Dynamics of dislocation densities in a bounded channel. Part II: existence of weak solutions to a singular Hamilton-Jacobi/parabolic strongly coupled system

H. Ibrahim *, M. Jazar [1], R. Monneau *

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Abstract

We study a strongly coupled system consisting of a parabolic equation and a singular Hamilton-Jacobi equation in one space dimension. This system describes the dynamics of dislocation densities in a material submitted to an exterior applied stress. Our system is a natural extension of that studied in [15] where the applied stress was set to be zero. The equations are written on a bounded interval with Dirichlet boundary conditions and require special attention to the boundary. We prove a result of global existence of a solution. The method of the proof consists in considering first a parabolic regularization of the full system, and then passing to the limit. For this regularized system, a result of global existence and uniqueness of a solution has been given in [16]. We show some uniform bounds on this solution which uses in particular an entropy estimate for the densities.

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1 Introduction

1.1 Physical motivation and setting of the problem

In [12], Groma, Czikor and Zaiser have proposed a model describing the dynamics of dislocation densities. Dislocations are defects in crystals that move when a stress field is applied on the material. These defects are one of the main explanations of the elastoviscoplasticity behavior of metals (see [8] and [9] for various models relating dislocations and elastoviscoplastic properties of metals). This model has been introduced in order to describe the possible accumulation of dislocations on the boundary layer of a bounded channel. Because of the motion of dislocations inside the bounded crystal, they can reach its boundary. Therefore the model has to describe carefully the behavior of these dislocations close to the boundary. Dislocations are distinguished by the sign of their Burgers vector $\pm \vec{b}$ (see [13] for a description of the Burgers vector). More precisely, let us call θ^+ and θ^-, the densities of the positive and negative dislocations respectively. In particular, the repulsive interactions between dislocations of the same Burgers vector is a microscopic explanation of the formation of the boundary layers at the surface of the crystal. This repulsive force is described at the mesoscopic level by a diffusion term introduced in the equations satisfied by the dislocation densities. For

$$x \in I := (-1, 1),$$

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and $t \in (0, T)$, for some time $T > 0$, the non-negative quantities $\theta^+(x, t)$ and $\theta^-(x, t)$ are governed by the following system (see [12]):

$$\begin{cases} \theta_t^+ = \left[\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau \right) \theta^+ \right]_x & \text{in} \quad I \times (0, T), \\ \theta_t^- = \left[-\left(\dfrac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau \right) \theta^- \right]_x & \text{in} \quad I \times (0, T), \end{cases} \tag{1.1}$$

where τ is the applied shear stress which is assumed to be constant. Here the term $\tau_b = \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-}$ is called the back stress and can be interpreted as the contribution to the stress of the short-range interactions between dislocations. This back stress can be seen as a diffusion term in the above equations. If $\tau = 0$ and $\theta^- = 0$ (resp. $\theta^+ = 0$) then θ^+ (resp. θ^-) satisfies the usual heat equation. More generally, the back stress is proportional to the gradient of the effective dislocation density $\theta^+ - \theta^-$, with a diffusion coefficient which is $\frac{1}{\theta^+ + \theta^-}$. In fact, system (1.1) is a model for a 2D channel with coordinates (x, y) that is invariant in the y-direction (see Figure 1). The channel is bounded by walls that are impenetrable by dislocations (i.e., the plastic deformation in the
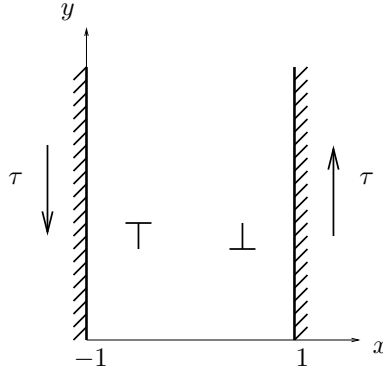

Figure 1: Geometry of the crystal.

walls is zero). In this case the boundary conditions are represented by the zero flux condition, i.e.

$$\frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau = 0, \quad \text{at} \quad x = \pm 1. \tag{1.2}$$

For related literature, let us mention the work of Groma-Balogh [11], where the back stress was neglected. For the model described in [11], we refer the reader to [5, 6] for a one-dimensional mathematical and numerical study, and to [4] for a two-dimensional existence result. The special case $\tau = 0$ for system (1.1) has been studied in [15], where a result of existence and uniqueness has been proved. In the present paper we study the case for general constant $\tau \in \mathbb{R}$.

1.2 Setting of the problem

We consider an integrated form of (1.1) and we let

$$\rho_x^\pm = \theta^\pm, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-,$$

to obtain, for special values of the constants of integration, the following system in terms of ρ and κ:

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{on} \quad I \times (0, T) \\ \rho_t = \rho_{xx} - \tau \kappa_x & \text{on} \quad I \times (0, T), \end{cases} \tag{1.3}$$

with the initial conditions:

$$\kappa(x, 0) = \kappa^0(x) \quad \text{and} \quad \rho(x, 0) = \rho^0(x). \tag{1.4}$$

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To formulate heuristically the boundary conditions at the walls located at $x = \pm 1$, we first suppose that $\kappa_x \neq 0$ at $x = \pm 1$. We recall that the dislocation fluxes at the walls must be zero, which require (1.2). Rewriting system (1.3) in terms of ρ, κ and $\Phi = \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-} - \tau$, we get

$$\begin{cases} \kappa_t = \rho_x \Phi, \\ \rho_t = \kappa_x \Phi. \end{cases} \tag{1.5}$$

From (1.2) and (1.5), we deduce that

$$\rho_t(x,.) = \kappa_t(x,.) = 0 \quad \text{for} \quad x = \pm 1. \tag{1.6}$$

In this case, we consider the following boundary conditions:

$$\begin{cases} \kappa(x,.) = x \quad & \text{for} \quad x = \pm 1 \\ \rho(x,.) = 0 \quad & \text{for} \quad x = \pm 1, \end{cases} \tag{1.7}$$

where we have taken the zero normalization for ρ on the boundary of the interval.

The non-negativity of $\theta^\pm \geq 0$ reduces in terms of ρ and κ to the following condition:

$$\kappa_x \geq |\rho_x|, \tag{1.8}$$

and hence a natural assumption to be considered concerning the initial conditions ρ^0 and κ^0 is to satisfy

$$\kappa_x^0 \geq |\rho_x^0| \quad \text{on} \quad I. \tag{1.9}$$

As indicated above, problem (1.3), (1.4) and (1.7), in the case $\tau = 0$, has been studied in [15], where a result of existence and uniqueness is given using the viscosity/entropy solution framework. Let us just mention that in this situation, system (1.3) becomes decoupled and easier to be handled.

1.3 Statement of the main result

Remark that the first equation of system (1.3) can be formally rewritten $\kappa_t = \rho_t \rho_x / \kappa_x$ which shows the singularity as κ_x goes to zero. Nevertheless, for this system we have the following result.

Theorem 1.1 (Global existence of a solution). *Let $\rho^0, \kappa^0 \in C^\infty(\bar{I})$ satisfying (1.9),*

$$\kappa^0(\pm 1) = \pm 1, \quad \rho^0(\pm 1) = 0, \tag{1.10}$$

and the additional conditions:

$$\rho_x^0(x) = \rho_{xx}^0(x) = \kappa_x^0(x) = \kappa_{xx}^0(x) = 0 \quad \text{for} \quad x = \pm 1. \tag{1.11}$$

Then there exists (ρ, κ) such that for every $T > 0$:

$$(\rho, \kappa) \in (C(\bar{I} \times [0,T]))^2 \quad \text{and} \quad \rho \in C^1(I \times (0,T)),$$

is a solution of (1.3), (1.4) and (1.7). Moreover, this solution satisfies (1.8) in the distributional sense, i.e.

$$\kappa_x \geq |\rho_x| \quad \text{in} \quad \mathcal{D}'(I \times (0,T)). \tag{1.12}$$

However, the solution has to be interpreted in the following sense:

1. κ is a viscosity solution of $\kappa_t \kappa_x = \rho_t \rho_x$ in $I_T := I \times (0,T)$,

2. ρ is a distributional solution of $\rho_t = \rho_{xx} - \tau \kappa_x$ in I_T,

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3. the initial and boundary conditions are satisfied pointwisely.

Remark 1.2 *The C^∞ regularity of ρ^0 and κ^0, together with (1.11) seems to be essentially technical and are related to an existence result for a regularized system (see Section 3, Theorem 3.1), that we use to prove Theorem 1.1.*

Theorem 1.1 states the global existence of a solution. Remark that the uniqueness of the solution (even for a short time analysis) remains an open question, which seems difficult to handle in the general case. Moreover, we do not know any counter-example to the uniqueness of the solution of system (1.3), (1.4) and (1.7). Indeed, even in the special case $\tau = 0$, the uniqueness remains an open question except for some particular initial data (see [15]). Let us stress on the fact that the vanishing of the quantity κ_x is one of the main difficulties for dealing with the uniqueness of the solution.

1.4   Organization of the paper

This paper is organized as follows: in Section 2, we present the strategy of the proof. In Section 3, we present the tools needed throughout this work. This includes some miscellaneous results for parabolic equations, a brief recall to the definition and the stability result of viscosity solutions, and a brief recall to Orlicz spaces. In Section 4, we show how to choose the regularized solution. An entropy inequality used to determine some uniform bounds on the regularized solution is presented in Section 5. Further uniform bounds and convergence arguments are done in Section 6. Section 7 is devoted to the proof of our main result: Theorem 1.1. In Section 8, some numerical simulations related to our physical model are presented. Finally, Section 9 is an appendix where we show the proofs of some technical results.

