathfrak{R}F_0 = \mathfrak{J}F_0$.*

Remark 1.4. The definition of Guerand's operator $\mathfrak{J}$ is recalled in Section 4, see Definition 4.3.

1.2.2 The heterogeneous multidimensional setting

If dimension is larger than or equal to 2, then the relaxation operator can be defined by freezing tangential variables. More precisely, if $x \in \partial\Omega$ and n denotes the outward unit normal, then $p \in \mathbb{R}^d$ is split into $p = p' - rn$ for $p' \perp n$ and $r \in \mathbb{R}$. Then

$$\bar{H}(r) = H(t, x, p' - rn) \quad \text{and} \quad \bar{F}_0(r) = F_0(t, x, p' - rn).$$

We then define $\mathfrak{R}F_0(t, x, p', r) = \mathfrak{R}\bar{F}_0(r)$ where the relaxation operator in the right hand side is computed with respect to the coercive Hamiltonian $\bar{H}$ and defined in (1.8).

We remark that the multi-dimensional relaxation operators can be written as,

$$\begin{cases} \underline{R}F_0(t, x, p) & = \sup_{\rho \geq 0} \ (F_0 \wedge H)(t, x, p - \rho n), \\ \overline{R}F_0(t, x, p) & = \inf_{\rho \leq 0} \ (F_0 \vee H)(t, x, p - \rho n). \end{cases} \tag{1.9}$$

1.3 The Neumann and Dirichlet problems

We now turn to the study of weak viscosity solutions of the Neumann problem.

Theorem 1.5 (From Neumann to Godunov)**.** *Any function $u \colon (0, T) \times \Omega \to \mathbb{R}$ is a weak solution of Neumann problem (1.2) if and only if it is a strong solution of*

$$\begin{cases} u_t + H(t, x, Du) = 0, & t \in (0, T), x \in \Omega, \\ u_t + N(t, x, Du) = 0, & t \in (0, T), x \in \partial\Omega \end{cases}$$

where N is the classical Godunov flux associated to the Hamiltonian $\rho \mapsto H(t,x,p-\rho n)$,

$$N(t,x,p) = \begin{cases} \max\left\{H(t,x,p-\rho n) : \rho \in [0, p\cdot n(x) + h(t,x)]\right\} & \text{if } p\cdot n(x) + h(t,x) \geq 0, \\ \min\left\{H(t,x,p-\rho n) : \rho \in [p\cdot n(x) + h(t,x), 0]\right\} & \text{if } p\cdot n(x) + h(t,x) \leq 0. \end{cases}$$

We remark that in dimension 1 (taking $\Omega = (0,+\infty)$ to simplify), Theorem 1.5 can be expressed in terms of Godunov's flux. Indeed, when H and h do not depend on (t,x), we get $N(p) = G(h,p)$, where G is the classical Godunov's flux defined later in (1.10), and the weak Neumann boundary condition is relaxed in $u_t + G(h,u_x) = 0$. This formulation seems very natural; indeed, at the level of the conservation law, it is expected that the spatial derivative $v := u_x$ (at least formally) is an entropy solution of

$$\begin{cases} v_t + H(v)_x = 0 \quad , & \text{for} \quad x > 0, \\ v(t,0) \in \mathcal{G}_h, & \text{for a.e.} \quad t \in (0,+\infty). \end{cases}$$

where the set $\mathcal{G}_h$ is given by[1]

$$\mathcal{G}_h = \{p \in \mathbb{R}, \quad H(p) = G(h,p)\}.$$

It is easy to check that we have

$$\mathcal{G}_h = \{p \in \mathbb{R}, \quad \{\text{sign}(p-k) - \text{sign}(h-k)\}\cdot\{H(p) - H(k)\} \leq 0 \quad \text{for all } k \in \mathbb{R}\}$$

which is nothing else that the well-known Bardos-Leroux-Nedelec (BLN) condition. This (BLN) condition that has been identified in [5], as the natural effective condition associated to the desired Dirichlet condition for scalar conservation laws, in the vanishing viscosity limit.

In our weak/strong terminology, this shows in this example, that (BLN) condition is a strong boundary condition associated to the weak Dirichlet boundary condition $v(t,0) = h$. Here the Dirichlet condition can not always be satisfied strongly. In other words, in this example, we see that relaxation of the boundary condition at the Hamilton-Jacobi level, selects the right choice of the effective boundary condition that is indeed satisfied strongly by a solution.

We refer the reader to Subsection 6.2, for a further discussion on the relation between Hamilton-Jacobi equations with boundary conditions and scalar conservation laws with (Dirichlet type) boundary conditions.

Notice that the Neumann problem has been adressed independently by P.-L. Lions and P. Souganidis [26] in the monodimensional case and the second author with V. D. Nguyen [20] in the case where the Hamiltonian is convex and the domain Ω is a half-space. In both works, the geometric setting corresponds to junctions and the junction conditions of Kirchoff type can be handled. These conditions generalize the Neumann boundary condition to the junction setting.

As far as the Dirichlet problem is concerned, the relaxed boundary condition turns out to be an obstacle problem.

Theorem 1.6 (Dirichlet to boundary obstacle problem)**.** *Consider a function $u\colon (0,T)\times\Omega \to \mathbb{R}$ which is weakly continuous at (t,x) for all $t > 0$ and $x \in \partial\Omega$, i.e.*

$$u^*(t,x) = \limsup_{(s,y)\to(t,x), y\in\Omega} u(s,y),$$

Then u is a weak solution of Dirichlet problem (1.3) if and only if it is a strong solution of

$$\begin{cases} u_t + H(t,x,Du) = 0, & t \in (0,T), x \in \Omega, \\ \max\{u - g, u_t + H_-(t,x,Du)\} = 0, & t \in (0,T), x \in \partial\Omega \end{cases}$$

[1]Notice that it is possible to show (similarly to the proof of Lemma 6.1 below) that $u(t,x) = px + \lambda t$ is a weak Neumann solution to (1.2), if and only if $\lambda = -H(p)$ and

$$(p-h)\cdot\{G(h,p) - H(p)\} \geq 0$$

which is easily seen to be equivalent to $p \in \mathcal{G}_h$.

where $n\colon \partial\Omega \to \mathbb{R}^d$ is the outward unit normal vector field and

$$H_-(t,x,p) = \inf_{\rho\le 0} H(t,x,p-\rho n(x)).$$

1.4 Godunov's relaxation

We show that relaxation is directly related to the classical Godunov's flux. For the sake of simplicity, we present it in the monodimensional setting. We recall that this "numerical" flux is defined for $p,q\in\mathbb{R}$ by

$$G(q,p) = \begin{cases} \max_{[p,q]} H & \text{if } p\le q,\\ \min_{[q,p]} H & \text{if } p\ge q. \end{cases} \tag{1.10}$$

Theorem 1.7 (Relaxation coincides with Godunov's relaxation)**.** *Assume $H, F_0\colon \mathbb{R}\to\mathbb{R}$ are continuous, H is coercive and F_0 is non-increasing and semi-coercive. Then for any $p\in\mathbb{R}$, there is one and only one $\lambda\in\mathbb{R}$ such that there exists $q\in\mathbb{R}$ with $\lambda = F_0(q) = G(q,p)$. If F_0G denotes the map $p\mapsto\lambda$, then it coincides with the relaxation operator,*

$$\mathfrak{R}F_0 = F_0G.$$

Remark 1.8. For some technical reasons that will appear in the proof of this result, it makes more sense to define the action of Godunov's flux G on the right of F_0 (rather than on the left).

1.5 Comments

Self-relaxed boundary conditions. We will see that the relaxed boundary condition cannot be further relaxed, *i.e.* it satisfies $\mathfrak{R}(\mathfrak{R}F_0) = \mathfrak{R}F_0$. When a function F_0 satisfies $F_0 = \mathfrak{R}F_0$, then we say that it is *self-relaxed*.

The lower non-increasing envelope of the Hamiltonian. The lower non-increasing envelope H_- of H satisfies semi-coercivity condition (1.5), it is self-relaxed, and for any boundary function F_0 satisfying (1.4), we have

$$\mathfrak{R}F_0 \ge H_-.$$

In other words, H_- is the minimal self-relaxed boundary function. It corresponds to the natural condition that appears for state contraint problems with convex Hamiltonians, and can be seen as a sort of generalization of it to the case of non-convex and coercive Hamiltonian. The previous inequality implies that every continuous weak F_0-subsolution is indeed a strong H_--subsolution (see Proposition 3.12). This explains (at least for a junction with a single branch) the observation made by P.-L. Lions and P. Souganidis [25] that only the supersolution condition has to be checked at the junction point. In other words, it is sufficient to check that the function is a weak (or strong) H_--supersolution.

Weak continuity condition. Notice that when F_0 does not satisfy the semi-coercivity condition, it is necessary to impose a weak continuity condition on the boundary,

$$\forall (t_0,x_0)\in(0,+\infty)\times\partial\Omega,\quad u^*(t_0,x_0) = \limsup_{(s,y)\to(t_0,x_0),\, y\in\Omega} u(s,y).$$

to ensure that the conclusion of Theorem 1.1 holds true. If none of these conditions is satisfied, then the conclusion may be wrong, as shown in the counter-example 3.16 below. It is due to J. Gerrand. Such a weak continuity condition appears for instance in the work by G. Barles and B. Perthame [10] in which they prove comparison principle for discontinuous viscosity solutions of the Dirichlet problem (in the stationary case). Such a condition also appears in [17] and in subsequent papers.

The stationary case. A version of Theorem 1.1 is still valid without changes in the definition of $\mathfrak{R}F_0$ for stationary equations like

$$\begin{cases} u + H(x,Du) = 0, & \text{for} \quad x\in\Omega,\\ u + F_0(x,Du) = 0, & \text{for} \quad x\in\partial\Omega \end{cases}$$

with adapted assumptions on H, F_0, and naturally adapted definitions of weak and strong viscosity solutions.

1.6 Review of literature and known results

Boundary conditions for viscosity solutions. The Dirichlet problem is considered in the first papers dealing with viscosity solutions, see [13, 14, 12]. We mentioned above that the weak continuity condition first appears in [10] where the authors prove a comparison principle for discontinuous viscosity solutions of HJ equations with Dirichlet boundary conditions. In this article, the boundary condition is imposed in the generalized sense recalled earlier.

The state-constraint condition is a boundary condition that has been identified early in the literature when Hamiltonians are convex. H. M. Soner [30] proved a general uniqueness result by constructing a special test function pushing contact points inside the domain. As far as the Neumann boundary condition is concerned, it has been first adressed by P.-L. Lions [24] for Hamiltonians that are not necessarily convex.

This first result for the Neumann boundary condition has been generalized later by G. Barles [6]. In this work, he also constructed a test function à la Soner. The Neumann boundary condition is easily interpreted in the optimal control setting.

Convex Hamiltonians and optimal control. In 2007, A. Bressan and Y. Hong studied optimal control problems on stratified domains. The case of junctions is the simplest geometric setting of stratified domains. For such a geometry, two groups of authors studied convex Hamilton-Jacobi equations: Y. Achdou, F. Camilli, A. Cutri and N. Tchou [1] on the one hand and the second and third authors together with H. Zidani [19] on the other hand. At the same time, in the two domains setting, G. Barles, A. Briani and E. Chasseigne [7, 8] developped an intermediate approach mixing PDE and optimal control tools for convex Hamiltonians.

In the monodimensional setting, solutions of a HJ equation are naturally associated with solutions of the corresponding scalar conservation law. In the two domains setting, B. Andreianov, K. H. Karlsen, N. H. Risebro [3] developped a theory for existence and uniqueness from which the second and third authors took inspiration to write [17]. We also mention the work by B. Andreianov and K. Sbihi [4] for the one domain problem in great generality.

Later, the second and third authors [17] introduced the notion of flux-limited solutions and cook up a PDE method generalizing the method of doubling of variables to prove comparison principles. The case of networks is treated in [17] while [18] is concerned with multi-dimensional junctions. They observed that the state constraint boundary conditions can be interpreted in terms of flux limiters, see [17, Proposition 2.15]. J. Guerand treated the multidimensional case of state constraints in [16]. We also mention that the second author together with V. D. Nguyen [20] addressed the case of parabolic equations degenerating to Hamilton-Jacobi equations at the (multi-dimensional) junction.

In [28], Z. Rao, A. Siconolfi and H. Zidani adopted a pure optimal control approach to deal with accumulation of components. More recently, A. Siconolfi [29] proposed another PDE method based on the notion of maximal subsolutions under trace constraints to prove a comparison principle on networks without loops. Hamiltonians are convex and depend on the space variable and the uniqueness result holds true for uniformly continuous sub/supersolutions.

Motivated by the study of a homogeneization problem, the notion of flux-limited solutions has also been extended by Y. Achdou and C. Le Bris [2] for a convex HJ problem in $\mathbb{R}^d \setminus \{0\}$ supplemented with a condition at the origin.

These works have been extended mainly for optimal control problems on stratified domains by Barles, Chasseigne [11] (see also the recent work of Jerhaoui, Zidani [23]), and recently by the same authors in a book Barles, Chasseigne [9] which is a reference book on the topic, including boundary conditions, junction problems in any dimensions, stratified problems, in particular in relation with optimal control problems and convex Hamiltonians.

Non-convex Hamiltonians. J. Guerand [15] proved comparison principles for non-convex HJ equations of evolution type posed in the half real line. She adressed both the coercive and non-coercive cases. In order to prove such uniqueness results, she introduced a relaxation operator $\mathfrak{J}$ and proved the equivalence between weak and strong solutions.

P.-L. Lions and P. Souganidis [25, 26] also studied Hamilton-Jacobi equations posed on junctions in the non-convex case. In particular, they introduced a blow-up method to prove the comparison principle between

bounded uniformly continuous sub- and super-solutions.

