Pointwise estimates for Laplace equation. Applications to the free boundary of the obstacle problem with Dini coefficients

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Abstract

In this paper we are interested in pointwise regularity of solutions to elliptic equations. In a first result, we prove that if the modulus of mean oscillation of Δu at the origin is Dini (in L^p average), then the origin is a Lebesgue point of continuity (still in L^p average) for the second derivatives D^2u. We extend this pointwise regularity result to the obtacle problem for the Laplace equation with Dini right hand side at the origin. Under these assumptions, we prove that the solution to the obstacle problem has a Taylor expansion up to the order 2 (in the L^p average). Moreover we get a quatitative estimate of the error in this Taylor expansion for regular points of the free boundary. In the case where the right hand side is moreover double Dini at the origin, we also get a quatitative estimate of the error for singular points of the free boundary.

Our method of proof is based on some decay estimates obtained by contradiction, using blow-up arguments and Liouville Theorems. In the case of singular points, our method uses moreover a refined monotonicity formula.

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1 Introduction

1.1 The Laplace equation

In this paper, we are interested in the pointwise regularity properties of solutions to elliptic problems. We first consider the solutions to the following Laplace equation

$$(1.1)\qquad \begin{cases} \Delta u = f \quad \text{in} \quad B_1 \\ \\ f \in L^p(B_1) \quad \text{and} \quad f(0) = 0 \end{cases}$$

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where in $\mathbb{R}^n$, we denote by $B_r = B_r(0)$ the open ball of radius r and of center the origin 0. Here $p \in (1, +\infty)$, and we only assume that 0 is a Lebesgue point of f, in order to define $f(0)$ when it is necessary.

It is well-known that, if f is Hölder continuous on the ball B_1, then this is also true for the second derivatives of the solution u (see for instance Gilbarg, Trudinger [26] for a classical proof of this result based on the potential theory).
Let us introduce the following modulus of continuity of f on the ball B_1:

$$\overline{\sigma}(r) = \sup_{|x-y|\leq r,\ x,y\in B_1} |f(x) - f(y)| \tag{1.2}$$

Definition 1.1 (Dini modulus of continuity / double Dini)
A modulus of continuity $\overline{\sigma}$ is said Dini if it satisfies the following integral condition:

$$\int_0^1 \frac{\overline{\sigma}(r)}{r}\, dr < +\infty \tag{1.3}$$

It is said double Dini if

$$\int_0^1 \frac{dr}{r} \left(\int_0^r \frac{ds}{s}\, \overline{\sigma}(s)\right) < +\infty \tag{1.4}$$

It is also well-known that if $\overline{\sigma}$ is Dini, then the second derivatives of the solution u are continuous in any ball striclty contained in B_1, with a modulus of continuity which is proportional to $\int_0^r \overline{\sigma}(s)ds/s + r\int_r^1 \overline{\sigma}(s)ds/s^2$. For proofs based on potential theory, see Hartman, Wintner [28], Matiichuk, Eidel'man [35], Burch [4], and for a proof based on Dini-Campanato spaces and explicit approximations of the solution by polynomials, see for instance Kovats [33]. For similar results for the regularity of solutions of elliptic systems in divergence form based on the harmonic approximation lemma, see Duzaar, Gastel [21], Duzaar, Gastel, Mingione [22], and also Wolf [43] with a different approach.

The previous results were obtained assuming a modulus of continuity in an open set. Here we change the point of view, and only want to consider *pointwise modulus of mean oscillation*. For any $p \in (1, +\infty)$, let us introduce the following modulus of mean oscillation (in L^p average) of the function f *at the origin*:

$$\tilde{\sigma}_p(\rho) = \sup_{r\in(0,\rho]} \inf_{c\in\mathbb{R}} \left(\frac{1}{|B_r|}\int_{B_r} |f(x) - c|^p\right)^{\frac{1}{p}} \tag{1.5}$$

Let us denote by $\tilde{\mathcal{P}}_2$ the set of polynomials of degree less or equal to 2, and let us set

$$\tilde{M}(u,\rho) = \sup_{r\in(0,\rho]} \left(\inf_{P\in\tilde{\mathcal{P}}_2} \left(\frac{1}{r^{n+2p}}\int_{B_r} |u - P|^p\right)^{\frac{1}{p}}\right)$$

Theorem 1.2 (Pointwise BMO estimates for the Laplace equation)
Let $p \in (1, +\infty)$ be given. Then there exist $\alpha \in (0, 1]$ and constants $C > 0$, $r_0 \in (0, 1)$, such that, given a function $u \in L^p(B_1)$ satisfying (1.1) with a modulus of mean oscillation $\tilde{\sigma}_p$ defined in (1.5), we have

i) Pointwise BMO estimate

$$\tilde{M}(u,1) \le C\left\{\left(\int_{B_1} |u|^p\right)^{\frac{1}{p}} + \left(\int_{B_1} |f|^p\right)^{\frac{1}{p}} + \tilde{\sigma}_p(1)\right\} \tag{1.6}$$

ii) Pointwise VMO estimate
Moreover, we have

$$\left(\tilde{\sigma}_p(r) \longrightarrow 0 \quad as \quad r \to 0^+\right) \quad \Longrightarrow \quad \left(\tilde{M}(u,r) \longrightarrow 0 \quad as \quad r \to 0^+\right)$$

iii) Pointwise control on the solution
Finally, if $\tilde{\sigma}_p$ is **Dini***, then $\tilde{M}(u,\cdot)$ is* **Dini***, and there exists a harmonic polynomial P_0 of degree less or equal to 2, such that for every $r \in (0,r_0)$:*

$$\left(\frac{1}{|B_r|}\int_{B_r} \left|\frac{u(x)-P_0(x)}{r^2}\right|^p\right)^{\frac{1}{p}} \le C\left(\tilde{M}_0 r^\alpha + \int_0^r \frac{\tilde{\sigma}_p(s)}{s}\,ds + r^\alpha \int_r^1 \frac{\tilde{\sigma}_p(s)}{s^{1+\alpha}}\,ds\right) \tag{1.7}$$

and

$$P_0(x) = a + b\cdot x + \frac{1}{2}{}^t x \cdot c \cdot x$$

with

$$|a| + |b| + |c| \le C\tilde{M}_0 \quad and \quad \tilde{M}_0 = \int_0^1 \frac{\tilde{\sigma}_p(s)}{s}\,ds + \left(\int_{B_1} |u|^p\right)^{\frac{1}{p}} + \left(\int_{B_1} |f|^p\right)^{\frac{1}{p}}$$

Here $b\cdot x$ denotes the scalar product between the vectors b and x.

Remark 1.3 *In other words, Theorem 1.2 iii) implies in particular (using elliptic estimates) that we have a Lebesgue point of continuity of the second derivatives D^2u (in the L^p average) if Δu has a Dini modulus of mean oscillation (in the L^p average) at the same point.*

Remark 1.4 *A straightforward corollary of Theorem 1.2 gives in particular that the second derivatives D^2u are Hölder continuous in an open set Ω, if Δu is Hölder continuous in Ω.*

Theorem 1.2 gives a kind of Taylor expansion up to the second order with a quantitative estimate of the rest in the L^p norm. This notion of continuity of derivatives seems quite natural and is related to the notion of approximate derivatives (for $p=1$, see section 2.9 of Federer [25], for $p=n$ see Caffarelli [10], see also Campanato spaces [18], the generalized Campanato-John-Nirenberg spaces in [12], the t_2^p class of Calderon, Zygmund [17], or the notion of generalized derivatives in Diederich [20]). For a characterization of such pointwise regularity in terms of wavelet coefficients, we refer the interested reader to Jaffard [29] (see also Jaffard, Meyer [30]).

We would like to emphasis that the result of Theorem 1.2 is completely *pointwise*, which does not seem so usual in the literature (see for instance the article of Simon [40] obtaining Schauder estimates by a scaling argument joint to compactness). Even if part of this result is somehow contained in a proof of Kovats [33] in the special case $p=+\infty$ (see the proof of his Lemma 2.1), our method of proof is completely different. Here we do not use an explicit construction of approximate polynomials, but on the contrary prove the result by contradiction, using a blow-up argument. The consequence is that we do not recover the best exponent $\alpha = 1$. Nevertheless, we think that our method of proof is quite flexible, and present below new consequences for the obstacle problem.

1.2 The model obstacle problem

In this article we are in particular interested in the regularity of the free boundary for solutions to obstacle problems. The model problem is the following. We consider bounded functions u, satisfying in the unit ball B_1, for $p \in (\max(n/2, 1), +\infty)$

$$(1.8) \qquad \begin{cases} \left.\begin{array}{l} \Delta u = f(x) \cdot 1_{\{u>0\}} \\ \\ u \geq 0 \end{array}\right| \quad \text{in} \quad B_1 \\ \\ u, f \ \in L^p(B_1) \quad \text{and} \quad f(0) = 1 \\ \\ 0 \in \partial \{u > 0\} \end{cases}$$

where $1_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$ which is equal to 1 if $u > 0$ and 0 if $u = 0$. From classical elliptic estimates joint to Sobolev imbbedings with our assumption $p > n/2$, every solution u is in particular continuous, which allows us to consider the boundary of the open set $\{u > 0\}$. Here $\partial \{u > 0\}$ is called the free boundary. We assume that 0 is a Lebesgue point of f in order to define $f(0)$.

Let us introduce the following *pointwise modulus of continuity* (in L^p average) of the function f *at the origin*:

$$(1.9) \qquad \sigma_p(\rho) = \sup_{r \in (0,\rho]} \left(\frac{1}{|B_r|} \int_{B_r} |f(x) - f(0)|^p \right)^{\frac{1}{p}}$$

We have the following general regularity result.

Proposition 1.5 (Quadratic growth)
Let $p \in (\max(n/2, 1), +\infty)$. Then there exists a constant $C > 0$ such that if u is a solution of (1.8) with σ_p bounded given by (1.9), then

$$\forall x \in B_{1/2}, \quad 0 \leq u(x) \leq C_1 |x|^2 \quad \textit{with} \quad C_1 = C \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma_p(1) + 1 \right\}$$

Let us mention that a certain Wiener criterion has been established for the continuity of the solution for the two-obstacle problem with irregular obstacles. We refer the interested reader to the work of Dal Maso, Mosco, Vivaldi [19] where the right hand side of the equation is estimated in the Kato space, and to Kilpeläinen, Ziemer [31] for related results for nonlinear operators.

When we assume furthermore that $p \geq 2n/(n+1)$, it is interesting to present the following preliminary result which distinguishes if 0 is a *degenerate*, *regular* or a *singular* point of the free boundary.

Theorem 1.6 (Definition of degenerate/regular/singular points by the monotonicity formula)
Given a solution u of (1.8), we define for $r \in (0,1)$

$$\Phi(r) = \frac{1}{r^{n+2}} \int_{B_r} \frac{1}{2}|\nabla u|^2 + u - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 \tag{1.10}$$

for some $p \in (\max(n/2,1),+\infty)$ with $p \geq 2n/(n+1)$. If the modulus of continuity σ_p defined in (1.9) is **Dini***, then Φ has a limit at $r=0$, that we denote by $\Phi(0^+)$. Moreover there exists a constant $\alpha = \alpha(n) > 0$, such that either*
i) $\Phi(0^+) = 0$ and then the point 0 is called a **degenerate point***, or*
ii) $\Phi(0^+) = \alpha$ and then the point 0 is called a **regular point***, or*
iii) $\Phi(0^+) = 2\alpha$ and then the point 0 is called a **singular point***.*
Moreover in the special case where $f \equiv 1$, the function Φ is nondecreasing in r.

Remark 1.7 *Under the present assumptions, it is possible to build examples (see Section 5) with degenerate points. On the contrary, if we assume moreover that $f \geq \delta_0 > 0$, then it is classical that 0 can not be a degenerate point (see Caffarelli [7], Blank [5]).*

Let us recall that this result is originally due to Weiss [42] for $f \equiv 1$ (see also Monneau [36] for a version for some Dini modulus of continuity, and Petrosyan, Shahgholian [38] for a similar monotonicity formula for double Dini modulus of continuity, but for obstacle problems with no sign condition on the solution).

Let us introduce the following quantity (which is finite by Proposition 1.5)

$$M_{reg}(u,\rho) = \sup_{r\in(0,\rho]} \left(\inf_{P\in\mathcal{P}_{reg}} \left(\frac{1}{r^{n+2p}} \int_{B_r} |u-P|^p \right)^{\frac{1}{p}} \right)$$

where

$$\mathcal{P}_{reg} = \left\{ P, \quad \exists \nu \in \mathbf{S}^{n-1}, \quad P(x) = \frac{1}{2}\max(0, x\cdot\nu)^2 \right\}$$

Our main results are the following three statements in the regular case and in the singular case.

Theorem 1.8 (Modulus of continuity at a regular point of the free boundary)
Let $p \in (\max(n/2,1),+\infty)$. There exist $\alpha \in (0,1]$ and constants $C > 0$, $M_0, r_0 \in (0,1)$ such that, given a function u satisfying (1.8), we have the following property.
If the modulus of continuity σ_p defined in (1.9) is assumed **Dini***, and if*

$$M_{reg}(u,r_0) \leq M_0$$

then there exists $P_0 \in \mathcal{P}_{reg}$ such that for every $r \in (0,r_0)$:

$$\left(\frac{1}{|B_r|} \int_{B_r} \left| \frac{u(x)-P_0(x)}{r^2} \right|^p \right)^{\frac{1}{p}} \leq C \left(M_{reg}(u,r_0)\, r^\alpha + \int_0^r \frac{\sigma_p(s)}{s}\, ds + r^\alpha \int_r^1 \frac{\sigma_p(s)}{s^{1+\alpha}}\, ds \right) \tag{1.11}$$

Remark 1.9 *With the same methods of proof, it would be possible to get a similar estimate for any $p \in (1,+\infty)$, but under the stronger assumption that the coefficient of the right hand side of the equation is bounded from above and from below, i.e. $0 < \delta_0 \leq f \leq 1/\delta_0$.*

Theorem 1.10 (Uniqueness of the blow-up limit at singular points) *Let $p \in (\max(n/2, 1), +\infty)$ with $p \geq 2$. If the modulus of continuity σ_p defined in (1.9) is assumed* **Dini**, *then there exists a non-negative polynomial P_0 homogeneous of degree 2 satisfying $\Delta P_0 = 1$, such that*

$$\left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{u(x)-P_0(x)}{r^2}\right|^2\right)^{\frac{1}{2}} \longrightarrow 0 \quad as \quad r \longrightarrow 0 \tag{1.12}$$

Let us define the set

$$\mathcal{P}_{sing} = \left\{P, \quad \exists Q \in \mathbb{R}^{n\times n}_{sym}, \quad P = \frac{1}{2}{}^t x \cdot Q \cdot x, \quad \text{trace }(Q) = 1, \quad Q \geq 0\right\}$$

We also define

$$M_{sing}(u,\rho) = \sup_{r\in(0,\rho]}\left(\inf_{P\in\mathcal{P}_{sing}}\left(\frac{1}{r^{n+4}}\int_{B_r}|u-P|^2\right)^{\frac{1}{2}}\right)$$

Then we have

Theorem 1.11 (Modulus of continuity at a singular point of the free boundary) *Let $p \in (n/2, +\infty)$ with $p \geq 2$. There exists $\alpha \in (0, 1]$ and constants $C > 0$, $M_0, r_0 \in (0, 1)$ such that, given a function u satisfying (1.8), we have the following property.*
If the modulus of continuity σ_p defined in (1.9) is assumed **double Dini**, *and if*

$$M_{sing}(u, r_0) \leq M_0$$

then there exists $P_0 \in \mathcal{P}_{sing}$ such that for every $r \in (0, r_0)$:
(1.13)

$$\left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{u(x)-P_0(x)}{r^2}\right|^2\right)^{\frac{1}{2}} \leq C\left(M_{sing}(u,r_0)r^\alpha + \int_0^r \frac{\Sigma_p(s)}{s}\,ds + r^\alpha\int_r^1 \frac{\Sigma_p(s)}{s^{1+\alpha}}\,ds\right)$$

with

$$\Sigma_p(s) = \sigma_p(s) + \int_0^s dt\ \frac{\sigma_p(t)}{t}$$

Remark 1.12 *In particular the boundary $\partial\{P_0 > 0\}$ can be interpreted as the tangent j-dimensional subspace to the free boundary at the origin 0. Here $j = n - 1$ for regular points, and $j = \dim \operatorname{Ker} Q$ for singular points.*

Remark 1.13 *Theorem 1.11 remains true if we replace $\Sigma_p(s)$ by*

$$\tilde{\Sigma}_p(s) = \sigma_p(s) + \left(\int_0^s dt\ \frac{\sigma_p^2(t)}{t}\right)^{\frac{1}{2}}$$

assuming only that $\tilde{\Sigma}_p$ is Dini. This is a sharper result, because we always have $\tilde{\Sigma}_p \leq C\Sigma_p$ and for instance for $\sigma_p(s) = |\ln s|^{-\frac{3}{2}}$, we have that $\tilde{\Sigma}_p$ is Dini while Σ_p is not.

Let us emphasis again that these results are *pointwise*, and seem the first pointwise results for the obstacle problem, up to our knowledge. The regularity of the free boundary can be easily deduced from these theorems (see Theorem 8.3).

Concerning regular points, let us mention slightly more precise results on the regularity of the free boundary when the modulus of continuity is not only controled at the origin 0, but controled at any points (see Blank [5] for sharp results). See the previous works of Caffarelli [6, 7, 8, 9], Weiss [42], and Caffarelli, Karp, Shahgholian [13] for Lipschitz coefficients, and also the work of Caffarelli, Kinderlehrer [14] for some related estimates on the modulus of continuity of the solution or of its gradient. Let us mention that very recently, similar regularity results have been obtained in Petrosyan, Shahgholian [38] for the regular points of the free boundary, for an obstacle problem with no sign assumption on the solution. These results are obtained under geometric and energetic conditions and the assumption that $\int_0^1 dr\, \frac{\overline{\sigma}(r)\ln\frac{1}{r}}{r}$ is finite, which can easily be seen to be equivalent to the double Dini assumption. See also Lee, Shahgholian [34] for regularity results for fully nonlinear obstacle problems.

Concerning singular points, the first result of regularity has been proved by Caffarelli [8] using the monotonicity formula of Alt, Caffarelli, Friedman [1], and this result has been generalized for Lipschitz coefficients (and for an obstacle problem without sign assumption on the solution) by Caffarelli, Shahgholian [16]. For the classical obstacle problem, pointwise regularity results of the singular set have been obtained for double Dini coefficients in Monneau [36]. This result was based on a monotonicity formula devoted to singular points (see also Monneau [37]). In the proof of Theorem 1.10, this monotonicity formula for singular points has been refined, which allows us to get the result assuming the modulus of continuity to be only Dini.

