Stability of travelling waves in a model for conical flames in two space dimensions

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Abstract. This paper deals with the question of the stability of conical-shaped solutions of a class of reaction-diffusion equations in $I\!R^2$. One first proves the existence of travelling waves solutions with conical-shaped level sets, generalizing earlier results by Bonnet, Hamel and Monneau [9], [19]. One then gives a characterization of the global attractor of these semilinear parabolic equations under some conical asymptotic conditions. Lastly, the global stability of the travelling waves solutions is proved.

1 Introduction and main results

This paper deals with the question of the global stability of the solutions ϕ of the following semilinear elliptic equation

$$\Delta\phi - c\partial_y\phi + f(\phi) = 0, \quad 0 < \phi < 1 \ \text{ in } I\!R^2, \tag{1.1}$$

under the following type of conical conditions at infinity

$$\begin{cases} \lim\limits_{y\to+\infty} \inf\limits_{\mathcal{C}_+(y,\pi-\alpha)} \phi = 1, \\ \lim\limits_{y\to-\infty} \sup\limits_{\mathcal{C}_-(y,\alpha)} \phi = 0. \end{cases} \tag{1.2}$$

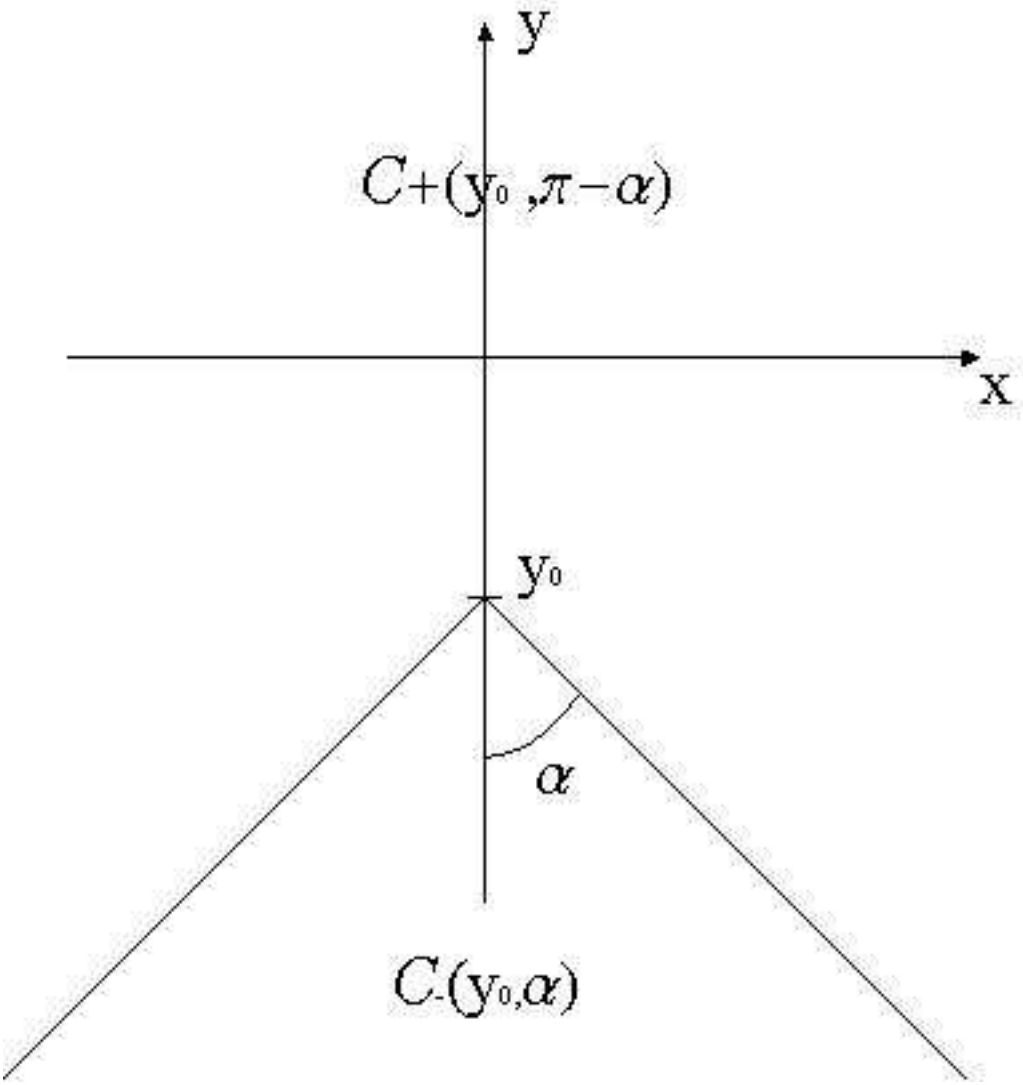

Figure 1: Upper and lower cones

Throughout the paper, the notation $\partial_y \phi$ (as well as ϕ_y) means the partial derivative of the function ϕ with respect to the variable y. For any $y_0 \in I\!R$ and any $0 \leq \beta \leq \pi$, the lower and upper cones $\mathcal{C}_\pm(y_0, \beta)$ are defined by

$$\mathcal{C}_\pm(y_0, \beta) = \{(x, y) = (0, y_0) + \rho(\cos\varphi, \sin\varphi),\ \rho \geq 0,\ |\varphi \mp \pi/2| \leq \beta\}.$$