1.7 Organisation of the article

In Section 2, we present the main properties of the relaxation operator $\mathfrak{R}$ and introduce the notion of characteristic points. In Section 3, we discuss relations between weak and strong (viscosity) solutions and propose a new proof of Theorem 1.1 (see Theorem 3.14). In this section, we also discuss existence and stability of weak viscosity solutions. In Section 4, we recall Guerand's relaxation formula, and show that it is equivalent to the new relaxation formula (Theorem 1.3). In Section 5, we introduce Godunov's relaxation formula, and show that it is equivalent to the new relaxation formula (Theorem 1.7). In Section 6, we treat the case of Neumann and Dirichlet boundary conditions and prove Theorems 1.5 and 1.6. We also discuss the link between the relaxation operator for HJ equations and scalar conservation laws.

Notation. For $a, b \in \mathbb{R}$, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

2 Relaxation operators and characteristic points

We recall that we always assume that H, F_0 satisfy (1.4). In this section, we discuss properties of the relaxation operators. For clarity, time, space and tangential variables are omitted throughout this section.

2.1 Relaxation operators

We begin by some properties on the sub and super-relaxation operators. We recall that there are defined respectively in (1.6) and (1.7) and we refer to Figure 1 for a representation of the action of these operators on the function F_0.

Lemma 2.1 (First properties of the operators $\underline{R}$ and $\overline{R}$)**.** *Assume (1.4). Then the functions $\underline{R}F_0$ and $\overline{R}F_0$ are continuous, nonincreasing and semi-coercice, and*

$$F_0 \wedge H \leq \underline{R}F_0 \leq F_0 \leq \overline{R}F_0 \leq F_0 \vee H$$

and $\underline{R}(\underline{R}F_0) = \underline{R}F_0$ and $\overline{R}(\overline{R}F_0) = \overline{R}F_0$.

Remark 2.2. We will use repeatedly the following easy consequences of this lemma: $\{F_0 \leq H\} \subset \{\underline{R}F_0 = F_0\}$ and $\{F_0 \geq H\} \subset \{\overline{R}F_0 = F_0\}$.

Proof. We only justify the properties satisfied by $\underline{R}F_0$ since proofs for $\overline{R}F_0$ are similar. We have by definition

$$(F_0 \wedge H)(p) \leq \underline{R}F_0(p) = \sup_{q \geq p}(F_0 \wedge H)(q) \leq \sup_{q \geq p} F_0(q) = F_0(p)$$

where we have used the monotonicity of F_0. Moreover, by construction, $\underline{R}F_0$ is nonincreasing and continuous. The fact that H is coercive and F_0 is semi-coercive implies that $F_0 \wedge H$ is also semi-coercive, and then $\underline{R}F_0$ is semi-coercive.

We set $F := \underline{R}F_0$. On the one hand, by coercivity of H, there exists some minimal $q^* \geq p$ such that

$$F(p) = \underline{R}F_0(p) = (F_0 \wedge H)(q^*).$$

Since $\underline{R}F$ is non-increasing, $q^* \geq p$ and $F(q^*) \geq (F_0 \wedge H)(q^*)$, we have

$$F(p) \geq \underline{R}F(p) \geq \underline{R}F(q^*) \geq (F \wedge H)(q^*) \geq (F_0 \wedge H)(q^*) = F(p).$$

The previous inequalities imply in particular that $F = \underline{R}F$. □

We now define the *relaxation operator*

$$(\mathfrak{R}F_0)(p) := \begin{cases} \underline{R}F_0(p) & \text{if} \quad F_0(p) \geq H(p) \\ \overline{R}F_0(p) & \text{if} \quad F_0(p) \leq H(p). \end{cases} \tag{2.1}$$

In particular, it satisfies $|\mathfrak{R}F_0 - H| \leq |F_0 - H|$ as we will show next. In this sense, we see that $\mathfrak{R}F_0$ is closer to H than F_0 itself.

Lemma 2.3 (Nice properties of the operator $\mathfrak{R}$)**.** *Assume* (1.4)*. The function $F := \mathfrak{R}F_0$ is well-defined, continuous, non-increasing, semi-coercive and satisfies*

$$\begin{cases} \underline{R}F = F = \overline{R}F \\ \mathfrak{R}F = F \end{cases}$$

$$F = \overline{R}(\underline{R}F_0) = \underline{R}(\overline{R}F_0)$$

$$\begin{cases} F_0 \leq H & \Longrightarrow \quad F_0 \leq \mathfrak{R}F_0 \leq H, \\ F_0 \geq H & \Longrightarrow \quad F_0 \geq \mathfrak{R}F_0 \geq H. \end{cases} \tag{2.2}$$

For H_- given by $H_-(p) = \inf_{q \leq p} H(q)$, the function $F_1 := F_0 \vee H_-$ is semi-coercive and satisfies

$$\mathfrak{R}F_0 = \mathfrak{R}F_1.$$

Proof. The proof is split in several steps.

Step 1: preliminaries. We first notice that from Lemma 2.1, we have

$$\begin{cases} F_0(p) \leq H(p) & \Longrightarrow \quad \underline{R}F_0 = F_0 \leq \overline{R}F_0 \leq H & \text{at} \quad p \\ F_0(p) \geq H(p) & \Longrightarrow \quad \overline{R}F_0 = F_0 \geq \underline{R}F_0 \geq H & \text{at} \quad p. \end{cases} \tag{2.3}$$

This implies that

$$F_0(p) = H(p) \quad \Longrightarrow \quad \overline{R}F_0 = F_0 = \underline{R}F_0 = H \quad \text{at} \quad p.$$

Hence the definition of $\mathfrak{R}F_0$ is equivalent to the following one,

$$(\mathfrak{R}F_0)(p) := \begin{cases} \underline{R}F_0(p) & \text{if} \quad F_0(p) > H(p), \\ F_0(p) & \text{if} \quad F_0(p) = H(p), \\ \overline{R}F_0(p) & \text{if} \quad F_0(p) < H(p). \end{cases}$$

In particular we see that $F := \mathfrak{R}F_0$ is continuous, non-increasing and semi-coercive.

Step 2: Effect of the operators on $F = \mathfrak{R}F_0$. We have

$$\underline{R}F_0 \leq F = \mathfrak{R}F_0 \leq \overline{R}F_0.$$

Hence, thanks to Lemma 2.1 and the previous step,

$$\begin{aligned} F \geq \underline{R}F = \mathfrak{R}F \geq \underline{R}(\underline{R}F_0) = \underline{R}F_0 = F \quad &\text{in} \quad \{F \geq H\} \\ F \leq \overline{R}F = \mathfrak{R}F \leq \overline{R}(\overline{R}F_0) = \overline{R}F_0 = F \quad &\text{in} \quad \{F \leq H\}. \end{aligned}$$

This implies that $\mathfrak{R}F = F$. Moreover (2.3) implies that $\{F \leq H\} \subset \{\underline{R}F = F\}$ and $\{F \geq H\} \subset \{\overline{R}F = F\}$. We thus also get $\underline{R}F = F = \overline{R}F$.

Step 3: $\overline{R}(\underline{R}F_0) = F$ and $\underline{R}(\overline{R}F_0) = F$. We only prove the first equality since the proof of the second one is very similar. It amounts to prove that

$$\overline{R}(\underline{R}F_0) = \begin{cases} \underline{R}F_0 & \text{in} \quad \{F_0 > H\} \\ F_0 & \text{in} \quad \{F_0 = H\} \\ \overline{R}F_0 & \text{in} \quad \{F_0 < H\}. \end{cases}$$

The equality in the set $\{F_0 = H\}$ follows directly from Lemma 2.1.

To check this equality in $\{F_0 > H\}$, we recall that Lemma 2.1 implies that $\{F_0 > H\} \subset \{\underline{R}F_0 \geq H\}$, and then by Remark 2.2, we get

$$\overline{R}(\underline{R}F_0) = \underline{R}F_0 \quad \text{on} \quad \{F_0 > H\}.$$

To check this equality in $\{F_0 < H\}$, we consider some maximal interval $(a, b) \subset \{F_0 < H\}$. Assume first that $a > -\infty$. In this case, we have $F_0(a) = H(a)$. Recalling that $\{F_0 \leq H\} \subset \{\underline{R}F_0 = F_0\}$ (see Lemma 2.1), we get that for $p \in (a, b)$,

$$\begin{aligned} \overline{R}(\underline{R}F_0)(p) &= \inf_{q \leq p} (\underline{R}F_0 \vee H)(q) \\ &= \min \left\{ \inf_{q \in [a,p]} (\underline{R}F_0 \vee H)(q), H(a) \right\} \\ &= \min \left\{ \inf_{q \in [a,p]} (F_0 \vee H)(q), H(a) \right\} \\ &= \inf_{q \leq p} (F_0 \vee H)(q) \\ &= \overline{R}F_0(p). \end{aligned}$$

Assume now that $a = -\infty$. Then the same computation works with $a = -\infty$, $F_0(a) = H(a) = +\infty$, and $[a, p]$ replaced by $(-\infty, p]$.

Step 4: Proof of (2.2). Combining (2.3) and the fact that $\mathfrak{R}F = \overline{R}(\underline{R}F_0) = \underline{R}(\overline{R}F_0)$, we get the desired result.

Step 5: properties of F_1. We have

$$\overline{R}F_0(p) = \inf_{q \leq p} (F_0 \vee H)(q)$$

and since $H_- \leq H$, the function $F_1 = F_0 \vee H_-$ satisfies

$$\overline{R}F_1(p) = \inf_{q \leq p} ((F_0 \vee H_-) \vee H)(q) = \inf_{q \leq p} (F_0 \vee H)(q) = \overline{R}F_0(p).$$

Hence $\mathfrak{R}F_1 = \underline{R}(\overline{R}F_1) = \underline{R}(\overline{R}F_0) = \mathfrak{R}F_0$. Finally F_1 inherits semi-coercivity from H_-. □

We now have the following tools.

Lemma 2.4 (Optimality and local properties of $\underline{R}F_0$)**.** *Let $p \in \mathbb{R}$.*

(i) (Optimality properties) *Let $q \geq p$ be minimal such that*

$$\underline{R}F_0(p) = (F_0 \wedge H)(q).$$

If $F_0(p) \geq H(p)$ then

$$\begin{cases} F_0(q) \geq H(q) \\ \underline{R}F_0 = H(q) \quad \text{in} \quad [p, q] \\ H < H(q) \quad \text{in} \quad [p, q). \end{cases}$$

(ii) (Local properties) *If $\underline{R}F_0(p) > H(p)$ then $\underline{R}F_0$ is constant in $[p - \varepsilon, p + \varepsilon)$ for some $\varepsilon > 0$.*

Lemma 2.5 (Optimality and local properties of $\overline{R}F_0$)**.** *Let $p \in \mathbb{R}$.*

(i) (Optimality properties) *Let $q \leq p$ be maximal such that*

$$\overline{R}F_0(p) = (F_0 \vee H)(q).$$

If $F_0(p) \leq H(p)$, then

$$\begin{cases} F_0(q) \leq H(q) \\ \overline{R}F_0 = H(q) \quad \text{in} \quad [q,p] \\ H > H(q) \quad \text{in} \quad (q,p]. \end{cases}$$

(ii) (Local properties) *If $\overline{R}F_0(p) < H(p)$ then $\overline{R}F_0$ is constant in $(p-\varepsilon, p+\varepsilon]$ for some $\varepsilon > 0$.*

As an immediate consequence of Lemmas 2.4 and 2.5 (using moreover definition (2.1)), we get

Corollary 2.6 (Local properties of $\mathfrak{R}F_0$)**.** *If $\mathfrak{R}F_0(p) \neq H(p)$, then $\mathfrak{R}F_0$ is constant in a neighbourhood of p.*

We only do the proof of Lemma 2.4 since the proof of Lemma 2.5 is very similar.

Proof of Lemma 2.4. The proof is split in two steps.
Optimality properties. We assume that $F_0(p) \geq H(p)$ and $q \geq p$ is minimal such that

$$\underline{R}F_0(p) = (F_0 \wedge H)(q).$$

Using the coercivity of H and the monotonicity of F_0, let us define $q_0 \in [p, +\infty)$ such that

$$q_0 := \sup\left\{q' \geq p, \quad F_0 \geq H \text{ in } [p, q']\right\}.$$

It satisfies $F_0(q_0) = H(q_0) = \underline{R}F_0(q_0)$ (see Lemma 2.1) and $q_0 \geq p$.
We observe first that $q \in [p, q_0]$. Indeed,

$$\begin{aligned} \underline{R}F_0(p) &= \max\left(\max_{q' \in [p,q_0]} H(q'), \underline{R}F_0(q_0)\right) \\ &= \max\left(\max_{q' \in [p,q_0]} H(q'), H(q_0)\right) \\ &= \max_{q' \in [p,q_0]} H(q'). \end{aligned}$$

We thus conclude that the maximum is reached for $q' \in [p, q_0]$ and since q is minimal, we get $q \in [p, q_0]$.
The fact that $q \leq q_0$ implies that $F_0(q) \geq H(q)$.
Since $H(q) = \max_{[p,q_0]} H$, we also get from the minimality of q that $H < H(q)$ in $[p, q)$.
To finish with, monotonicity of $\underline{R}F_0$ implies that for any $q' \in [p, q)$,

$$\underline{R}F_0(q) \leq \underline{R}F_0(q') \leq \underline{R}F_0(p) = (F_0 \wedge H)(q) = H(q) \leq \underline{R}F_0(q).$$

This series of inequalities yields that $\underline{R}F_0$ is constant, equal to $H(q)$.