The proof of theorems 1.2, 1.8 and 1.11 are based on a decay estimate (see Propositions 2.5, 6.2 and 7.4), similar to other decay estimates obtained for dynamical systems converging to stable states (see for instance Simon [39]). Our proof of this decay estimate is done by contradiction, and uses blow-up techniques like in Caffarelli [7] and the stability of the obstacle problem. Our approach is strongly inspired on the one hand from the epiperimetric inequality given in Weiss [42], and on the other hand on blow-up techniques and Caccioppoli inequalities as used in Evans [23], Evans, Gariepy [24], and finally on classical results for Dini-continuity results for solutions of elliptic equations or systems (see the references cited in subsection 1.1).

Remark 1.14 *From our proofs, we can check that all the previous estimates are still true (with different constants), if we replace* $\sigma_p(s)$ *by* $\sigma_p(\gamma s)$ *for a fixed constant* $\gamma \in (0, 1]$.

1.3 Organization of the article

In Section 2, we prove a fundamental decay estimate (Proposition 2.5) for the Laplace equation and give the proof of a weak version of Theorem 1.2, namely Theorem 2.1, and finally give the proof of Theorem 1.2. The proof of the Theorem also uses some general large scales estimates (Lemma 2.9 and Proposition 2.7) that are proved in Section 3.

In Section 4, we start the study of the obstacle problem, giving the proof of the growth estimate Proposition 1.5 and the monotonicity formula Theorem 1.6. In Section 5, we study degenerate points of the free boundary and give an example of such points. In Section 6, we study the regular points of the free boundary and prove Theorem 1.8, also based on a decay estimate (Proposition 6.2).
Section 7 is devoted to the study of singular points of the free boundary. We first prove two monotonicity formulas (one for double Dini coefficients and another one for Dini coefficients assuming moreover $p \geq 2$). We then show Theorem 1.10 on the uniqueness of the blow-up limits. The quantitative estimate Theorem 1.11 is proved using a decay estimate. This decay estimate (Proposition 7.4) is in particular based on new Liouville type results.
In Section 8, we give some applications to the regularity of the free boundary for general second order linear elliptic operators. In the Appendix (Section 9), we also give an application to the regularity of solutions to fully nonlinear elliptic equations.

2 A decay estimate for Laplace equation and proof of Theorem 1.2

2.1 Proof of a weak version of Theorem 1.2

We will start to prove a weak version of Theorem 1.2 (namely Theorem 2.1) whose proof is slightly simpler and enlights the method we use. Moreover this method of proof will be directly adapted later for the obstacle problem. The proof of Theorem 1.2 will be done in Subsection 2.4 and will consist in an adaptation of the proof of the following result:

Theorem 2.1 (Pointwise modulus of continuity for the Laplace equation)
Let $p \in (1, +\infty)$ given. Then there exist $\alpha \in (0,1]$ and constants $C > 0$, $r_0 \in (0,1)$, such that, given a function $u \in L^p(B_1)$ satisfying (1.1) with a modulus of continuity σ_p defined in (1.9) which is assumed **Dini***, then there exists a harmonic polynomial P_0 of degree less or equal to 2, such that for every $r \in (0, r_0)$:*

$$\left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{u(x)-P_0(x)}{r^2}\right|^p\right)^{\frac{1}{p}} \leq C\left(M_0 r^\alpha + \int_0^r \frac{\sigma_p(s)}{s}\,ds + r^\alpha \int_r^1 \frac{\sigma_p(s)}{s^{1+\alpha}}\,ds\right) \tag{2.14}$$

and

$$P_0(x) = a + b\cdot x + \frac{1}{2}{}^t x\cdot c\cdot x$$

with

$$|a|+|b|+|c| \leq CM_0 \quad \textit{and} \quad M_0 = \int_0^1 \frac{\sigma_p(s)}{s}\,ds + \left(\int_{B_1}|u|^p\right)^{\frac{1}{p}}$$

For the reader's convenience, we recall the equation (1.1) satisfied by u, namely

$$\left\{\begin{array}{l} \Delta u = f \quad \text{in} \quad B_1 \\ \\ f \in L^p(B_1) \quad \text{and} \quad f(0) = 0 \end{array}\right. \tag{2.15}$$

We will use the following

Definition 2.2 (Quantities M and N)
We introduce the following set of functions

$$\mathcal{P}_2 = \{P, \quad \textit{with} \quad P \quad \textit{polynomial of degree less or equal to 2 such that} \quad \Delta P = 0\}$$

and define

$$M(u,\rho) = \sup_{r\in(0,\rho]} N(u,r) \qquad \textit{with} \qquad N(u,r) = \inf_{P\in\mathcal{P}_2} \left(\frac{1}{r^{n+2p}} \int_{B_r} |u-P|^p\right)^{\frac{1}{p}}$$

When there is no ambiguity on the choice of function u, we simply denote these quantities by $M(\rho)$ and $N(r)$.

Let us remark that M and N give a measure of the distance between the function u and the set $\mathcal{P}_2$ which contains the possible limit behaviour of the solution at the origin.

At this stage, it is not clear if M is finite or not. Nevertheless, we have the following property which will be proved at the end of Subsection 2.3. We claim the following

Proposition 2.3 (Finiteness of M)
There exists a constant $C > 0$ such that

$$M(u,1) \le C \left\{ \left(\int_{B_1} |u|^p\right)^{\frac{1}{p}} + \sigma_p(1) \right\} \tag{2.16}$$

Remark 2.4 *Proposition 2.3 is sharp, in view of the following example. In dimension $n = 2$, let us consider $P(x) = x_1^2 - x_2^2$ for $x = (x_1, x_2)$. Then $u(x) = P(x)\ln|x|$ satisfies $\Delta u(x) = 4P(x)/|x|^2$. Therefore Δu is bounded, while D^2u is not.*

Then we have the following cornerstone result which will be proved in Subsection 2.3.

Proposition 2.5 (Decay estimate in a smaller ball)
Given $p \in (1,+\infty)$, there exist constants $C_0 > 0$, $r_0, \lambda, \mu \in (0,1)$ (depending only on p and dimension n) such that for every functions u and f satisfying (1.1) with a modulus of continuity σ_p given by (1.9), then we have the following property

$$\forall r \in (0,r_0), \quad M(u,\lambda r) < \mu M(u,r) \quad \textit{or} \quad M(u,r) < C_0\sigma_p(r) \tag{2.17}$$

Remark 2.6 *Here the problem is linear, so $M(u,r)$ does not need to be small to satisfy the decay estimate.*

Contrarily to Proposition 2.5, the following result does not depend on the particular PDE that we study, but can be considered as a routine result and will be proved in Section 3.

Proposition 2.7 (Modulus of continuity of the solution up to the second order)
Let us consider any function u which satisfies (2.17) with constants $C_0 > 0$, $r_0, \lambda, \mu \in (0,1)$, and a Dini modulus of continuity σ_p. Let us define $\alpha = \ln\mu/\ln\lambda$. Then there exist $P_0 \in \mathcal{P}_2$ and a constant $C_0' > 0$ depending only on C_0, r_0, λ, μ, such that for every $\rho \in (0, \lambda r_0/2)$, we have

$$\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} dy\ |u-P_0|^p\right)^{\frac{1}{p}} \le C_0' \left\{ M(u,r_0)\ \rho^\alpha + \int_0^\rho \frac{\sigma_p(r)}{r}\ dr + \rho^\alpha \int_\rho^{r_0} \frac{\sigma_p(r)}{r^{1+\alpha}}\ dr \right\} \tag{2.18}$$

Proof of Theorem 2.1

The proof of Theorem 2.1 follows from estimates on $M(u,r_0)$ and on P_0 that we establish successively.

Estimate on $M(u,r_0)$

Because the right hand side of the inequality (2.18) is non-increasing with respect to α, it is sufficient to replace α by $\min(1,\alpha)$. Let us choose r_1 such that $C_0' r_1^\alpha \le 1/2$ and $r_1 \le \lambda r_0/2$. Then we have

$$M(u,r_0) \le M(u,r_1) + \sup_{\rho\in[r_1,r_0]} \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u|^p \right)^{\frac{1}{p}}$$

and

$$M(u,r_1) = N(u,\rho_0) \le \left(\frac{1}{\rho_0^{n+2p}} \int_{B_{\rho_0}} |u-P_0|^p \right)^{\frac{1}{p}}$$

for some $\rho_0 \in (0,r_1]$ for which we deduce from (2.18) that

$$M(u,r_1) \le 2C_0' \left\{ \int_0^{\rho_0} \frac{\sigma_p(r)}{r}\, dr + \rho_0^\alpha \int_{\rho_0}^{r_0} \frac{\sigma_p(r)}{r^{1+\alpha}}\, dr \right\}$$

Because the right hand side is a non-decreasing function of ρ_0, we deduce that there exists a constant $C_1 > 0$ depending only on C_0', r_0, α, n, p such that

$$M(u,r_0) \le C_1 \left(\int_0^{r_0} \frac{\sigma_p(r)}{r}\, dr + \left(\int_{B_{r_0}} |u|^p \right)^{\frac{1}{p}} \right)$$

Estimate on P_0

Let us remark that for some ρ_0 (for instance $\rho_0 = \lambda r_0/4$) we have

$$\left(\frac{1}{\rho_0^{n+2p}} \int_{B_{\rho_0}} |P_0|^p \right)^{\frac{1}{p}} \le \left(\frac{1}{\rho_0{}^{n+2p}} \int_{B_{\rho_0}} |u|^p \right)^{\frac{1}{p}} + \left(\frac{1}{\rho_0{}^{n+2p}} \int_{B_{\rho_0}} |u-P_0|^p \right)^{\frac{1}{p}}$$

Then from (2.18), we deduce that for some constant $C_2 > 0$

$$\left(\frac{1}{\rho_0^{n+2p}} \int_{B_{\rho_0}} |P_0|^p \right)^{\frac{1}{p}} \le C_2 \left(\int_0^{\rho_0} \frac{\sigma_p(r)}{r}\, dr + \left(\int_{B_{\rho_0}} |u|^p \right)^{\frac{1}{p}} \right)$$

Finally, if $P_0(x) = a + b\cdot x + \frac{1}{2}{}^t x \cdot c \cdot x$, it can be easily checked that there exists a constant C_3 (independent of P_0) such that

$$|a| + |b| + |c| \le C_3 \left(\frac{1}{\rho_0^{n+2p}} \int_{B_{\rho_0}} |P_0|^p \right)^{\frac{1}{p}}$$

This implies the result and ends the proof of the Theorem.

2.2 Preliminary results

Before performing the proof of Proposition 2.5, we need two lemmata. We first state and prove the following Caccioppoli type estimate.

Lemma 2.8 (Caccioppoli type estimate)
Let $\zeta \in C_0^\infty(\mathbb{R}^n)$ with supp $\zeta \subset B_R(0)$ with $R > 0$. Let $P \in \mathcal{P}_2$ and u be a solution of (1.1) and σ_p defined in (1.9) for some $p \in (1, +\infty)$. Then we have for $W = (u - P)|u - P|^{\frac{p}{2}-1}$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$$\int_{\mathbb{R}^n} \frac{4(p-1)}{p^2}\zeta^2|\nabla W|^2 \leq \int_{\mathbb{R}^n} \frac{4}{p-1}W^2|\nabla\zeta|^2 + 2|B_R|\sigma_p(R)\left(\frac{1}{|B_R|}\int_{B_R}\zeta^{2p'}|W|^2\right)^{\frac{1}{p'}} \quad (2.19)$$

Proof of Lemma 2.8
On the ball B_1, we have

$$-\Delta u + f = 0 \quad \text{and} \quad -\Delta P = 0 = f(0)$$

for every $P \in \mathcal{P}$. Taking the difference of the equations, and multiplying by $\zeta^2 w|w|^{p-2}$ for $w = u - P$, we get

$$\int_{\mathbb{R}^n} -\zeta^2 w|w|^{p-2}\Delta w = \int_{B_R} -\zeta^2 w|w|^{p-2}(f(x) - f(0))$$

An integration by parts shows that we get with $W = w|w|^{\frac{p}{2}-1}$

$$\int_{\mathbb{R}^n} -\zeta^2 w|w|^{p-2}\Delta w = \int_{\mathbb{R}^n} \frac{4(p-1)}{p^2}\zeta^2|\nabla W|^2 + \frac{4}{p}\zeta W\ \nabla\zeta\cdot\nabla W$$

Therefore for $\lambda = \frac{4(p-1)}{p^2}$, we get with $\frac{1}{p} + \frac{1}{p'} = 1$

$$\begin{aligned}
&\int_{\mathbb{R}^n} \lambda\zeta^2|\nabla W|^2 \\
&\leq \int_{\mathbb{R}^n} -\frac{4}{p}\zeta W\ \nabla\zeta\cdot\nabla W + \int_{B_R}\zeta^2|W|^{\frac{2(p-1)}{p}}|f(x) - f(0)| \\
&\leq \int_{\mathbb{R}^n} \frac{1}{2}\left\{\lambda\zeta^2|\nabla W|^2 + \frac{16}{p^2}\lambda^{-1}W^2|\nabla\zeta|^2\right\} + |B_R|\sigma_p(R)\left(\frac{1}{|B_R|}\int_{B_R}\left(\zeta^2|W|^{\frac{2(p-1)}{p}}\right)^{p'}\right)^{\frac{1}{p'}} \\
&\leq \int_{\mathbb{R}^n} \frac{1}{2}\left\{\lambda\zeta^2|\nabla W|^2 + \frac{4}{p-1}W^2|\nabla\zeta|^2\right\} + |B_R|\sigma_p(R)\left(\frac{1}{|B_R|}\int_{B_R}\zeta^{2p'}|W|^2\right)^{\frac{1}{p'}}
\end{aligned}$$

Substracting the term $\frac{1}{2}\int_{\mathbb{R}^n}\lambda\zeta^2|\nabla W|^2$ to the left hand side, this gives (2.19). This ends the proof of the Lemma.

We will also use the following result which shows that we can control a distance between the function u and a particular element of $\mathcal{P}_2$, once we control an integral of $M(u, s)/s$:

Lemma 2.9 (Control of u by M)
Let us assume that the set $\mathcal{P}$ is either the set of harmonic polynomials of degree less or equal to 2, or that each element of $\mathcal{P}$ is homogeneous of degree 2 and that for each $a \geq 0$, the set $K_a = \left\{P \in \mathcal{P}, \quad |P|_{L^p(B_1)} \leq a\right\}$ is compact.
For $p \in (1,+\infty)$, there exists a constant $C_1 > 0$ (which only depends on p and the dimension n) such that if

$$N(u,1) = \left(\int_{B_1} |u - P_1|^p\right)^{\frac{1}{p}} \quad \textit{for some} \quad P_1 \in \mathcal{P}$$

and if u is defined in $B_{2\rho}$ with $\rho \geq 1$, then we have

$$\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_1|^p\right)^{\frac{1}{p}} \leq C_1 \int_1^{2\rho} \frac{M(u,s)}{s}\, ds$$

This result will be proved in Section 3.

2.3 Proof of the decay estimate Proposition 2.5

Proof of Proposition 2.5
We perform the proof by contradiction in two Steps. For simplicity, we fix the exponent p, and set

$$\sigma(r) = \sigma_p(r)$$

By the way, we will have to consider sequences of modulus of continuity σ that we denote by $(\sigma_m)_m$ indexed by m, with no possible confusion.

Step 1: A priori estimates on a sequence v_m
If the Proposition is false, then there exist sequences $(r_m)_m$, $(C_m)_m$, $(\lambda_m)_m$, $(\mu_m)_m$, $(f_m)_m$, $(u_m)_m$, $(\sigma_m)_m$ such that

$$\left\{\begin{array}{l} r_m, \lambda_m \longrightarrow 0 \\ C_m \longrightarrow +\infty \\ \mu_m \longrightarrow 1 \end{array}\right.$$

and

$$M(u_m, r_m) \geq C_m \sigma_m(r_m) \quad \text{and} \quad M(u_m, \lambda_m r_m) \geq \mu_m M(u_m, r_m) \tag{2.20}$$

From Proposition 2.3, $M(u_m,\cdot)$ is bounded (and non-decreasing). Therefore there exists $\rho_m \in (0, \lambda_m r_m]$, such that $N(u_m,\rho_m)$ is arbitrarily close to $M(u_m,\lambda_m r_m)$ and satisfies for instance

$$\frac{M(u_m,\lambda_m r_m)}{1+1/m} \leq N(u_m,\rho_m) =: \varepsilon_m$$

with

$$N(u_m,\rho_m) = \left(\frac{1}{\rho_m^{n+2p}} \int_{B_{\rho_m}} |u_m - P_m|^p\right)^{\frac{1}{p}} \quad \text{for some} \quad P_m \in \mathcal{P}_2$$

We now apply Lemma 2.9 to $u_m^{\rho_m}(x) = u_m(\rho_m \cdot x)/\rho_m^2$, $P_m^{\rho_m}(x) = P_m(\rho_m \cdot x)/\rho_m^2$ and get for every $s \in (1, s_m/2)$ with $s_m = r_m/\rho_m \geq 1/\lambda_m \longrightarrow +\infty$:

$$
(2.21)\qquad
\begin{aligned}
\left(\frac{1}{(s\rho_m)^{n+2p}} \int_{B_{s\rho_m}} |u_m - P_m|^p\right)^{\frac{1}{p}} &= \left(\frac{1}{s^{n+2p}} \int_{B_s} |u_m^{\rho_m} - P_m^{\rho_m}|^p\right)^{\frac{1}{p}} \\
&\leq C_1 \int_1^{2s} ds' \, \frac{M(u_m^{\rho_m}, s')}{s'} \\
&\leq C_1 \int_1^{2s} ds' \, \frac{M(u_m, s'\rho_m)}{s'} \\
&\leq C_1 \frac{(1+1/m)\varepsilon_m}{\mu_m} \ln(2s)
\end{aligned}
$$

We now define the renormalized function

$$
v_m(y) := \frac{1}{\varepsilon_m \rho_m^2} \left(u_m - P_m\right)(\rho_m y)
$$

which satisfies

$$
\Delta v_m = g_m \quad \text{with} \quad g_m(y) = \frac{f_m(\rho_m y)}{\varepsilon_m}
$$

with for fixed $R > 0$

$$
(2.22)\qquad \left(\frac{1}{|B_R|} \int_{B_R} |g_m|^p\right)^{\frac{1}{p}} = \frac{\sigma_m\left(\rho_m R\right)}{\varepsilon_m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty
$$

Indeed, from (2.20), for $s \in (0, s_m)$, we deduce that

$$
\sigma_m(s\rho_m) \leq \sigma_m(s_m\rho_m) = \sigma_m(r_m) \leq \frac{\varepsilon_m(1+1/m)}{\mu_m C_m}
$$

and therefore for every $s \in (0, s_m)$ we have

$$
(2.23)\qquad \frac{\sigma_m(\rho_m s)}{\varepsilon_m} \leq \frac{1+1/m}{\mu_m C_m} \longrightarrow 0
$$

Moreover we have (because here $\mathcal{P}_2$ is a vector space)

$$
(2.24)\qquad \inf_{P \in \mathcal{P}_2} \left(\int_{B_1} |v_m - P|^p\right)^{\frac{1}{p}} = 1,
$$

and for $s \in (1, s_m/2)$

$$
(2.25)\qquad \left(\frac{1}{s^{n+2p}} \int_{B_s} |v_m|^p\right)^{\frac{1}{p}} \leq \frac{C_1(1+1/m)}{\mu_m} \ln(2s) \longrightarrow C_1 \ln(2s)
$$

where the limits are taken as m goes to infinity.
We now apply the Caccioppoli type estimate (2.19) to v_m, we get for supp $\zeta \subset B_R$ and $W_m = v_m |v_m|^{\frac{p}{2}-1}$
(2.26)
$$\int_{\mathbb{R}^n} \frac{4(p-1)}{p^2} \zeta^2 |\nabla W_m|^2 \leq \int_{\mathbb{R}^n} \frac{4}{p-1} W_m^2 |\nabla \zeta|^2 + 2|B_R| \frac{\sigma_m(\rho_m R)}{\varepsilon_m} \left(\frac{1}{|B_R|} \int_{B_R} \zeta^{2p'} |W_m|^2 \right)^{\frac{1}{p'}}$$

Step 2: Convergence of the sequence v_m

From (2.22),(2.26) and (2.25), we get that for every $R > 0$, there exists a constant $C_R > 0$ such that uniformly in m we have

$$|W_m|_{H^1(B_R)} \leq C_R$$

Then, up to extracting a subsequence, we can assume that

$$\begin{cases} W_m \longrightarrow W_\infty = v_\infty |v_\infty|^{\frac{p}{2}-1} & \text{in} \quad L^2_{loc}(\mathbb{R}^n) \quad \text{and a.e. in} \quad \mathbb{R}^n \\ W_m \longrightarrow W_\infty & \text{weakly in} \quad H^1_{loc}(\mathbb{R}^n) \end{cases}$$

where v_∞ has to be seen as the limit of v_m. More precisely, remark that we have the following convergence for all $R > 0$:

$$|v_m|^p_{L^p(B_R)} = |W_m|^2_{L^2(B_R)} \longrightarrow |W_\infty|^2_{L^2(B_R)} = |v_\infty|^p_{L^p(B_R)}$$

and then
$$v_m \to v_\infty \quad \text{in} \quad L^p_{loc}(\mathbb{R}^n)$$
This implies in particular

$$\inf_{P \in \mathcal{P}_2} \left(\int_{B_1} |v_\infty - P|^p \right)^{\frac{1}{p}} = 1, \tag{2.27}$$

and

$$\left(\frac{1}{s^{n+2p}} \int_{B_s} |v_\infty|^p \right)^{\frac{1}{p}} \leq C_1 \ln(2s) \quad \text{for every} \quad s \geq 1 \tag{2.28}$$

and by (2.22), we deduce that v_∞ satisfies

$$\Delta v_\infty = 0 \quad \text{in} \quad \mathbb{R}^n$$

Together with (2.28), we see that $v_\infty \in \mathcal{S}'$ (the dual of the Schwarz space) and then v_∞ is a polynomial, whose degree is less or equal to 2 by (2.28). Therefore this is in contradiction with (2.27).
This ends the proof of the Proposition.