2   Strategy of the proof

The main difficulty we have to face is to work with the equation

$$\kappa_t \kappa_x = \rho_t \rho_x. \tag{2.1}$$

Since ρ solves itself a parabolic equation (see the second equation of (1.3)), we expect enough regularity on ρ (indeed ρ is C^1), and then we need a framework where the equation involving κ is stable under approximation. This property is naturally satisfied in the framework of viscosity solutions (see for instance [2] and the references therein). Then, assuming $\kappa_x \geq 0$, we interpret κ as the viscosity solution of (2.1). Assuming (1.9), we will indeed show that

$$M := \kappa_x - |\rho_x| \geq 0.$$

This is formally true because M satisfies:

$$M_t = bM_x + cM,$$

with

$$b = \tau \operatorname{sgn}(\rho_x) - \frac{\rho_x \rho_{xx}}{\kappa_x^2}, \quad c = \frac{\rho_{xx}^2}{\kappa_x^2} - \frac{\rho_{xxx} \operatorname{sgn}(\rho_x)}{\kappa_x},$$

where for suitable boundary conditions, we can (again formally) see that $M \geq 0$. In order to justify the computations on M, we modify the system and we consider the following parabolic regularization for $\varepsilon > 0$ small enough:

$$\begin{cases} \kappa_t^\varepsilon = \varepsilon \kappa_{xx}^\varepsilon + \dfrac{\rho_x^\varepsilon \rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} - \tau \rho_x^\varepsilon & \text{in} \quad I \times (0, \infty) \\[2mm] \rho_t^\varepsilon = (1 + \varepsilon)\rho_{xx}^\varepsilon - \tau \kappa_x^\varepsilon & \text{in} \quad I \times (0, \infty), \end{cases} \tag{2.2}$$

with the initial conditions:

$$\kappa^\varepsilon(x, 0) = \kappa^{0,\varepsilon}(x), \quad \rho^\varepsilon(x, 0) = \rho^{0,\varepsilon}(x), \tag{2.3}$$

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where $\kappa^{0,\varepsilon}$ and $\rho^{0,\varepsilon}$ are some regularizations of κ^0 and ρ^0 respectively, and the same boundary conditions:

$$\begin{cases} \kappa^\varepsilon(x,.) = x & \text{for} \quad x = \pm 1 \\ \rho^\varepsilon(x,.) = 0 & \text{for} \quad x = \pm 1. \end{cases} \tag{2.4}$$

The system (2.2) formally reduces to (1.3) for $\varepsilon = 0$. Notice that system (2.2) can be viewed as the following natural parabolic regularization of system (1.3) (written in terms of $\theta^{\pm,\varepsilon}$ with a similar change of unknown functions $\theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}$):

$$\theta_t^{\pm,\varepsilon} = \varepsilon\theta_{xx}^{\pm,\varepsilon} \pm \left(\left(\frac{\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau \right) \theta^{\pm,\varepsilon} \right)_x.$$

This particular choice of regularization will guarantee later good *a priori* estimates (like the entropy estimate), and will also be good enough to make survive a comparison principle for the quantity $\kappa_x - |\rho_x| > 0$ (see (3.5)). This comparison principle is a key property to allow the global existence of solutions of the regularized system (2.2). Let us mention that among several regularizations that we have tried, the regularization (2.2) is the only one that was good enough to make the proof of Theorem 1.1.

System (2.2), (2.3) and (2.4) has (under some conditions on the initial and boundary data) a unique smooth global solution (see [16, Theorem 1.1]) for $\alpha \in (0,1)$:

$$(\rho^\varepsilon, \kappa^\varepsilon) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{I} \times [0,\infty)) \cap C^\infty(\bar{I} \times (0,\infty)).$$

This result will be recalled in the forthcoming section (see Section 3, Theorem 3.1). The next step is to find some uniform bounds (independent of ε) on this solution; this is done in particular via:

(1) an entropy inequality shown to be valid for our regularized model (2.2);

(2) a bound on $\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon$ uniformly in ε.

In fact, (1) guarantees the global uniform-in-time control of the modulus of continuity in space of our regularized solution, while (2) guarantees the local uniform-in-space control of the modulus of continuity in time. The entropy inequality can be easily understood. For instance, for $\varepsilon = 0$ and $\tau = 0$, we can formally check that the entropy of the dislocation densities

$$\theta^\pm = \frac{\kappa_x \pm \rho_x}{2},$$

defined by:

$$S(t) = \int_I \sum_\pm \theta^\pm(.,t) \log(\theta^\pm(.,t)),$$

satisfies:

$$\frac{dS(t)}{dt} = -\int_I \frac{(\theta_x^+ - \theta_x^-)^2}{\theta^+ + \theta^-} \leq 0,$$

therefore we get $S(t) \leq S(0)$ which controls the entropy uniformly in time. Finally, we need to pass to the limit $\varepsilon \to 0$ after multiplying the first equation of (2.2) by κ_x^ε. Having enough control on the regularized solutions, we can find a solution of the limit equation using in particular the stability of viscosity solutions of Hamilton-Jacobi equations. However, the passage to the limit in the second equation of (2.2) is done in the distributional sense.

3 Tools: miscellaneous results on parabolic equations, viscosity solution, and Orlicz spaces

3.1 Miscellaneous results on parabolic equations

We first fix some notations. Denote

$$I_T := I \times (0,T), \quad \overline{I_T} := \bar{I} \times [0,T], \quad \partial^p I_T := I \cup (\partial I \times [0,T]), \quad \text{and} \quad \|\cdot\|_{L^p(\Omega)} = \|\cdot\|_{p,\Omega}.$$

Define the parabolic Sobolev space $W_p^{2,1}(I_T)$, $1 < p < \infty$ by:

$$W_p^{2,1}(I_T) := \left\{ u \in L^p(I_T); \ (u_t, u_x, u_{xx}) \in (L^p(I_T))^3 \right\}.$$

We start with a result of global existence and uniqueness of smooth solutions of the regularized system (2.2), with the initial and boundary conditions (2.3) and (2.4).

Theorem 3.1 *(Global existence for the regularized system, [16, Theorem 1.1]). Let $0 < \alpha < 1$ and $0 < \varepsilon < 1$. Let $\rho^{0,\varepsilon}$, $\kappa^{0,\varepsilon}$ satisfying:*

$$\rho^{0,\varepsilon}, \kappa^{0,\varepsilon} \in C^\infty(\bar{I}), \quad \rho^{0,\varepsilon}(\pm 1) = 0, \quad and \quad \kappa^{0,\varepsilon}(\pm 1) = \pm 1, \tag{3.1}$$

$$\begin{cases} (1+\varepsilon)\rho_{xx}^{0,\varepsilon} = \tau \kappa_x^{0,\varepsilon} & on \quad \partial I \\ (1+\varepsilon)\kappa_{xx}^{0,\varepsilon} = \tau \rho_x^{0,\varepsilon} & on \quad \partial I, \end{cases} \tag{3.2}$$

and

$$\kappa_x^{0,\varepsilon} > |\rho_x^{0,\varepsilon}| \quad on \quad \bar{I}. \tag{3.3}$$

Then there exists a unique global solution

$$(\rho^\varepsilon, \kappa^\varepsilon) \in C^{3+\alpha, \frac{3+\alpha}{2}}(\bar{I} \times [0, \infty)) \cap C^\infty(\bar{I} \times (0, \infty)), \tag{3.4}$$

of the system (2.2), (2.3) and (2.4). Moreover, this solution satisfies :

$$\kappa_x^\varepsilon > |\rho_x^\varepsilon| \quad on \quad \bar{I} \times [0, \infty). \tag{3.5}$$

Remark 3.2 *Conditions (3.2) are natural here. Indeed, the regularity (3.4) of the solution of equation (2.2) with boundary conditions (2.3) and (2.4) imply in particular condition (3.2).*

Remark 3.3 *(Uniform L^∞ bound on ρ^ε and κ^ε). We remark, from the boundary conditions (2.4) and from the inequality (3.5), that:*

$$\|\rho^\varepsilon\|_{L^\infty(\bar{I} \times [0, \infty))} \leq 1 \quad and \quad \|\kappa^\varepsilon\|_{L^\infty(\bar{I} \times [0, \infty))} \leq 1. \tag{3.6}$$

We now present two technical lemmas that will be used in the proof of Theorem 1.1. The proofs of these lemmas will be given in the Appendix.

Lemma 3.4 *(Control of the modulus of continuity in time uniformly in ε). Let $p > 3$, and $u^\varepsilon \in W_p^{2,1}(I_T)$. Suppose furthermore that the sequences*

$$(u^\varepsilon)_\varepsilon \quad and \quad (f^\varepsilon)_\varepsilon = (u_t^\varepsilon - \varepsilon u_{xx}^\varepsilon)_\varepsilon,$$

are locally bounded in I_T uniformly for $\varepsilon \in (0, 1)$. Then for every $V \subset\subset I_T$, there exist two constants $c > 0$, $\varepsilon_0 > 0$ depending on V, and $0 < \beta < 1$ such that for all $0 < \varepsilon < \varepsilon_0$:

$$\frac{|u^\varepsilon(x, t+h) - u^\varepsilon(x, t)|}{h^\beta} \leq c, \quad \forall (x,t), (x, t+h) \in V.$$

Lemma 3.5 *(An interior estimate for the heat equation). Let $a \in C^\infty(I_T) \cap L^1(I_T)$ satisfying:*

$$a_t = a_{xx} \quad on \quad I_T,$$

then for any $V \subset\subset I_T$, an open set, we have:

$$\|a\|_{p,V} \leq c\|a\|_{1,I_T}, \quad \forall 1 < p < \infty,$$

where $c = c(p, V) > 0$ is a positive constant.