Local properties. Keeping in mind that $(F_0 \wedge H) \leq \underline{R}F_0 \leq F_0$, if $\underline{R}F_0(p) > H(p)$ then $F_0(p) \geq \underline{R}F_0(p) = H(q) > H(p)$, with q defined above. This implies that $q > p$ and so $\underline{R}F_0$ is constant in $[p, q]$. Using the monotonicity of F_0 and the continuity of H, we get also that there exists $\varepsilon > 0$ such that

$$H < H(q) \leq F_0 \quad \text{on } [p-\varepsilon, p].$$

Using the monotonicity of $\underline{R}F_0$, this implies, for all $p' \in [p-\varepsilon, p]$, that

$$\underline{R}F_0(q) \leq \underline{R}F_0(p') = \max(\sup_{q' \in [p',p]} (F_0 \wedge H)(q'), \underline{R}F_0(p)) \leq \max(H(q), \underline{R}F_0(q)) = \underline{R}F_0(q).$$

Hence $\underline{R}F_0$ is constant in $[p-\varepsilon, q]$, with $q > p$. This yields the desired local property. $\square$

Lemma 2.7 (Commutation of max/min with $\mathfrak{R}$)**.** *Assume that H is continuous and coercive. If F_a, F_b are continuous non-increasing, then*

$$\mathfrak{R}(F_a \wedge F_b) = (\mathfrak{R}F_a) \wedge (\mathfrak{R}F_b) \quad \textit{and} \quad \mathfrak{R}(F_a \vee F_b) = (\mathfrak{R}F_a) \vee (\mathfrak{R}F_b).$$

Proof. We only prove $\mathfrak{R}(F_a \wedge F_b) = (\mathfrak{R}F_a) \wedge (\mathfrak{R}F_b)$ (the proof of the other relation with the max is similar).
Step 1: commutation of min with $\overline{R}$. We have

$$\overline{R}(F_a \wedge F_b)(p) = \inf_{(-\infty,p]} (F_a \wedge F_b) \vee H = \inf_{(-\infty,p]} (F_a \vee H) \wedge (F_b \vee H) = \left\{ \inf_{(-\infty,p]} (F_a \vee H) \right\} \wedge \left\{ \inf_{(-\infty,p]} (F_b \vee H) \right\}$$

i.e.

$$\overline{R}(F_a \wedge F_b) = (\overline{R}F_a) \wedge (\overline{R}F_b).$$

Step 2: commutation of min with $\underline{R}$. We first notice that

$$\underline{R}(F_a \wedge F_b) \leq \underline{R}F_a, \underline{R}F_b \quad \text{i.e.} \quad \underline{R}(F_a \wedge F_b) \leq (\underline{R}F_a) \wedge (\underline{R}F_b).$$

Now we want to prove the reverse inequality. For $c = a, b$, let $q_c^* \geq p$ be minimal such that $\underline{R}F_c(p) = (F_c \wedge H)(q_c^*)$. Setting

$$q^* := q_a^* \wedge q_b^*,$$

we get using the monotonicities of F_a, F_b

$$H(q^*) \geq H(q_a^*) \wedge H(q_b^*), \quad F_a(q^*) \geq F_a(q_a^*), \quad F_b(q^*) \geq F_b(q_b^*).$$

Hence

$$\underline{R}(F_a \wedge F_b)(p) = \sup_{[p,+\infty)} F_a \wedge F_b \wedge H \geq (F_a \wedge F_b \wedge H)(q^*) \geq (F_a \wedge H)(q_a^*) \wedge (F_b \wedge H)(q_b^*) = (\underline{R}F_a) \wedge (\underline{R}F_b)(p)$$

which is the reverse inequality. Hence we conclude that

$$\underline{R}(F_a \wedge F_b) = (\underline{R}F_a) \wedge (\underline{R}F_b).$$

Step 3: conclusion. From Steps 1 and 2, we deduce that $\mathfrak{R} = \underline{R}\overline{R}$ also satisfies the same equality. □

2.2 Characteristic points

The following definition is concerned by the characteristic points. These characteristic points will be usefull in particular to reduce the set of test function in the definition of viscosity solutions (see Subsection 3.2.2)

Definition 2.8 (Characteristic points)**.** *(i) p is a* positive characteristic point *of F_0 if $H(p) = F_0(p)$ and $H > H(p)$ in $(p, p+\varepsilon)$ for some $\varepsilon > 0$. The set of positive characteristic points is denoted by $\chi^+(F_0)$.*

(ii) p is a negative characteristic point *of F_0 if $H(p) = F_0(p)$ and $H < H(p)$ in $(p-\varepsilon, p)$ for some $\varepsilon > 0$. The set of negative characteristic points is denoted by $\chi^-(F_0)$.*

(iii) The set of all characteristic points *is denoted by $\chi(F_0)$, i.e. $\chi(F_0) := \chi^+(F_0) \cup \chi^-(F_0)$.*

We present some example of characteristic points in Figure 2. We would like to point out that in the case d), the intersection point is not a characteristic point for F_0. Nevertheless, we will use this notion of characteristic point with the relaxation of F_0. In that case the left point of the plateau is in $\chi^-(\mathfrak{R}F_0)$ and the right point is in $\chi^+(\mathfrak{R}F_0)$.

In order to manipulate simply characteristic points, we use the notation introduced by J. Guerand in [15] and consider upper and lower points $p^\pm$ which only depend on p and H. The definition of $p^\pm$ is only related to the Hamiltonian H while characteristic points give information about the intersection of the graphs of H and of F_0. An illustration of these points is given in Figure 3

Definition 2.9 (Upper and lower points)**.** *Let $p \in \mathbb{R}$.*

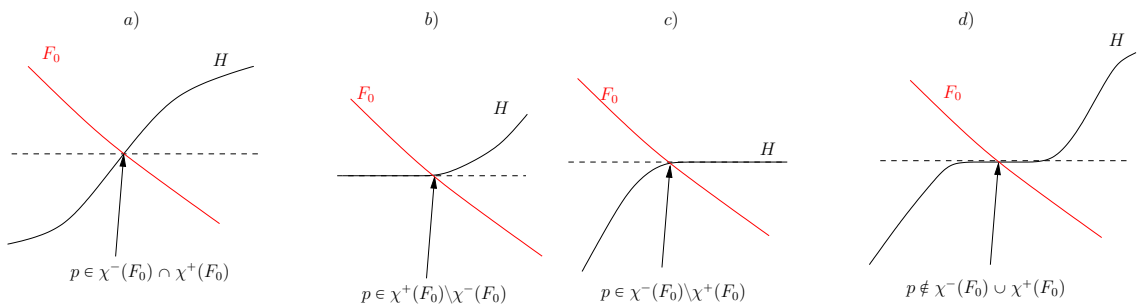

Figure 2: Characteristic points of F_0 (along H)

(i) If there exists $p_n \to p$ such that $p_n > p$ and $H(p_n) \leq H(p)$, then the upper point p^+ *is equal to p. If not,*

$$p^+ = \sup \{q > p : H > H(p) \text{ in } (p, q)\}.$$

(ii) If there exists $p_n \to p$ such that $p_n < p$ and $H(p_n) \geq H(p)$, then the lower point p^- *is equal to p. If not,*

$$p^- := \inf \{q < p : H < H(p) \text{ in } (q, p)\}.$$

Remark 2.10. The coercivity of H implies that $-\infty < p^- \leq p \leq p^+ \leq +\infty$.

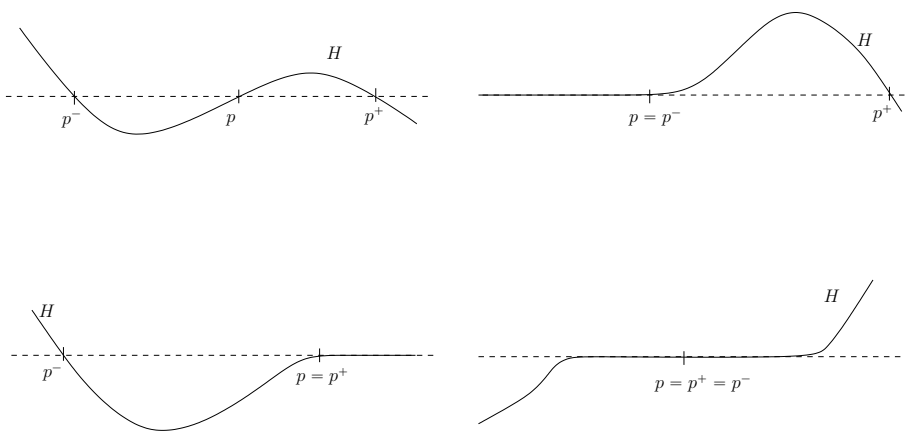

Figure 3: Points p^+ and p^- associated to p (and H)

Lemma 2.11 (Characteristic points and relaxation operators)**.** *Let $p \in \mathbb{R}$.*

(i) If $p \in \chi^+(\overline{R}F_0)$, then $\overline{R}F_0$ is constant and equal to $H(p) < H$ in (p, p^+). Moreover $H(p^+) = H(p)$ if $p^+ < +\infty$.

(ii) If $p \in \chi^-(\underline{R}F_0)$, then $\underline{R}F_0$ is constant and equal to $H(p) = H(p^-) > H$ in (p^-, p).

Proof. We only do the proof for negative characteristic points since the proof for positive ones is very similar.

Let $p \in \chi^-(\underline{R}F_0)$. Then $H(p) = \underline{R}F_0(p)$, $p^- < p$ and $H < H(p)$ in (p^-, p). For $p' \in (p^-, p)$, we then have $H(p') < H(p) = \underline{R}F_0(p) \leq \underline{R}F_0(p') \leq F_0(p')$. This implies

$$\underline{R}F_0(p^-) = \max(\sup_{[p^-,p]} H, \underline{R}F_0(p)) = \underline{R}F_0(p).$$

Since $\underline{R}F_0$ is non-increasing, this yields the desired result. $\square$

Corollary 2.12 (Property of $\mathfrak{R}F_0$)**.** *The function $\mathfrak{R}F_0$ satisfies*

$$\mathfrak{R}F_0 = \text{constant} = H(p) \quad \begin{cases} \text{in } [p^-, p] & \text{if} \quad p \in \chi^-(\mathfrak{R}F_0), \\ \text{in } [p, p^+] \cap \mathbb{R} & \text{if} \quad p \in \chi^+(\mathfrak{R}F_0). \end{cases}$$

Remark 2.13. In Corollary 2.12, we only need $[p, p^+] \cap \mathbb{R}$ instead of $[p, p^+]$ in the special case where $p^+ = +\infty$.

Proof. We only do the proof for negative characteristic points since the proof for positive ones is similar.

Let $F_1 = \overline{R}F_0$. In particular $\mathfrak{R}F_0 = \underline{R}F_1$. If $p \in \chi^-(\mathfrak{R}F_0) = \chi^-(\underline{R}F_1)$, then Lemma 2.11 implies that for $p' \in (p^-, p)$, we have in particular $F_1(p') \geq \underline{R}F_1(p') > H(p')$. This implies that $(p^-, p) \subset \{F_1 > H\}$. Moreover,

$$\{F_1 > H\} = \{\overline{R}F_0 > H\} \subset \{F_0 > H\}$$

since $\{F_0 \leq H\} \subset \{\overline{R}F_0 \leq H\}$ by Lemma 2.1. By definition of $\mathfrak{R}F_0$, we have $\mathfrak{R}F_0 = \underline{R}F_0$ in $\{F_0 > H\}$ and in particular in (p^-, p). We conclude by Lemma 2.11 that $\mathfrak{R}F_0$ is constant and equal to $H(p)$ in (p^-, p). By continuity, we get the result in $[p^-, p]$. □

We also have another corollary of the previous results.

Corollary 2.14 (Values of $\mathfrak{R}F_0$ at its characteristic points)**.** *We have $\mathfrak{R}F_0 \leq F_0$ in $\chi^-(\mathfrak{R}F_0)$ and $\mathfrak{R}F_0 \geq F_0$ in $\chi^+(\mathfrak{R}F_0)$.*

Proof. We only do the proof for negative characteristic points since the proof for positive ones is similar.

Let $F = \mathfrak{R}F_0$ and $p \in \chi^-(F)$. This means

$$H < F(p) = H(p) \text{ in } (p^-, p) \neq \emptyset.$$

Since F is non-increasing, this implies that

$$H < F = \mathfrak{R}F_0 \text{ in } (p^-, p).$$

In other words, $(p^-, p) \subset \{F > H\}$. Lemma 2.3 implies that $\{F > H\} \subset \{F_0 > H\}$. Hence $(p^-, p) \subset \{F_0 > H\}$. By continuity of F_0 and H, we then get $F_0(p) \geq H(p)$ and by Lemma 2.3 $F_0(p) \geq \mathfrak{R}F_0(p)$. □

3 Viscosity solutions: properties, stability and existence

In this section, time, space and tangential variables are not omitted anymore. We first discuss the notion of viscosity solutions and then explain how to reduce the set of test functions for verifying that a function is indeed a strong viscosity solution. As an application, we get our first main result, see Theorem 1.1 in the introduction and Theorem 3.14 below.

3.1 Definitions of weak and strong viscosity solutions

We consider two notions of viscosity solutions for the boundary value problem (1.1). Weak viscosity solutions are useful to get existence since they are naturally stable. Strong viscosity solutions are useful to prove uniqueness.

Before defining weak and strong viscosity solutions of (1.1), we recall that a function φ touches a function u from above (resp. from below) in a set Q at a point $P_0 \in Q$ if $\varphi \geq u$ in Q (resp. $\varphi \leq u$ in Q) and $u = \varphi$ at P_0. We also recall that if a function u is locally bounded from below (resp. from above), then its lower semi-continuous envelope u_* (resp. upper semi-continuous envelope u^*) is the largest lower semi-continuous function lying below u (resp. smallest upper semi-continuous function lying above u).

In order to define weak and strong viscosity solutions of the three boundary value problems (1.1), (1.2) and (1.3), we consider a real-valued continuous function $L = L(t, x, v, p_0, p)$ such that

$$L\colon (0, +\infty) \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \text{ is non-decreasing in } v,\ p_0 \text{ and } p \cdot n(x). \tag{3.1}$$

The associated boundary value problem is the following one,

$$\begin{cases} u_t + H(t,x,Du) = 0, & t>0, x\in\Omega, \\ L(t,x,u,u_t,Du) = 0, & t>0, x\in\partial\Omega. \end{cases} \tag{3.2}$$

The corresponding functions L are respectively $L = p_0 + F_0(t,x,p)$, $L = v - g(t,x)$ and $L = p\cdot n(x) + h(t,x)$.