Remark 2.10 *Remark that in the proof of Proposition 2.5, the Caccioppoli estimate can be replaced by any reasonable bound that implies the compactness of the sequence in* $L^p_{loc}(\mathbb{R}^n)$.

Proof of Proposition 2.3

We will perform the proof in two steps. For simplicity, we fix the exponent p, and set

$$\sigma(r) = \sigma_p(r)$$

By the way, we will have to consider sequences of modulus of continuity σ that we denote by $(\sigma_m)_m$ indexed by m, with no possible confusion.

Step 1: finiteness of $M(u,1)$

The proof of the finiteness of $M(u,1)$ follows almost lines by lines the proof of Proposition 2.5 for the decay estimate on M.

Assumme that $M(u,1) = +\infty$. This implies that there is a sequence $(\rho_m)_m$ such that

$$N(u,\rho_m) \longrightarrow +\infty \quad \text{with} \quad \rho_m \longrightarrow 0$$

and

$$N(u,r) \leq N(u,\rho_m) \quad \text{for} \quad r \in (\rho_m, r_0)$$

Still defining $\varepsilon_m = N(u,\rho_m)$ (which this times goes to infinity), we see that

$$v_m(y) := \frac{1}{\varepsilon_m \rho_m^2} \left(u - P_m\right)(\rho_m y)$$

still satisfies

$$\inf_{P \in \mathcal{P}_2} \left(\int_{B_1} |v_m - P|^p \right)^{\frac{1}{p}} = 1 \tag{2.29}$$

and for $s \in (1, s_m/2)$ with this time $s_m = r_0/\rho_m \to +\infty$

$$\left(\frac{1}{s^{n+2p}} \int_{B_s} |v_m|^p \right)^{\frac{1}{p}} \leq C_1 \ln(2s) \tag{2.30}$$

where we have used the fact that the maximum of $M(u,\cdot)$ is reached at ρ_m in (2.21), and the Caccioppoli type estimate for supp $\zeta \subset B_R$ and $W_m = v_m |v_m|^{\frac{p}{2}-1}$

(2.31)

$$\int_{\mathbb{R}^n} \frac{4(p-1)}{p^2} \zeta^2 |\nabla W_m|^2 \leq \int_{\mathbb{R}^n} \frac{4}{p-1} W_m^2 |\nabla \zeta|^2 + 2|B_R| \frac{\sigma(R\rho_m)}{\varepsilon_m} \left(\frac{1}{|B_R|} \int_{B_R} \zeta^{2p'} |W_m|^2 \right)^{\frac{1}{p'}}$$

where we see directly this time that $\dfrac{\sigma(R\rho_m)}{\varepsilon_m} \to 0$, because $\varepsilon_m \to +\infty$ and σ is assumed finite. Finally, using the fact that $\varepsilon_m \to +\infty$, we get that the limit v_∞ of v_m is harmonic, and we get the contradiction following Step 2 of the proof of Proposition 2.5.

Step 2: bound on $M(u,1)$

We now know that $M(u,1)$ is bounded. Let us assume that the Proposition is false. Again, we can find sequences $(C_m)_m$, $(f_m)_m$, $(u_m)_m$, $(\sigma_m)_m$ such that

$$C_m \longrightarrow +\infty$$

and

$$M(u_m, 1) \geq C_m \left\{ \left(\int_{B_1} |u_m|^p \right)^{\frac{1}{p}} + \sigma_m(1) \right\}$$

Therefore there exists $\rho_m \in (0,1]$, such that $N(u_m,\rho_m)$ is arbitrarily close to $M(u_m,1)$. Therefore, we have $\rho_m \longrightarrow 0$ (because $N(u_m,r)$ is bounded by $C_{r_0}\left(\int_{B_1}|u_m|^p\right)^{\frac{1}{p}}$ for some constant $C_{r_0}>0$ for $r\geq r_0>0$). Consequently, we can choose ρ_m satisfying for instance

$$\frac{M(u_m,1)}{1+1/m}\leq N(u_m,\rho_m)=:\varepsilon_m$$

and

$$N(u,r)\leq N(u,\rho_m)\quad\text{for}\quad r\in(\rho_m,1)$$

We finally proceed as in Step 1 of the present proof, but defining

$$v_m(y):=\frac{1}{\varepsilon_m\rho_m^2}\left(u_m-P_m\right)(\rho_m y)$$

Moreover in (2.30), C_1 is replaced by $C_1(1+1/m)$, and in (2.31), $\dfrac{\sigma(R\rho_m)}{\varepsilon_m}$ can be replaced by $\dfrac{\sigma_m(1)}{\varepsilon_m}\leq\dfrac{1+1/m}{C_m}\to 0$. This ends the proof of Proposition 2.3.

2.4 Proof of Theorem 1.2

The proofs of Theorem 1.2 i), ii) and iii) are respectively an adaptation of the proofs of Proposition 2.3, Proposition 2.5 and Theorem 2.1. We give below the results that we have to adapt.

Step 1: preliminaries
First, we consider a constant c_r such that

$$\inf_{c\in\mathbb{R}}\int_{B_r}|f(x)-c|^p=\frac{1}{|B_r|}\int_{B_r}|f(x)-c_r|^p\ .$$

Then, we fix a polynomial P_* homogeneous of degree 2 satisfying $\Delta P_*=1$ (for instance $P_*(x)=\dfrac{x^2}{2n}$), and define

$$\hat{M}(u,\rho)=\sup_{r\in(0,\rho]}\hat{N}(u,r)\quad\text{with}\quad\hat{N}(u,r)=\inf_{P\in\mathcal{P}_2}\left(\frac{1}{r^{n+2p}}\int_{B_r}|u-P-c_rP_*|^p\right)^{\frac{1}{p}}$$

where $\mathcal{P}_2$ is still the set of harmonic polynomials of degree less or equal to 2. The renormalized function is then

$$v_m(y):=\frac{1}{\varepsilon_m\rho_m^2}\left(u-P_m-c_{\rho_m}P_*\right)(\rho_m y)$$

with $P_m\in\mathcal{P}_2$ realizing the infimum in the definition of $\hat{N}(u,\rho_m)$.

Step 2: proof of the analogue of Proposition 2.5
Assuming first that $\hat{M}(u,\cdot)$ is finite, we prove a decay estimate similar to Proposition 2.5, with $M(u,\cdot)$ and σ_p respectively replaced by $\hat{M}(u,\cdot)$ and $\tilde{\sigma}_p$ (remark that this decay estimate implies in particular the VMO estimate ii) of Theorem 1.2). To prove this decay estimate,

we check easily that the proof of Lemma 2.9 applies perfectly in our case. Indeed, it is sufficient to work with the polynomials $\hat{P}_r = P_r + c_r P_*$ which gives with the same constant C_1 the result for $\rho \geq 1$ (and then for u rescalled):

$$\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_1 - c_1 P_*|^p \right)^{\frac{1}{p}} \leq C_1 \int_1^{2\rho} \frac{\hat{M}(u,s)}{s}\, ds$$

Then the rest of the proof is similar with the choice

$$g_m(y) = \frac{f_m(\rho_m y) - c_{\rho_m}}{\varepsilon_m}$$

which introduces an additional factor $1 + \ln R$ (for $R \geq 1$) in the analogue of (2.22) which does not affect the conclusion of the proof.

Step 3: proof of the analogue of Proposition 2.3
The proof of the analogue of Proposition 2.3 is the same, except that here $N(u_m, r)$ is bounded by $C_{r_0}\left(\left(\int_{B_1} |u_m|^p \right)^{\frac{1}{p}} + \left(\int_{B_1} |f_m|^p \right)^{\frac{1}{p}} \right)$ for some constant $C_{r_0} > 0$ for $1 \geq r \geq r_0 > 0$.

Step 4: proof of the analogue of Theorem 2.1
Lemmata 3.3 and 3.4 are still true with $M(u,\cdot)$ and σ_p replaced by $\hat{M}(u,\cdot)$ and $\tilde{\sigma}_p$. Noticing that

$$\tilde{M}(u,r) \leq \hat{M}(u,r) \ ,$$

we get (1.7) (i.e. the analogue of Proposition 2.7) as previously as a consequence of Lemma 3.5 applied to $\tilde{M}(u,\cdot)$ with $\mathcal{P} = \tilde{\mathcal{P}}_2$.

Finally, in the rest of the proof of Theorem 1.2 consisting to control the coefficients of P_0, the only change appears applying (1.7) instead of (2.18). Therefore we use the bound (1.6) to get the estimate on the coefficients of the polynomial. This makes appear an additional term $\left(\int_{B_1} |f|^p \right)^{\frac{1}{p}}$, and finishes the proof of Theorem 1.2.

3 General large scale estimates : proof of Lemma 2.9 and Proposition 2.7

3.1 Proof of Lemma 2.9

Before proving Lemma 2.9, we need the following easy result:

Lemma 3.1 (Larger ball/smaller ball)
There exists a constant $C_2 \geq 1$ only depening on p and the dimension n such that for every polynomial P of degree less or equal to 2, we have for any $r \geq 1$:

$$\left(\frac{1}{r^{n+2p}} \int_{B_r} |P|^p \right)^{\frac{1}{p}} \leq C_2 \left(\int_{B_1} |P|^p \right)^{\frac{1}{p}} \tag{3.32}$$

Proof of Lemma 3.1
We simply remark that if $P(x) = a + b \cdot x + \frac{1}{2} {}^t x \cdot c \cdot x$, then there exists a constant $C_0 > 0$ such that

$$\left(\frac{1}{r^{n+2p}} \int_{B_r} |P|^p \right)^{\frac{1}{p}} \le C_0 \left(\frac{|a|}{r^2} + \frac{|b|}{r} + |c| \right)$$

On the other hand there exists a constant $C_1 > 0$ (easily checked by contradiction) such that

$$|a| + |b| + |c| \le C_1 \left(\int_{B_1} |P|^p \right)^{\frac{1}{p}}$$

Putting together these two inequalities, we get the result for $r \ge 1$, which ends the proof of the Lemma.

Proof of Lemma 2.9
For every $r > 0$, we have

$$N(r) = \left(\frac{1}{r^{n+2p}} \int_{B_r} |u - P_r|^p \right)^{\frac{1}{p}} \quad \text{for some} \quad P_r \in \mathcal{P}$$

Then for $\alpha \in (1, 2]$ we have
(3.33)

$$\begin{aligned}
\left(\frac{1}{r^{n+2p}} \int_{B_r} |P_{\alpha r} - P_r|^p \right)^{\frac{1}{p}} &\le \left(\frac{1}{r^{n+2p}} \int_{B_r} |u - P_r|^p \right)^{\frac{1}{p}} + \left(\frac{1}{r^{n+2p}} \int_{B_r} |u - P_{\alpha r}|^p \right)^{\frac{1}{p}} \\
&\le \left(\frac{1}{r^{n+2p}} \int_{B_r} |u - P_r|^p \right)^{\frac{1}{p}} + \alpha^{\frac{n+2p}{p}} \left(\frac{1}{(\alpha r)^{n+2p}} \int_{B_{\alpha r}(0)} |u - P_{\alpha r}|^p \right)^{\frac{1}{p}} \\
&\le \alpha^{\frac{n+2p}{p}} \left(N(r) + N(\alpha r) \right) \\
&\le C_0 M(\alpha r) \quad \text{with} \quad C_0 = 2^{\frac{n+3p}{p}}
\end{aligned}$$

In the case where the elements $P \in \mathcal{P}$ are 2-homogeneous, i.e. satisfy $P(rx) = r^2 P(x)$, we simply have

$$\left(\int_{B_1} |P_{2r} - P_r|^p \right)^{\frac{1}{p}} = \left(\frac{1}{r^{n+2p}} \int_{B_r} |P_{2r} - P_r|^p \right)^{\frac{1}{p}} \le C_0 M(2r)$$

In the case where $\mathcal{P}$ is the set of harmonic polynomials of degree less or equal to 2, we deduce from (3.32) and a rescaling that for $\rho \ge r > 0$:

$$\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_{2r} - P_r|^p \right)^{\frac{1}{p}} \le C_2 \left(\frac{1}{r^{n+2p}} \int_{B_r} |P_{2r} - P_r|^p \right)^{\frac{1}{p}} \le C M(2r) \quad \text{with} \quad C = C_2 C_0$$

Similarly, we get

$$\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_r - P_1|^p \right)^{\frac{1}{p}} \le C_2 \left(\frac{1}{r^{n+2p}} \int_{B_r} |P_r - P_1|^p \right)^{\frac{1}{p}} \le C M(1)$$

where we have used (3.33) with $1 = \alpha r$ and $\alpha = 1/r \in (1, 2]$.

Now for every $\rho \geq 1$, we write $\rho = 2^k r$ with an integer $k \geq 1$ and $r \in [\frac{1}{2}, 1)$. Then we have

$$(3.34) \qquad \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_1|^p \right)^{\frac{1}{p}} \leq \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_\rho|^p \right)^{\frac{1}{p}} + \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_\rho - P_1|^p \right)^{\frac{1}{p}}$$

We get
(3.35)

$$\begin{aligned} \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_\rho - P_1|^p \right)^{\frac{1}{p}} \quad &\leq \quad \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_r - P_1|^p \right)^{\frac{1}{p}} + \sum_{j=1}^{k} \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |P_{2^j r} - P_{2^{j-1} r}|^p \right)^{\frac{1}{p}} \\ &\leq \quad CM(1) + C \sum_{j=1}^{k} M(2^j r) \end{aligned}$$

From (3.34)-(3.35), we deduce that:

$$\begin{aligned} \left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_1|^p \right)^{\frac{1}{p}} \quad &\leq \quad M(\rho) + C \sum_{j=1}^{k} M(2^j r) + CM(1) \\ &\leq \quad 3C \sum_{j=1}^{k} M(2^j r) \\ &\leq \quad 6C \sum_{j=1}^{k} \frac{M(2^j r)}{2^{j+1} r} \left(2^{j+1} r - 2^j r \right) \\ &\leq \quad 6C \int_{2r}^{2^{k+1} r} \frac{M(s)}{s} \, ds \\ &\leq \quad 6C \int_{1}^{2\rho} \frac{M(s)}{s} \, ds \end{aligned}$$

which ends the proof of the Lemma with $C_1 = 6C$.

3.2 Proof of Proposition 2.7

We will prove the following result which will imply Proposition 2.7 because Proposition 2.5 shows that we can choose the threshold $M_0 = +\infty$:

Proposition 3.2 (Modulus of continuity of the solution up to the second order) *Let us assume that the set $\mathcal{P}$ is either the set of harmonic polynomials of degree less or equal to 2, or that each element of $\mathcal{P}$ is homogeneous of degree 2 and that for each $a \geq 0$, the set*

$K_a = \left\{P \in \mathcal{P}, \quad |P|_{L^p(B_1)} \le a\right\}$ is compact.
For $p \in (1,+\infty)$, let us consider any function u which satisfies

$$\forall r \in (0,r_0), \quad (M(u,r) \le M_0) \Longrightarrow (M(u,\lambda r) < \mu M(u,r) \quad \text{or} \quad M(u,r) < C_0\sigma_p(r))$$

for some constants $M_0, C_0 > 0$, $r_0, \lambda, \mu \in (0,1)$, and a Dini modulus of continuity σ_p. Let us define $\alpha = \ln\mu/\ln\lambda$. If $M(u,r_0) \le M_0$, then there exist $P_0 \in \mathcal{P}$ and a constant $C_0' > 0$ depending only on C_0, r_0, λ, μ, such that for every $\rho \in (0, \lambda r_0/2)$, we have

$$\left(\frac{1}{\rho^{n+2p}}\int_{B_\rho} dy\ |u-P_0|^p\right)^{\frac{1}{p}} \le C_0'\left\{M(u,r_0)\ \rho^\alpha + \int_0^\rho \frac{\sigma_p(r)}{r}\ dr + \rho^\alpha \int_\rho^{r_0} \frac{\sigma_p(r)}{r^{1+\alpha}}\ dr\right\}$$

Before proving Proposition 3.2, we will need several Lemmata. In all what follows, we will set

$$\sigma(r) = \sigma_p(r)$$

Lemma 3.3 (Decay estimate of M)
Under the assumptions of Proposition 3.2, we have for every $r \in (0, \lambda r_0]$,

$$M(u,r) \le \max\left(C_2 r^\alpha, \quad C_0 r^\alpha \sup_{\rho\in[r,\lambda r_0]} \frac{\sigma(\rho)}{\rho^\alpha}\right)$$

with $\alpha = \ln\mu/\ln\lambda$ and $C_2 = M(u,r_0)/(\lambda r_0)^\alpha$.