3.2 Viscosity solution: definition and stability result

Let $\Omega \subset \mathbb{R}^n$ be an open domain, and consider the following Hamilton-Jacobi equation:

$$H(x, u(x), Du(x), D^2u(x)) = 0, \quad \forall x \in \Omega, \tag{3.7}$$

where $H : \Omega \times \mathbb{R} \times \mathbb{R}^n \times M_{sym}^{n \times n} \to \mathbb{R}$ is a continuous mapping.

Definition 3.6 *(Viscosity solution of Hamilton-Jacobi equations). A continuous function $u : \Omega \mapsto \mathbb{R}$ is a viscosity sub-solution of (3.7) if for any $\phi \in C^2(\Omega; \mathbb{R})$ and any local maximum $x_0 \in \Omega$ of $u - \phi$, one has*

$$H(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

Similarly, u is a viscosity super-solution of (3.7), if at any local minimum point $x_0 \in \Omega$ of $u - \phi$, one has

$$H(x_0, u(x_0), D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

Finally, if u is both a viscosity sub-solution and a viscosity super-solution, then u is called a viscosity solution.

To get a "non-empty" and useful definition, it is usually assumed that H is elliptic (see [2]). This notion of ellipticity will be indirectly used in Section 7. In fact, this definition is used for interpreting solutions of the first equation of (1.3) in the viscosity sense. This will be shown in Section 5. To be more precise, in the case where $\Omega = I_T$, we say that u is a viscosity solution of the Dirichlet problem (3.7) with $u = \zeta \in C(\partial^p I_T)$ if:

(1) $u \in C(\overline{I_T})$,

(2) u is a viscosity solution of (3.7) in I_T,

(3) $u = \zeta$ on $\partial^p I_T$.

For a better understanding of the viscosity interpretation of boundary conditions of Hamilton-Jacobi equations, we refer the reader to [2, Section 4.2]. We now state a stability result for viscosity solutions of Hamilton-Jacobi equations.

Theorem 3.7 *(Stability of viscosity solutions, [2, Lemma 2.3]). Suppose that, for $\varepsilon > 0$, $u^\varepsilon \in C(\Omega)$ is a viscosity sub-solution (resp. super-solution) of the equation*

$$H^\varepsilon(x, u^\varepsilon, Du^\varepsilon, D^2u^\varepsilon) = 0 \quad in \quad \Omega,$$

where $(H^\varepsilon)_\varepsilon$ is a sequence of continuous functions. If $u^\varepsilon \to u$ locally uniformly in Ω and if $H^\varepsilon \to H$ locally uniformly in $\Omega \times \mathbb{R} \times \mathbb{R}^n \times M_{sym}^{n \times n}$, then u is a viscosity sub-solution (resp. super-solution) of the equation:

$$H(x, u, Du, D^2u) = 0 \quad in \quad \Omega.$$

3.3 Orlicz spaces: definition and properties

We recall the definition of an Orlicz space and some of its properties (for details see [1]). A real valued function $\Psi : [0, \infty) \to \mathbb{R}$ is called a Young function if

$$\Psi(t) = \int_0^t \psi(s)ds,$$

where $\psi : [0, \infty) \to [0, \infty)$ satisfying:

- $\psi(0) = 0$, $\psi > 0$ on $(0, \infty)$, $\psi(t) \to \infty$ as $t \to \infty$;

- ψ is non-decreasing and right continuous at any point $s \geq 0$.

Let Ψ be a Young function. The Orlicz class $K_\Psi(I)$ is the set of equivalence classes of real-valued measurable functions u on I satisfying

$$\int_I \Psi(|u(x)|)dx < +\infty.$$

Definition 3.8 (Orlicz spaces). *The Orlicz space $L_\Psi(I)$ is the linear span of $K_\Psi(I)$ supplemented with the Luxemburg norm*

$$\|u\|_{L_\Psi(I)} = \inf \left\{ k > 0; \ \int_I \Psi \left(\frac{|u(x)|}{k} \right) \leq 1 \right\}, \tag{3.8}$$

and with this norm, the Orlicz space is a Banach space.

The function

$$\Phi(t) = \int_0^t \phi(s)ds, \quad \phi(s) = \sup_{\psi(t) \leq s} t,$$

is called the complementary Young function of Ψ. An example of such pair of complementary Young functions is the following:

$$\Psi(s) = (1+s)\log(1+s) - s \quad \text{and} \quad \Phi(s) = e^s - s - 1. \tag{3.9}$$

We now state a lemma giving two useful properties of Orlicz spaces that will be used in the proof of Lemma 5.4.

Lemma 3.9 (Norm control and Hölder inequality, [17]). *If $u \in L_\Psi(I)$ for some Young function Ψ, then we have:*

$$\|u\|_{L_\Psi(I)} \leq 1 + \int_I \Psi(|u(x)|)dx. \tag{3.10}$$

Moreover, if $v \in L_\Phi(I)$, Φ being the complementary Young function of Ψ, then we have the following Hölder inequality:

$$\left| \int_I uvdx \right| \leq 2\|u\|_{L_\Psi(I)}\|v\|_{L_\Phi(I)}. \tag{3.11}$$

4 The regularized problem

As we have already mentioned, we will use a parabolic regularization of (1.3), and a result of global existence of this regularized system from [16] (see Theorem 3.1). In order to use this result, we need to give a special attention to the conditions on the initial data of the approximated system $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ (see (3.1), (3.2) and (3.3)). This section aims to show how to choose the suitable initial data $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ in order to benefit Theorem 3.1. Let ρ^0 and κ^0 be the functions given in Theorem 1.1. Set

$$\rho^{0,\varepsilon}(x) = \frac{\rho^0(x) + \varepsilon\tau\phi(x)}{(1+\varepsilon)^2} \quad \text{and} \quad \kappa^{0,\varepsilon}(x) = \frac{\kappa^0(x) + \varepsilon x}{1+\varepsilon}, \tag{4.1}$$

with the function ϕ defined by:

$$\phi(x) = \begin{cases} \frac{1}{4\tau^2}[1 - \cos\tau(x^2 - 1)] & \text{if } \tau \neq 0 \\ 0 & \text{if } \tau = 0. \end{cases} \tag{4.2}$$

The function ϕ enjoys some properties that are shown in the following lemma.

Lemma 4.1 (Properties of ϕ)
The function ϕ given by (4.2) satisfies the following properties:

(P1) $\phi, \phi'|_{\partial I} = 0$;

(P2) $\phi''|_{\partial I} = 1$;

(P3) $|\phi'(x)| < 1/|\tau|$ for $x \in \bar{I}$.

Proof. Straightforward computations. □

Form the above lemma, and from the construction of $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ (see (4.1)) together with the properties enjoyed by ρ^0 and κ^0 (see (1.10) and (1.11)), we write down some properties of $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$.

Lemma 4.2 (Properties of $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$)
The functions $\rho^{0,\varepsilon}$ and $\kappa^{0,\varepsilon}$ given by (4.1) , satisfy the following properties:

(P4) $\rho^{0,\varepsilon}(\pm 1) = \pm 1, \quad and \quad \kappa^{0,\varepsilon}(\pm 1) = 0$;

(P5) $(1+\varepsilon)\kappa_{xx}^{0,\varepsilon}\big|_{\partial I} = \tau \rho_x^{0,\varepsilon}\big|_{\partial I} \quad and \quad (1+\varepsilon)\rho_{xx}^{0,\varepsilon}\big|_{\partial I} = \tau \kappa_x^{0,\varepsilon}\big|_{\partial I}$;

(P6) $\kappa_x^{0,\varepsilon} \geq |\rho_x^{0,\varepsilon}| + \dfrac{\varepsilon(1 - |\tau||\phi'|)}{1+\varepsilon} > |\rho_x^{0,\varepsilon}|$.

Proof. Straightforward computations. □

Remark 4.3 (The regularized solution $(\rho^\varepsilon, \kappa^\varepsilon)$). *Properties (P4)-(P5)-(P6) of Lemma 4.2 implies condition (3.1)-(3.2)-(3.3) of Theorem 3.1. In this case, call*

$$(\rho^\varepsilon, \kappa^\varepsilon), \tag{4.3}$$

the solution of (2.2), (2.3) and (2.4), given in Theorem 3.1, with the initial conditions

$$\rho(x,0) = \rho^{0,\varepsilon} \quad and \quad \kappa(x,0) = \kappa^{0,\varepsilon},$$

that are given by (4.1).

5 Entropy inequality

Proposition 5.1 (Entropy inequality). *Let $(\rho^\varepsilon, \kappa^\varepsilon)$ be the regular solution given by (4.3). Define $\theta^{\pm,\varepsilon}$ by:*

$$\theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}, \tag{5.1}$$

then the quantity $S(t)$ given by:

$$S(t) = \int_I \sum_\pm \theta^{\pm,\varepsilon}(x,t) \log \theta^{\pm,\varepsilon}(x,t) dx, \tag{5.2}$$

satisfies for every $t \geq 0$:

$$S(t) \leq S(0) + \frac{\tau^2 t}{2}. \tag{5.3}$$

Proof. From (3.5), we know that $\kappa_x^\varepsilon > |\rho_x^\varepsilon|$, hence $\theta^{\pm,\varepsilon} > 0$, and the term $\log(\theta^{\pm,\varepsilon})$ is well defined. Also from the regularity (3.4) of the solution $(\rho^\varepsilon, \kappa^\varepsilon)$, we know that $\theta^{\pm,\varepsilon}(.,t) \in C(\bar{I})$ for all $t \geq 0$, hence the term $S(t)$ is well defined. We differentiate system (2.2) with respect to x, and we write it in terms of $\theta^{\pm,\varepsilon}$, we get:

$$\begin{cases} \theta_t^{+,\varepsilon} = \left[\left(\dfrac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau \right) \theta^{+,\varepsilon} + \varepsilon\theta_x^{+,\varepsilon} \right]_x \\[4mm] \theta_t^{-,\varepsilon} = \left[-\left(\dfrac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau \right) \theta^{-,\varepsilon} + \varepsilon\theta_x^{-,\varepsilon} \right]_x. \end{cases} \tag{5.4}$$