Definition 3.1 (Weak viscosity solutions)**.** *Let $Q = (0,+\infty)\times\Omega$ and $u : Q \to \mathbb{R}$.*

(i) Let u be upper semi-continuous. We say that u is a weak L-subsolution *of* (1.1) *if for any point $P_0 = (t_0,x_0)\in Q$, and any C^1 function φ touching u from above, then*

$$\begin{aligned} &\text{if } x_0\in\Omega, \quad \varphi_t + H(t,x,D\varphi) \le 0 && \text{at } P_0 \\ &\text{if } x_0\in\partial\Omega, \quad \text{either} \quad \varphi_t + H(t,x,D\varphi)\le 0 \quad \text{or} \quad L(t,x,\varphi,\varphi_t,D\varphi)\le 0, && \text{at } P_0. \end{aligned}$$

(ii) Let u be lower semi-continuous. We say that u is a weak L-supersolution *of* (1.1) *if for any point $P_0 = (t_0,x_0)\in Q$, and any C^1 function φ touching u from below, then*

$$\begin{aligned} &\text{if } x_0\in\Omega, \quad \varphi_t + H(t,x,D\varphi) \ge 0, && \text{at } P_0 \\ &\text{if } x_0\in\partial\Omega, \quad \text{either} \quad \varphi_t + H(t,x,D\varphi)\ge 0 \quad \text{or} \quad L(t,x,\varphi,\varphi_t,D\varphi)\ge 0, && \text{at } P_0. \end{aligned}$$

(iii) Let u be locally bounded. We say that u is a weak L-solution *(weak viscosity solution) of* (1.1)*, if u^* is a weak L-subsolution of* (1.1)*, and u_* is a weak L-supersolution of* (1.1)*.*

Definition 3.2 (Strong viscosity solutions)**.** *Let $Q := (0,+\infty)\times\Omega$ and $u : Q\to\mathbb{R}$.*

(i) Let u be upper semi-continuous. We say that u is a strong L-subsolution *of* (1.1) *if for any point $P_0 = (t_0,x_0)\in Q$, and any C^1 function φ touching u from above, then*

$$\begin{aligned} &\text{if } x_0\in\Omega, \quad \varphi_t + H(t,x,D\varphi) \le 0 && \text{at } P_0 \\ &\text{if } x_0\in\partial\Omega, \quad L(t,x,\varphi,\varphi_t,D\varphi)\le 0, && \text{at } P_0. \end{aligned}$$

(ii) Let u be lower semi-continuous. We say that u is a strong L-supersolution *of* (1.1) *if for any point $P_0 = (t_0,x_0)\in Q$, and any C^1 function φ touching u from below, then*

$$\begin{aligned} &\text{if } x_0>0, \quad \varphi_t + H(t,x,D\varphi) \ge 0, && \text{at } P_0 \\ &\text{if } x_0=0, \quad L(t,x,\varphi,\varphi_t,D\varphi)\ge 0, && \text{at } P_0. \end{aligned}$$

(iii) Let u be locally bounded. We say that u is a strong L-solution *(strong viscosity solution) of* (1.1)*, if u^* is a strong F_0-subsolution of* (1.1)*, and u_* is a strong F_0-supersolution of* (1.1)*.*

Remark 3.3. In the case where $L = u_t + F_0(t,x,Du)$, weak/strong L-sub/super-solutions are simply called weak/strong F_0-sub/super-solutions.

3.2 Reducing the set of test functions

3.2.1 Critical normal slopes and weak continuity

We consider the equation without the boundary condition,

$$u_t + H(t,x,Du) = 0 \quad \text{in } Q \tag{3.3}$$

where we recall that Q denotes $(0,+\infty)\times\Omega$ and Ω is a C^1 domain of $\mathbb{R}^d$. The regularity of the domain amounts to assume that for all $x_0\in\partial\Omega$, there exists $r_0>0$ such that

$$\Omega\cap B_{r_0}(x_0) = \{(x',x_d)\in B_{r_0}(x_0) : x_d > \gamma(x')\} \tag{3.4}$$

for some C^1 function $\gamma\colon \mathbb{R}^{d-1}\to\mathbb{R}$ such that $\gamma(x_0') = 0$ and $D'\gamma(x_0') = 0$ where D' denotes the derivative with respect to x'. In particular, $n(x_0) = (0,-1)\in\mathbb{R}^{d-1}\times\mathbb{R}$. The following lemma is proved in [16] for Hamiltonians that do not depend on (t,x) and that have convex sub-level sets. The reader can check that neither the (t,x) dependency nor the quasi-convex assumption play a role in the proof.

Lemma 3.4 (Critical normal slope for supersolutions – [16])**.** *Assume that H is continuous and coercive and $\partial\Omega$ is C^1. Let $u : Q \to \mathbb{R}$ be lower semi-continuous. Assume that u is a viscosity supersolution of (3.3) and let φ be a test function touching u from below at $P_0 := (t_0, x_0)$ with $t_0 > 0$ and $x_0 \in \partial\Omega$. Let γ be a C^1 function and $r_0 > 0$ such that (3.4) holds true. Then the* critical normal slope *defined by*

$$\overline{p} := \sup\{p \in \mathbb{R}, \quad \exists r \in (0, r_0), \quad \varphi(t, x) + p(x_d - \gamma(x')) \leq u(t, x) \quad \textit{for all} \quad (t, x) \in B_r(t_0, x_0) \cap Q\}$$

is non-negative. If it is finite ($\overline{p} < +\infty$) then $\varphi_t + H(t, x, D\varphi - \overline{p}n(x_0)) \geq 0$ at P_0.

Remark 3.5. In the case where Ω is a half space (*i.e.* when $\partial\Omega$ is a hyperplane) and the Hamiltonian is quasi-convex, this lemma is proved in [20, Lemma 3.4].

We now get a similar result for subsolutions. In this case, the critical normal slope is necessarily finite.

Lemma 3.6 (Critical normal slope for subsolutions – [16])**.** *Assume that H is continuous and coercive and $\partial\Omega$ is C^1. Let $u : Q \to \mathbb{R}$ be upper semi-continuous. Assume that u is a viscosity supersolution of (3.3) and let φ be a test function touching u from below at $P_0 := (t_0, x_0)$ with $t_0 > 0$ and $x_0 \in \partial\Omega$. Let γ be a C^1 function and $r_0 > 0$ such that (3.4) holds true. Then the* critical normal slope *defined by*

$$\overline{p} := \inf\{p \in \mathbb{R}, \quad \exists r \in (0, r_0), \quad \varphi(t, x) + p(x_d - \gamma(x')) \geq u(t, x) \quad \textit{for all} \quad (t, x) \in B_r(t_0, x_0) \cap Q\}$$

is non-positive. If

$$u^*(t_0, 0) = \limsup_{(s,y)\to(t_0,0),\quad y>0} u(s, y) \tag{3.5}$$

then it is finite ($\underline{p} > -\infty$) and $\varphi_t + H(t, x, D\varphi - \underline{p}n(x_0)) \leq 0$ at P_0.

Remark 3.7. In the case where Ω is a half space (*i.e.* when $\partial\Omega$ is a hyperplane) and the Hamiltonian is quasi-convex, this lemma is proved in [20, Lemma 3.4].

Notice that Condition (3.5) is always satisfied for subsolutions of (3.2) when H is coercive and L is semi-coercive.

Lemma 3.8 (Weak continuity of weak subsolutions)**.** *Assume that H and L are continuous, H is coercive and $\lambda \mapsto L(t, x, v, p_0, p - \lambda n(x))$ is non-increasing and semi-coercive for all (t, x, v, p_0, p),*

$$\inf_{p' \perp n(x)} L(t, x, v, p_0, p' + \lambda n(x)) \to +\infty \ \textit{as} \ \lambda \to +\infty.$$

If u is a weak L-subsolution of (3.2), then for all $t > 0$, we have

$$u^*(t, x) = \limsup_{(s,y)\to(t,x), y\in\Omega} u(s, y).$$

Proof. In the case where Ω is a half-space, the result corresponds to [20, Lemma 2.3]. The reader can check that the convexity of sub-level sets of H are not used in this proof and that the only needed assumptions are the ones from the statement.

In the case where Ω is a C^1 domain, we consider $x_0 \in \partial\Omega$ and $r > 0$ and a C^1 function $\gamma\colon \mathbb{R}^{d-1} \to \mathbb{R}$ such that (3.4) holds true. We reduce to the case of the half-space by considering the function $\bar{u}(t, x)$ defined by $\bar{u}(t, x) = u(t, x', \gamma(x') + x_d)$. It is a weak $\bar{L}$-subsolution of (3.2) in an open ball centered $(t_0, x_0', 0)$ intersected with $\{x_d > 0\}$ with $\bar{H}$ and $\bar{L}$ given

$$\begin{aligned}\bar{L}(t, x', x_d, v, p_0, p', p_d) &= L(t, x', x_d + \gamma(x'), v, p_0, p' - D'\gamma(x')p_d, p_d)\\ \bar{H}(t, x, p) &= H(t, x', x_d + \gamma(x'), p' - D'\gamma(x')p_d, p_d).\end{aligned}$$

One can choose $r > 0$ such that $|D'\gamma(x')| < 1/2$ in $B_r(t_0, x_0)$. With such a choice at hand, we have $|p' - D'\gamma(x')p_d| + |p_d| \geq |p'| + \frac{1}{2}|p_d|$ and this ensures the coercivity of $\bar{H}$. Moreover, the assumption on L implies that $\bar{L}$ is semi-coercive. The weak continuity of $\bar{u}$ at $(t_0, x_0', 0)$ implies the weak continuity of u at (t_0, x_0', x_d^0). □

3.2.2 Reduction of the set of test functions

In the two following results, we do not assume that F_0 is semi-coercive.

Proposition 3.9 (Reducing the set of test functions for strong subsolutions)**.** *Assume that H, F_0 satisfy (1.4). Let $u : Q \to \mathbb{R}$ be upper semi-continuous and be a subsolution of (3.3) in $Q \cap B_r(t_0, x_0)$ with $x_0 \in \partial\Omega$ with r and γ such that (3.4) holds true. We assume that*

$$u^*(t_0, x_0) = \limsup_{(s,y)\to(t_0,x_0), y\in\Omega} u(s, y).$$

We then consider the class of test functions of the form

$$\varphi(t, x) = \psi(t, x') + p_d x_d \tag{3.6}$$

with ψ continuously differentiable in (t, x') and p_d a negative characteristic point of $q_d \mapsto \underline{R}F_0(t_0, x_0, p_0', q_d)$ where $p_0' = D'\psi(t_0, x_0')$.

If for any φ of the form (3.6) touching u from above at $P_0 = (t_0, x_0)$, we have

$$\varphi_t + \underline{R}F_0(t, x, D\varphi) \le 0 \quad at \quad P_0$$

then u is a strong $\underline{R}F_0$-subsolution of (1.1) at P_0.

Proof. Let ϕ be an arbitrary test function touching u from above at $P_0 = (t_0, x_0)$ with $t_0 > 0$. Let $\lambda := -\phi_t(P_0)$. We want to show that

$$\underline{R}F_0(t, x, D\phi) \le \lambda \quad \text{at} \quad P_0. \tag{3.7}$$

Let $\underline{p} \in (-\infty, 0]$ be given by Lemma 3.6. In particular, $H(t, x, D\phi - \underline{p}n(x)) \le \lambda$ at P_0. Let $D\phi(P_0) = (p_0', p_d^0)$ and $\underline{p}_d^0 := p_d^0 + \underline{p}$. Let us drop the (t_0, x_0, p_0') dependency for clarity. We thus know that $H(\underline{p}_d^0) \le \lambda$.

If $\underline{R}F_0(p_d^0) \le H(\underline{p}_d^0)$, then we get (3.7). We are left with treating the case $\underline{R}F_0(p_d^0) > H(\underline{p}_d^0)$. In this case, we have $\underline{R}F_0(\underline{p}_d^0) \ge \underline{R}F_0(p_d^0) > H(\underline{p}_d^0)$ and Lemma 2.4 implies that $\underline{R}F_0$ is constant in $[\underline{p}_d^0, \underline{p}_d^0 + \varepsilon)$ for some $\varepsilon > 0$. From the coercivity of H and the monotonicity of F, we also deduce that there exists some $p_* > \underline{p}_d^0$ such that

$$\underline{R}F_0 = \text{const} > H \quad \text{on} \quad [\underline{p}_d^0, p_*)$$

with $\text{const} = \underline{R}F_0(p_*) = H(p_*)$. In other words, p_* is a negative characteristic point of $\underline{R}F_0$: $p_* \in \chi^-(\underline{R}F_0)$.

We now write $p_* = \underline{p}_d^0 + \delta = p_d^0 + (\underline{p} + \delta)$ for some $\delta > 0$. Moreover, the definition of $\underline{p}$ from Lemma 3.6 implies that there exists $r_0 > 0$ such that we have

$$\phi(t, x) + (\underline{p} + \delta/2)(x_d - \gamma(x')) \ge u(t, x) \text{ in } B_{r_0}(t_0, x_0) \cap Q.$$

Moreover,

$$\phi(t, x) \le \phi(t, x', \gamma(x')) + (p_d^0 + \delta/2)(x_d - \gamma(x')) \text{ in } B_{r_1}(t_0, x_0) \cap Q$$

for some $r_1 < r_0$. Hence,

$$\varphi(t, x) := \phi(t, x', \gamma(x')) + \underbrace{(p_d^0 + \underline{p} + \delta)}_{p_*}(x_d - \gamma(x')) \ge u(t, x) \text{ in } B_{r_1}(t_0, x_0) \cap Q.$$

By assumption, we have $\lambda \ge \underline{R}F_0(p_*) = \underline{R}F_0(\underline{p}_d^0) \ge \underline{R}F_0(p_d^0)$ which in turn yields (3.7). □

As far as strong supersolutions are concerned, it is not necessary to impose a weak continuity assumption, and we show similarly the following result.