Proof of Lemma 3.3
If $r \le \lambda r_0$, we write it $r = \lambda^k r_1$ with an integer $k \ge 1$ and $r_1 \in (\lambda r_0, r_0]$. Then we have

$$\begin{aligned} M(u,r) &\le \max\left(C_0\sigma(r), \mu M(u,r/\lambda)\right) \\ &\le \max\left(C_0\sigma(r), C_0\mu\sigma(r/\lambda), \mu^2 M(u,r/\lambda^2)\right) \\ &\le \max\left(C_0\sigma(r), C_0\mu\sigma(r/\lambda), C_0\mu^2\sigma(r/\lambda^2), ..., C_0\mu^{k-1}\sigma(r/\lambda^{k-1}), \mu^k M(u,r/\lambda^k)\right) \end{aligned}$$

Now for $\rho = r/\lambda^j$ with $j \ge 1$, we have on the one hand

$$\begin{aligned} \mu^j\sigma(r/\lambda^j) &= \sigma(\rho)e^{j\ln\mu} \\ &= \sigma(\rho)e^{\ln(r/\rho)\frac{\ln\mu}{\ln\lambda}} \\ &= \frac{\sigma(\rho)}{\rho^\alpha} r^\alpha \end{aligned}$$

where $\alpha = \ln\mu/\ln\lambda$. On the other hand, we have

$$\begin{aligned}
\mu^k M(u, r/\lambda^k) &\le \mu^k M(u, r_1) \\
&\le \mu^k M(u, r_0) \\
&= C_2 \mu^k (\lambda r_0)^\alpha \\
&\le C_2 \mu^k r_1^\alpha \\
&= C_2 \mu^k \left(\frac{r}{\lambda^k}\right)^\alpha \\
&= C_2 r^\alpha
\end{aligned}$$

Therefore we deduce that

$$M(u, r) \le \max\left(C_0 r^\alpha \sup_{\rho\in[r,\lambda r_0]} \frac{\sigma(\rho)}{\rho^\alpha}, \quad C_2 r^\alpha \right)$$

which ends the proof of the Lemma.

Lemma 3.4 (Decay estimate)
Under the assumptions of Proposition 3.2, there exists $C = C(\lambda, \mu, C_0) > 0$ *such that for* $R \le r_0$ *we have*

$$\int_0^{\lambda R} \frac{M(u,r)}{r}\, dr \quad \le \quad M(u, r_0)\, \frac{1}{\alpha} \left(\frac{R}{r_0}\right)^\alpha + C \left(\int_0^{\lambda R} \frac{\sigma(r)}{r}\, dr + R^\alpha \int_{\lambda R}^{r_0} \frac{\sigma(r)}{r^{1+\alpha}}\, dr \right)$$

with $\alpha = \ln\mu/\ln\lambda$.

Proof of Lemma 3.4
We first remark that for $r \le \lambda r_0$ we have

$$\begin{aligned}
\sup_{\rho\in[r,\lambda r_0]} \frac{\sigma(\rho)}{\rho^\alpha} &= \frac{\sigma(\rho_0)}{\rho_0^\alpha} && \text{for some} \quad \rho_0 \in [r, \lambda r_0] \\
&\le \frac{1}{\rho_0^\alpha} \frac{1}{t\rho_0} \int_{\rho_0}^{\rho_0 + t\rho_0} \sigma(\rho)\, d\rho && \text{with} \quad t = \frac{1-\lambda}{\lambda} > 0 \\
&\le \frac{1}{t\lambda^{1+\alpha}} \int_{\rho_0}^{\frac{\rho_0}{\lambda}} \frac{\sigma(\rho)}{\rho^{1+\alpha}}\, d\rho && \text{with} \quad t = \frac{1-\lambda}{\lambda} > 0 \\
&\le C_3 \int_r^{r_0} \frac{\sigma(\rho)}{\rho^{1+\alpha}}\, d\rho && \text{with} \quad C_3 = \frac{1}{(1-\lambda)\lambda^\alpha} > 0
\end{aligned}$$

We deduce that

$$\int_0^{\lambda R} \frac{M(u,r)}{r}\, dr \quad \le \quad C_2 \int_0^{\lambda R} r^{\alpha-1}\, dr + C_0 C_3 J$$

with

$$
\begin{aligned}
J &:= \int_0^{\lambda R} dr\ r^{\alpha-1} \left(\int_r^{r_0} \frac{\sigma(\rho)}{\rho^{1+\alpha}}\ d\rho \right) \\
&= \int_0^{\lambda R} dr\ \frac{r^\alpha}{\alpha} \frac{\sigma(r)}{r^{1+\alpha}}\ dr + [A(r)]_0^{\lambda R} \quad \text{with} \quad A(r) = \frac{r^\alpha}{\alpha} \left(\int_r^{r_0} \frac{\sigma(\rho)}{\rho^{1+\alpha}}\ d\rho \right) \\
&= \frac{1}{\alpha} \int_0^{\lambda R} \frac{\sigma(r)}{r}\ dr + A(\lambda R)
\end{aligned}
$$

where for the second line we have used integration by parts, and for the third line the fact that $A(0^+) = 0$, comming from the dominated convergence theorem applied to

$$
\alpha A(r) = \int_0^{r_0} h_r(\rho) \left(\frac{\sigma(\rho)}{\rho} \right)\ d\rho \quad \text{with} \quad h_r(\rho) := 1_{\{\rho \geq r\}} \left(\frac{r}{\rho} \right)^\alpha
$$

with $0 \leq h_r(\rho) \leq 1$ and $h_r(\rho) \longrightarrow 0$ for a.e. $\rho \in [0, r_0]$ as $r \longrightarrow 0$.
We get the result with $C = C_0 C_3 / \alpha$.

Lemma 3.5 (Modulus of continuity of the solution up to the second order)
If u is defined in B_{r_0}, then there exists $P_0 \in \mathcal{P}$ such that for every $\rho \in (0, r_0/2)$, we have

$$
\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} dy\ |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{2\rho} \frac{M(u,r)}{r}\ dr
$$

Proof of Lemma 3.5
We assume that u is defined on B_{r_0}. From Lemma 2.9 applied to $u^r(x) = u(rx)/r^2$, we get

$$
N(u_r, 1) = \left(\int_{B_1} |u^r - P^r|^p \right)^{\frac{1}{p}} \quad \text{for some} \quad P^r \in \mathcal{P}
$$

and for $2\gamma r \leq r_0$ with $\gamma \geq 1$, we have

$$
\left(\frac{1}{\gamma^{n+2p}} \int_{B_\gamma(0)} |u^r - P^r|^p \right)^{\frac{1}{p}} \leq C_1 \int_1^{2\gamma} \frac{M(u^r, s)}{s}\ ds
$$

A change of variables with $\rho = \gamma r$ and $P^r(x) = P_r(rx)/r^2$ allows to see that (using $M(u^r, s) = M(u, rs)$)

$$
\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_r|^p \right)^{\frac{1}{p}} \leq C_1 \int_r^{2\rho} \frac{M(u,t)}{t}\ dt
$$

Now for $\rho \in (0, r_0/2)$ fixed, we can pass to the limit as r goes to zero, and up to extraction of a subsequence we can assume that $P_r \longrightarrow P_0 \in \mathcal{P}$, and we get

$$
\left(\frac{1}{\rho^{n+2p}} \int_{B_\rho} |u - P_0|^p \right)^{\frac{1}{p}} \leq C_1 \int_0^{2\rho} \frac{M(u,t)}{t}\ dt
$$

This ends the proof.

We are now ready to prove Proposition 3.2.
Proof of Proposition 3.2
Just apply Lemma 3.5 and Lemma 3.4. We get

$$\left(\frac{1}{\rho^{n+2p}}\int_{B_\rho} dy\ |u-P_0|^p\right)^{\frac{1}{p}} \leq C_1\left\{M(u,r_0)\ \frac{1}{\alpha}\left(\frac{2\rho}{\lambda r_0}\right)^\alpha + C\left(\int_0^{2\rho}\frac{\sigma(r)}{r}\ dr + \left(\frac{2\rho}{\lambda}\right)^\alpha\int_{2\rho}^{r_0}\frac{\sigma(r)}{r^{1+\alpha}}\ dr\right)\right\}$$

which implies the result with $\sigma(r)=\sigma_p(r)$. This ends the proof of the Proposition.

4 General results for the obstacle problem: proof of Proposition 1.5 and of Theorem 1.6

We recall that we are interested in solution u of (1.8), that we recall for the convenience of the reader for $p\in(\max(n/2,1),+\infty)$

$$\text{(4.36)}\qquad \begin{cases} \left.\begin{array}{l}\Delta u = f(x)\cdot 1_{\{u>0\}}\\[1ex] u\geq 0\end{array}\right| \quad \text{in} \quad B_1\\[2ex] f\in L^p(B_1) \quad \text{and} \quad f(0)=1\\[2ex] 0\in\partial\{u>0\}\end{cases}$$

In this section we will prove estimates on the pointwise quadratic growth and on the classification in degenerate/regular/singular points using the “monotonicity formula”.

4.1 Quadratic growth of the solution

Proof of Proposition 1.5
For simplicity, we fix the exponent p, and set

$$\sigma(r)=\sigma_p(r)$$

By the way, we will have to consider sequences of modulus of continuity σ that we denote by $(\sigma_m)_m$ indexed by m, with no possible confusion.
Let us first remark that defining

$$g=f\cdot 1_{\{u>0\}}$$

we have with obvious notation for the corresponding modulus of continuity

$$\sigma^g\leq\sigma^f+|f(0)|$$

therefore we can apply Proposition 2.3 with finite modulus of continuity σ^g, and conclude that

$$\text{(4.37)}\quad \forall r\in(0,1),\quad \exists P_r\in\mathcal{P}_2,\quad \left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{u-P_r}{r^2}\right|^p\right)^{\frac{1}{p}}\leq C\left\{\left(\int_{B_1}|u|^p\right)^{\frac{1}{p}}+\sigma(1)+1\right\}$$

with

$$\mathcal{P}_2 = \left\{ P(x) = a + b \cdot x + \frac{1}{2} {}^t x \cdot c \cdot x, \quad (a,b,c) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}_{sym}, \quad \text{trace } (c) = 0 \right\}$$

Let us write

$$P_r(x) = a_r + b_r \cdot x + \frac{1}{2} {}^t x \cdot c_r \cdot x$$

Step 1 : estimate on a_r/r^2
Let us now remark that

$$\Delta u = g \quad \text{in} \quad B_1$$

and then by the classical $W^{2,p}$ elliptic estimates and the Sobolev imbbedings, we get that there exists a constant $C_1 > 0$ such that for every $x, y \in B_{1/2}$:

$$\begin{aligned} |u(x) - u(y)| &\leq C_1 |x-y|^\alpha \left\{ \left(\frac{1}{|B_1|} \int_{B_1} |u|^p \right)^{\frac{1}{p}} + \left(\frac{1}{|B_1|} \int_{B_1} |g|^p \right)^{\frac{1}{p}} \right\} \\ &\leq C_1 |x-y|^\alpha \left\{ \left(\frac{1}{|B_1|} \int_{B_1} |u|^p \right)^{\frac{1}{p}} + \left(\frac{1}{|B_1|} \int_{B_1} |f(x) - f(0)|^p \right)^{\frac{1}{p}} + |f(0)| \right\} \end{aligned}$$

for $\alpha = \min \left(1, 2 - \dfrac{n}{p} \right)$. Let us now set for P_r realizing the infimum in the Definition 2.2 of $N(u,r)$

$$w_r(x) = \frac{(u - P_r)(rx)}{r^2}$$

Applying the previous result to w_r (and a rescaling in the ball B_r), we get that for every $x, y \in B_{1/2}$, we have

$$|w_r(x) - w_r(y)| \leq C|x-y|^\alpha \left\{ N(u,r) + \left(\frac{1}{|B_r|} \int_{B_r} |f(x) - f(0)|^p \right)^{\frac{1}{p}} + |f(0)| \right\}$$

We then write (using the fact that $u(0) = 0$, and then $-\dfrac{a_r}{r^2} = w_r(0) = w_r(x) - (w_r(x) - w_r(0))$) for $\rho \in (0, 1/2)$

$$\begin{aligned} \frac{|a_r|}{r^2} &\leq \left(\frac{1}{|B_\rho|} \int_{B_\rho} |w_r - w_r(0)|^p \right)^{\frac{1}{p}} + \left(\frac{1}{|B_\rho|} \int_{B_\rho} |w_r|^p \right)^{\frac{1}{p}} \\ &\leq C\rho^\alpha \left\{ N(u,r) + \sigma(r) + 1 \right\} + |B_\rho|^{-\frac{1}{p}} N(u,r) \end{aligned}$$

Using the bound given by Proposition 2.3 on $M(u,r) \geq N(u,r)$, we deduce that there exists a constant $C_0 > 0$ such that

$$\frac{|a_r|}{r^2} \leq C_0 \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma(1) + 1 \right\} \tag{4.38}$$

Step 2 : estimate on $|b_r|/r + |c_r|$
We want to prove that there exists a constant $C_0' > 0$ such that

$$\frac{|b_r|}{r} + |c_r| < C_0' \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma(1) + 1 \right\}$$

If it is false, then there exist sequences $(r_m)_m$, $(C_m)_m$, $(u_m)_m$, $(f_m)_m$, $(\sigma_m)_m$ such that

$$r_m \longrightarrow 0 \quad \text{and} \quad C_m \longrightarrow +\infty$$

and

$$M_m := \frac{|b_{r_m}|}{r_m} + |c_{r_m}| \geq C_m \left\{ \left(\int_{B_1} |u_m|^p \right)^{\frac{1}{p}} + \sigma_m(1) + 1 \right\}$$

Let us define

$$v_m(x) = \frac{u_m(r_m x)}{r_m^2 M_m} \geq 0 \quad \text{and} \quad P_m(x) = \frac{P_{r_m}(r_m x)}{r_m^2 M_m}$$

Then we have (from (4.37))

$$\left(\frac{1}{|B_1|} \int_{B_1} |v_m - P_m|^p \right)^{\frac{1}{p}} \leq \frac{C}{C_m} \longrightarrow 0$$

and

$$\frac{|a_{r_m}|}{r_m^2 M_m} \leq \frac{C_0}{C_m} \longrightarrow 0$$

We deduce, up to extracting a subsequence, that P_m converges to a harmonic polynomial P_∞ which can be written

$$P_\infty(x) = b_\infty \cdot x + \frac{1}{2} {}^t x \cdot c_\infty \cdot x \quad \text{with} \quad |b_\infty| + |c_\infty| = 1$$

By contruction $v_m \geq 0$ also converges to P_∞ in B_1, and then $P_\infty \geq 0$ in B_1. Therefore $P_\infty = 0$. Contradiction.

Step 3 : Conclusion
Therefore we conclude that there exists a constant $C_0'' > 0$ such that

$$\sup_{B_r} \frac{|P_r|}{r^2} \leq C_0'' \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma(1) + 1 \right\}$$

and then there exists $C_3 > 0$ such that

$$\forall r \in (0,1), \quad \left(\frac{1}{|B_r|} \int_{B_r} \left| \frac{u}{r^2} \right|^p \right)^{\frac{1}{p}} \leq C_3 \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma(1) + 1 \right\}$$

Then applying interior $W^{2,p}$ elliptic estimates and Sobolev imbeddings joined to a scaling argument, we get that there exists a constant $C_4 > 0$ such that

$$\forall r \in (0,1), \quad \left| \frac{u}{r^2} \right|_{L^\infty(B_{r/2})} \leq C_4 \left\{ \left(\frac{1}{|B_r|} \int_{B_r} \left| \frac{u}{r^2} \right|^p \right)^{\frac{1}{p}} + \left(\frac{1}{|B_r|} \int_{B_r} |\Delta u|^p \right)^{\frac{1}{p}} \right\}$$

This finally leads to the existence of a constant $C_5 > 0$ such that

$$\forall x \in B_{1/2}, \quad |u(x)| \leq |x|^2 C_5 \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma(1) + 1 \right\}$$

This ends the proof of the Proposition.

Corollary 4.1 (Bound on U)
Let $p \in (\max(n/2, 1), +\infty)$ with $p \geq 2n/(n+1)$. Then there exists constants $C > 0$ such that if u is a solution of (1.8) with σ_p given by (1.9), then $U = x \cdot \nabla u - 2u$ satisfies for $\frac{1}{p} + \frac{1}{p'} = 1$

$$\forall r \in (0, 1/2), \quad \left(\frac{1}{B_r} \int_{B_r} \left| \frac{U}{r^2} \right|^{p'} \right)^{\frac{1}{p'}} \leq C \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma_p(1) + 1 \right\}$$

Proof of Corollary 4.1
Let us define

$$u_r(x) = \frac{u(rx)}{r^2}, \quad f_r(x) = f(rx)$$

We have $\Delta u_r = f_r \cdot 1_{\{u_r > 0\}}$. From classical interior $W^{2,p}$ elliptic estimates and Sobolev imbbedings applied to u_r, we get for $r \in (0, 1/2)$ the existence of a constant $C > 0$ such that (because $p \geq 2n/(n+1)$)

$$\left(\frac{1}{B_1} \int_{B_1} |u_r|^{p'} \right)^{\frac{1}{p'}} + \left(\frac{1}{B_1} \int_{B_1} |\nabla u_r|^{p'} \right)^{\frac{1}{p'}} \leq C \left\{ \left(\frac{1}{|B_1|} \int_{B_1} |u_r|^p \right)^{\frac{1}{p}} + \sigma_p(r) + 1 \right\}$$

Therefore

$$\begin{aligned} \left(\frac{1}{B_1} \int_{B_1} |x \cdot \nabla u_r - 2u_r|^{p'} \right)^{\frac{1}{p'}} &\leq 3C \left\{ \left(\frac{1}{|B_1|} \int_{B_1} |u_r|^p \right)^{\frac{1}{p}} + \sigma_p(r) + 1 \right\} \\ &\leq C_2 \left\{ \left(\frac{1}{|B_1|} \int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma_p(1) + 1 \right\} \end{aligned}$$

for some constant $C_2 > 0$ where we have used Proposition 1.5 for the last line. This is exactly the expected result for $U = x \cdot u - 2u$, which ends the proof of the Corollary.