We first remark that:

$$\pm \left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau \right) \theta^{\pm,\varepsilon} + \varepsilon\theta_x^{\pm,\varepsilon} = \frac{\kappa_t^\varepsilon \pm \rho_t^\varepsilon}{2}.$$

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Since κ_t^ε and ρ_t^ε are zeros on $\partial I \times [0, \infty)$, then

$$\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{+,\varepsilon} + \varepsilon\theta_x^{+,\varepsilon} = -\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{-,\varepsilon} + \varepsilon\theta_x^{-,\varepsilon} = 0 \text{ on } \partial I \times [0, \infty). \tag{5.5}$$

Using (5.5), we compute for $t \geq 0$:

$$
\begin{aligned}
S'(t) &= \sum_\pm \int_I \theta_t^{\pm,\varepsilon} \log(\theta^{\pm,\varepsilon}) + \theta_t^{\pm,\varepsilon}, \\
&= \sum_\pm \int_I \mp\left(\frac{(\theta^{+,\varepsilon} - \theta^{-,\varepsilon})_x}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta_x^{\pm,\varepsilon} - \varepsilon\frac{\left(\theta_x^{\pm,\varepsilon}\right)^2}{\theta^{\pm,\varepsilon}}, \\
&= \int_I -\frac{\left(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right)^2}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} + \tau(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}) - \varepsilon\left(\frac{\left(\theta_x^{+,\varepsilon}\right)^2}{\theta^{+,\varepsilon}} + \frac{\left(\theta_x^{-,\varepsilon}\right)^2}{\theta^{-,\varepsilon}}\right),
\end{aligned}
$$

where we have integrated by parts in the second line. By Young's Inequality, we have:

$$\left|\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right| \leq \frac{1}{|\tau|}\frac{\left(\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}\right)^2}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} + \frac{|\tau|}{4}(\theta^{+,\varepsilon} + \theta^{-,\varepsilon}),$$

and hence

$$S'(t) \leq \frac{\tau^2}{4}\int_I (\theta^{+,\varepsilon} + \theta^{-,\varepsilon}).$$

Moreover, we have from (2.4), that

$$\int_I (\theta^{+,\varepsilon}(.,t) + \theta^{-,\varepsilon}(.,t)) = \int_I \kappa_x^\varepsilon(.,t) = \kappa^\varepsilon(1,t) - \kappa^\varepsilon(-1,t) = 2,$$

and therefore

$$S'(t) \leq \frac{\tau^2}{2}.$$

Integrating the previous inequality from 0 to t, we get (5.3). $\qquad\square$

An immediate corollary of Proposition 5.1 is the following:

Corollary 5.2 (Special control of κ_x^ε)
For all $t \geq 0$, we have:

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t))dx \leq S(0) + \frac{\tau^2 t}{2} + 2, \tag{5.6}$$

where S is given by (5.2).

The proof of Corollary 5.2 comes from the following inequality.

Lemma 5.3 *For every $x, y > 0$, we have:*

$$(x + y)\log(x + y) \leq x\log(x) + y\log(y) + x\log(2) + y. \tag{5.7}$$

Proof. Direct computations. $\qquad\square$

Proof of Corollary 5.2. From (5.1), it follows that

$$\kappa_x^\varepsilon = \theta^{+,\varepsilon} + \theta^{-,\varepsilon} > 0.$$

Then we have for $t \geq 0$:

$$
\begin{aligned}
\int_I \kappa_x^\varepsilon \log \kappa_x^\varepsilon &= \int_I (\theta^{+,\varepsilon} + \theta^{-,\varepsilon}) \log(\theta^{+,\varepsilon} + \theta^{-,\varepsilon}) \\
&\leq \int_I \theta^{+,\varepsilon} \log(\theta^{+,\varepsilon}) + \theta^{-,\varepsilon} \log(\theta^{-,\varepsilon}) + \theta^{+,\varepsilon} \log 2 + \theta^{-,\varepsilon} \\
&\leq \left(\int_I \theta^{+,\varepsilon} \log(\theta^{+,\varepsilon}) + \theta^{-,\varepsilon} \log(\theta^{-,\varepsilon}) \right) + \log 2 + 1 \\
&\leq S(t) + 2.
\end{aligned}
$$

Here we have used Lemma 5.3 with $x = \theta^{+,\varepsilon}$ and $y = \theta^{-,\varepsilon}$ for the second line, and we have used for the third line, the fact that

$$
\int_I \theta^{\pm,\varepsilon} = \frac{1}{2} \int_I \kappa_x^\varepsilon \pm \rho_x^\varepsilon = \frac{1}{2}[\kappa^\varepsilon(1,.) - \kappa^\varepsilon(-1,.)] = 1.
$$

Using (5.3), the result follows. □

Lemma 5.4 *(Control of the modulus of continuity in space)*
Let $u \in C^1(I)$, $u_x > 0$, satisfying

$$
\int_I u_x \log(u_x) \leq c_1,
$$

for some positive constant c_1, then we have for any x, $x + h \in I$ with $h > 0$:

$$
|u(x+h) - u(x)| \leq \frac{2(c_1 + 1 + \log 2)}{|\log h|}. \tag{5.8}
$$

Proof. Let $x, x + h \in I$.

Step 1. $(u_x \in L_\Psi(x, x+h)$ with Ψ given in (3.9))

We compute

$$
\begin{aligned}
\int_x^{x+h} \Psi(u_x) &\leq \int_I (1 + u_x) \log(1 + u_x) - u_x \\
&\leq \int_I u_x \log(u_x) + \log 2 \leq c_1 + \log 2,
\end{aligned}
$$

where we have used (5.7) in the second line inequality. Hence from (3.10), we get

$$
\|u_x\|_{L_\Psi(x,x+h)} \leq c_1 + 1 + \log 2.
$$

Step 2. (Estimating the modulus of continuity)

It is easy to check that the function 1 lies in $L_\Phi(x, x+h)$ for Φ given by (3.9), and that $\|1\|_{L_\Phi(x,x+h)} \leq -\frac{1}{\log h}$. Therefore, by Hölder inequality (3.11), we obtain:

$$
\begin{aligned}
|u(x+h) - u(x)| &= \left| \int_x^{x+h} u_x.1 \right| \\
&\leq 2\|u_x\|_{L_\psi(x,x+h)} \|1\|_{L_\Phi(x,x+h)} \leq \frac{2(c_1 + 1 + \log 2)}{|\log h|},
\end{aligned}
$$

and the result follows. □

Remark 5.5 *As mentioned to us by Jérôme Droniou, it is possible to estimate directly the quantity $|u(x + h) - u(x)| \leq \frac{A}{|\log h|}$ by splitting the integral $\int_x^{x+h} u_x$ on the set where u_x is bigger and lower than λ, and then optimizing on the parameter λ.*

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6 An interior estimate

In this section, we give an interior estimate for the term

$$A^\varepsilon = \rho_x^\varepsilon - \tau\kappa^\varepsilon. \tag{6.1}$$

that will be used in the passage to the limit as ε goes to zero in the regularized system. We start by deriving an equation satisfied by A^ε:

$$A_t^\varepsilon = (1+\varepsilon)A_{xx}^\varepsilon - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon. \tag{6.2}$$

We now show an interior L^p estimate concerning the term A^ε. This estimate gives a control on the local L^p norm of A^ε by its global L^1 norm over I_T, and it will be used in the following section. More precisely, we have the following lemma.

Lemma 6.1 (Interior L^p estimate). *Let $0 < \varepsilon < 1$ and $1 < p < \infty$. Then the quantity A^ε given by (6.1) satisfies:*

$$\|A^\varepsilon\|_{p,V} \le c\left(\|A^\varepsilon\|_{1,I_T} + 1\right), \tag{6.3}$$

where V is an open subset of I_T such that $V \subset\subset I_T$, and $c = c(p,V) > 0$ is a constant independent of ε.

Proof. Throughout the proof, the term $c = c(p,V) > 0$ is a positive constant independent of ε, and it may vary from line to line. A simple computation gives:

$$
\begin{aligned}
-\tau\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon &= -\tau\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}(\rho_{xx}^\varepsilon - \tau\kappa_x^\varepsilon) \\
&= -\tau\frac{\rho_x^\varepsilon\rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} + \tau^2\rho_x^\varepsilon \\
&= -\tau(\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon).
\end{aligned}
\tag{6.4}
$$

Define $\bar\kappa^\varepsilon$ as the unique solution of

$$
\begin{cases}
\bar\kappa_t^\varepsilon = (1+\varepsilon)\bar\kappa_{xx}^\varepsilon + \kappa^\varepsilon & \text{on} \quad I_T, \\
\bar\kappa^\varepsilon = 0 & \text{on} \quad \partial^p I_T,
\end{cases}
\tag{6.5}
$$

where the existence and uniqueness of this equation is a direct consequence of the L^p theory for parabolic equations (see for instance [18, Theorem 9.1]) using in particular the fact that $\kappa^\varepsilon \in C^1(\overline{I_T})$. Moreover, from the regularity (3.4) of κ^ε, we can deduce that $\bar\kappa^\varepsilon \in C^\infty(I_T)$. Let

$$\bar{A}^\varepsilon = -\tau(\bar\kappa_t^\varepsilon - \varepsilon\bar\kappa_{xx}^\varepsilon), \quad \text{with} \quad a^\varepsilon = A^\varepsilon - \bar{A}^\varepsilon.$$