Proposition 3.10 (Reducing the set of test functions for strong supersolutions)**.** *Assume that H, F_0 satisfy (1.4). Let $u : Q \to \mathbb{R}$ be lower semi-continuous and be a viscosity supersolution of (3.3) in $Q \cap B_r(t_0, x_0)$ with $x_0 \in \partial\Omega$ with r and γ such that (3.4) holds true.*

We then consider the class of test functions of the form

$$\varphi(t,x) = \psi(t,x') + p_d x_d \tag{3.8}$$

with ψ continuously differentiable and p_d a positive characteristic point of $q_d \mapsto \overline{R}F_0(t_0, x_0, p_0', q_d)$ where $p_0' = D'\psi(t_0, x_0')$.

If for any φ of the form (3.8) touching u from below at $P_0 = (t_0, x_0)$, we have

$$\varphi_t + \overline{R}F_0(t, x, D\varphi) \geq 0 \quad at \quad P_0$$

then u is a strong $\overline{R}F_0$-supersolution of (1.1) at P_0.

3.3 Weak F_0-solutions are strong $\mathfrak{R}F_0$-solutions

Lemma 3.11 (Weak F_0 sub/super-solutions are strong $\underline{R}F_0/\overline{R}F_0$ sub/super-solutions)**.** *(i) Let $u : Q \to \mathbb{R}$ be upper semi-continuous. Then u is a weak F_0-subsolution of (1.1) if and only if u is a strong $\underline{R}F_0$-subsolution of (1.1) .*

(ii) Let $u\colon Q \to \mathbb{R}$ be lower semi-continuous. Then u is a weak F_0-supersolution of (1.1) if and only if u is a strong $\overline{R}F_0$-supersolution of (1.1).

Proof. We only prove the result for subsolutions since the case of supersolutions is treated similarly.

Weak implies strong. Assume that u is a weak F_0-subsolution. Consider a test function ϕ touching u from above at $P_0 = (t_0, x_0)$ with $t_0 > 0$ and $x_0 \in \partial\Omega$. Let $r_0 > 0$ and $\gamma \in C^1(\mathbb{R}^{d-1})$ such that (3.4) holds true. Then for any $\bar{q} \geq 0$, consider

$$\varphi(t,x) := \phi(t,x) + \bar{q}(x_d - \gamma(x'))$$

which is also touching u from above at P_0. Then, either the equation or the boundary condition is satisfied at P_0,

$$\varphi_t + (F_0 \wedge H)(t, x, D\varphi) \leq 0 \quad \text{at} \quad P_0.$$

We used the fact that $D'\gamma(x_0') = 0$. With $p := D\phi(P_0)$, the previous inequality reads,

$$\phi_t(P_0) + (F_0 \wedge H)(p - \bar{q}n(x_0)) \leq 0.$$

Because $\bar{q} \geq 0$ is arbitrary and recalling the definition of $\underline{R}F_0$ in (1.9), the previous inequality implies that u is a strong $\underline{R}F_0$-subsolution.

Strong implies weak. Assume that u is a strong $\underline{R}F_0$-subsolution. Consider a test function φ touching u from above at $P_0 = (t_0, x_0)$ with $t_0 > 0$ and $x_0 \in \partial\Omega$. Then we have

$$\varphi_t(P_0) + \underline{R}F_0(t, x, p) \leq 0 \quad \text{with} \quad p := D\varphi(P_0).$$

Because $\underline{R}F_0 \geq (F_0 \wedge H)$, we deduce that

$$\varphi_t(P_0) + (F_0 \wedge H)(t, x, p) \leq 0$$

which shows that u is a weak F_0-subsolution. □

Even if Lemma 3.11 gives a full characterization of weak solutions in terms of strong solutions, it is not completely satisfactory, because we may have $\underline{R}F_0 < \overline{R}F_0$, and we would like to have the same boundary function. This is achieved in the following two results (for subsolutions and for supersolutions) where the common boundary function is $\mathfrak{R}F_0$.

Proposition 3.12 (Weak F_0-subsolutions are strong $\mathfrak{R}F_0$-subsolutions)**.** *Assume that H, F_0 satisfy (1.4). Consider an upper semi-continuous function $u : Q \to \mathbb{R}$.*

(i) If u is a weak F_0-subsolution of (1.1) and if for all $t > 0$ and $x_0 \in \partial\Omega$,

$$u^*(t, x_0) = \limsup_{(s,y)\to(t,x_0), y\in\Omega} u(s, y) \tag{3.9}$$

then u is a strong $\mathfrak{R}F_0$-subsolution of (1.1).

(ii) If u is a strong $\mathfrak{R}F_0$-subsolution of (1.1), then u is a weak F_0-subsolution of (1.1).

Proof. Let $F := \mathfrak{R}F_0$.

Let u be a weak F_0-subsolution of (1.1) satisfying the weak continuity condition (3.9). Consider a test function φ touching u from above at $P_0 = (t_0, x_0)$ with $t_0 > 0$ and $x_0 \in \partial\Omega$. Setting $p := D\varphi(P_0)$ and $\lambda := -\varphi_t(P_0)$, we have

$$(F_0 \wedge H)(t_0, x_0, p) \leq \lambda.$$

Since we have $F = \underline{R}F$ (see Lemma 2.3), we know from Proposition 3.9 that we can assume that $p = (p', p_d)$ where p_d is a negative characteristic point of $q_d \mapsto F(t_0, x_0, p' - q_d n(x_0))$. From Corollary 2.14, we deduce that $H(t_0, x_0, p) = F(t_0, x_0, p) \leq F_0(t_0, x_0, p)$ and then $F(t_0, x_0, p) \leq \lambda$ which shows that u is a strong F-subsolution.

If we assume now that u is a strong F-subsolution, because $F = \overline{R}\underline{R}F_0 \geq \underline{R}F_0$, we deduce that u is also a strong $\underline{R}F_0$-subsolution. Then (ii) of Lemma 3.11 shows that u is a weak F_0-subsolution. □

Similarly, we show the following result.

Proposition 3.13 (Weak F_0-supersolutions are strong $\mathfrak{R}F_0$-supersolutions)**.** *Assume that H, F_0 satisfy (1.4). Consider a lower semi-continuous function $u : Q \to \mathbb{R}$. Then u is a weak F_0-supersolution of (1.1) if and only if u is a strong $\mathfrak{R}F_0$-supersolution of (1.1).*

As a corollary of Lemma 3.8, and of Propositions 3.12, 3.13, we get the following equivalence between weak F_0-solutions and strong $\mathfrak{R}F_0$-solutions.

Theorem 3.14 (Weak F_0-solutions are strong $\mathfrak{R}F_0$-solutions)**.** *Assume that H, F_0 satisfy (1.4). Assume that one of the following two conditions is satisfied:*

(i) either F_0 satisfies the semi-coercivity condition (1.5),

(ii) or u is weakly continuous at the boundary $\partial\Omega$, i.e. it satisfies (3.9).

Then a function $u : Q \to \mathbb{R}$ is a weak F_0-solution if and only if u is a strong $\mathfrak{R}F_0$-solution.

Remark 3.15. This result under assumption (i) is exactly the same result as in [15, Theorem 1.3], when we use our identification result Theorem 1.3.

Counter-example 3.16. When we have neither the semi-coercivity of F_0, nor the weak continuity of the solution u, then u can be a weak F_0-solution without being a strong F-solution for $F := \mathfrak{R}F_0$, as shows the following counter-example. We consider $\Omega = (0, +\infty)$ and

$$H(p) := |p|, \quad F_0 \equiv 0, \quad F(p) = (\mathfrak{R}F_0)(p) = \max(-p, 0)$$

where F is semi-coercive, and for all $t > 0$, we consider

$$u(t, x) = \begin{cases} 1 & \text{if} \quad x = 0 \\ 0 & \text{if} \quad x > 0. \end{cases}$$

One can check that u is a (discontinuous) weak F_0-solution, but is not a strong $\mathfrak{R}F_0$-solution, neither a weak $\mathfrak{R}F_0$-solution.

On the contrary, for instance the function

$$v(t, x) = \begin{cases} -1 & \text{if} \quad x = 0, \\ 0 & \text{if} \quad x > 0 \end{cases}$$

is both a (discontinuous) weak F_0-solution, and a strong $\mathfrak{R}F_0$-solution (and then also a weak $\mathfrak{R}F_0$-solution).

3.4 Existence and stability of weak solutions

Given $T > 0$, we consider the following problem,

$$\begin{cases} u_t + H(t,x,Du) = 0 & \text{in} \quad (0,T)\times\Omega \\ u_t + F_0(t,x,Du) = 0 & \text{on} \quad (0,T)\times\partial\Omega \end{cases} \tag{3.10}$$

supplemented with the following initial condition

$$u(0,\cdot) = u_0 \quad \text{in} \quad \{0\}\times\Omega. \tag{3.11}$$

We have the following results. Their proofs are standard, so we skip it.

Proposition 3.17 (Stability of weak solutions by infimum/suppremum)**.** *Assume that H, F_0 satisfy (1.4). Let $\mathcal{A}$ be a non-empty set and let $(u_a)_{a\in\mathcal{A}}$ be a family of weak F_0-subsolutions (resp. weak F_0-supersolutions) of (3.10). Let us assume that*

$$u := \sup_{a\in\mathcal{A}} u_a \quad (\textit{resp.} \quad u := \inf_{a\in\mathcal{A}} u_a)$$

is locally bounded on $(0,T)\times\overline{\Omega}$. Then u^* is a weak F_0-subsolution (resp. u_* is weak F_0-supersolution) of (3.10).

Proposition 3.18 (Stability of weak solutions by half-relaxed limits)**.** *Assume that H, F_0 satisfy (1.4). Let $(u_\varepsilon)_\varepsilon$ be a family of weak F_0-subsolutions (resp. weak F_0-supersolutions) of (3.10). Let us assume that the half-relaxed limit*

$$u := \limsup_{\varepsilon\to 0}{}^* u^\varepsilon \quad (\textit{resp.} \quad u := \liminf_{\varepsilon\to 0}{}_* u^\varepsilon)$$

is locally bounded on $(0,T)\times\overline{\Omega}$. Then u is a weak F_0-subsolution (resp. weak F_0-supersolution) of (3.10).

Finally, we have the following existence result.

Theorem 3.19 (Existence of weak solutions)**.** *Assume that H, F_0 satisfy (1.4) and Ω is bounded and that the initial data $u_0 : \overline{\Omega} \to \mathbb{R}$ is uniformly continuous. Then there exists a function $u\colon [0,T)\times\overline{\Omega}\to\mathbb{R}$ that is a weak F_0-solution of (3.10)-(3.11) satisfying for some constant $C_T > 0$*

$$|u(t,x) - u_0(x)| \le C_T \quad \textit{for all} \quad (t,x)\in[0,T)\times\overline{\Omega}.$$

Remark 3.20. The boundedness of Ω can be removed if one assumes for instance that

$$\sup_{t\in(0,T),x\in\Omega,p\in B_R} |H(t,x,p)| + |F_0(t,x,p)| < +\infty$$

for all $R > 0$.

Such a result is proved by using Perron's method. We recall that this method was introduced for viscosity solution by H. Ishii in [22]). Here we skip the proof since it is completely similar to the proof of [17, Theorem 2.14].

4 Guerand's approach

This section is devoted to the proof of Theorem 1.3. We first recall the definition of Guerand's relaxation operator.

4.1 Guerand's relaxation operator

The definition of Guerand's relaxation operator relies on the notion of limiter points. We split the set of limiter points A_{F_0} into two subsets $A^+_{F_0}$ and $A^-_{F_0}$.

Definition 4.1 (Positive and negative limiter points). *(i) A real number p is a* positive limiter point *of F_0 if $p^+ > p$ and $H(p) \geq F_0(p)$ and for all $q \in \mathbb{R}$,*

$$H(p) > H(q) \geq F_0(q) \Rightarrow (q^-, q^+) \cap (p, p^+) = \emptyset.$$

The set of all positive limiter points is denoted by $A^+_{F_0}$.

(ii) A real number p is a negative limiter point *of F_0 if $p^- < p$ and $H(p) \leq F_0(p)$ and for all $q \in \mathbb{R}$,*

$$F_0(q) \geq H(q) > H(p) \Rightarrow (q^-, q^+) \cap (p^-, p) = \emptyset.$$

The set of all negative limiter points is denoted by $A^-_{F_0}$.

(iii) The set of all positive and negative limiter points is denoted by A_{F_0}.

Remark 4.2. Remark that $A_{F_0} = \bigcup_{\alpha \in I} \{p_\alpha\}$ where I is at most countable. Moreover, open intervals (p^-_α, p^+_α) are disjoint, see [15, Lemma 3.7].

Definition 4.3 (Guerand's relaxation operator). *We set for $p \in \mathbb{R}$*

$$(\mathfrak{J}F_0)(p) := \begin{cases} H(p_\alpha) & \text{if } p \in [p^-_\alpha, p^+_\alpha] \text{ for some } p_\alpha \in A_{F_0}, \\ H(p) & \text{elsewhere.} \end{cases}$$

Remark 4.4. In [15], $\mathfrak{J}F_0$ is denoted by $F_{A_{F_0}}$.

Proposition 4.5 (Property of $\mathfrak{J}F_0$, [15]). *The function $\mathfrak{J}F_0$ is well-defined, continuous and non-increasing.*

4.2 Relaxation operators coincide

In order to prove that $\mathfrak{R}F_0$ and $\mathfrak{J}F_0$ coincide, we first prove that it is the case for limiter and characteristic points.