4.2 Liouville Theorem and monotonicity formula

The following result, proved by Caffarelli [7] and Weiss [42], classifies the possible blow-up limits:

Theorem 4.2 (Liouville theorem)
Let us consider a function u^0 which is solution of

$$\begin{cases} \Delta u^0 = 1_{\{u^0>0\}} \quad \textit{in} \quad \mathbb{R}^n \\ \\ u^0 \geq 0 \\ \\ u^0(\lambda x) = \lambda^2 u^0(x) \quad \textit{for any} \quad \lambda > 0 \end{cases}$$

Then either
i) **(degenerate case)** $u^0(x) \equiv 0$, *or*
ii) **(regular case)** *there exists* $\nu \in \mathbf{S}^{n-1}$ *such that*

$$u^0(x) = \frac{1}{2} \max\left(0, x \cdot \nu\right)^2$$

or
iii) **(singular case)** *there exists a symmetric matrix* $Q \in \mathbb{R}^{n\times n}_{sym}$ *with* $Q \geq 0$, *trace*$(Q) = 1$ *such that*

$$u^0(x) = \frac{1}{2} {}^t x \cdot Q \cdot x$$

Sketch of the proof of Theorem 1.6
It is possible to compute (see [36]):

$$\frac{d\Phi}{dr}(r) = \frac{2}{r^n} \int_{\partial B_r} \left|\frac{U(x)}{r^2}\right|^2 - \frac{2}{r^{n+3}} \int_{B_r} U\left(f(x) - f(0)\right) \tag{4.39}$$

where $r = |x|$ in the integral on the boundary ∂B_r, and

$$U = x \cdot \nabla u - 2u$$

It can be easily checked that this computation is justified for $p \geq 2n/(n+1)$ which garanties in particular that $\nabla u \in L^2(\partial B_r)$ and $\nabla u \cdot \nabla U \in L^1_{loc}(B_1)$. We have

$$\frac{1}{|B_r| r^3} \int_{B_r} |U\left(f(x) - f(0)\right)| \quad \leq \quad \left(\frac{1}{|B_r|} \int_{B_r} \left|\frac{U(x)}{r^2}\right|^{p'}\right)^{\frac{1}{p'}} \cdot \frac{\sigma_p(r)}{r}$$

The bound of Corollary 4.1 implies that there exists a constant $C > 0$ such that

$$\frac{2}{r^{n+3}} \int_{B_r} |U\left(f(x) - f(0)\right)| \quad \leq \quad C \frac{\sigma_p(r)}{r}$$

and then for $r > t > 0$

$$\Phi(r) - \Phi(t) \geq -C \int_t^r \frac{\sigma_p(s)}{s}\, ds$$

This implies that $\Phi(r)$ has a limit $\Phi(0^+)$ at $r = 0$.
The rest of the proof is similar to what is done in the constant coefficient case (see [42] and [7]). This ends the proof of the Theorem.

5 Degenerate points of the free boundary

We first start with the following result

Lemma 5.1 (Nondegeneracy)
Let $p \in (\max(n/2, 1), +\infty)$. Then there exists a constant $C_0 > 0$ such that if u satisfies for some $R > 0$:

$$\begin{cases} \Delta u = f \cdot 1_{\{u>0\}} & \text{in} \quad B_R \\ \\ u \geq 0 \end{cases}$$

and if $B_d(x_0) \subset B_R$ for some $d > 0$, then for

$$\lambda := C_0 d^{2-\frac{n}{p}} R^{\frac{n}{p}} \left(\frac{1}{|B_R|} \int_{B_R} |f(x) - f(0)|^p \right)^{\frac{1}{p}}$$

we have

$$(u(x_0) > 2\lambda) \quad \Longrightarrow \quad \left(\sup_{\partial B_d(x_0)} u \quad \geq \quad \frac{d^2}{2n} - 2\lambda \right)$$

Proof of Lemma 5.1
Let us consider the solution v of

$$\begin{cases} \Delta v = f(x) - f(0) & \text{in} \quad B_d(x_0) \\ \\ v = 0 & \text{on} \quad \partial B_d(x_0) \end{cases}$$

From the classical $W^{2,p}$ elliptic estimates and the Sobolev imbbedings for $p > n/2$, we deduce that there exists a constant $C_0 > 0$ such that (rescaling back to the unit ball)

$$|v|_{L^\infty(B_d(x_0))} \leq C_0 d^2 \left(\frac{1}{|B_d|} \int_{B_d(x_0)} |f(x) - f(0)|^p \right)^{\frac{1}{p}} \leq \lambda$$

We now define

$$w(x) = u(x) - \frac{|x - x_0|^2}{2n} - v(x)$$

which satisfies

$$\Delta w = 0 \quad \text{in} \quad \omega := \{u > 0\} \cap B_d(x_0)$$

then from the maximum principle, we have

$$w(x_0) \leq \sup_{\partial \omega} w$$

On the one hand, we have

$$w(x_0) = u(x_0) - v(x_0) \geq u(x_0) - |v|_{L^\infty(B_d(x_0))} \geq u(x_0) - \lambda$$

and

$$w(x) = -\frac{|x - x_0|^2}{2n} - v(x) < \lambda \quad \text{on} \quad (\partial \{u > 0\}) \cap B_d(x_0)$$

Therefore, while $u(x_0) > 2\lambda$, we get that

$$\sup_{\partial\omega} w = \sup_{\{u>0\}\cap\partial B_d(x_0)} w \leq \sup_{\partial B_d(x_0)} u - \frac{d^2}{2n} + |v|_{L^\infty(B_d(x_0))}$$

and therefore

$$\sup_{\partial B_d(x_0)} u \geq \frac{d^2}{2n} - 2|v|_{L^\infty(B_d(x_0))}$$

which ends the proof of the Lemma.

Proposition 5.2 (Growth at a degenerate point)
Let $p \in (\max(n/2, 1), +\infty)$ with $p \geq 2n/(n+1)$. There exists a constant $C > 0$ such that if 0 is a degenerate point for u, then there exists r_0 (here depending on u) such that

$$\forall x \in B_{r_0}, \quad 0 \leq u(x) \leq C|x|^2\sigma_p(2|x|)$$

Proof of Proposition 5.2
Assume that the Proposition is false. Then there exists sequences $(C_m)_m$, $(x_m)_m$ such that

$$C_m \longrightarrow +\infty \quad \text{and} \quad x_m \longrightarrow 0$$

and

$$u(x_m) \geq C_m|x_m|^2\sigma_p(2|x_m|)$$

Apply Lemma 5.1 with $|x_m| = d_m = R_m/2$. Then we get that there exists a constant $C_1 > 0$ such that

$$\sup_{\partial B_{d_m}(x_m)} u \quad \geq \quad \frac{d_m^2}{2n} - C_1 d_m^2 \sigma_p(2d_m)$$

and then

$$\sup_{\partial B_{R_m}} u \quad \geq \quad \frac{R_m^2}{8n} - \frac{C_1}{4} R_m^2 \sigma_p(R_m)$$

Then we see that up to extraction of a subsequence, we have

$$u_m(x) = \frac{u(R_m x)}{R_m^2} \longrightarrow u_\infty(x) \not\equiv 0$$

On the other hand, if we note $\Phi_u(r)$ the expression (1.10) associated to the function u, we get

$$\Phi_u(rR_m) \longrightarrow \Phi(0^+) = 0$$

and

$$\Phi_u(rR_m) = \Phi_{u_m}(r) \longrightarrow \Phi_{u_\infty}(r)$$

which implies that $u_\infty \equiv 0$. Contradiction. This ends the proof of the Proposotion.

Example of a degenerate point
For $p > (\max(n/2, 1), +\infty)$, we build here a function $f \in L^p(B_1)$ with σ_p Dini such that u solves in B_1

$$\Delta u = f \cdot 1_{\{u>0\}} \quad \text{and} \quad u \geq 0$$

and the origin 0 is a degenerate point for u (see Figure 1).
Let $\zeta \in C_c^\infty(\mathbb{R}^n)$ such that supp $\zeta \subset B_{1/4}$, $\zeta \geq 0$ and $\zeta \not\equiv 0$. We set $\gamma = 1 + |\Delta\zeta|_{L^\infty(\mathbb{R}^n)}$ and for $\lambda > 0$

$$\zeta_\lambda(x) = \lambda^2 \zeta\left(\frac{x}{\lambda}\right)$$

We also set $x = (x_1, x')$ with $x' = (x_2, ..., x_n)$. Given a non-increasing sequence $(\lambda_k)_k$ with $0 < \lambda_k \leq 1$, we set

$$u(x) = \sum_{k\geq 1} 4^{\frac{n}{p}} \zeta_{\lambda_k 2^{-k}}\left(x_1 - 2^{-k}, x'\right)$$

and define for $x^k = (2^{-k}, 0, ..., 0)$

$$\Omega = \bigcup_{k\geq 1} B_{\lambda_k 2^{-k}}\left(x^k\right)$$

Let us define with $f(0) = 1$

$$f = \begin{cases} \Delta u & \text{in} \quad B_1 \cap \Omega \\ \\ 1 & \text{in} \quad B_1 \backslash \Omega \end{cases}$$

the union of disjoint balls. Then we compute for $K \geq 1$

$$\begin{aligned} &\frac{1}{|B_{2^{-K}}|} \int_{B_{2^{-K}}} |f(x) - f(0)|^p \\ &\leq \frac{1}{|B_{2^{-K}}|} \sum_{k\geq K} 4^n \gamma^p \left(\lambda_k 2^{-k}\right)^n |B_{1/4}| \\ &\leq \gamma^p \lambda_K^n 2^{nK} \sum_{k\geq K} 2^{-nk} \\ &\leq \mu^p \lambda_K^n \quad \text{with} \quad \mu^p = \gamma^p (1 - 2^{-n})^{-1} \end{aligned}$$

Therefore

$$\left(\frac{1}{|B_{2^{-K}}|} \int_{B_{2^{-K}}} |f(x) - f(0)|^p\right)^{\frac{1}{p}} \leq \mu \lambda_K^{\frac{n}{p}}$$

Hence σ_p is Dini, if we choose the sequence $(\lambda_k)_k$ such that

$$\sum_{K\geq 1} \lambda_K^{\frac{n}{p}} < +\infty$$

(for instance for a geometric sequence $(\lambda_K)_K$).

6 Regular points of the free boundary and proof of Theorem 1.8

6.1 Proof of Theorem 1.8

Here we adapt the proof of Theorem 2.1, and replace the set $\mathcal{P}_2$ by

$$\mathcal{P}_{reg} = \left\{P, \quad \exists \nu \in \mathbf{S}^{n-1}, \quad P = \frac{1}{2}\max\left(0, x \cdot \nu\right)^2\right\}$$

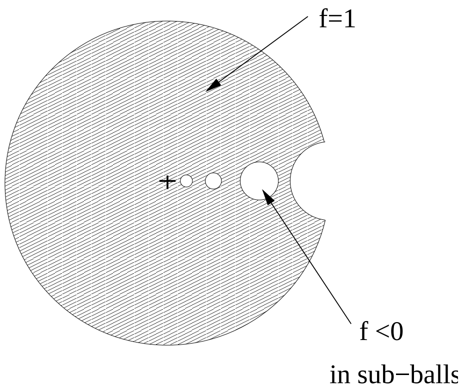

Figure 1: Construction of a function f such that the origin is a degenerate point

keeping the same notation for $M(u,\rho)$ and $N(u,r)$ (see Definition 2.2). Here the set $\mathcal{P}_{reg}$ contains the possible limit behaviours of the solution at the origin when the origin is a regular point.

We claim the following

Proposition 6.1 (Finiteness of M)
Let us assume that $p \in (\max(n/2,1),+\infty)$. Then there exists a constant $C>0$ such that

$$\forall r \in (0,1), \quad M(u,r) \le C\left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma_p(1) + 1 \right\}$$

Proof of Proposition 6.1
This is a straightforward consequence of Proposition 1.5.

Then we have the following cornerstone result which will be proved in subsection 6.3.

Proposition 6.2 (Decay estimate in a smaller ball)
Given $p \in (n/2,+\infty)$, there exist constants $M_0, C_0 > 0$, $r_0, \lambda, \mu \in (0,1)$ (depending only on p and dimension n) such that for every functions u and f satisfying (1.8) with a modulus of continuity σ_p given by (1.9), then we have the following property
(6.40)

$$\forall r \in (0,r_0), \quad (M(u,r) \le M_0) \Longrightarrow (M(u,\lambda r) < \mu M(u,r) \quad or \quad M(u,r) < C_0 \sigma_p(r))$$

Remark 6.3 *Proposition 6.2 is similar to Proposition 2.5 for Laplace equation. One important difference is that we have to introduce a threshold M_0 for the obstacle problem, because it is a nonlinear problem. Indeed, $M(u,r)$ has to be smaller than the threshold M_0, in order to be able to claim that it satisfies the decay estimate. As an illustration, the reader can simply think to functions that are blow-up limits u^0 at singular points, like for instance $u^0(x) = \dfrac{x^2}{2n}$. Then $M(u^0,r)$ is a positive constant independent of r, which implies in particular that $M(u^0,r) > M_0$.*

Proof of Theorem 1.8
The proof of Theorem 1.8 follows exactly the lines of the proof of Theorem 2.1, where Proposition 2.7 is replaced by Proposition 3.2. This ends the proof of the Theorem.

6.2 Preliminary results

To prove Proposition 6.2, we will need the following Caccioppoli estimate for the obstacle problem.

Lemma 6.4 (Caccioppoli type estimate)
Lemma 2.8 is still valid for solutions u of (1.8) for P satisfying

$$-\Delta P + 1_{\{P>0\}} = 0 \quad \text{and} \quad P \geq 0 \tag{6.41}$$

Proof of Lemma 6.4
On the ball B_1, we have

$$-\Delta u + f(x) \cdot 1_{\{u>0\}} = 0$$

and let us consider any P satisfying

$$-\Delta P + 1_{\{P>0\}} = 0$$

Taking the difference of these two equations, and multiplying by $\zeta^2 w|w|^{p-2}$ for $w = u - P$, we get

$$\int_{\mathbb{R}^n} -\zeta^2 w|w|^{p-2}\Delta w + \zeta^2 w|w|^{p-2}\left(1_{\{u>0\}} - 1_{\{P>0\}}\right) = \int_{B_R} -\zeta^2 w|w|^{p-2}(f(x) - f(0))1_{\{u>0\}}$$

The rest of the proof is identical to the one of Lemma 2.8, simply taking into account the fact that $w\left(1_{\{u>0\}} - 1_{\{P>0\}}\right) \geq 0$. This ends the proof of the Lemma.

We will also use the following result

Lemma 6.5 (Weak nondegeneracy)
Let us fix $R > 0$ and $p \in (\max(n/2, 1), +\infty)$. Let us consider sequences of functions $(u_m)_m$, $(f_m)_m$ such that

$$\left\{\begin{array}{ll} \left.\begin{array}{l} \Delta u_m = f_m(x) \cdot 1_{\{u_m>0\}} \\ \\ u_m \geq 0 \end{array}\right| & \text{in} \quad B_R \\ f_m(0) = 1, \quad \text{and} \quad \left(\dfrac{1}{|B_R|}\displaystyle\int_{B_R} |f_m(x) - f_m(0)|^p\right)^{\frac{1}{p}} =: \tau_m \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty & \end{array}\right.$$

Let us assume that u_m converges to u_∞ in $L^\infty_{loc}(B_R)$. Let us consider a compact K contained in the interior of the coincidence set $\{u_\infty = 0\}$. Then there exists a constant $C > 0$ (which depends in particular on K and R, but is independent of τ_m) such that

$$u_m \leq C\tau_m \quad \text{in} \quad K$$

Proof of Lemma 6.5
Let us assume that the Lemma is false. Then we can find a sequence of points $(x_m)_m$ a sequence $(C_m)_m$ such that (up to extraction)

$$C_m \longrightarrow +\infty$$

$$u_m(x_m) > C_m\tau_m, \quad x_m \in K$$

Let us choose d such that $0 < d < \mathrm{dist}(K, \{u_\infty > 0\})$. Then we can apply Lemma 5.1 which states that

$$(u_m(x_m) > 2\lambda_m) \quad \Longrightarrow \quad \left(\sup_{\partial B_d(x_m)} u_m \quad \geq \quad \frac{d^2}{2n} - 2\lambda_m\right)$$

with

$$\lambda_m = C_0 d^{2-\frac{n}{p}} R^{\frac{n}{p}} \left(\frac{1}{|B_R|}\int_{B_R} |f_m(x) - f_m(0)|^p\right)^{\frac{1}{p}} \leq C_1\tau_m \quad \text{with} \quad C_1 = C_0 d^{2-\frac{n}{p}} R^{\frac{n}{p}}$$

Passing to the limit as $m \to +\infty$, we get (up to extraction of a subsequence) that

$$x_m \longrightarrow x_\infty \in K$$

and

$$\sup_{\partial B_d(x_\infty)} u_\infty \geq \frac{d^2}{2n}$$

Contradiction because $u_\infty = 0$ in a neighbourhood of K. This ends the proof of the Lemma.

6.3 Proof of the decay estimate Proposition 6.2

Proof of Proposition 6.2

We perform the proof by contradiction in three Steps. For simplicity, we fix the exponent p, and set

$$\sigma(r) = \sigma_p(r)$$

By the way, we will have to consider sequences of modulus of continuity σ that we denote by $(\sigma_m)_m$ indexed by m, with no possible confusion.

Step 1: A priori estimates on a sequence v_m

If the Proposition is false, then there exist sequences $(M_m)_m$, $(r_m)_m$, $(C_m)_m$, $(\lambda_m)_m$, $(\mu_m)_m$, $(f_m)_m$, $(u_m)_m$, $(\sigma_m)_m$ such that

$$\begin{cases} M_m, r_m, \lambda_m \longrightarrow 0 \\ C_m \longrightarrow +\infty \\ \mu_m \longrightarrow 1 \end{cases}$$

and

$$M_m \geq M(u_m, r_m) \geq C_m\sigma_m(r_m) \quad \text{and} \quad M(u_m, \lambda_m r_m) \geq \mu_m M(u_m, r_m) \tag{6.42}$$

Let us recall that by assumption $M(u_m, r_m)$ is bounded by M_m which goes to zero. Therefore there exists $\rho_m \in (0, \lambda_m r_m]$, such that $N(u_m, \rho_m)$ is arbitrarily close to $M(u_m, \lambda_m r_m)$ and satisfies for instance

$$\frac{M(u_m, \lambda_m r_m)}{1 + 1/m} \leq N(u_m, \rho_m) =: \varepsilon_m \leq M(u_m, r_m) \leq M_m \longrightarrow 0$$

with

$$\varepsilon_m := N(u_m,\rho_m) = \left(\frac{1}{\rho_m^{n+2p}}\int_{B_{\rho_m}} |u_m - P_m|^p\right)^{\frac{1}{p}} \quad \text{for some} \quad P_m \in \mathcal{P}_{reg}$$

Now for every $s \in (0, s_m)$ with $s_m = r_m/\rho_m \geq 1/\lambda_m \longrightarrow +\infty$, we have

$$\inf_{P\in\mathcal{P}} \left(\frac{1}{(s\rho_m)^{n+2p}}\int_{B_{s\rho_m}} |u_m - P_m|^p\right)^{\frac{1}{p}} \leq \frac{\varepsilon_m(1+1/m)}{\mu_m}$$

Let us set

$$u_m^{\rho_m}(x) = u_m(\rho_m \cdot x)/\rho_m^2$$

We now define the renormalized function

$$v_m(y) := \frac{1}{\varepsilon_m\rho_m^2}\left(u_m - P_m\right)(\rho_m y)$$

which satisfies

$$\text{(6.43)} \qquad \left\{\begin{array}{rcll} \Delta v_m & = & g_m & \text{in} \quad \{u_m^{\rho_m} > 0\} \cap \{P_m > 0\} \\ v_m & = & 0 & \text{in} \quad \{u_m^{\rho_m} = 0\}^0 \cap \{P_m = 0\}^0 \end{array}\right.$$

where for a set A, we denote by A^0 its interior, and with

$$g_m(y) = \frac{f_m(\rho_m y) - f_m(0)}{\varepsilon_m}$$

which satisfies for fixed $R \in (0, s_m)$ (as in (2.22))

$$\text{(6.44)} \qquad \left(\frac{1}{|B_R|}\int_{B_R} |g_m|^p\right)^{\frac{1}{p}} = \frac{\sigma_m\left(\rho_m R\right)}{\varepsilon_m} \leq \frac{1+1/m}{\mu_m C_m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty$$

Moreover as in Step 1 of the proof of Propositon 2.5, we get

$$\text{(6.45)} \qquad \inf_{P\in\mathcal{P}} \left(\int_{B_1} \left|v_m - \left(\frac{P-P_m}{\varepsilon_m}\right)\right|^p\right)^{\frac{1}{p}} = 1,$$

and for $s \in (0, s_m)$:

$$\text{(6.46)} \qquad \inf_{P\in\mathcal{P}} \left(\frac{1}{s^{n+2p}}\int_{B_s} \left|v_m - \left(\frac{P-P_m}{\varepsilon_m}\right)\right|^p\right)^{\frac{1}{p}} \leq \frac{1+1/m}{\mu_m} \longrightarrow 1$$

and for $s \in (1, s_m/2)$

$$\text{(6.47)} \qquad \left(\frac{1}{s^{n+2p}}\int_{B_s} |v_m|^p\right)^{\frac{1}{p}} \leq \frac{C_1(1+1/m)}{\mu_m}\ln(2s) \longrightarrow C_1\ln(2s)$$

where the limits are taken as m goes to infinity.