We calculate:

$$
\begin{aligned}
\bar{A}_t^\varepsilon &= -\tau[\bar\kappa_{tt}^\varepsilon - \varepsilon\bar\kappa_{xxt}^\varepsilon] \\
&= -\tau[(1+\varepsilon)\bar\kappa_{xxt}^\varepsilon + \kappa_t^\varepsilon - \varepsilon((1+\varepsilon)\bar\kappa_{xxxx}^\varepsilon + \kappa_{xx}^\varepsilon)] \\
&= -\tau(1+\varepsilon)(\bar\kappa_{xxt}^\varepsilon - \varepsilon\bar\kappa_{xxxx}^\varepsilon) - \tau(\kappa_t^\varepsilon - \varepsilon\kappa_{xx}^\varepsilon) \\
&= (1+\varepsilon)\bar{A}_{xx} - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon,
\end{aligned}
$$

where for the first two line, we have used (6.5), and for the last line, we have used (6.4). In this case, we obtain:

$$
\begin{aligned}
a_t^\varepsilon &= A_t^\varepsilon - \bar{A}_t^\varepsilon \\
&= (1+\varepsilon)A_{xx}^\varepsilon - \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon - (1+\varepsilon)\bar{A}_{xx} + \frac{\tau\rho_x^\varepsilon}{\kappa_x^\varepsilon}A_x^\varepsilon \\
&= (1+\varepsilon)(A_{xx}^\varepsilon - \bar{A}_{xx}^\varepsilon) = (1+\varepsilon)a_{xx}^\varepsilon,
\end{aligned}
$$

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where for the first line, we have used the equation (6.2). We apply Lemma 3.5 to the function a^ε, after doing parabolic rescaling of the form $\tilde{a}^\varepsilon(x,t) = a^\varepsilon\left(x, \frac{t}{1+\varepsilon}\right)$, we get:

$$\|a^\varepsilon\|_{p,V} \le c(1+\varepsilon)^{1-\frac{1}{p}}\|a^\varepsilon\|_{1,I_T},$$

and since $0 < \varepsilon < 1$, we finally obtain

$$\|a^\varepsilon\|_{p,V} \le c\|a^\varepsilon\|_{1,I_T}.$$

From the definition of a^ε, and the last inequality, we finally deduce that:

$$\|A^\varepsilon\|_{p,V} \le c(\|A^\varepsilon\|_{1,I_T} + \|\bar{A}^\varepsilon\|_{p,I_T}). \tag{6.6}$$

In order to complete the proof, we need to control the term $\|\bar{A}^\varepsilon\|_{p,I_T}$ in (6.6). We use the equation (6.5) satisfied by $\bar{\kappa}^\varepsilon$ to obtain:

$$\begin{aligned}
\|\bar{A}^\varepsilon\|_{p,I_T} &= \tau\|\bar{\kappa}^\varepsilon_t - \varepsilon\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} \\
&= \tau\|\bar{\kappa}^\varepsilon_{xx} + \kappa^\varepsilon\|_{p,I_T} \\
&\le c(\|\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} + \|\kappa^\varepsilon\|_{p,I_T}).
\end{aligned} \tag{6.7}$$

The L^p estimates for parabolic equations (see [16, Lemma 2.7]) applied to (6.5) gives:

$$\|\bar{\kappa}^\varepsilon_{xx}\|_{p,I_T} \le \frac{c}{1+\varepsilon}\|\kappa^\varepsilon\|_{p,I_T},$$

then (6.7), together with the fact that $0 \le \kappa^\varepsilon \le 1$ (see Remark 3.3), implies that:

$$\|\bar{A}^\varepsilon\|_{p,I_T} \le c\|\kappa^\varepsilon\|_{p,I_T} \le cT^{1/p},$$

hence the result follows. $\qquad\square$

7 Proof of the main theorem

At this stage, we are ready to present the proof of our main result (Theorem 1.1). This depends essentially on the passage to the limit in the family of solutions $(\rho^\varepsilon, \kappa^\varepsilon)$ of system (2.2). Since $\kappa^\varepsilon_x \ne 0$, we multiply the first equation of (2.2) by κ^ε_x and we rewrite system (2.2) in terms of A^ε, we obtain:

$$\begin{cases} \kappa^\varepsilon_t \kappa^\varepsilon_x = \varepsilon\kappa^\varepsilon_x\kappa^\varepsilon_{xx} + \rho^\varepsilon_x A^\varepsilon & \text{on} \quad I_T \\ \rho^\varepsilon_t = \varepsilon\rho^\varepsilon_{xx} + A^\varepsilon_x & \text{on} \quad I_T. \end{cases} \tag{7.1}$$

We will pass to the limit in the framework of viscosity solutions for the first equation of (7.1), and in the distributional sense for the second equation. We start with the following proposition.

Proposition 7.1 (Local uniform convergence)
The sequences $(\rho^\varepsilon)_\varepsilon$, $(\rho^\varepsilon_x)_\varepsilon$, $(\kappa^\varepsilon)_\varepsilon$, $(A^\varepsilon)_\varepsilon$ and $(A^\varepsilon_x)_\varepsilon$ converge (up to extraction of a subsequence) locally uniformly in I_T as ε goes to zero.

Proof. Let V be an open compactly contained subset of I_T. The constants that will appear in the proof are all independent of ε. However, they may depend on other fixed parameters including V. The idea is to give an ε-uniform control of the modulus of continuity in space and in time of the quantities mentioned in Proposition (7.1), which gives the local uniform convergence. The ε-uniform control on the space modulus of continuity will be derived from the Corollary 5.2 and Lemma 5.4, while the ε-uniform control on the time modulus of continuity will be derived from Lemma 3.4. The proof is divided into five steps.

Step 1. (Convergence of A^ε and A^ε_x)

From (3.5), we know that $\left\|\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}\right\|_\infty \leq 1$. We apply the interior L^p, $p > 1$, estimates for parabolic equations (see for instance [19, Theorem 7.13, page 172]) to the term A^ε satisfying (6.2), we obtain:

$$\|A^\varepsilon\|_{W_p^{2,1}(V)} \leq c_2 \|A^\varepsilon\|_{p,V'}, \tag{7.2}$$

where V' is any open subset of I_T satisfying $V \subset\subset V' \subset\subset I_T$. The constant $c_2 = c_2(p, \tau, V, V')$ can be chosen independent of ε first by applying a parabolic rescaling of (6.2), and then using the fact that the factor multiplied by A_{xx}^ε in (6.2) satisfying $1 \leq 1 + \varepsilon \leq 2$. At this point, we apply Lemma 6.1 for A^ε on V', we get:

$$\|A^\varepsilon\|_{p,V'} \leq c_3(\|A^\varepsilon\|_{1,I_T} + 1), \tag{7.3}$$

and hence the above two equations (7.2) and (7.3) give:

$$\|A^\varepsilon\|_{W_p^{2,1}(V)} \leq c_4(\|A^\varepsilon\|_{1,I_T} + 1). \tag{7.4}$$

We estimate the right hand side of (7.4) in the following way:

$$
\begin{aligned}
\|A^\varepsilon\|_{1,I_T} &= \int_{I_T} |\rho_x^\varepsilon - \tau \kappa^\varepsilon| \\
&\leq \int_{I_T} \kappa_x^\varepsilon + \tau |\kappa^\varepsilon| \\
&\leq (2 + \tau)T,
\end{aligned}
$$

where we have used the fact that $|\rho_x^\varepsilon| < \kappa_x^\varepsilon$ (see (3.5) of Theorem 3.1) in the second line, and the fact that $0 \leq |\kappa^\varepsilon| \leq 1$ (see Remark 3.3) in the last line. Therefore, inequality (7.4) implies:

$$\|A^\varepsilon\|_{W_p^{2,1}(V)} \leq c_5, \quad 1 < p < \infty. \tag{7.5}$$

We use the above inequality for $p > 3$. In this case, the Sobolev embedding in Hölder spaces (see [16, Lemma 2.8]) gives:

$$W_p^{2,1}(V) \hookrightarrow C^{1+\alpha, \frac{1+\alpha}{2}}(V), \quad \alpha = 1 - 3/p$$

and hence (7.5) implies:

$$\|A^\varepsilon\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(V)} \leq c_6, \tag{7.6}$$

which guarantees the equicontinuity and the equiboundedness of $(A^\varepsilon)_\varepsilon$ and $(A_x^\varepsilon)_\varepsilon$. By the Arzela-Ascoli Theorem (see for instance [3]), we finally obtain

$$A^\varepsilon \longrightarrow A \quad \text{and} \quad A_x^\varepsilon \longrightarrow A_x, \tag{7.7}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 2. (Convergence of κ^ε)

We control the modulus of continuity of κ^ε in space and in time, locally uniformly in ε.

Step 2.1. (Control of the modulus of continuity in time)

The first equation of (7.1) gives:

$$\kappa_t^\varepsilon = \varepsilon \kappa_{xx}^\varepsilon + \frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon} A_x^\varepsilon,$$

and hence, using the fact that $\left\|\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}\right\|_\infty \leq 1$, together with (7.6), we get:

$$\|\kappa_t^\varepsilon - \varepsilon \kappa_{xx}^\varepsilon\|_{\infty,V} \leq \left\|\frac{\rho_x^\varepsilon}{\kappa_x^\varepsilon}\right\|_{\infty,V} \|A_x\|_{\infty,V} \leq c_6. \tag{7.8}$$

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Also, by (3.6), we have:
$$\|\kappa^\varepsilon\|_{\infty,V} \leq 1.$$

This uniform bound on κ^ε together with (7.8) permit to use Lemma 3.4 to conclude that

$$|\kappa^\varepsilon(x,t) - \kappa^\varepsilon(x,t+h)| \leq c_7 h^\beta, \quad (x,t),(x,t+h) \in V, \quad 0 < \beta < 1, \tag{7.9}$$

which controls the modulus of continuity of κ^ε with respect to t uniformly in ε. We now move to control the modulus of continuity in space.