Proposition 4.6 (Limiter points coincide with characteristic points of the relaxed function). *We have $\chi^\pm(\mathfrak{R}F_0) = A^\pm_{F_0}$. In other words, the characteristic points of the relaxed function coincide with the limiter points of the original function.*

Proof. We only do the proof for negative characteristic points since the proof for positive ones is very similar. Let $F = \mathfrak{R}F_0$.

Step 1: negative characteristic points are negative limiter points. Let $p \in \chi^-(F)$. We have in particular $p^- < p$ and $H(p) = F(p)$. Then Corollary 2.12 implies that

$$F = \mathfrak{R}F_0 = \text{constant} = H(p) \text{ in } [p^-, p]. \tag{4.1}$$

We argue by contradiction and assume that $p \notin A^-_{F_0}$. This means that there exists some $q \in \mathbb{R}$ such that

$$F_0(q) \geq H(q) > H(p) \quad \text{and} \quad (q^-, q^+) \cap (p^-, p) \neq \emptyset. \tag{4.2}$$

Then (4.2) and (2.2) imply in particular

$$F(q) \geq H(q) > H(p) = F(p) = F(p^-). \tag{4.3}$$

This implies in particular that $q < p^-$.

We next prove that $p > q^+$. In order to do so, we first justify the fact that $p \notin [q, q^+]$. Assume by contradiction that $p \in [q, q^+]$. Then this implies $H(p) \geq H(q)$, which contradicts (4.3). Then $p \notin [q, q^+]$. If $p \leq q$ then by monotonicity we have $F(p) \geq F(q)$ that contradicts (4.3). Hence $p > q^+$.

We deduce from (4.2) and $q < p^-$ and $p > q^+$ that

$$q^- \leq q < p^- \leq q^+ < p.$$

This implies that $H(p) = H(p^-) > H(q) = H(q^+)$, but this is in contradiction with (4.3). Hence $p \in A^-_{F_0}$.

Step 2: negative limiter points are negative characteristic points. For $p \in A^-_{F_0}$, we have,

$$\begin{aligned} &p^- < p \\ &H < H(p) \le F_0(p) \le F_0 \text{ in } (p^-, p) \\ &F(p) = \mathfrak{R}F_0(p) = \underline{R}F_0(p) \ge H(p). \end{aligned} \tag{4.4}$$

From (i) of Lemma 2.4, we know that there exists $q \ge p$ minimal such that

$$\underline{R}F_0(p) = (F_0 \wedge H)(q) \tag{4.5}$$

$$\text{with} \qquad \begin{cases} F_0(q) \ge H(q) & \\ \underline{R}F_0 = \text{constant} = H(q) & \text{in } [p,q], \\ H(q) > H & \text{in } [p,q). \end{cases}$$

Hence by monotonicity of F_0, we have $F_0 \ge H$ on $[p,q]$, and then

$$F = \underline{R}F_0 = \text{constant} = H(q) > H \quad \text{on} \quad [p,q).$$

Combined with (4.4), this implies

$$H < H(q) \quad \text{on} \quad (p^-, q) \quad \text{with} \quad p^- < p \le q.$$

We can now consider the lower point q^- associated with q. We deduce from the previous inequality that

$$q^- \le p^- < p \le q.$$

In particular $(q^-, q^+) \cap (p^-, p) \neq \emptyset$.

If $q > p$, then we have $F_0(q) \ge H(q) > H(p)$, in contradiction with the fact that $p \in A^-_{F_0}$.

We thus conclude that $q = p$, then (4.5) shows that $F(p) = H(p)$. Combined with (4.4), this yields $p \in \chi^-(F)$. □

We can now state and prove that the two relaxation operators are in fact the same one.

Theorem 4.7 (Relaxation operators coincide)**.** *We assume that H is continuous and coercive, and that F_0 is continuous, nonincreasing, and semi-coercive. Then $\mathfrak{R}F_0 = \mathfrak{J}F_0$.*

Proof. We set $E = E^- \cup E^+$ with

$$E^- := \bigcup_{\alpha \in I} [p^-_\alpha, p_\alpha] \quad \text{and} \quad E^+ := \bigcup_{\alpha \in I} [p_\alpha, p^+_\alpha]$$

where I is an at most countable set (see Proposition 4.5) such that

$$A_{F_0} = \bigcup_{\alpha \in I} \{p_\alpha\}$$

with $p^-_\alpha \le p_\alpha \le p^+_\alpha$ and $p^-_\alpha < p^+_\alpha$. We also set $F := \mathfrak{R}F_0$.

Step 1: relaxation operators coincide in E. We only prove the result in E^- since it can be obtained in E^+ similarly. In the case where $p_\alpha \in A^-_{F_0}$, Proposition 4.6 implies that $p_\alpha \in \chi^-(F)$, that is to say $p^-_\alpha < p_\alpha$ and, using also Corollary 2.12,

$$H < F(p_\alpha) = H(p_\alpha) = H(p^-_\alpha) = \mathfrak{J}F_0 \text{ in } (p^-_\alpha, p_\alpha).$$

Since F is non-increasing, we have

$$\mathfrak{R}F_0 = F > H \text{ in } (p^-_\alpha, p_\alpha).$$

This implies that

$$H < F = \underline{R}F_0 \leq F_0 \text{ in } (p_\alpha^-, p_\alpha). \tag{4.6}$$

Thanks to the continuity of H, F and $\underline{R}F_0$, we deduce from (4.6) that

$$H(p_\alpha) = F(p_\alpha) = \underline{R}F_0(p_\alpha).$$

Hence

$$\begin{aligned} F(p_\alpha^-) &= \underline{R}F_0(p_\alpha^-) \\ &= \max\left\{ \sup_{q' \in [p_\alpha^-, p_\alpha)} (F_0 \wedge H)(q'), \ \sup_{q' \geq p_\alpha} (F_0 \wedge H)(q') \right\} \\ &\leq \max(H(p_\alpha), \underline{R}F_0(p_\alpha)) \\ &= F(p_\alpha). \end{aligned}$$

From the monotonicity of F, we deduce that

$$F = \text{constant} = H(p_\alpha) = \mathfrak{J}F_0 \text{ in } [p_\alpha^-, p_\alpha].$$

Step 2: $\{F \neq H\}$ is contained in E. If $p \in \{F \neq H\}$, then we know from Corollary 2.6 that there exists $\varepsilon > 0$ such that

$$F = \text{constant in } (p - \varepsilon, p + \varepsilon).$$

We can then consider the largest interval $(a, b) \ni p$ such that

$$(a, b) \subset \{F \neq H\}.$$

Then F is constant in (a, b). The fact that F is semi-coercive implies that $a > -\infty$. We distinguish two cases.

If $F(p) > H(p)$, then from the coercivity of H and the monotonicity of F, we have $a, b \in \mathbb{R}$ and

$$H(a) = F(a) = F(p) = F(b) = H(b) > H \text{ in } (a, b).$$

This implies that $b \in \chi^-(F) = A_{F_0}^-$ and $a := b^-$ and in turn $(a, b) \subset E^-$. In particular, $p \in E^-$ in this case.

If $F(p) < H(p)$, we can then argue as in the previous case and get, thanks to Proposition 4.6, that

$$a \in \chi^+(F) = A_{F_0}^+, \quad b = a^+ \in \mathbb{R} \cup \{+\infty\}$$

and thus $(a, b) \subset E^+$. In particular, $p \in E^+$ in this case.

Step 3: conclusion. We proved that $F = \mathfrak{J}F_0$ in E and also that $F = H$ outside E. Since $\mathfrak{J}F_0 = H$ outside E too (by definition), we thus get $F = \mathfrak{J}F_0$ everywhere. □

5 Godunov fluxes

5.1 Definition of Godunov fluxes

We still consider a coercive and continuous Hamiltonian H and we recall the standard Godunov flux associated to H defined by

$$G(q, p) = \begin{cases} \max\limits_{[p,q]} H & \text{if } p \leq q, \\ \min\limits_{[q,p]} H & \text{if } p \geq q. \end{cases}$$

In particular, G is non-decreasing in the first variable and non-increasing in the second one. Moreover, we have $G(p, p) = H(p)$. We define next the action of the Godunov flux on a semi-coercive, continuous and non-increasing function F_0.

Proposition 5.1 (Godunov's operator)**.** *Assume that F_0 is semi-coercive, continuous and non-increasing and that H is continuous and coercive. Let $p \in \mathbb{R}$, then the following properties hold true.*

(i) There exists at least one $q \in \mathbb{R}$ such that $F_0(q) = G(q,p)$. The common value is denoted by λ_q.

(ii) The value λ_q defined above is independent on q. We denote this unique value by $\lambda = \lambda(p) =: (F_0 G)(p)$

Proof. We first prove (i). Given $p \in \mathbb{R}$, the function $\phi(q) = F_0(q) - G(q,p)$ is continuous and non-increasing. On the one hand, if $q \leq p$, then $G(q,p) \leq H(p)$ and $\phi(q) \geq F_0(q) - H(p)$. Using that F_0 is semi-coercive, we deduce that

$$\lim_{q \to -\infty} \phi(q) = +\infty.$$

On the other hand, if $q \geq p$, using that $F_0(q) \leq F_0(p) < +\infty$, the fact that $G(q,p) = \max_{[p,q]} H \geq H(q)$ and the fact that H is coercive, we deduce that

$$\lim_{q \to +\infty} \phi(q) = -\infty.$$

Since ϕ is continuous and non-increasing, we deduce the existence of a q such that $\phi(q) = 0$, that is to say that $F_0(q) = G(q,p)$.

We now turn to (ii). By contradiction, assume that there exist q_1 and q_2 such that

$$\lambda_{q_1} = F_0(q_1) = G(q_1,p) > \lambda_{q_2} = F_0(q_2) = G(q_2,p).$$

Since F_0 is non-increasing, we deduce that $q_1 < q_2$. Using that G is non-decreasing in its first argument, we deduce that $G(q_1,p) \leq G(q_2,p)$ which is a contradiction. □

The goal is now to prove that $\mathfrak{R}F_0 = F_0 G$. More precisely, we have the following theorem.

Theorem 5.2 (Relaxation operator coincide with Godunov's operator)**.** *Assume that F_0 is semi-coercive, continuous and non-increasing and that H is continuous and coercive. Then*

$$\mathfrak{R}F_0 = F_0 G.$$

In order to prove this theorem, we need to introduce the Godunov semi-fluxes. This is done in the next section. The proof of Theorem 5.2 is postponed until Subsection 5.3.

5.2 Godunov semi-fluxes

We introduce the Godunov semi-fluxes, $\underline{G}$ and $\overline{G}$, which are set-valued applications defined by

$$\underline{G}(q,p) = \begin{cases} \{-\infty\} & \text{if } q < p, \\ [-\infty, H(p)] & \text{if } q = p, \\ \left\{ \max\limits_{[p,q]} H \right\} & \text{if } q > p \end{cases}$$

and

$$\overline{G}(q,p) = \begin{cases} \left\{ \min\limits_{[q,p]} H \right\} & \text{if } q < p, \\ [H(p), +\infty] & \text{if } q = p, \\ \{+\infty\} & \text{if } q > p. \end{cases}$$

As before, we can define the action of these semi-fluxes on non-increasing semi-coercive continuous functions.

Proposition 5.3 (Lower Godunov operator $F_0\underline{G}$)**.** *Assume that F_0 is semi-coercive, continuous and non-increasing and that H is continuous and coercive. Let $p \in \mathbb{R}$. We define the sets*

$$\underline{Q} := \{q \in \mathbb{R}, F_0(q) \in \underline{G}(q,p)\} \quad \text{and} \quad \underline{\Lambda} := \{F_0(q),\ q \in \underline{Q}\}.$$

Then the following properties hold true.

(i) The set $\underline{Q}$ is non-empty and contained in $[p, +\infty[$.

(ii) The set $\underline{\Lambda}$ is reduced to a singleton that we denote by $\{(F_0\underline{G})(p)\}$.

Proof. We first prove (i). In order to do so, we distinguish two cases.

Suppose first that $F_0(p) > H(p)$. In that case, we remark that $\underline{G}(q,p) = \{G(q,p)\}$ for all $q > p$. Then, the proof is the same as the one of Proposition 5.1. Indeed, if we define $\phi(q) = F_0(q) - G(q,p)$, then $\phi(p) = F_0(p) - G(p,p) = F_0(p) - H(p) > 0$ and so the zero of ϕ defined in the proof of Proposition 5.1 is greater than p and satisfies the desired condition.

Suppose now that $F_0(p) \leq H(p)$. In that case, we remark that $p \in \underline{Q}$ since $\underline{G}(p,p) = [-\infty, H(p)]$.

The proof of (ii) follow the same lines as the one of (ii) from Proposition 5.1. □

In the same way, we have the following proposition concerning $\overline{G}$. Since the proof is similar to the previous one, we skip it.

Proposition 5.4 (Upper Godunov operator $F_0\overline{G}$)**.** *Assume that F_0 is semi-coercive, continuous and non-increasing and that H is continuous and coercive. Let $p \in \mathbb{R}$. We define the sets*

$$\overline{Q} := \{q \in \mathbb{R}, F_0(q) \in \overline{G}(q,p)\} \quad \text{and} \quad \overline{\Lambda} := \{F_0(q),\ q \in \overline{Q}\}.$$

Then the following properties hold true

(i) The set $\overline{Q}$ is non-empty and contained in $]-\infty, p]$.

(ii) The set $\overline{\Lambda}$ is reduced to a singleton that we denote by $\{(F_0\overline{G})(p)\}$.

In order to compose semi-Godunov operators, we first need to make sure that $F_0\underline{G}$ satisfy the same assumptions as F_0.