We now apply the Caccioppoli type estimate (6.4) to v_m, and we get for supp $\zeta \subset B_R$ and $W_m = v_m|v_m|^{\frac{p}{2}-1}$
(6.48)

$$\int_{\mathbb{R}^n} \frac{4(p-1)}{p^2}\zeta^2|\nabla W_m|^2 \le \int_{\mathbb{R}^n} \frac{4}{p-1}W_m^2|\nabla\zeta|^2 + 2|B_R|\frac{\sigma_m(\rho_m R)}{\varepsilon_m}\left(\frac{1}{|B_R|}\int_{B_R}\zeta^{2p'}|W_m|^2\right)^{\frac{1}{p'}}$$

Step 2: Convergence of the sequence v_m

From (6.44)-(6.45)-(6.46)-(6.47)-(6.48), we get as in Step 2 of the proof of Propositon 2.5 that up to extracting a subsequence, we have

$$\begin{cases} W_m \longrightarrow W_\infty = v_\infty|v_\infty|^{\frac{p}{2}-1} & \text{in} \quad L^2_{loc}(\mathbb{R}^n) \quad \text{and a.e. in} \quad \mathbb{R}^n \\ W_m \longrightarrow W_\infty & \text{weakly in} \quad H^1_{loc}(\mathbb{R}^n) \end{cases}$$

where v_∞ has to be seen as the limit of v_m.
Up to tilting the coordinates, we can assume that the function P_m is fixed with

$$P_m = P_\infty = \frac{1}{2}\left(\max(0,x_1)\right)^2, \quad \forall m$$

For any $\beta = (\beta_1, ..., \beta_n) \in \mathbb{R}^n$ and $x = (x_1, ..., x_n)$, let us define

$$q_\beta(x) = (\beta \cdot x)\cdot \max(0,x_1).$$

Then we introduce the following set

$$T_{P_\infty}\mathcal{P} = \{q, \quad \exists \beta \in \mathbb{R}^n, \quad \text{such that} \quad q = q_\beta \quad \text{with} \quad \beta_1 = 0\}$$

which can be interpretated as the tangent space to the set $\mathcal{P}$ at the point P_∞, which justifies the notation.
Now for every $q_\beta \in T_{P_\infty}\mathcal{P}$, we set $\nu_m = \dfrac{e_1+\varepsilon_m\beta}{|e_1+\varepsilon_m\beta|}$ and define

$$\tilde{P}_m := \frac{1}{2}\left(\max\left(0, \nu_m\cdot x\right)\right)^2$$

for which we have

$$\frac{\tilde{P}_m - P_\infty}{\varepsilon_m} \longrightarrow q_\beta$$

Then (6.45) implies

(6.49)
$$\inf_{q\in T_{P_\infty}\mathcal{P}}\left(\int_{B_1}|v_\infty - q|^p\right)^{\frac{1}{p}} = 1$$

Moreover (6.46) implies

(6.50)
$$\inf_{q\in T_{P_\infty}\mathcal{P}}\left(\frac{1}{s^{n+2p}}\int_{B_s}|v_\infty - q|^p\right)^{\frac{1}{p}} \le 1 \quad \text{for every} \quad s>0$$

And (6.47) implies

(6.51)
$$\left(\frac{1}{s^{n+2p}}\int_{B_s}|v_\infty|^p\right)^{\frac{1}{p}} \le C_1\ln(2s) \quad \text{for every} \quad s\ge 1$$

and (6.48) implies

$$\int_{\mathbb{R}^n} \frac{4(p-1)}{p^2}\zeta^2|\nabla W_\infty|^2 \le \int_{\mathbb{R}^n} \frac{4}{p-1}W_\infty^2|\nabla\zeta|^2 \tag{6.52}$$

We have $u_m^{\rho_m} - P_m = \varepsilon_m v_m$, and $\Delta u_m^{\rho_m} = f_m(\rho_m\cdot)\cdot 1_{\{u_m^{\rho_m}>0\}}$ where the right hand side is bounded in $L^p_{loc}(\mathbb{R}^n)$. Then by classical elliptic estimates, $u_m^{\rho_m}$ is bounded in $W^{2,p}_{loc}(\mathbb{R}^n)$, and then by Sobolev imbeddings, $u_m^{\rho_m}$ converges (up to extraction of some subsequence) to its limit P_∞ in $L^\infty_{loc}(\mathbb{R}^n)$ because $p > n/2$. We deduce from (6.43) that v_∞ satisfies the first line of the following equalities

$$\left\{\begin{array}{llll} \Delta v_\infty & = & 0 & \text{in} \quad \{P_\infty > 0\} = \{y_1 > 0\} \\ & & & \\ v_\infty & = & 0 & \text{in} \quad \{P_\infty = 0\}^0 = \{y_1 < 0\} \end{array}\right. \tag{6.53}$$

To state the second line, we simply apply Lemma 6.5 to the sequence of functions $u_m^{\rho_m}$ with $\tau_m = \sigma_m(\rho_m R)$ and deduce that for any compact K of $\{P_\infty = 0\}^0 \cap B_R$, there exists a constant $C > 0$ such that

$$v_m \le C\frac{\sigma_m(\rho_m R)}{\varepsilon_m} \longrightarrow 0.$$

Step 3: Identification of the limit v_∞ and contradiction

From (6.53) and the fact that $v_\infty|v_\infty|^{\frac{p}{2}-1}$ belongs to $H^1_{loc}(\mathbb{R}^n)$ we deduce that

$$v_\infty = 0 \quad \text{on} \quad \{y_1 = 0\}\,. \tag{6.54}$$

Because v_∞ is harmonic in the half space $\{y_1 > 0\}$, we deduce from the regularity theory that v_∞ is analytic on $\{y_1 \ge 0\}$. Therefore we easily check that the function

$$\tilde{v}_\infty(y) = \left\{\begin{array}{ll} v_\infty(y) & \text{if} \quad y_1 \ge 0 \\ & \\ -\; v_\infty(-y_1, y_2, ..., y_n) & \text{if} \quad y_1 < 0 \end{array}\right.$$

satisfies

$$\Delta\tilde{v}_\infty = 0 \quad \text{in} \quad \mathbb{R}^n$$

From (6.51), we see that $\tilde{v}_\infty \in \mathcal{S}'$ (the dual of the Schwarz space) and then $\tilde{v}_\infty$ is a polynomial whose degree is less or equal to 2, still from (6.51). Moreover from (6.50) we deduce that $\tilde{v}_\infty$ is homogeneous of degree 2.
From (6.54), we deduce with $y' = (y_2, ..., y_n)$ that

$$\tilde{v}_\infty(y_1, y') = \tilde{v}_\infty(0, y') + y_1\frac{\partial\tilde{v}_\infty}{\partial y_1}(0, y') + \frac{1}{2}\,y_1^2\,\frac{\partial\tilde{v}_\infty}{\partial y_1^2}(0, y')$$

with $\tilde{v}_\infty(0, y') = 0$, $\dfrac{\partial\tilde{v}_\infty}{\partial y_1}(0, y') = \beta\cdot y$ for some $\beta \in \mathbb{R}^n$ with $\beta_1 = 0$, $\dfrac{\partial\tilde{v}_\infty}{\partial y_1^2}(0, y') = constant$ $= \Delta\tilde{v}_\infty = 0$, i.e.

$$\tilde{v}_\infty(y_1, y') = y_1\cdot(\beta\cdot y)$$

and

$$v_\infty = p_\beta$$

This gives a contradiction with (6.49).
This ends the proof of the Proposition.

7 Singular points of the free boundary and proof of Theorem 1.11

7.1 Monotonicity formula for singular points and proof of Theorem 1.10

Proposition 7.1 (Monotonicity formula for singular points)
Let $p \in (\max(n/2,1),+\infty)$ with $p \geq 2n/(n+1)$. There exists a constant $C>0$. For any matrix $Q \in \mathbb{R}^{n\times n}_{sym}$ with trace $Q=1$ and $Q \geq 0$, we set $v(x) = \frac{1}{2}{}^t x \cdot Q \cdot x$. Then, for any solution u of (1.8), we have

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r}(u-v)^2\right) = g(r) + \frac{4}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U(x)}{|x|^2}\right|^2 \tag{7.55}$$

for $U = x\cdot\nabla u - 2u$ and

$$-g(r) \leq C\frac{\Sigma_p(r)}{r}F_{p'}(r) \quad \text{with} \quad \Sigma_p(r) = \sigma_p(r) + \int_0^r \frac{\sigma_p(s)}{s}\, ds$$

and for $\frac{1}{p}+\frac{1}{p'}=1$

$$F_{p'}(r) = \left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{u-v}{r^2}\right|^{p'}\right)^{\frac{1}{p'}} + \sup_{\rho\in(0,r]}\left(\frac{1}{|B_\rho|}\int_{B_\rho}\left|\frac{U}{\rho^2}\right|^{p'}\right)^{\frac{1}{p'}}$$

Moreover there exists a constant $C_0>0$ such that

$$\forall r \in (0,1/2), \quad F_{p'}(r) \leq C_0\left\{\left(\int_{B_1}|u|^p\right)^{\frac{1}{p}} + \sigma_p(1) + 1\right\}$$

Proof of Proposition 7.1
From the appendix of [36], we have

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r}(u-v)^2\right) = \frac{2}{r}\left(\Phi(r)-\Phi(0^+)\right) + \frac{2}{r^{n+3}}\int_{B_r\cap\{u>0\}}(u-v)(f-f(0)) + \frac{2}{r^{n+3}}\int_{B_r\cap\{u=0\}} v$$

From (4.39), we deduce (7.55) with

$$g(r) = -\frac{4}{r}\int_0^r \frac{ds}{s^{n+3}}\int_{B_s} U\left(f(x)-f(0)\right) + \frac{2}{r^{n+3}}\int_{B_r\cap\{u>0\}}(u-v)(f-f(0)) + \frac{2}{r^{n+3}}\int_{B_r\cap\{u=0\}} v \tag{7.56}$$

for $U = x\cdot\nabla u - 2u$. The result follows from Proposition 1.5 and Corollary 4.1 (noticing moreover that the last term in g is non-negative because $v \geq 0$). This ends the proof of the Proposition.

Corollary 7.2 (Uniqueness of the blow-up limits)
Under the assumptions of Proposition 7.1, let us consider

$$u^\varepsilon(x) = \frac{u(\varepsilon x)}{\varepsilon^2}$$

If we assume that σ_p is double Dini, then u^ε converges (uniformly on compact sets) to a unique limit $u^0 = v$ as ε goes to zero, for some v as in Proposition 7.1.

Proof of Corollary 7.2
Let us set

$$\Psi_u^v(r) = \frac{1}{r^{n+3}} \int_{\partial B_r} (u-v)^2$$

Let us call u^0 one of the blow-up limits of u^ε. Then, using the 2-homogeneity of u^0, we get

$$\Psi_u^{u^0}(\varepsilon r) = \Psi_{u^\varepsilon}^{u^0}(r) \longrightarrow \Psi_{u^0}^{u^0}(r) = 0 \tag{7.57}$$

where the convergence happens for a suitable subsequence. On the other hand, from Proposition 7.1, we deduce that for double Dini σ_p, the following limit exists

$$\lim_{\rho \to 0} \Psi_u^{u^0}(\rho) = \Psi_u^{u^0}(0^+)$$

We conclude that

$$\Psi_u^{u^0}(0^+) = 0$$

and then the convergence in (7.57) happens for the whole sequence as ε goes to zero. The convergence on compact sets of $\mathbb{R}^n$ follows. This ends the proof of the Corollary.

When we assume moreover that $p \geq 2$, we get a better estimate than in Proposition 7.1, namely

Proposition 7.3 (Monotonicity formula for singular points for $p \geq 2$)
Let $p \in (\max(n/2, 1), +\infty)$ with $p \geq 2$. There exists a constant $C > 0$. For any matrix $Q \in \mathbb{R}^{n\times n}_{sym}$ with trace $Q = 1$ and $Q \geq 0$, we set $v(x) = \frac{1}{2} {}^t x \cdot Q \cdot x$. Then, for any solution u of (1.8), we have

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}} \int_{\partial B_r} (u-v)^2\right) = h(r) + \frac{2}{r} \int_{B_r} \frac{1}{|x|^n} \left|\frac{U(x)}{|x|^2}\right|^2 \tag{7.58}$$

for $U = x \cdot \nabla u - 2u$ and

$$-h(r) \leq C \left\{ \frac{1}{r} \int_0^r \frac{\sigma_p^2(s)}{s}\, ds + \frac{\sigma_p(r)}{r} \left(\frac{1}{|B_r|} \int_{B_r} \left|\frac{u-v}{r^2}\right|^{p'} \right)^{\frac{1}{p'}} \right\}$$

and for $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover there exists a constant $C_0 > 0$ such that

$$\forall r \in (0, 1/2), \quad \left(\frac{1}{|B_r|} \int_{B_r} \left|\frac{u-v}{r^2}\right|^{p'} \right)^{\frac{1}{p'}} \leq C_0 \left\{ \left(\int_{B_1} |u|^p \right)^{\frac{1}{p}} + \sigma_p(1) + 1 \right\}$$

Proof of Proposition 7.3
We start with

$$h(r) = g(r) + \frac{2}{r} \int_{B_r} \frac{1}{|x|^n} \left|\frac{U(x)}{|x|^2}\right|^2 \tag{7.59}$$

where g is given in (7.56). We really have to work on the following term (first term in g given in (7.56), using the fact that $p' \le 2$:

$$\begin{aligned}
\frac{4}{r}\int_0^r \frac{ds}{s^{n+3}}\int_{B_s} U\left(f(x)-f(0)\right) &\le \frac{4|B_1|}{r}\int_0^r \frac{ds}{s}\ \sigma_p(s)\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{p'}\right)^{\frac{1}{p'}}\\
&\le \frac{4|B_1|}{r}\int_0^r \frac{ds}{s}\ \sigma_p(s)\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{2}\right)^{\frac{1}{2}}\\
&\le \frac{2|B_1|}{r}\left\{\frac{1}{\varepsilon}\int_0^r \frac{ds}{s}\ \sigma_p^2(s)+\varepsilon\int_0^r\frac{ds}{s}\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{2}\right)\right\}
\end{aligned}$$

for any $\varepsilon > 0$ that we will choose later small enough.
We now claim that

$$\int_0^r \frac{ds}{s}\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{2}\right) \le \frac{1}{(n+4)|B_1|}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U(x)}{|x|^2}\right|^2 \tag{7.60}$$

Choosing now ε small enough, we see that the term $\frac{2|B_1|\varepsilon}{r}\int_0^r \frac{ds}{s}\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{2}\right)$ is controled by the term $\frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U(x)}{|x|^2}\right|^2$ introduced in the definition (7.59) of h. The expected bound on $-h$ then follows easily.
We only need to prove (7.60). We compute (with obvious notation)

$$\begin{aligned}
\int_0^r \frac{ds}{s}\ \left(\frac{1}{|B_s|}\int_{B_s}\left|\frac{U}{s^2}\right|^{2}\right) &= \frac{1}{|B_1|}\int_{\mathbf{S}^{n-1}} d\theta\ \int_0^r \frac{ds}{s^{n+5}}\int_0^s d\rho\ \rho^{n-1}|U(\rho,\theta)|^2\\
&= \frac{1}{|B_1|}\int_{\mathbf{S}^{n-1}} d\theta\ \int_0^r d\rho\ \rho^{n-1}|U(\rho,\theta)|^2\int_\rho^r \frac{ds}{s^{n+5}}\\
&\le \frac{1}{(n+4)|B_1|}\int_{\mathbf{S}^{n-1}} d\theta\ \int_0^r \frac{d\rho}{\rho^5}|U(\rho,\theta)|^2\\
&= \frac{1}{(n+4)|B_1|}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U(x)}{|x|^2}\right|^2
\end{aligned}$$

This ends the proof of the Proposition.

As a corollary, we see that Corollary 7.2 holds if we assume moreover $p \ge 2$, but relax the condition on σ_p to be only Dini. This is the following proof.

Proof of Theorem 1.10
This is a consequence of Proposition 7.3, noting in particular that

$$\int_0^1 \frac{dr}{r}\ \left(\int_0^r \frac{\sigma_p^2(s)}{s}\ ds\right) \le \left(\int_0^1 \frac{\sigma_p(s)}{s}\ ds\right)^2 < +\infty$$

This ends the proof of the Theorem.

7.2 Proof of Theorem 1.11

Here we adapt the proof of Theorem 1.8, and replace the set $\mathcal{P}_{reg}$ by the set

$$\mathcal{P}_{sing} = \left\{ P, \quad \exists Q \in \mathbb{R}^{n\times n}_{sym}, \quad P = \frac{1}{2}{}^t x \cdot Q \cdot x, \quad \text{trace } (Q) = 1, \quad Q \geq 0 \right\}$$

keeping the same notation for $M(u,\rho)$ and $N(u,r)$ (see Definition 2.2). Here the set $\mathcal{P}_{sing}$ contains the possible limit behaviours of the solution at the origin when the origin is a singular point. Again M is finite for $p \in (\max(n/2,1),+\infty)$ by Proposition 1.5.

Then we have the following cornerstone result which will be proved in subsection 7.4.