Step 2.2 (An ε-uniform bound on $S(0)$)

Recall the definition (5.2) of $S(t)$:

$$S(t) = \int_I \sum_\pm \theta^{\pm,\varepsilon}(x,t) \log \theta^{\pm,\varepsilon}(x,t) dx,$$

with

$$\theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}.$$

Hence

$$S(0) = \int_I \frac{\kappa_x^{0,\varepsilon} + \rho_x^{0,\varepsilon}}{2} \log\left(\frac{\kappa_x^{0,\varepsilon} + \rho_x^{0,\varepsilon}}{2}\right) + \int_I \frac{\kappa_x^{0,\varepsilon} - \rho_x^{0,\varepsilon}}{2} \log\left(\frac{\kappa_x^{0,\varepsilon} - \rho_x^{0,\varepsilon}}{2}\right).$$

Using the elementary identities $x \log x \leq x^2$ and $(x \pm y)^2 \leq 2(x^2 + y^2)$, we compute:

$$\begin{aligned}
S(0) &\leq \int_I \left(\frac{\kappa_x^{0,\varepsilon} + \rho_x^{0,\varepsilon}}{2}\right)^2 + \int_I \left(\frac{\kappa_x^{0,\varepsilon} - \rho_x^{0,\varepsilon}}{2}\right)^2 \\
&\leq \|\rho_x^{0,\varepsilon}\|_{2,I}^2 + \|\kappa_x^{0,\varepsilon}\|_{2,I}^2.
\end{aligned} \tag{7.10}$$

From (4.1), we know that:

$$|\rho_x^{0,\varepsilon}| = \left|\frac{\rho_x^0 + \varepsilon\tau\phi'}{(1+\varepsilon)^2}\right| \leq \frac{|\rho_x^0| + \varepsilon}{(1+\varepsilon)^2} \leq |\rho_x^0| + 1,$$

and

$$|\kappa_x^{0,\varepsilon}| = \left|\frac{\kappa_x^0 + \varepsilon}{1+\varepsilon}\right| \leq |\kappa_x^0| + 1.$$

Using the above two inequalities into (7.10), we deduce that:

$$S(0) \leq 2(\|\rho_x^0\|_{2,I}^2 + \|\kappa_x^0\|_{2,I}^2 + 2).$$

Step 2.3. (Control of the modulus of continuity in space and conclusion)

We use the uniform bound obtained for $S(0)$ in Step 2.1, together with the special control (5.6) of κ_x^ε given in Corollary 5.2, we get for all $0 \leq t \leq T$:

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t)) dx \leq 2(\|\rho_x^0\|_{2,I}^2 + \|\kappa_x^0\|_{2,I}^2 + 2) + \frac{\tau^2 T}{2} + 2,$$

therefore

$$\int_I \kappa_x^\varepsilon(x,t) \log(\kappa_x^\varepsilon(x,t)) dx \leq c_8, \quad \forall 0 \leq t \leq T. \tag{7.11}$$

Inequality (7.11) permits to use Lemma 5.4, hence we obtain:

$$|\kappa^\varepsilon(x+h,t) - \kappa^\varepsilon(x,t)| \leq \frac{c_9}{|\log h|}, \quad (x,t),(x+h,t) \in I_T, \tag{7.12}$$

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Inequalities (7.9) and (7.12) give the equicontinuity of the sequence $(\kappa^\varepsilon)_\varepsilon$ on V, and again by the Arzela-Ascoli Theorem, we get:

$$\kappa^\varepsilon \to \kappa, \tag{7.13}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 3. (Convergence of ρ^ε)

As in step 2, we control the modulus of continuity of ρ^ε in space and in time, locally uniformly with respect to ε.

Step 3.1. (Control of the modulus of continuity in time)

The second equation of (7.1) gives:

$$\rho_t^\varepsilon - \varepsilon \rho_{xx}^\varepsilon = A_x^\varepsilon,$$

hence, from (7.6), we deduce that:

$$\|\rho_t^\varepsilon - \varepsilon \rho_{xx}^\varepsilon\|_{\infty,V} \le c_6,$$

and from (3.6), we have:

$$\|\rho^\varepsilon\|_{\infty,V} \le 1.$$

The above two inequalities permit to use Lemma 3.4, we finally get:

$$|\rho^\varepsilon(x,t) - \rho^\varepsilon(x,t+h)| \le c_7 h^\beta, \quad (x,t),(x,t+h) \in V, \quad 0 < \beta < 1, \tag{7.14}$$

which controls the modulus of continuity of ρ^ε with respect to t uniformly in ε.

Step 3.2. (Control of the modulus of continuity in space and conclusion)

The control of the space modulus of continuity is based on the following observation. From (3.5), we know that $|\rho_x^\varepsilon| \le \kappa_x^\varepsilon$ on I_T. Using this inequality, we get, for every $(x,t),(x+h,t) \in I_T$:

$$|\rho^\varepsilon(x+h,t) - \rho^\varepsilon(x,t)| \le \int_x^{x+h} |\rho_x^\varepsilon(y,t)| dy \le \int_x^{x+h} \kappa_x^\varepsilon(y,t) dy \le |\kappa^\varepsilon(x+h,t) - \kappa^\varepsilon(x,t)|.$$

Inequality (7.12) gives immediately that:

$$|\rho^\varepsilon(x+h,t) - \rho^\varepsilon(x,t)| \le \frac{c_9}{|\log h|}, \quad (x,t),(x+h,t) \in I_T. \tag{7.15}$$

From (7.14) and (7.15), we deduce that:

$$\rho^\varepsilon \to \rho, \tag{7.16}$$

up to a subsequence, uniformly on V as $\varepsilon \to 0$.

Step 4. (Convergence of ρ_x^ε and conclusion)

In fact, this follows from Step 1, Step 2, and the fact that

$$\rho_x^\varepsilon = A^\varepsilon + \tau \kappa^\varepsilon \to \rho_x, \tag{7.17}$$

uniformly on V as $\varepsilon \to 0$. In this case, we also deduce that

$$A = \rho_x - \tau \kappa.$$

The proof of Proposition 7.1 is done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We now move to the proof of the main result.

Proof of Theorem 1.1. We first remark that κ^ε is a viscosity solution of the first equation of (7.1):

$$\kappa_t^\varepsilon \kappa_x^\varepsilon - \varepsilon \kappa_x^\varepsilon \kappa_{xx}^\varepsilon - \rho_x^\varepsilon A_x^\varepsilon = 0 \quad \text{on} \quad I_T. \tag{7.18}$$

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Remark 7.2 *The equation (7.18) can be viewed as the following Hamilton-Jacobi equation of second order:*

$$H^\varepsilon(X, D\kappa^\varepsilon, D^2\kappa^\varepsilon) = 0, \quad X = (x, t) \in I_T \tag{7.19}$$

with

$$D\kappa^\varepsilon = (\kappa_x^\varepsilon, \kappa_t^\varepsilon) \quad and \quad D^2\kappa^\varepsilon = \begin{pmatrix} \kappa_{xx}^\varepsilon & \kappa_{xt}^\varepsilon \\ \kappa_{tx}^\varepsilon & \kappa_{tt}^\varepsilon \end{pmatrix},$$

where H^ε is the Hamiltonian function given by:

$$\begin{aligned} H^\varepsilon : I_T \times \mathbb{R}^2 \times M_{sym}^{2\times2} &\longrightarrow \mathbb{R} \\ (X, p, M) &\longmapsto H^\varepsilon(X, p, M) = p_1 p_2 - \varepsilon p_1 M_{11} - \rho_x^\varepsilon(X) A_x^\varepsilon(X), \end{aligned} \tag{7.20}$$

$p = (p_1, p_2)$ *and* $M = (M_{ij})_{i,j=1,2}$.

From (7.7) and (7.17), we deduce that $(H^\varepsilon)_\varepsilon$ converges locally uniformly in $I_T \times \mathbb{R}^2 \times M_{sym}^{2\times2}$ to the function H given by:

$$\begin{aligned} H : I_T \times \mathbb{R}^2 \times M_{sym}^{2\times2} &\longrightarrow \mathbb{R} \\ (X, p, M) &\longmapsto H(X, p, M) = p_1 p_2 - \rho_x(X) A_x(X). \end{aligned} \tag{7.21}$$

This, together with the local uniform convergence of κ^ε to κ (see 7.13), and the fact that κ^ε is a viscosity solution of (7.18), permit to use the stability of viscosity solutions (see Theorem 3.7), which proves that κ is a viscosity solution of

$$H(X, D\kappa, D^2\kappa) = \kappa_t \kappa_x - \rho_x A_x = 0 \quad \text{in} \quad I_T. \tag{7.22}$$

We now pass to the limit $\varepsilon \to 0$ in the second equation of (7.1), we obtain

$$\rho_t = A_x \in C(I_T) \quad \text{in} \quad \mathcal{D}'(I_T). \tag{7.23}$$

From (7.22) and (7.23), we get:

1. κ is a viscosity solution of $\kappa_t \kappa_x = \rho_t \rho_x$ in I_T;

2. ρ is a distributional solution of $\rho_t = \rho_{xx} - \tau\kappa_x$ in I_T.