Lemma 5.5 (Properties of $F_0\underline{G}$ and $F_0\overline{G}$)**.** *Under the same assumptions, $F_0\underline{G}$ and $F_0\overline{G}$ are non-increasing, continuous and semi-coercive.*

Proof. We do the proof only for $F_0\underline{G}$, the one for $F_0\overline{G}$ being similar.

We first show that $F_0\underline{G}$ is non-increasing. Let $p_1 > p_2$ and q_1, q_2 be such that $(F_0\underline{G})(p_i) = F_0(q_i) \in \underline{G}(q_i, p_i)$ for $i \in \{1,2\}$. In particular, since $q_i \in \underline{Q}$, we have $q_i \geq p_i$ thanks to (i) from Proposition 5.3.

We assume by contradiction that $(F_0\underline{G})(p_1) > (F_0\underline{G})(p_2)$. This implies $F_0(q_1) > F_0(q_2)$ and in particular $q_2 > q_1 \geq p_1 > p_2$. Hence $\underline{G}(q_2,p_2) = \{G(q_2,p_2)\}$ and so $F_0(q_2) = G(q_2,p_2) \geq G(q_1,p_1) \geq G(p_1,p_1)$. The inequalities follow from monotonicity properties of G in both variables. If $q_1 > p_1$, then $F_0(q_1) = G(q_1,p_1)$ and we get a contradiction: $F_0(q_1) \leq F_0(q_2)$. If $q_1 = p_1$, then $\underline{G}(p_1,q_1) = [-\infty, H(p_1)]$ from which we get $F_0(q_1) \leq H(p_1) = G(p_1,p_1) \leq F_0(q_2)$ and we get the same contradiction.

We now prove that $F_0\underline{G}$ is semi-coercive. Let $M > 0$. There exists p_0 such that for every $p < p_0$, $H(p) \geq M$ and $F_0(p) \geq M$. Let $p < p_0$. Proposition 5.3 implies that there exists $q \geq p$ such that $(F_0\underline{G})(p) = F_0(q) \in \underline{G}(q,p)$. If $q = p$, then $(F_0\underline{G})(p) = F_0(p) \geq M$. If $q > p$, then $(F_0\underline{G})(p) = G(q,p) \geq H(p) \geq M$. This shows that $F_0\underline{G}$ is semi-coercive.

We now prove that $F_0\underline{G}$ is continuous. Let $p_n \to p$ and $q_n \geq p_n$ be such that $(F_0\underline{G})(p_n) = F_0(q_n) \in \underline{G}(q_n,p_n)$. From the coercivity of H, we get that $(q_n)_n$ is bounded: indeed, either $q_n = p_n$ or $F_0(p_n) \geq F_0(q_n) = G(q_n,p_n) \geq H(q_n)$. Hence, up to extract a subsequence (still denoted by $(q_n)_n$), we have $q_n \to q_0 \geq p$.

Assume first that $q_{n_j} = p_{n_j}$ along a subsequence $\{n_j\}$. In this case $F_0(p_{n_j}) = F_0(q_{n_j}) \leq H(p_{n_j})$. This implies that $F_0(p) \leq H(p)$ and so $F_0(p) \in \underline{G}(p,p)$. This means that $p \in \underline{Q}$ and $F_0(p) = (F_0\underline{G})(p)$ and

$$(F_0\underline{G})(p_{n_j}) = F_0(q_{n_j}) \to F_0(p) = (F_0\underline{G})(p).$$

Assume now that $q_n > p_n$ for n large enough, then $(F_0\underline{G})(p_n) = F_0(q_n) = G(q_n, p_n)$. Since $q_n \to q_0 \geq p$ and $F_0(q_n) \leq F_0(p_n)$, we get $F_0(q_0) \leq F_0(p)$ and $F_0(q_0) = G(q_0, p)$.

If $q_0 = p$ then $F_0(p) = H(p) \in \underline{G}(p,p)$. If $q_0 > p$ then $F_0(q_0) \in \underline{G}(q_0, p)$. In both cases, $q_0 \in \underline{Q}$ and thus $F_0(q_0) = (F_0\underline{G})(p)$. We thus proved that $F_0\underline{G}(p_n) = F_0(q_n) \to F_0(q_0) = F_0\underline{G}(p)$. This implies that indeed the whole sequence $\{(F_0\underline{G})(p_n)\}$ converges to $(F_0\underline{G})(p)$. □

We now want to prove that the action of $\underline{G}$ on the action of $\overline{G}$ on F_0 is in fact the action of G on F_0.

Proposition 5.6 (Composition of Godunov semi-fluxes)**.** *We have* $(F_0\overline{G})\underline{G} = F_0 G = (F_0\underline{G})\overline{G}$.

In order to prove this proposition, the following lemma is needed.

Lemma 5.7 (Key composition result)**.** *(i) For all $(q,p) \in \mathbb{R}^2$, there exists $q' \in \mathbb{R}$ such that $\overline{G}(q,q') \cap \underline{G}(q',p) \neq \emptyset$. Moreover, for such a real number q', we have $\overline{G}(q,q') \cap \underline{G}(q',p) = \{G(q,p)\}$.*

(ii) For all (q,p), there exists $q' \in \mathbb{R}$ such that $\underline{G}(q,q') \cap \overline{G}(q',p) \neq \emptyset$. Moreover, for such a real number q', we have $\underline{G}(q,q') \cap \overline{G}(q',p) = \{G(q,p)\}$.

Proof. We only prove (i) since the proof of (ii) follows the same reasoning.

We first show that $\overline{G}(q,q') \cap \underline{G}(q',p)$ is either empty or equal to the singleton $\{G(q,p)\}$.

Remark that the intersection can only contain real numbers, but neither $+\infty$ nor $-\infty$. Hence, if the intersection is not empty, then $p \leq q'$ and $q \leq q'$. We now distinguish four cases.

Case 1: $p = q = q'$. In that case $\underline{G}(p,p) = [-\infty, H(p)]$ and $\overline{G}(p,p) = [H(p), +\infty]$ and so the intersection is reduced to a singleton of element $H(p) = G(p,p) = G(q,p)$.

Case 2: $p < q = q'$. In that case $\overline{G}(q,q') = [H(q), +\infty]$ and $\underline{G}(q',p) = \{G(q',p)\} = \{G(q,p)\}$. Since $q \geq p$, we have $G(q,p) \geq G(q,q) = H(q)$ and so the intersection is non-empty and then reduced to $G(q,p)$.

Case 3: $q < p = q'$. In that case $\overline{G}(q,q') = \{G(q,p)\}$ and $\underline{G}(q',p) = [-\infty, H(p)]$. Since $q \leq p$, we have $G(q,p) \leq G(p,p) = H(p)$ and so the intersection is reduced to $G(q,p)$.

Case 4: $q < q'$ and $p < q'$. In that case $\overline{G}(q,q') = \{G(q,q')\}$ and $\underline{G}(q',p) = \{G(q',p)\}$. If the interscetion is not empty, then $G(q,q') = G(q',p)$, which means that

$$\max_{[p,q']} H = \min_{[q,q']} H,$$

i.e. H is constant on $[\max(q,p), q']$. If $p < q$, this implies in particular that

$$G(q,p) = \max_{[p,q]} H = \max_{[p,q']} H = G(q',p).$$

Similarly if $p > q$, we get $G(q,p) = G(q,q')$. In the last case $p = q$, we get

$$G(q,p) = G(q,q') = G(q',p).$$

We now prove that we can always find a q' such that the intersection is non empty. If $p = q$, we can take $q' = p = q$ as in Case 1. If $p < q$, we can take $q' = q$ as in Case 2, while if $p > q$, we can take $q' = p$ as in Case 3. □

We are now able to prove Proposition 5.6.

Proof of Proposition 5.6. Let $F_1 = F_0\overline{G}$. We use successively the definition of $F_0 G$, (i) from Lemma 5.7, the definitions of $F_0\overline{G}$ and of $F_1\underline{G}$ to write,

$$\begin{aligned}
\{F_0 G(p)\} &= \{F_0(q) \text{ for some } q \text{ s.t. } F_0(q) \in G(q,p)\} \\
&= \{F_0(q) \text{ for some } q \text{ and } q' \text{ s.t. } F_0(q) \in \overline{G}(q,q') \cap \underline{G}(q',p)\} \\
\{F_1(q')\} &= \{F_0\overline{G}(q')\} = \{F_0(q) \text{ for some } q \text{ s.t. } F_0(q) \in \overline{G}(q,q')\} \\
\{F_1\underline{G}(p)\} &= \{F_1(q') \text{ for some } q' \text{ s.t. } F_1(q') \in \underline{G}(q',p)\} \\
&= \{F_0(q) \text{ for some } q \text{ and } q' \text{ s.t. } F_0(q) \in \overline{G}(q,q') \cap \underline{G}(q',p)\}.
\end{aligned}$$

This implies that $F_0 G(p) = F_1\underline{G}(p) = (F_0\overline{G})\underline{G}$.

Using (ii) from Lemma 5.7, we can follow the same reasoning and get $F_0 G(p) = (F_0\underline{G})\overline{G}$. □

5.3 Relaxation and Godunov fluxes

The proof of Theorem 5.2 is a direct consequence of the following proposition which makes the link between the semi-relaxation of F_0 and the actions of the Godunov semi-fluxes on F_0.

Proposition 5.8 (Semi-relaxations and Godunov's semi-fluxes)**.** *Assume that F_0 is semi-coercive, continuous and non-increasing and that H is continuous and coercive. Then $F_0\underline{G} = \underline{R}F_0$ and $F_0\overline{G} = \overline{R}F_0$.*

Proof. We only prove that $F_0\underline{G} = \underline{R}F_0$ since the proof of the other equality is similar. Let p and $q' \geq p$ be such that

$$(F_0\underline{G})(p) = F_0(q') \in \underline{G}(q', p).$$

If $q' = p$, then $(F_0\underline{G})(p) = F_0(p) \leq H(p)$. Using Lemma 2.1, we deduce that

$$\underline{R}F_0(p) = F_0(p) = F_0\underline{G}(p).$$

If $q' > p$, then $F_0(q') = G(q', p) = \max\limits_{[p,q']} H$. In particular $F_0(q') \geq H(q')$ and by Lemma 2.1, we have $\underline{R}F_0(q') \leq F_0(q')$. Recall also that

$$\underline{R}F_0(p) = \max\left(\sup_{[p,q']}(F_0 \wedge H), \underline{R}F_0(q')\right).$$

Since F_0 is non-increasing, we have for all $q \in [p, q']$,

$$F_0(q) \geq F_0(q') = \max_{[p,q']} H.$$

In particular,

$$\sup_{q\in[p,q']}(F_0 \wedge H)(q) = \max_{q\in[p,q']} H(q) = F_0(q') \geq \underline{R}F_0(q')$$

and we finally get

$$\underline{R}F_0(p) = F_0(q') = (F_0\underline{G})(p).$$

□

We now turn to the proof of Theorem 5.2.

Proof of Theorem 5.2. Lemma 5.5 implies that $F_0\overline{G}$ satisfies the assumptions of Proposition 5.8. Using Proposition 5.8 first to F_0 and then to $\hat{F}_0 = F_0\overline{G}$, we have

$$\underline{R}(\overline{R}F_0) = \underline{R}(F_0\overline{G}) = (F_0\overline{G})\underline{G}.$$

Using Lemma 2.3 and Proposition 5.6, we then get $\mathfrak{R}F_0 = F_0G$. □

6 The Neumann and Dirichlet problems

6.1 Strong solutions for the Neumann problem

This subsection is devoted to the proof of Theorem 1.5.

Proof of Theorem 1.5. The proof is split in several steps.
Step 1: the condition N is self-relaxed. We recall that

$$N(t,x,p) = \begin{cases} \max\left\{H(t,x,p-\rho n) : \rho \in [0, p\cdot n(x) + \rho_0]\right\} & \text{if } p\cdot n(x) + \rho_0 \geq 0, \\ \min\left\{H(t,x,p-\rho n) : \rho \in [p\cdot n(x) + \rho_0, 0]\right\} & \text{if } p\cdot n(x) + \rho_0 \leq 0 \end{cases}$$

with $\rho_0 = h(t,x)$. For $p = p' - \rho n$ with $p' \perp n$, it is convenient to consider $H_0(\rho) = H(t,x,p'-\rho n)$ and $N_0\colon \rho \mapsto N(t,x,p'-\rho n)$. In particular,

$$N_0(\rho) = \begin{cases} \min\limits_{[\rho_0,\rho]} H_0 & \text{if } \rho \geq \rho_0, \\ \max\limits_{[\rho,\rho_0]} H_0 & \text{if } \rho \leq \rho_0. \end{cases}$$

In other words, $N_0(\rho) = G(\rho_0,\rho)$ where G denotes the Godunov flux function. We remark that N_0 is *self-relaxed* in the sense that $\mathfrak{R}N_0 = N_0$. Indeed, we remark that

$$(H_0(\rho) - N_0(\rho))(\rho - \rho_0) \geq 0.$$

In particular, thanks to Lemma 2.1, we know that $\underline{R}N_0 = N_0$ in $(\rho_0, +\infty) \subset \{N_0 \leq H_0\}$. For $\rho \leq \rho_0$, we write

$$\begin{aligned} \underline{R}N_0(\rho) &= \max_{q \geq \rho} (N_0 \wedge H_0)(q) \\ &= \left(\max_{q \in [\rho,\rho_0]} H_0(q) \right) \vee \underline{R}N_0(\rho_0) \\ &= N_0(\rho) \vee N_0(\rho_0) \\ &= N_0(\rho). \end{aligned}$$

Hence $\underline{R}N_0 = N_0$ in $\mathbb{R}$. Similarly, $\overline{R}N_0 = N_0$ and $\mathfrak{R}N_0 = N_0$.

We observe next that negative characteristic points of N_0 are contained in $(-\infty, \rho_0]$. Indeed, if $\rho > \rho_0$ and $N_0(\rho) = H_0(\rho)$, then $N_0(\rho) = \min_{[\rho_0,\rho]} H_0$ and in particular, $H_0 \geq N_0(\rho) = H_0(\rho)$ in $[\rho_0,\rho]$. In particular, ρ is not a negative characteristic point of N_0.