Proposition 7.4 (Decay estimate in a smaller ball)
Given $p \in (\max(n/2,1),+\infty)$ with $p \geq 2$, there exist constants $M_0, C_0 > 0$, $r_0, \lambda, \mu \in (0,1)$ (depending only on p and dimension n) such that for every functions u and f satisfying (1.8) with a modulus of continuity σ_p given by (1.9), then we have the following property
(7.61)

$$\forall r \in (0,r_0), \quad (M(u,r) \leq M_0) \Longrightarrow \Big(M(u,\lambda r) < \mu M(u,r) \quad \textit{or} \quad M(u,r) < C_0 \tilde{\Sigma}_p(r) \Big)$$

where

$$\tilde{\Sigma}_p(r) = \sigma_p(r) + \left(\int_0^r ds \ \frac{\sigma_p^2(s)}{s} \right)^{\frac{1}{2}}$$

Remark 7.5 *Proposition 7.4 is similar to Proposition 6.2 for regular points. The quantity $M(u,r)$ has to be smaller than the threshold M_0, in order to be able to claim that it satisfies the decay estimate. As an illustration, the reader can simply think to functions that are blow-up limits u^0 at regular points, like for instance $u^0(x) = \frac{1}{2}(\max(0,x_1))^2$. Then $M(u^0,r)$ is a positive constant independent of r, which implies in particular that $M(u^0,r) > M_0$.*

Proof of Theorem 1.11
The proof of Theorem 1.11 follows exactly the lines of the proof of Theorem 1.8. This ends the proof of the Theorem.

7.3 Preliminary results

To prove Proposition 7.4, we will need the following results

Lemma 7.6 (Liouville result (I))
Let $v_\infty \in H^1_{loc}(\mathbb{R}^n)$ satisfying $\Delta v_\infty \leq 0$ in $\mathbb{R}^n$ and harmonic in $\mathbb{R}^n \setminus \{x_1 = 0\}$. If v_∞ is homogeneous of degree 2, then $\Delta v_\infty = 0$.

Proof of Lemma 7.6
By assumption, we know that Δv_∞ is a non-positive measure μ supported in $\{x_1 = 0\}$ (which

is moreover invariant by dilations). Now let P be a C^2 and 2-homogeneous function, and $\psi \in C_c^\infty([0,+\infty))$ with $\psi \geq 0$ and $\Psi(x) = \psi(|x|)$. Then we compute

$$\begin{aligned} -<\mu, \Psi P> &= <-\Delta v_\infty, \Psi P> \\ &= \int_{\mathbb{R}^n} \nabla v_\infty \cdot \nabla (\Psi P) \\ &= \int_{\mathbb{R}^n} \Psi \nabla v_\infty \cdot \nabla P + P \nabla v_\infty \cdot \nabla \Psi \\ &= \int_{\mathbb{R}^n} -\Psi v_\infty \Delta P - v_\infty \nabla \Psi \cdot \nabla P + P \nabla v_\infty \cdot \nabla \Psi \\ &= \int_{\mathbb{R}^n} -\Psi v_\infty \Delta P \end{aligned}$$

where in the last line we have used the homogeneity which implies that $x \cdot \nabla P(x) = 2P(x)$ and the same property for v_∞. Let us choose $P(x) = -(n-1)x_1^2 + x_2^2 + ... + x_n^2$. Then $\Delta P = 0$ and we get

$$-<\mu, \Psi P> = 0$$

This implies that supp $\mu \subset \{0\}$, and then

$$\mu = -c\delta_0 \quad \text{for some} \quad c \geq 0$$

Finally the invariance of μ by dilations implies that $c = 0$ and $\mu = 0$, i.e. $\Delta v_\infty = 0$. This ends the proof of the Lemma.

Lemma 7.7 (Liouville result (II))
Let v_∞ be a harmonic polynomial in $\mathbb{R}^n$, homogeneous of degree 2, and satisfying the following conditions for some $\overline{k}_\infty \in \{1, ..., n\}$

$$(7.62) \quad \begin{cases} v_\infty \geq 0 \quad on \quad \{x_i = 0, \quad i = 1, ..., \overline{k}_\infty\} \\ \displaystyle\int_{B_1} v_\infty \left(\sum_{i=1}^n \gamma_i x_i^2 \right) \leq 0 \quad for\ every \quad \begin{cases} \gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{R}^{\overline{k}_\infty} \times [0,+\infty)^{n-\overline{k}_\infty} \\ with \quad \displaystyle\sum_{i=1}^n \gamma_i = 0 \end{cases} \\ \displaystyle\int_{B_1} v_\infty x_i x_j = 0 \quad for \quad i = 1, ..., \overline{k}_\infty, \quad j = 1, ..., n, \quad j \neq i \end{cases}$$

Then $v_\infty = 0$.

We first prove the following result

Lemma 7.8 (Scalar product)
Let V_n the space of symmetric $n \times n$ matrices Q such that trace $Q = 0$. Then there exists $\beta > 0$ such that

$$\forall Q \in V_n, \quad \frac{1}{|B_1|} \int_{B_1} \left({}^t x \cdot Q \cdot x \right)^2 = \beta \cdot trace \left({}^t Q \cdot Q \right)$$

Proof of Lemma 7.8
Given a matrix $Q \in V_n$, we set

$$(Q,Q) = \frac{1}{|B_1|}\int_{B_1} \left(\frac{1}{2}{}^t x \cdot Q \cdot x\right)^2$$

We then diagonalize the matrix Q in an orthonormal basis with coordinates $(x_1, ..., x_n)$, and with eigenvalues $(\lambda_1, ..., \lambda_n)$. Therefore we can compute

$$(Q,Q) = \frac{1}{|B_1|}\int_{B_1} \left(\sum_{i=1}^n \lambda_i x_i^2\right)^2 = \sum_{i,j=1}^n a_{ij}\lambda_i\lambda_j$$

where

$$a_{ij} = b + (a-b)\delta_{ij} \quad \text{with} \quad b = \frac{1}{|B_1|}\int_{B_1} x_1^2 x_2^2, \quad a = \frac{1}{|B_1|}\int_{B_1} x_1^4$$

We deduce that

$$(Q,Q) = b\left(\sum_{i=1}^n \lambda_i\right)^2 + (a-b)\sum_{i=1}^n \lambda_i^2$$

and then using the fact that $\sum_{i=1}^n \lambda_i = 0$, and $\beta = a - b$

$$(Q,Q) = \beta \text{ trace}\left({}^tQ \cdot Q\right)$$

Finally $\beta > 0$ because (Q,Q) is trivially non-zero for non-zero Q. This ends the proof of the Lemma.

Proof of Lemma 7.7
We first remark that we can write $v_\infty(x) = \frac{1}{2}{}^t x \cdot C \cdot x$ with $C \in V_n$ (using the notation V_n as in Lemma 7.8 to denote the symmetric $n \times n$ matrices with trace equal to 0). From (7.62), and the fact that the matrices $\frac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)$ for $i \neq j$ are orthogonal for the scalar product studied in Lemma 7.8, we see that we can write

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with A a diagonal matrix $\overline{k}_\infty \times \overline{k}_\infty$ and B a non-negative matrix $(n - \overline{k}_\infty) \times (n - \overline{k}_\infty)$. Let choose $\gamma = (\gamma', \gamma'') \in (-\infty, 0)^{\overline{k}_\infty} \times (0, +\infty)^{n-\overline{k}_\infty}$ in (7.62) with identical constant coordinates $\gamma' = (-\gamma_A, ..., -\gamma_A)$ and $\gamma'' = (\gamma_B, ..., \gamma_B)$, satisfying $\gamma_A > 0$, $\gamma_B > 0$, and $-\overline{k}_\infty \gamma_A + (n - \overline{k}_\infty)\gamma_B = 0$. Then we get from (7.62)

$$-\gamma_A (\text{trace } A) + \gamma_B \ (\text{trace } B) \leq 0$$

Because trace $C = 0$ and trace $B \geq 0$, we deduce that trace $A \leq 0$, and then trace $A = 0 =$ trace B. Therefore $B = 0$. Choosing now any $\gamma = (\gamma', \gamma'')$ in (7.62) with $\gamma'' = 0$ and $\sum_{i=1}^{\overline{k}_\infty} \gamma_i' = 0$, we deduce that

$$\forall \overline{A} \in V_{\overline{k}_\infty}, \quad \text{trace } {}^tA\overline{A} = 0$$

which implies that $A = 0$, because A itself satisfies trace $A = 0$ belongs to $V_{\overline{k}_\infty}$. This ends the proof of the Lemma.

7.4 Proof of the decay estimate Proposition 7.4

Proof of Proposition 7.4

We perform the proof by contradiction in five Steps. For simplicity, we set

$$\sigma(r) = \sigma_p(r)$$

and

$$\tilde{\Sigma}(r) = \sigma(r) + \left(\int_0^r ds\ \frac{\sigma^2(s)}{s}\right)^{\frac{1}{2}} \tag{7.63}$$

By the way, we will have to consider sequences of modulus of continuity σ that we denote by $(\sigma_m)_m$ indexed by m, with no possible confusion.
Similarly, we will associate their $(\tilde{\Sigma}_m)_m$, using formula (7.63).

Step 1: A priori estimates on a sequence v_m

If the Proposition is false, then there exist sequences $(M_m)_m$, $(r_m)_m$, $(C_m)_m$, $(\lambda_m)_m$, $(\mu_m)_m$, $(f_m)_m$, $(u_m)_m$, $(\sigma_m)_m$ such that

$$\begin{cases} M_m, r_m, \lambda_m \longrightarrow 0 \\ C_m \longrightarrow +\infty \\ \mu_m \longrightarrow 1 \end{cases}$$

and

$$M_m \geq M(u_m, r_m) \geq C_m \tilde{\Sigma}_m(r_m) \quad \text{and} \quad M(u_m, \lambda_m r_m) \geq \mu_m M(u_m, r_m) \tag{7.64}$$

Let us recall that by assumption $M(u_m, r_m)$ is bounded by M_m which goes to zero. Therefore there exists $\rho_m \in (0, \lambda_m r_m]$, such that $N(u_m, \rho_m)$ is arbitrarily close to $M(u_m, \lambda_m r_m)$ and satisfies for instance

$$\frac{M(u_m, \lambda_m r_m)}{1 + 1/m} \leq N(u_m, \rho_m) =: \varepsilon_m \leq M(u_m, r_m) \leq M_m \longrightarrow 0$$

with

$$\varepsilon_m := N(u_m, \rho_m) = \left(\frac{1}{\rho_m^{n+4}} \int_{B_{\rho_m}} |u_m - P_m|^2\right)^{\frac{1}{2}} \quad \text{for some} \quad P_m \in \mathcal{P}_{sing}$$

Now for every $s \in (0, s_m)$ with $s_m = r_m/\rho_m \geq 1/\lambda_m \longrightarrow +\infty$, we have

$$\inf_{P \in \mathcal{P}_{sing}} \left(\frac{1}{(s\rho_m)^{n+4}} \int_{B_{s\rho_m}} |u_m - P_m|^2\right)^{\frac{1}{2}} \leq \frac{\varepsilon_m (1 + 1/m)}{\mu_m}$$

Let us set

$$u_m^{\rho_m}(x) = u_m(\rho_m \cdot x)/\rho_m^2$$

We now define the renormalized function

$$v_m(y) := \frac{1}{\varepsilon_m \rho_m^2} (u_m - P_m)(\rho_m y)$$

which satisfies

$$\Delta v_m = g_m \quad \text{in} \quad \{u_m^{\rho_m} > 0\} \tag{7.65}$$

and

$$\Delta v_m \leq g_m \cdot 1_{\{u_m^{\rho_m}>0\}} \quad \text{in} \quad \mathbb{R}^n \tag{7.66}$$

with

$$g_m(y) = \frac{f_m(\rho_m y) - f_m(0)}{\varepsilon_m}$$

which satisfies for fixed $R \in (0, s_m)$ (as in (2.22))

$$\left(\frac{1}{|B_R|}\int_{B_R} |g_m|^p\right)^{\frac{1}{p}} = \frac{\sigma_m\,(\rho_m R)}{\varepsilon_m} \leq \frac{1+1/m}{\mu_m C_m} \longrightarrow 0 \quad \text{as} \quad m \longrightarrow +\infty \tag{7.67}$$

Moreover as in Step 1 of the proof of Propositon 2.5, we get

$$\inf_{P\in\mathcal{P}_{sing}} \left(\int_{B_1} \left|v_m - \left(\frac{P-P_m}{\varepsilon_m}\right)\right|^2\right)^{\frac{1}{2}} = 1, \tag{7.68}$$

and for $s \in (0, s_m)$:

$$\inf_{P\in\mathcal{P}_{sing}} \left(\frac{1}{s^{n+2p}}\int_{B_s} \left|v_m - \left(\frac{P-P_m}{\varepsilon_m}\right)\right|^2\right)^{\frac{1}{2}} \leq \frac{1+1/m}{\mu_m} \longrightarrow 1 \tag{7.69}$$

and for $s \in (1, s_m/2)$

$$\left(\frac{1}{s^{n+2p}}\int_{B_s} |v_m|^2\right)^{\frac{1}{2}} \leq \frac{C_1(1+1/m)}{\mu_m}\ln(2s) \longrightarrow C_1\ln(2s) \tag{7.70}$$

where the limits are taken as m goes to infinity.

We now apply the Caccioppoli type estimate (6.4) to v_m with $p=2$, and get for supp $\zeta \subset B_R$

$$\int_{\mathbb{R}^n} \zeta^2|\nabla v_m|^2 \leq \int_{\mathbb{R}^n} 4v_m^2|\nabla\zeta|^2 + 2|B_R|\frac{\sigma_m(\rho_m R)}{\varepsilon_m}\left(\frac{1}{|B_R|}\int_{B_R}\zeta^4|v_m|^2\right)^{\frac{1}{2}} \tag{7.71}$$

Step 2: Convergence of the sequence v_m

From (7.67)-(7.68)-(7.69)-(7.70)-(7.71), we get as in Step 2 of the proof of Propositon 2.5 that up to extracting a subsequence, we have

$$\begin{cases} v_m \longrightarrow v_\infty & \text{in} \quad L^2_{loc}(\mathbb{R}^n) \quad \text{and a.e. in} \quad \mathbb{R}^n \\ v_m \longrightarrow v_\infty & \text{weakly in} \quad H^1_{loc}(\mathbb{R}^n) \end{cases}$$

Then, because equality is achieved in (7.68) for $P = P_m$, this implies

$$\left(\int_{B_1} |v_\infty|^2\right)^{\frac{1}{2}} = 1, \tag{7.72}$$

and (7.70) implies

$$(7.73)\qquad \left(\frac{1}{s^{n+4}}\int_{B_s}|v_\infty|^2\right)^{\frac{1}{2}} \le C_1\ln(2s) \quad \text{for every} \quad s\ge 1$$

and (7.71) implies

$$(7.74)\qquad \int_{\mathbb{R}^n}\zeta^2|\nabla v_\infty|^2 \le \int_{\mathbb{R}^n} 4v_\infty^2|\nabla\zeta|^2$$

Up to extracting a subsequence we can assume that P_m converges to some P_∞.
We have $u_m^{\rho_m} - P_m = \varepsilon_m v_m$, and $\Delta u_m^{\rho_m} = f_m(\rho_m\cdot)\cdot 1_{\{u_m^{\rho_m}>0\}}$ where the right hand side is bounded in $L^p_{loc}(\mathbb{R}^n)$, then by classical elliptic estimates, $u_m^{\rho_m}$ is bounded in $W^{2,p}_{loc}(\mathbb{R}^n)$, and then by Sobolev imbeddings, $u_m^{\rho_m}$ converges (up to extraction of some subsequence) to its limit P_∞ in $L^\infty_{loc}(\mathbb{R}^n)$ because $p>n/2$. Then we deduce from (7.65) that v_∞ satisfies

$$(7.75)\qquad \Delta v_\infty = 0 \quad \text{in} \quad \{P_\infty>0\}$$

Moreover we also deduce from (7.66) that

$$(7.76)\qquad \Delta v_\infty \le 0 \quad \text{in} \quad \mathbb{R}^n$$

Step 3: The limit differential
We define the following limit differential (up to a subsequence):

$$\overline{\partial}_{P_\infty}\mathcal{P}_{sing} = \left\{q=\frac{1}{2}{}^t x\cdot Q\cdot x,\quad \exists Q\in\mathbb{R}^{n\times n}_{sym},\quad \text{trace }(Q)=0,\quad \exists \overline{P}_m\in\mathcal{P}_{sing}\quad\text{with}\quad \frac{\overline{P}_m-P_m}{\varepsilon_m}\longrightarrow q\right\}$$

By construction $\overline{\partial}_{P_\infty}\mathcal{P}_{sing}$ contains the origin and is convex, because $\mathcal{P}_{sing}$ is convex. Moreover it is easy to check that $\overline{\partial}_{P_\infty}\mathcal{P}_{sing}$ is closed.