Inequality (1.12) can be easily obtained by testing against a nonnegative function $\phi \in C_0^\infty(I_T)$. Finally, let us show how to retrieve the initial and boundary conditions. Indeed, the local uniform convergence $(\rho^\varepsilon, \kappa^\varepsilon) \to (\rho, \kappa)$, together with the uniform control of the modulus of continuity of these solutions:

- with respect to x near $\partial I \times [0, T]$ by (7.12);

- with respect to t near $I \times \{t = 0\}$, away from 0 and 1 by (7.9),

and the fact that $\kappa^{0,\varepsilon} \to \kappa^0$, $\rho^{0,\varepsilon} \to \rho^0$ uniformly in $\bar{I}$, show that $(\rho, \kappa) \in (C(\overline{I_T}))^2$, so the initial and boundary conditions are satisfied pointwisely, and the proof of the main result is done. $\qquad\square$

8 Application: simulations for the evolution of elastoviscoplastic materials

Motivated by the simulation of the elastoviscoplastic behavior that are formulated by the model of Groma, Csikor and Zaiser [12], this section is devoted to write down the equations of the displacement vector u inside the crystal when a constant exterior shear stress τ (supposed to be positive $\tau > 0$ in this section) is applied on the boundary walls (see Figure 1). Also, at the end of this section, we present some numerical simulations revealing the evolution of an elastoviscoplastic crystal. We consider a 2-dimensional crystal (Figure 1) with the displacement vector:

$$u = (u_1, u_2) : \mathbb{R}^2 \longmapsto \mathbb{R}^2.$$

We use indifferently the following notations for the coordinates: $x = x_1$ and $y = x_2$. By letting (e_1, e_2) the corresponding orthonormal basis, we define the total strain by:

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right) \quad \text{with} \quad \partial_j u_i = \frac{\partial u_i}{\partial x_j}, \quad i,j = 1,2.$$

This total strain could be decomposed into two parts as follows:

$$\varepsilon_{ij}(u) = \varepsilon_{ij}^e(u) + \varepsilon_{ij}^p, \tag{8.1}$$

where $\varepsilon^e(u)$ is the elastic strain and ε^p is the plastic strain which is given by:

$$\varepsilon_{ij}^p = \rho \varepsilon_{ij}^0, \tag{8.2}$$

with

$$\varepsilon_{ij}^0 = \frac{1}{2}(1 - \delta_{ij}),$$

in the special case of a single slip system where dislocations move following the Burgers vector $\vec{b} = e_1$ (δ_{ij} is the usual Kronecker symbol). Here ρ is the resolved plastic strain. The stress field σ inside the crystal is given by:

$$\sigma_{ij} = 2\mu\varepsilon_{ij}^e(u) + \lambda\delta_{ij}\left(\sum_{k=1,2} \varepsilon_{kk}^e(u)\right), \tag{8.3}$$

with $\lambda, \mu > 0$ are the constants of Lamé coefficients of the crystal that are assumed (for simplification) to be isotropic. This stress field σ has to satisfy the equation of elasticity:

$$\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{8.4}$$

Here, as we have already mentioned in the introduction, we suppose that the distribution of dislocations and the displacement are both invariant by translation in the y-direction. As a conclusion, the functions $\rho(x,t)$, $\kappa(x,t)$ and $u(x,t)$ satisfy the following coupled system:

$$\begin{cases} \displaystyle\sum_{j=1,2} \frac{\partial \sigma_{ij}}{\partial x_j} &= 0 & \text{on} \quad I \times (0,\infty) \\[3mm] \sigma_{ij} &= 2\mu\varepsilon_{ij}^e(u) + \lambda\delta_{ij}\left(\displaystyle\sum_{k=1,2} \varepsilon_{kk}^e(u)\right) & \text{on} \quad I \times (0,\infty) \\[3mm] \varepsilon_{ij}^e &= \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right) - \rho\varepsilon_{ij}^0 & \text{on} \quad I \times (0,\infty) \\[2mm] \kappa_t \kappa_x &= \rho_t \rho_x & \text{on} \quad I \times (0,\infty) \\[2mm] \rho_t &= \rho_{xx} - \tau\kappa_x & \text{on} \quad I \times (0,\infty), \end{cases} \tag{8.5}$$

with the boundary conditions (taking into account the discussion in Subsection 1.2):

$$\begin{cases} \kappa(x,.) = x & \text{for} \quad x \in \partial I \\ \rho(x,.) = 0 & \text{for} \quad x \in \partial I \\ \sigma \cdot n = \pm\tau e_2, \ n = \pm e_1 & \text{for} \quad x = \pm 1, \end{cases} \tag{8.6}$$

where τ is the constant shear stress field applied on the walls. Taking into account the fact that $\partial_2\rho = \partial_2 u = 0$, equation (8.4) can be reformulated as:

$$\begin{cases} (\lambda + 2\mu)\partial_{11} u_1 = 0 \\ \mu\partial_{11} u_2 = \mu\partial_1\rho, \end{cases} \tag{8.7}$$

and equation (8.3) gives:

$$\sigma = \mu(\partial_1 u_2 - \rho)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \partial_1 u_1 \begin{pmatrix} \lambda + 2\mu & 0 \\ 0 & \lambda \end{pmatrix}. \tag{8.8}$$

Calculation of u. In this paragraph we drop the time dependence. We first calculate the value of the displacement u on the boundary walls. Taking into account the boundary condition (8.6), we get:

$$\sigma = \begin{pmatrix} 0 & \tau \\ \tau & a \end{pmatrix} \quad \text{on} \quad \partial I, \tag{8.9}$$

where a is an unknown coefficient. Using (8.8) and (8.9), we find that $a = 0$, and we derive the following conditions on the boundary:

$$\begin{cases} \partial_1 u_1 = 0 & \text{on} \quad \partial I, \\ \mu(\partial_1 u_2 - \rho) = \tau & \text{on} \quad \partial I. \end{cases} \tag{8.10}$$

Equation (8.7) leads to the following two equations inside I:

$$\begin{cases} \partial_1 u_1 = 0 & \text{on} \quad I \\ \mu(\partial_1 u_2 - \rho) = \tau & \text{on} \quad I. \end{cases} \tag{8.11}$$

Combining (8.11) with the following condition at $x_1 = 0$:

$$u_1(0) = u_2(0) = 0,$$

finally lead to:

$$\begin{cases} u_1(x_1) = 0, & x_1 \in I \\ u_2(x_1) = \dfrac{\tau}{\mu}x_1 + \displaystyle\int_0^{x_1} \rho(x)dx, & x_1 \in I. \end{cases} \tag{8.12}$$

Formal computation of the long time solution of system (1.3). All computations that will be done here are formal. We seek to calculate long time solutions for system (1.3). For this reason, we first calculate the long time solution (stationary solution, $\kappa_t^\varepsilon = \rho_t^\varepsilon = 0$) of the regularized ε-system (2.2), and then we pass to the limit $\varepsilon \to 0$. Doing some computations, the long time ($t \to \infty$) solution for the ε-system is given by:

$$\rho^\varepsilon(x) = B\left(\cosh\left(\frac{\tau x}{1+\varepsilon}\right) - \cosh\left(\frac{\tau}{1+\varepsilon}\right)\right) \quad \text{and} \quad \kappa^\varepsilon(x) = B\sinh\left(\frac{\tau x}{1+\varepsilon}\right), \tag{8.13}$$

with $B = 1/\sinh\left(\frac{\tau}{1+\varepsilon}\right)$. Passing (again formally) to the limit as $\varepsilon \to 0$ in (8.13), and using (8.12), we can compute the long time displacement u_2 inside the material. In fact, we have:

$$u_2(x_1) = \left(\frac{\tau}{\mu} - \frac{\cosh\tau}{\sinh\tau}\right)x_1 + \frac{\sinh(\tau x_1)}{\tau\sinh\tau} \quad \text{for} \quad x_1 \in I. \tag{8.14}$$

Therefore, the displacement given in (8.14) seems to be a good candidate for the possible long time limit of the displacement u_2, solution of (8.5) and (8.6). This is a conjecture and remains an open problem. In the next paragraph, we will do some numerical simulations of solutions to system (8.5) and (8.6) in order to compare our numerical results with the above candidate (8.14).

Numerical simulations. The displacement u is numerically computed by discretizing system (1.3) using an upwind scheme. The space and time steps in this scheme are well chosen in order to satisfy a CFL condition (for the details, see [14, Appendix]). We choose the values $\mu = 1.6$, $\tau = 0.4$ and the initial conditions $\kappa^0(x) = x$ and $\rho^0(x) = 0$. In Figure 2, we show successively the initial state of the crystal at time $t = 0$ without any applied stress, then the instantaneous (elastic) deformation of the crystal when we apply the shear stress $\tau > 0$ at time $t = 0^+$. The deformation of the crystal evolves in time and finally converges numerically to some

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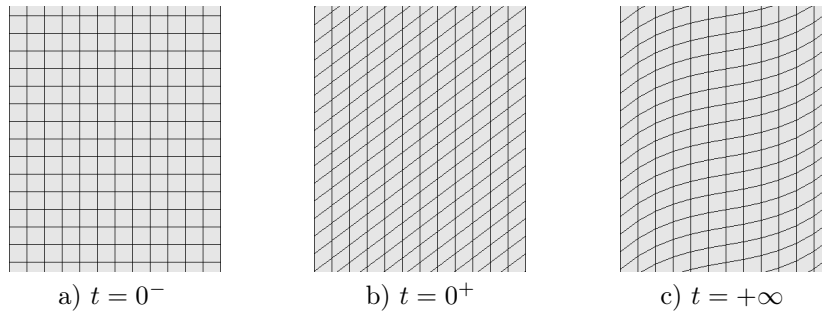a) $t = 0^-$ b) $t = 0^+$ c) $t = +\infty$

Figure 2: Deformation of a slab for model (1.3).

particular deformation which is shown on the last picture after a very long time. This kind of behavior is called elastoviscoplasticity in mechanics. Moreover, on the last picture of Figure 2, we observe the presence of boundary layer deformations. This effect is directly related to the introduction of the back stress $\tau_b = \frac{\theta_x^+ - \theta_x^-}{\theta^+ + \theta^-}$ in the model (1.1).