Step 2: weak solutions of the Neumann problem are strong N-solutions. We only treat the case of weak subsolutions since weak supersolutions can be treated similarly.

Let $u\colon Q \to \mathbb{R}$ be a weak solution of (1.2). Then Lemma 3.8 implies that u is weakly continuous.

Thanks to Proposition 3.9, we only consider a C^1 test function ϕ touching u^* from above at $P_0 = (t_0, x_0)$ with $x_0 \in \partial\Omega$ of the form

$$\phi(t,x) = \psi(t,x') + \rho x_d$$

for a negative characteristic point ρ of N_0 (recall that $\underline{R}N_0 = N_0$). In particular, $\rho \leq \rho_0$.

Consider $r > 0$ and $\gamma \in C^1(\mathbb{R}^{d-1})$ such that (3.4) holds true. Then we have the viscosity inequality,

$$\phi_t + H(t,x,D\phi) = \min\left(\phi_t + H(t,x,D\phi), \rho_0 - \rho\right) \leq 0 \quad \text{at} \quad P_0.$$

For $p = D\phi(P_0)$ and $R \in [0, \rho_0 - \rho]$, the function $\varphi(t,x) = \phi(t,x) + R(x_d - \gamma(x'))$ is still a test function for u at P_0. Since $D'\gamma(x_0') = 0$ and $R + \rho \leq \rho_0$,

$$\phi_t(t_0,x_0) + \max_{R \in [0,\rho_0-\rho]} H(t_0,x_0,D\phi(t_0,x_0) - Rn(x_0)) \leq 0.$$

Since $\rho = -\frac{\partial\phi}{\partial n}(t_0,x_0)$, this precisely means $\phi_t + N(t_0,x_0,D\phi) \leq 0$ at P_0.

Step 3: strong N-solutions are weak solutions of the Neumann problem. We show it for strong N-subsolutions since the proof for strong N-supersolutions is similar. Assume that u is a strong N-subsolution. Let φ be a C^1 test function touching u^* from above at $P_0 = (t_0, 0)$. Letting $\lambda := \varphi_t(P_0)$ and $p := D\varphi(P_0)$, we have

$$\lambda + N(t_0,x_0,p) \leq 0.$$

If $\rho = -p \cdot n(x_0) \leq \rho_0 = h(t_0,x_0)$, then $N(t_0,x_0,p) = N_0(\rho) \geq H_0(\rho) = H(t_0,x_0,p)$, which implies

$$\lambda + H(t_0,x_0,p) \leq 0.$$

In particular,

$$\min(\lambda + H(t_0,x_0,p), h(t_0,x_0) + p \cdot n(x_0)) \leq 0.$$

If $\rho = -p \cdot n(x_0) > \rho_0 = h(t_0,x_0)$, the previous inequality also holds true. □

6.2 Connection with scalar conservation laws

In this subsection, we would like to make a link between the relaxation operator $\mathfrak{R}F_0$ and the theory of boundary conditions for scalar conservation laws[2]. To this end, we consider a linear function,

$$u(t,x) = px + \lambda t.$$

It is straightforward to check that it is a weak viscosity solution of (1.1) if and only if $\lambda = -H(p)$ and

$$(\underline{R}F_0)(p) \leq H(p) = -\lambda \leq (\overline{R}F_0)(p). \tag{6.1}$$

Then we have

Lemma 6.1 (Relation with the germ)**.** *Assume (1.1). An element $p \in \mathbb{R}$ satisfies (6.1) if and only if p is an element of the set (which is called a germ)*

$$\mathcal{G} = \{q \in \mathbb{R}, \quad H(q) = \mathfrak{R}F_0(q)\}. \tag{6.2}$$

Remark 6.2. For the notion of germ and its properties (maximal germs, complete germs) we refer the reader to [3].

Remark 6.3. The fact that $\mathfrak{R}F_0$ is nonincreasing provides to the set $\mathcal{G}$ the property to be a germ for H. Moreover it is possible to check that this germ is maximal if and only if it is of the form of (6.2) for some suitable F_0. With some further work, it is also possible to show that the germ $\mathcal{G}$ is complete for instance if $H \in C^1$ (but it is out of the scope of this paper).

Proof. Recall from Lemma 2.1 and Remark 2.2 that

$$\begin{cases} (\overline{R}F_0)(p) \begin{cases} = F_0(p) & \text{if} \quad F_0(p) \geq H(p) \\ \in [F_0(p), H(p)] & \text{if} \quad F_0(p) \leq H(p) \end{cases} \\ \\ (\underline{R}F_0)(p) \begin{cases} \in [H(p), F_0(p)] & \text{if} \quad F_0(p) \geq H(p) \\ = F_0(p) & \text{if} \quad F_0(p) \leq H(p) \end{cases} \\ \\ \underline{R}F_0 \leq \overline{R}F_0 \\ \\ (\overline{R}F_0)(p) = H(p) = (\underline{R}F_0)(p) \quad \text{if} \quad F_0(p) = H(p). \end{cases}$$

Hence we deduce that (6.1) is equivalent to

$$-\lambda = H(p) = \left\{ \begin{array}{ll} (\overline{R}F_0)(p) & \text{if} \quad F_0(p) \leq H(p) \\ (\underline{R}F_0)(p) & \text{if} \quad F_0(p) \geq H(p) \end{array} \right\} = (\mathfrak{R}F_0)(p)$$

i.e. $p \in \mathcal{G} := \{H = \mathfrak{R}F_0\}$ which ends the proof of the lemma. □

6.3 Strong solutions for the Dirichlet problem

In this subsection, we compute the relaxed Dirichlet boundary condition.

Proof of Theorem 1.6. Let $u\colon Q \to \mathbb{R}$ be a weak viscosity subsolution of (1.3). Let ϕ be a C^1 test function touching u^* from above at $P_0 = (t_0, x_0)$ with $x_0 \in \partial\Omega$. Then we have

$$\min(u^* - g, \phi_t + H(t, x, D\phi)) \leq 0 \quad \text{at } P_0.$$

[2]Morally if u is a strong $\mathfrak{R}F_0$-solution that is Lipschitz continuous, then the function $v := u_x$ is expected to be an entropy solution of

$$\begin{cases} v_t + H(v)_x = 0, & \text{on} & (0,+\infty)_t \times (0,+\infty)_x, \\ v(t,0) \in \mathcal{G}, & \text{for a.e.} & t \in (0,+\infty) \end{cases}$$

where $v(t,0)$ is a strong (quasi)-trace of v in the sense of Panov [27]. It is possible to prove it, if H is C^1 and H' is not constant on every interval of positive length. But it requires some additional work which is out of the scope of the present paper.

This boundary condition can be interpreted as follows,

$$\phi_t + \min(u^* - g - \lambda, H(t, x, D\phi) \leq 0 \quad \text{at } P_0$$

where $\lambda = \phi_t(t_0, x_0)$ (recall that we look at pointwise inequality and that only the behavior in the normal gradient is taken into account). We can argue similarly for weak viscosity supersolutions of (1.3) and we conclude that the Dirichlet condition can be interpreted as a dynamic boundary condition with $F_0(p) = u^*(t_0, x_0) - g(t_0, x_0) - \lambda =: A_0$.

Recalling the definition of $\underline{R}F_0$ and $\overline{R}F_0$, see (1.9), we compute,

$$\begin{aligned}
\underline{R}F_0(p) &= \sup_{\rho \geq 0} \min(A_0, H(t_0, x_0, p - \rho n(x_0)) \\
&= \min(A_0, \sup_{\rho \geq 0} H(t_0, x_0, p - \rho n(x_0)) \\
&= A_0 \\
\overline{R}F_0(p) &= \inf_{\rho \leq 0} \max(A_0, H(t_0, x_0, p - \rho n(x_0)) \\
&= \max(A_0, \inf_{\rho \leq 0} H(t_0, x_0, p - \rho n(x_0)) \\
&= \max(A_0, H_-(t_0, x_0, p)).
\end{aligned}$$

Recalling now the definition of the relaxation operator, see (1.8),

$$\begin{aligned}
\mathfrak{R}F_0(p) &= \begin{cases} A_0 & \text{if } H(t_0, x_0, p) \leq A_0, \\ \max(A_0, H_-(t_0, x_0, p)) & \text{if } H(t_0, x_0, p) \geq A_0 \end{cases} \\
&= \max(A_0, H_-(t_0, x_0, p)).
\end{aligned}$$

We used the fact that $H \geq H_-$ to get the last line. Recalling that $A_0 = u^*(t_0, x_0) - g(t_0, x_0) - \lambda$, Theorem 3.14 implies the conclusion of Theorem 1.6. □

Acknowledgements This research was funded, in whole or in part, by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010. For the purpose of open access, the author has applied a CC-BY public copyright licence to any Author Accepted Manuscript (AAM) version arising from this submission.

References

[1] Y. Achdou, F. Camilli, A. Cutrì, and N. Tchou, *Hamilton-Jacobi equations constrained on networks*, NoDEA, Nonlinear Differ. Equ. Appl., 20 (2013), pp. 413–445.

[2] Y. Achdou and C. Le Bris, *Homogenization of some periodic Hamilton-Jacobi equations with defects.* working paper or preprint, Nov. 2022.

[3] B. Andreianov, K. H. Karlsen, and N. H. Risebro, *A theory of L^1-dissipative solvers for scalar conservation laws with discontinuous flux*, Arch. Ration. Mech. Anal., 201 (2011), pp. 27–86.

[4] B. Andreianov and K. Sbihi, *Well-posedness of general boundary-value problems for scalar conservation laws*, Trans. Am. Math. Soc., 367 (2015), pp. 3763–3806.

[5] C. Bardos, A. Y. le Roux, and J.-C. Nédélec, *First order quasilinear equations with boundary conditions*, Comm. Partial Differential Equations, 4 (1979), pp. 1017–1034.

[6] G. Barles, *Nonlinear Neumann boundary conditions for quasilinear degenerate elliptic equations and applications*, J. Differ. Equations, 154 (1999), pp. 191–224.

[7] G. Barles, A. Briani, and E. Chasseigne, *A Bellman approach for two-domains optimal control problems in $\mathbb{R}^N$*, ESAIM, Control Optim. Calc. Var., 19 (2013), pp. 710–739.

[8] ———, *A Bellman approach for regional optimal control problems in $\mathbb{R}^N$*, SIAM J. Control Optim., 52 (2014), pp. 1712–1744.

[9] G. Barles and E. Chasseigne, *An Illustrated Guide of the Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities.* working paper or preprint, Apr. 2023.

[10] G. Barles and B. Perthame, *Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations*, Appl. Math. Optim., 21 (1990), pp. 21–44.

[11] E. Chasseigne and G. Barles, *(Almost) everything you always wanted to know about deterministic control problems in stratified domains*, Netw. Heterog. Media, 10 (2015), pp. 809–836.

[12] M. G. Crandall, L. C. Evans, and P.-L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 282 (1984), pp. 487–502.

[13] M. G. Crandall and P.-L. Lions, *Condition d'unicité pour les solutions généralisées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci., Paris, Sér. I, 292 (1981), pp. 183–186.

[14] ———, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), pp. 1–42.

[15] J. Guerand, *Effective nonlinear Neumann boundary conditions for 1d nonconvex Hamilton-Jacobi equations*, J. Differ. Equations, 263 (2017), pp. 2812–2850.

[16] ———, *Flux-limited solutions and state constraints for quasi-convex Hamilton-Jacobi equations in multidimensional domains*, Nonlinear Anal., 162 (2017), pp. 162–177.

[17] C. Imbert and R. Monneau, *Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks*, Ann. Sci. Éc. Norm. Supér. (4), 50 (2017), pp. 357–448.

[18] C. Imbert and R. Monneau, *Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case*, Discrete Contin. Dyn. Syst., 37 (2017), pp. 6405–6435.

[19] C. Imbert, R. Monneau, and H. Zidani, *A Hamilton-Jacobi approach to junction problems and application to traffic flows*, ESAIM, Control Optim. Calc. Var., 19 (2013), pp. 129–166.

[20] C. Imbert and V. D. Nguyen, *Effective junction conditions for degenerate parabolic equations*, Calc. Var. Partial Differential Equations, 56 (2017), pp. Paper No. 157, 27.

[21] H. Ishii, *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci. Eng., Chuo Univ., Ser. I, 28 (1985), pp. 33–77.

[22] ———, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J., 55 (1987), pp. 369–384.

[23] O. Jerhaoui and H. Zidani, *A general comparison principle for Hamilton Jacobi Bellman equations on stratified domains*, ESAIM, Control Optim. Calc. Var., 29 (2023), p. 38. Id/No 9.

[24] P.-L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke Math. J., 52 (1985), pp. 793–820.

[25] P.-L. Lions and P. E. Souganidis, *Viscosity solutions for junctions: well posedness and stability*, Rendiconti Lincei, 27 (2016), pp. 535–545.

[26] P.-L. Lions and P. E. Souganidis, *Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions*, Atti Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat., IX. Ser., Rend. Lincei, Mat. Appl., 28 (2017), pp. 807–816.

[27] E. Y. Panov, *Existence of strong traces for quasi-solutions of multidimensional conservation laws*, J. Hyperbolic Differ. Equ., 4 (2007), pp. 729–770.

[28] Z. Rao, A. Siconolfi, and H. Zidani, *Transmission conditions on interfaces for Hamilton-Jacobi-Bellman equations*, J. Differ. Equations, 257 (2014), pp. 3978–4014.

[29] A. Siconolfi, *Time-dependent Hamilton-Jacobi equations on networks*, J. Math. Pures Appl. (9), 163 (2022), pp. 702–738.

[30] H. M. Soner, *Optimal control with state-space constraint. I*, SIAM J. Control Optim., 24 (1986), pp. 552–561.