Then (7.68) implies

$$(7.77)\qquad \inf_{q\in\overline{\partial}_{P_\infty}\mathcal{P}_{sing}}\left(\int_{B_1}|v_\infty-q|^p\right)^{\frac{1}{p}} = 1$$

and (7.69) implies

$$(7.78)\qquad \inf_{q\in\overline{\partial}_{P_\infty}\mathcal{P}_{sing}}\left(\frac{1}{s^{n+2p}}\int_{B_s}|v_\infty-q|^p\right)^{\frac{1}{p}} \le 1 \quad\text{for every}\quad s>0$$

Step 4: 2-homogeneity of v_∞ and consequences
To deduce the 2-homogeneity of v_∞, we will make strong use of the monotonicity formula (Proposition 7.3) for singular points in the case $p\ge 2$.
Comming back at the level of the functions v_m, we remark that Proposition 7.3 implies

$$(7.79)\qquad \frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r} v_m^2\right) = h_m(r) + \frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_m(x)}{|x|^2}\right|^2$$

for $U_m = x\cdot\nabla v_m - 2v_m$ and (because $p' \le 2 \le p$)

$$-h_m(r) \le C\left\{\frac{1}{r}\int_0^r \frac{\sigma_m^2(\rho_m s)}{\varepsilon_m^2 s}\, ds + \frac{\sigma_m(\rho_m r)}{\varepsilon_m r}\ \left(\frac{1}{|B_r|}\int_{B_r}\left|\frac{v_m}{r^2}\right|^p\right)^{\frac{1}{p}}\right\}$$

In particular, we have with $s_m \longrightarrow +\infty$

$$\frac{\tilde{\Sigma}_m}{\varepsilon_m} = \frac{\sigma_m(\rho_m s_m)}{\varepsilon_m} + \left(\int_0^{s_m}\frac{\sigma_m^2(\rho_m s)}{\varepsilon_m^2 s}\, ds\right)^{\frac{1}{2}} \le \frac{(1+1/m)}{\mu_m C_m} \longrightarrow 0$$

Therefore using the notation $a^+ = \max(a,0)$, we get as $m \longrightarrow +\infty$

$$(-h_m)^+ \longrightarrow 0 \quad \text{uniformly on compact sets of} \quad (0,+\infty)$$

Let now $\varphi \in C_c^\infty(0,+\infty)$ such that $\varphi \ge 0$. Multiplying (7.79) by φ and integrating by parts, we get

(7.80)

$$-\int_0^{+\infty} dr\ \varphi'(r)\left(\frac{1}{r^{n+3}}\int_{\partial B_r} v_m^2\right) \ge \int_0^{+\infty} dr\ \varphi(r)\left\{\frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_m(x)}{|x|^2}\right|^2 - (-h_m(r))^+\right\}$$

Passing to the limit in (7.80), we get

$$-\int_0^{+\infty} dr\ \varphi'(r)\left(\frac{1}{r^{n+3}}\int_{\partial B_r} v_\infty^2\right) \ge \int_0^{+\infty} dr\ \varphi(r)\left\{\frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_\infty(x)}{|x|^2}\right|^2\right\} \tag{7.81}$$

where $U_\infty = x\cdot\nabla v_\infty - 2v_\infty$. This implies in particular that

$$\frac{d}{dr}\left(\frac{1}{r^{n+3}}\int_{\partial B_r} v_\infty^2\right) \ge \frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_\infty(x)}{|x|^2}\right|^2 \quad \text{in} \quad \mathcal{D}'(0,+\infty)$$

and for every $\lambda > 1$ and $s > 0$

$$\left(\frac{1}{(\lambda s)^{n+3}}\int_{\partial B_{\lambda s}} v_\infty^2\right) \ge \left(\frac{1}{s^{n+3}}\int_{\partial B_s} v_\infty^2\right) + \int_s^{\lambda s} dr\ \left\{\frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_\infty(x)}{|x|^2}\right|^2\right\} \tag{7.82}$$

Similarly, for any $q \in \overline{\partial}_{P_\infty}\mathcal{P}_{sing}$, we can do the same reasoning with v_m replaced by $v_m - q_m$ with $q_m = \dfrac{\overline{P}_m - P_m}{\varepsilon_m} \longrightarrow q$ for some $\overline{P}_m \in \mathcal{P}_{sing}$ which gives (7.82) with v_∞ replaced by $v_\infty - q$, namely

(7.83)

$$\left(\frac{1}{(\lambda s)^{n+3}}\int_{\partial B_{\lambda s}} (v_\infty - q)^2\right) \ge \left(\frac{1}{s^{n+3}}\int_{\partial B_s} (v_\infty - q)^2\right) + \int_s^{\lambda s} dr\ \left\{\frac{2}{r}\int_{B_r}\frac{1}{|x|^n}\left|\frac{U_\infty(x)}{|x|^2}\right|^2\right\}$$

We now remark that for $w_q = v_\infty - q$, we have

$$\Gamma_q(\rho) := \frac{1}{\rho^{n+4}}\int_{B_\rho} w_q^2 = \frac{1}{\rho^{n+4}}\int_0^\rho ds\ s^{n+3}\gamma_q(s) \quad \text{with} \quad \gamma_q(s) := \left(\frac{1}{s^{n+3}}\int_{\partial B_s} w_q^2\right)$$

Then by integration of (7.83) for $s \in (0,\rho)$, we get

$$\Gamma_q(\lambda\rho) \geq \Gamma_q(\rho) + \frac{1}{\rho^{n+4}} \int_0^\rho ds\ s^{n+3} \ \left(\int_s^{\lambda s} dr \ \left\{ \frac{2}{r} \int_{B_r} \frac{1}{|x|^n} \left| \frac{U_\infty(x)}{|x|^2} \right|^2 \right\} \right)$$

Taking the infimum for $q \in \overline{\partial}_{P_\infty} \mathcal{P}_{sing}$ for $\rho = 1$, we get

$$\inf_{q \in \overline{\partial}_{P_\infty} \mathcal{P}_{sing}} \left(\frac{1}{\lambda^{n+2p}} \int_{B_\lambda} |v_\infty - q|^2 \right)$$

$$\geq \inf_{q \in \overline{\partial}_{P_\infty} \mathcal{P}_{sing}} \left(\int_{B_1} |v_\infty - q|^2 \right) + \int_0^1 ds\ s^{n+3} \ \left(\int_s^{\lambda s} dr \ \left\{ \frac{2}{r} \int_{B_r} \frac{1}{|x|^n} \left| \frac{U_\infty(x)}{|x|^2} \right|^2 \right\} \right)$$

Using (7.77) and (7.78), we conclude that the second term of the last line is zero for any $\lambda > 1$, and then $U_\infty = 0$, i.e. v_∞ is homogeneous of degree 2.
Therefore, by Lemma 7.6, and the fact that v_∞ is superharmonic (see (7.76)), we deduce that v_∞ is harmonic on the whole space.

Step 5: Properties of the minimizer P_m and consequences

We write $P_\infty(x) = \frac{1}{2} {}^t x \cdot Q_\infty \cdot x$ with a diagonal matrix Q_∞ (up to rotations and symmetry of the coordinates)

$$Q_\infty = \begin{pmatrix} a_\infty^1 & 0 & 0 & ... & 0 \\ 0 & a_\infty^2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_\infty^n \end{pmatrix}$$

with $a_\infty^1 \geq a_\infty^2 \geq ... \geq a_\infty^n \geq 0$. We also recall that trace $Q_\infty = 1$. Similarly (still up to rotations and symmetry of the coordinates), we can assume that we can write $P_m(x) = \frac{1}{2} {}^t x \cdot Q_m \cdot x$ with a diagonal matrix Q_m (up to rotations)

$$Q_m = \begin{pmatrix} a_m^1 & 0 & 0 & ... & 0 \\ 0 & a_m^2 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & a_m^n \end{pmatrix}$$

with $a_m^1 \geq a_m^2 \geq ... \geq a_m^n \geq 0$, trace $Q_m = 1$ and

$$a_m^i \longrightarrow a_\infty^i \quad \text{for} \quad i = 1, ..., n$$

Let us call k_∞ the rank of the matrix Q_∞, and k_m the rank of the matrix Q_m which then satisfies $k_m \geq k_\infty$ (for m large enough).

The fact that $a_m^i = 0$ for $i = k_m + 1, ..., n$, joint to the non-negativity of u_m, then implies that

$$v_m \geq 0 \quad \text{on} \quad \{x_i = 0, \quad i = 1, ..., k_m\}$$

By hypothesis the energy for $q \in \varepsilon_m^{-1} \left(\mathcal{P}_{sing} - P_m\right)$

$$\mathcal{E}(q) := \int_{B_1} (v_m - q)^2$$

is minimal for $q=0$.

Because we have $a_m^i>0$ for $i=1,...,k_m$, we see that the first variation of this energy with respect to variations of the whole coefficients (constraint to stay non-negative with $a_m^1+...+a_m^n=1$), implies that

$$\int_{B_1} v_m \left(\sum_{i=1}^n \gamma_i x_i^2\right) \le 0 \quad \text{for every} \quad \begin{cases} \gamma=(\gamma_1,...,\gamma_n)\in \mathbb{R}^{k_m}\times[0,+\infty)^{n-k_m} \\ \text{with} \quad \displaystyle\sum_{i=1}^n \gamma_i = 0 \end{cases}$$

Let us also remark that defining for $i<j$

$$R_\theta(x)=(x_1,...,x_{i-1},x_i\cos\theta+x_j\sin\theta,x_{i+1},...,x_{j-1},x_j\cos\theta-x_i\sin\theta,x_{j+1},...,x_n)$$

and

$$q_\theta(x)=P_m(R_\theta(x))-P_m(x)$$

and derivating the energy $\mathcal{E}(q_\theta)$ with respect to θ in $\theta=0$, we get

$$\int_{B_1} v_m\left(a_m^i-a_m^j\right)x_ix_j=0 \quad \text{for} \quad i,j=1,...,n \tag{7.84}$$

Because $a_m^i>0$ for $i=1,...,k_m$, we see that $q_\eta(x)=\eta x_ix_j$ for $i,j\in\{1,...,k_m\}$ with $i\neq j$, satisfies $q_\eta\in\varepsilon_m^{-1}(\mathcal{P}_{sing}-P_m)$ for η small enough. The first variation of the energy with respect to η in $\eta=0$ gives

$$\int_{B_1} v_m x_ix_j=0 \quad \text{for} \quad i,j=1,...,k_m, \quad i\neq j \tag{7.85}$$

From equations (7.84)-(7.85), we deduce in particular that

$$\int_{B_1} v_m x_ix_j=0 \quad \text{for} \quad i=1,...,k_m, \quad j=1,...,n, \quad j\neq i \tag{7.86}$$

Let us call $\overline{k}_\infty$ the limit of the k_m (which satisfies $\overline{k}_\infty\ge k_\infty$). Passing to the limit, we get that v_∞ satisfies the assumptions of Lemma 7.7. We deduce that $v_\infty=0$. Contradiction with (7.72).
This ends the proof of the Proposition.

8 Applications

We now consider obstacle problems for more general elliptic operators. Given an open set $\Omega\in\mathbb{R}^n$, we consider solutions $u\in H^2(\Omega)$ to the following obstacle problem

$$\left\{\begin{array}{l} \displaystyle\sum_{i,j=1}^n a_{ij}(x)u_{ij}+\sum_i^n b_i(x)u_i+c(x)u=f(x)\cdot 1_{\{u>0\}} \\ \\ u\ge 0 \end{array}\right| \quad \text{in} \quad \Omega \tag{8.87}$$

where u_{ij} and u_i stand respectively for $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ and $\dfrac{\partial u}{\partial x_i}$. We assume that the coefficients a_{ij}, b_i, c, f are continuous and satisfy the following ellipticity/nondegeneracy condition

$$\exists \delta_1 > 0, \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \delta_1 |\xi|^2 \quad \text{and} \quad f(x) \geq \delta_1 \tag{8.88}$$

8.1 Reduction of the problem

Let us fix a point $x_0 \in \Omega \cap \partial\{u > 0\}$. Then, assuming that the coefficients a_{ij}, b_i, c, f are Dini, we can rewrite the first line of (8.87) as

$$\sum_{i,j=1}^n a_{ij}(x_0) u_{ij} = \tilde{f}_{x_0}(x) \cdot 1_{\{u>0\}}$$

where the function $\tilde{f}_{x_0}$ is defined by

$$\tilde{f}_{x_0}(x) = f(x) - \sum_{i,j=1}^n (a_{ij}(x) - a_{ij}(x_0)) u_{ij} - \sum_i^n (b_i(x) - b_i(x_0)) u_i - (c(x) - c(x_0)) u \ .$$

Thus, up to diagonalizing the matrix $(a_{ij}(x_0))_{ij}$ and changing the coordinates, we see that there exists a matrix A_{x_0} such that if we define $u_{x_0}(x) = u(x_0 + A_{x_0} \cdot x)$, $f_{x_0}(x) = \tilde{f}_{x_0}(x_0 + A_{x_0} \cdot x)$, then we have

$$\left\{ \begin{array}{l} \Delta u_{x_0} = f_{x_0}(x) \cdot 1_{\{u_{x_0} > 0\}} \\ u_{x_0} \geq 0 \\ \dfrac{1}{\delta_0} \geq f_{x_0} \geq \delta_0 \\ f_{x_0}(0) = 1 \\ 0 \in \partial\{u_{x_0} > 0\} \end{array} \right. \quad \text{in} \quad B_1 \tag{8.89}$$

Then we define

$$\sigma_{x_0}(\rho) = \sup_{r \in (0,\rho]} \left(\frac{1}{|B_r|} \int_{B_r} |f_{x_0}(x) - f_{x_0}(0)|^p \right)^{\frac{1}{p}}$$

In particular we get that σ_{x_0} is Dini.

Indeed, it is possible to apply the approach of Alt, Phillips [2], which shows that the second derivatives of u are bounded. Their method is based on the interior Schauder estimate, which has to be replaced by a similar estimate for Dini coefficients. This last estimate is for instance a consequence of the estimate of Theorem 2.1, using the classical perturbation method.

Remark 8.1 *It would be interesting to provide a pointwise proof that σ_{x_0} is Dini, avoiding the argument of Alt, Phillips.*

8.2 Regularity of the free boundary

We also introduce the following

Definition 8.2 (Dini continuity)
We say that a function f is Dini continuous, if its modulus of continuity given by (1.2) is Dini in the sense of Definition 1.1.

Then we have the following result

Theorem 8.3 (Regularity of the free boundary)
Under assumption (8.88), let us consider a solution u of (8.87) with **Dini continuous** *coefficients a_{ij}, b_i, c, f. Then we can write the free boundary as follows*

$$(\partial \{u > 0\}) \cap \Omega = \mathcal{R} \cup \mathcal{S}$$

i) where $\mathcal{R}$ is locally a C^1 hypersurface,
ii) and the singular set $\mathcal{S}$ of the free boundary can be written

$$\mathcal{S} = \bigcup_{j=0}^{n-1} \mathcal{S}_j$$

where each $\mathcal{S}_j$ is locally contained in a C^1 j-dimensional manifold.

Remark 8.4 *Here the sets $\mathcal{R}$ (resp. $\mathcal{S}$) stands for the set of regular points defined via Proposition 1.6 applied to the reduced problem (8.89).*

The first claim of this Theorem on the regular part of the free boundary was already proved (with a different proof for the Laplace operator) in Blank [5].
The second claim of the Theorem on the singular part of the free boundary was proved by Caffarelli [8] for Lipschitz coefficients, and by Monneau [36] for Hölder coefficients, including extensions for double Dini coeffifients).

Sketch of the proof of Theorem 8.3
We already know the uniqueness of the blow-up limit at every regular and singular points. Moreover these limits enjoy a stability property for neighbouring points (using respectively (2.14) and (7.58)). Consequently the map which associate its blow-up limit to each regular point is continuous. This is also the case for singular points. A further inspection of the structure of the blow-up limits $u^0(x) = \frac{1}{2} {}^t x \cdot Q \cdot x$ shows that we can consider the singular points $\mathcal{S}_k$ associated to a matrix Q of rank $k \leq n - 1$.
Finally, the rest of the proof is classical, see for instance Caffarelli [7], [9], Monneau [36] (see also Caffarelli, Shahgholian [16], Caffarelli, Petrosyan, Shahgholian [15], for similar obstacle problem without sign condition on the solution).
This ends the sketch of the proof of the Theorem.

9 Appendix : an application to fully nonlinear elliptic equations

Let us mention that $C^{2,\alpha}$ estimates are known for solutions of concave (or convex) fully nonlinear uniformly elliptic equations (see chapter 6 of Caffarelli, Cabre [11] or its generalizations in chapter 8, see also the original paper of Caffarelli [10]). For results with Dini conditions, see Kovats [32, 33], Zou, Chen [44], and Bao [3], and for the connection between the $C^{2,\alpha}$ regularity of $C^{1,1}$ solutions with the Liouville property, see Huang [27]. Here we show, without assuming concavity or convexity of the equation, a pointwise $C^{2,\alpha}$ estimate (in the L^p norm) assuming a pointwise C^2-Dini regularity (in the L^∞ norm).

Proposition 9.1 (C^2-Dini implies $C^{2,\alpha}$ for fully nonlinear equations)
Let us consider a function $u \in C^2(B_1)$ solution of

$$F(D^2u) = 0 \quad in \quad B_1$$

with $F \in C^2$ and uniformly elliptic.
Let us define for $p \in (1, +\infty)$ (and for P_0 a polynomial of degree less or equal to 2)

$$\hat{\omega}_p(r) = \sup_{\rho \in (0,r]} \left(\frac{1}{|B_\rho|} \int_{B_\rho} \left| D^2u - D^2P_0 \right|^p \right)^{\frac{1}{p}}$$

and

$$\hat{\omega}_\infty(r) = \sup_{\rho \in (0,r]} \sup_{B_\rho} \left| D^2u - D^2P_0 \right|$$

If $\int_0^1 \frac{\hat{\omega}_\infty(r)}{r}\, dr < +\infty$, then there exist constants $C > 0$ and $\alpha, r_1 \in (0, 1]$ such that

$$\forall r \in (0, r_1), \quad \hat{\omega}_p(r) \le C r^\alpha$$

Proof of Proposition 9.1
up to redefining u, we can assume that $D^2u(0) = 0$ and perform a Taylor expansion of

$$F(D^2u) = 0$$

as

$$0 = F(0) + F'(0)D^2u + O\left(|D^2u|^2\right)$$

i.e.

$$F'(0)D^2u = O\left(|D^2u|^2\right)$$

We deduce from Theorem 2.1, that for $r < r_0$ (and a suitable polynomial P_0 with $D^2P_0 = 0$)

$$\begin{aligned}\left(\frac{1}{|B_r|} \int_{B_r} \left| \frac{u(x) - P_0(x)}{r^2} \right|^p \right)^{\frac{1}{p}} &\le C \left(M r^\alpha + \int_0^r \frac{\hat{\omega}_p(s)\hat{\omega}_\infty(s)}{s}\, ds + r^\alpha \int_r^1 \frac{\hat{\omega}_p(s)\hat{\omega}_\infty(s)}{s^{1+\alpha}}\, ds \right) \\ &\le C \left(M r^\alpha + \hat{\omega}_p(r)\varepsilon(r) \right)\end{aligned}$$

with

$$\varepsilon(r) = \int_0^r \frac{\hat{\omega}_\infty(s)}{s}\, ds + r^\alpha \int_r^1 \frac{\hat{\omega}_\infty(s)}{s^{1+\alpha}}$$

Now recall that from classical interior $W^{2,p}$ elliptic estimate joined to a scaling argument, we have for $\lambda \in (0,1)$ and any $r \in (0,1)$

$$\left(\frac{1}{|B_{\lambda r}|}\int_{B_{\lambda r}} |D^2u - D^2P_0|^p\right)^{\frac{1}{p}} \le C_1 \left\{ \left(\frac{1}{|B_r|}\int_{B_r} \left|\frac{u(x)-P_0(x)}{r^2}\right|^p\right)^{\frac{1}{p}} + \left(\frac{1}{|B_r|}\int_{B_r} |F'(0)D^2u|^p\right)^{\frac{1}{p}} \right\}$$

where the constant $C_1 > 0$ depends on λ. Therefore, using the previous estimate and the equation to estimate the term $F'(0)D^2u$, we get for some constant $C_2 > 0$

$$\hat{\omega}_p(\lambda r) \le C_2 \left(Mr^\alpha + \hat{\omega}_p(r)\hat{\varepsilon}(r)\right) \quad \text{with} \quad \hat{\varepsilon}(r) = \varepsilon(r) + \hat{\omega}_\infty(r)$$

We now fix $\mu \in (0,1)$ and $r_0' \in (0,1]$ such that $C_2\hat{\varepsilon}(r_0') \le \mu/2$. Then we get

$$\forall r \in (0, r_0'), \quad \hat{\omega}_p(\lambda r) \le \mu\hat{\omega}_p(r) \quad \text{or} \quad \hat{\omega}_p(r) \le C_3 r^\alpha$$

with $C_3 = 2C_2M/\mu$. Applying Lemma 3.3 with $M(u,r)$ replaced by $\hat{\omega}_p(r)$, we get the result with $r_1 = \lambda r_0'$. This ends the proof of the Proposition.

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