In the last image of Figure 2, the long time displacement u_2 inside the material is coherent with our formal computation for the term u_2 (see equation (8.14)). See also Figure 3 below where we have shown both the analytical solution (given by (8.14)) and the numerical solution (given by (8.12) for the discrete solution (ρ, κ) associated to system (1.3)).
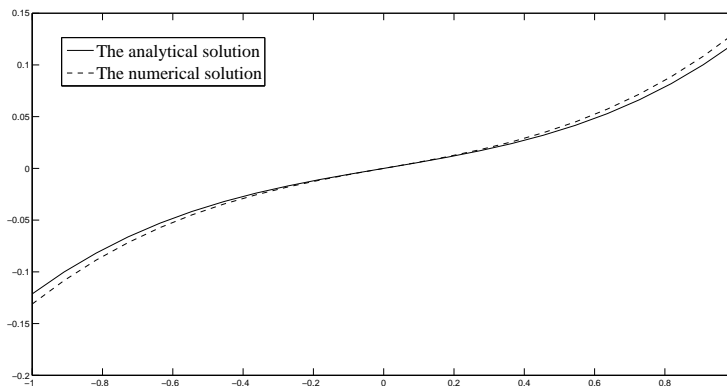

Figure 3: Analytical and numerical solutions (long time behavior).

9 Appendix

A1. Proof of Lemma 3.4 (control of the modulus of continuity in time)

Let V be a compactly contained subset of I_T. Throughout the proof, the constant c may take several values but only depending on V. Since $V \subset\subset I_T$, then there is a rectangular cube of the form

$$\mathcal{Q} = (x_1, x_2) \times (t_1, t_2),$$

such that $V \subset\subset \mathcal{Q} \subset\subset I_T$. In this case, there exists a constant ε_0, also depending on V such that for any

$$0 < \varepsilon < \varepsilon_0,$$

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and any $(x, t) \in V$, we have:
$$(x - 2\sqrt{\varepsilon}, x + 2\sqrt{\varepsilon}) \times \{t\} \subset \mathcal{Q}.$$

Moreover, for any $(x, t), (x, t + h) \in V$, we can always find two intervals $\mathcal{I}$ and $\mathcal{J}$ such that
$$(t, t + h) \subset \mathcal{I} \subset\subset \mathcal{J}, \quad \text{with} \quad \{x\} \times \mathcal{I} \subset \mathcal{Q} \quad \text{and} \quad \{x\} \times \mathcal{J} \subset \mathcal{Q}.$$

Let us indicate that these intervals might have different lengths depending on h and V but we always have
$$|\mathcal{J}|, |\mathcal{I}| \le |t_2 - t_1|.$$

Consider the following rescaling of the function u^ε defined by:
$$\tilde{u}^\varepsilon(x, t) = u^\varepsilon(\sqrt{\varepsilon}x, t). \tag{9.1}$$

This function satisfies
$$\tilde{u}_t^\varepsilon = \tilde{u}_{xx}^\varepsilon + \tilde{f}^\varepsilon, \quad (x, t) \in (0, 1/\sqrt{\varepsilon}) \times (0, T),$$
where $\tilde{f}^\varepsilon(x, t) = f^\varepsilon(\sqrt{\varepsilon}x, t)$. Take $(x_0, t_0), (x_0, t_0 + h)$ in V, and let
$$\mathcal{Q}_1 = (x_0 - \sqrt{\varepsilon}, x_0 + \sqrt{\varepsilon}) \times \mathcal{I} \quad \text{and} \quad \mathcal{Q}_2 = (x_0 - 2\sqrt{\varepsilon}, x_0 + 2\sqrt{\varepsilon}) \times \mathcal{J}.$$

These two cylinders are transformed by the above rescaling into
$$\tilde{\mathcal{Q}}_1 = \left(\frac{x_0}{\sqrt{\varepsilon}} - 1, \frac{x_0}{\sqrt{\varepsilon}} + 1 \right) \times \mathcal{I} \quad \text{and} \quad \tilde{\mathcal{Q}}_2 = \left(\frac{x_0}{\sqrt{\varepsilon}} - 2, \frac{x_0}{\sqrt{\varepsilon}} + 2 \right) \times \mathcal{J}.$$

We apply the interior L^p, $p > 3$, estimates for parabolic equations (see for instance [19, Theorem 7.13, page 172]) to the function $\tilde{u}^\varepsilon$ over the domains $\tilde{\mathcal{Q}}_1 \subset\subset \tilde{\mathcal{Q}}_2$, we get
$$\|\tilde{u}^\varepsilon\|_{W_p^{2,1}(\tilde{\mathcal{Q}}_1)} \le c(\|\tilde{u}^\varepsilon\|_{p, \tilde{\mathcal{Q}}_2} + \|\tilde{f}^\varepsilon\|_{p, \tilde{\mathcal{Q}}_2}). \tag{9.2}$$

Using the local ε-uniform boundedness of $(u^\varepsilon)_\varepsilon$, and $(f^\varepsilon)_\varepsilon$, we get:
$$\|\tilde{u}^\varepsilon\|_{L^p(\tilde{\mathcal{Q}}_2)}^p \le c \quad \text{and} \quad \|\tilde{f}^\varepsilon\|_{L^p(\tilde{\mathcal{Q}}_2)}^p \le c.$$

hence, inequality (9.2) implies:
$$\|\tilde{u}^\varepsilon\|_{W_p^{2,1}(\tilde{\mathcal{Q}}_1)} \le c. \tag{9.3}$$

We use the Sobolev embedding in Hölder spaces (see for instance [16, Lemma 2.8]):
$$W_p^{2,1}(\tilde{\mathcal{Q}}_1) \hookrightarrow C^{1+\alpha, \frac{1+\alpha}{2}}(\tilde{\mathcal{Q}}_1), \quad p > 3, \ \alpha = 1 - 3/p,$$

to obtain, from (9.3), that:
$$\|\tilde{u}^\varepsilon\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(\tilde{\mathcal{Q}}_1)} \le c,$$

and hence
$$\frac{|\tilde{u}^\varepsilon(x_0/\sqrt{\varepsilon}, t_0 + h) - \tilde{u}^\varepsilon(x_0/\sqrt{\varepsilon}, t_0)|}{h^{\frac{1+\alpha}{2}}} \le c,$$

then from (9.1),
$$\frac{|u^\varepsilon(x_0, t_0 + h) - u^\varepsilon(x_0, t_0)|}{h^{\frac{1+\alpha}{2}}} \le c.$$

Choosing $\beta = \frac{1+\alpha}{2}$ we get the desired result. $\qquad\square$

A2. Proof of Lemma 3.5 (An interior estimate for the heat equation)

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Recall that a is a solution of the heat equation on I_T,

$$a_t = a_{xx}.$$

The proof of lemma 3.5 is a direct computation using a mean value formula for solutions of the heat equations. Usually, basic mean value formulae of the solution of the heat equation are expressed through unbounded kernels (see for example [7, Theorem 1]), where a can be expressed as:

$$a(x_0, t_0) = (4\pi r^2)^{-1/2} \int_{\Omega_r(x_0, t_0)} a(x, t) \frac{(x_0 - x)^2}{4(t_0 - t)^2} dx dt. \tag{9.4}$$

Here, $(x_0, t_0) \in I_T$, $(x, t) \in \Omega_r(X_0)$, and $r > 0$ small enough in order to ensure that the parabolic ball of radius r:

$$\Omega_r(x_0, t_0) = \left\{ (x, t); \ t_0 - r^2 < t < t_0, \ (x - x_0)^2 < 2(t_0 - t) \log\left(\frac{r^2}{t_0 - t}\right) \right\} \subset I_T. \tag{9.5}$$

In our case, we need a mean value formula similar to (9.4) but with a bounded kernel on $\Omega_r(x_0, t_0)$. In [10], the authors have given such a representation formula for the solution of the heat equation. The following is a direct corollary of [10, Theorem 3.1]):

Corollary 9.1 *(Mean value formula with bounded kernels, [10, Theorem 3.1])*
Let $u \in C^2(\mathcal{D})$ be a solution of the heat equation:

$$u_t = u_{xx} \quad on \quad \mathcal{D},$$

where $\mathcal{D}$ is an open subset of $\mathbb{R}^2$ containing the modified unit parabolic ball $\Omega'_r(x_0, t_0)$, $r > 0$, with

$$\Omega'_r(x_0, t_0) = \left\{ (x, t); \ t_0 - r^2 < t < t_0, \ |x - x_0|^2 < 8(t_0 - t) \log\left(\frac{r^2}{t_0 - t}\right) \right\}.$$

Then we have:

$$u(x_0, t_0) = \frac{\bar{c}}{|\Omega'_r(x_0, t_0)|} \int_{\Omega'_r(x_0, t_0)} u(x, t) E\left(\frac{x - x_0}{r}, \frac{t - t_0}{r^2}\right) dx dt, \tag{9.6}$$

where $\bar{c} > 0$, $|\Omega'_r(x_0, t_0)| = \bar{c} r^3$, and the kernel E satisfies:

$$\|E(x, t)\|_{\infty, \Omega'_1(0,0)} \leq c, \tag{9.7}$$

and $c > 0$ is a fixed positive constant.

Remark 9.2 *In the above corollary, which is an application of [10, Theorem 3.1] in the case $m = 3$, an explicit expression of E can be given by:*

$$E(x, t) = \frac{\omega_3}{16\pi^2} \left(-x^2 + 8t \log(-t)\right)^{3/2} \left[\frac{x^2}{4t^2} + \frac{3(-x^2 + 8t \log(-t))}{20t^2}\right],$$

where ω_3 is the volume of the unit ball in $\mathbb{R}^3$. For a more general expression of E, we send the reader to [10, Equality (3.6) of Theorem 3.1].

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