We also use the following notation : for a function v of the 2D real variable (x, y), and for $(a, b) \in I\!R^2$, we denote by $\tau_{a,b} v$ the function

$$\tau_{a,b} v \ :\ (x, y) \mapsto v(x + a, y + b).$$

Another way of formulating the question of the stability of the solutions ϕ of (1.1-1.2) is to ask the question of the convergence to the travelling fronts

$\phi(x, y + ct)$, or to some translates of them, for the solutions $u(t, x, y)$ of the Cauchy problem

$$(1.3) \qquad \begin{cases} u_t = \Delta u + f(u), \quad t > 0, \ (x, y) \in I\!R^2, \\ u(0, x, y) = u_0(x, y) \text{ given}, \ 0 \leq u_0 \leq 1 \end{cases}$$

where u_0 is close, in some sense to be defined later, to a translate $\tau_{a,b}\phi$ of a solution ϕ of (1.1-1.2).

The function f is assumed to be of class $C^{1,\delta}$ in $[0, 1]$ (for some $\delta > 0$) and to have the following profile :

$$(1.4) \ \exists\, \theta \in (0, 1), \ f = 0 \text{ on } [0, \theta] \cup \{1\}, \ f > 0 \text{ on } (\theta, 1) \text{ and } f'(1^-) < 0.$$

For mathematical convenience, we extend f by 0 outside $[0, 1]$. Notice that, from standard elliptic estimates, any classical solution ϕ of (1.1) is actually of class $C^{2,\mu}(I\!R^2)$ for any $\mu \in [0, 1)$.

Equation (1.1) arises in models of equidiffusional premixed Bunsen flames. The function u is a normalized temperature and its level sets represent the profile of a conical-shaped Bunsen flame coming out of a thin elongated Bunsen burner (*see* Buckmaster and Ludford [12], Joulin [24], Sivashinsky [38], Williams [40]). The temperature of the unburnt gases is close to 0 and that of the burnt gases is close to 1, the hot zone being above the fresh zone. The real θ is called an ignition temperature, below which no reaction happens. The real c is the speed of the gases at the exit of the burner. Since the shape of the Bunsen flames is invariant with respect to the size of the Bunsen burner, one way of modelling these conical flames consists in setting equation (1.1) in the whole plane $I\!R^2$ together with asymptotic conical conditions of the type (1.2). The angle 2α then stands for the aperture of the tip of the flame.

In the onedimensional case, equation (1.1) and conditions at infinity (1.2) reduce to the ordinary differential equation

$$(1.5) \qquad \begin{cases} \phi_0'' - c_0\phi_0' + f(\phi_0) = 0 \\ \phi_0(-\infty) = 0, \quad \phi_0(+\infty) = 1. \end{cases}$$

It is well known (Aronson, Weinberger [2], Berestycki, Nicolaenko, Scheurer [6], Kanel' [25]) that there exists a unique solution (c_0, ϕ_0) of (1.5) such that $\phi_0(0) = \theta$ (the solutions of (1.5) are actually unique up to translation). Besides, the speed c_0 is positive and the function ϕ_0 is increasing. The function $\phi_0(y)$ is also a solution of the two-dimensional problem (1.1-1.2) in the particular case $\alpha = \pi/2$.

In the two-dimensional case with $\alpha \neq \pi/2$, the existence of solutions ϕ of (1.1-1.2) was proved by Hamel and Monneau [19] for some angles $\alpha \in (0, \pi/2)$ and some functions f satisfying (1.4) under some additional assumptions (*see* Theorem 1.8 in [19]). Existence of solutions of (1.1) under some conical conditions weaker than (1.2) was also proved by Bonnet and Hamel (*see* Theorem 1.1 in [9]).

The first result of this paper is to establish the existence of solutions of (1.1-1.2) for any angle $\alpha \in (0, \pi/2]$ and for any function f satisfying (1.4) :

Theorem 1.1 (Existence) *For every angle $\alpha \in (0, \pi/2]$ and for every function f satisfying (1.4), there exists a solution ϕ to (1.1-1.2), with $c = c_0/\sin\alpha$.*

Furthermore, it follows from Theorem 1.7 in [19] that the solutions (c, ϕ) of (1.1-1.2) are unique, in the sense that c is unique, and ϕ up to a translation in (x, y). The speed c is necessarily equal to $c = c_0/\sin\alpha$. Besides, any solution ϕ satisfies the following properties : 1) there exists a real x_0 such that ϕ is symmetric with respect to the vertical line $\{x = x_0\}$, 2) for any $\lambda \in (0, 1)$, the level set $\{\phi(x, y) = \lambda\}$ has two asymptots parallel to the half-lines $\{y = -\cot\alpha|x|,\ x \geq 0\}$ and $\{y = -\cot\alpha|x|,\ x \leq 0\}$, 3) there exist two reals $t_\pm$ such that for any sequence $x_n \to \pm\infty$, the functions $\phi_n(x, y) = \phi(x + x_n, y - |x_n|\cot\alpha)$ go to the planar fronts $\phi_0(\pm x\cos\alpha\ +\ y\sin\alpha\ + t_\pm)$ as $x_n \to \pm\infty$ in $C^2_{loc}(I\!R^2)$. The last two properties mean that any solution ϕ is asymptotically conical-shaped far away from the origin : namely, ϕ is asymptotically planar and asymptotically equal to two translates of the planar front ϕ_0 in the two directions of angle α with respect to the vector $-e_2 = (0, -1)$.

The formula $c = c_0/\sin\alpha$, which actually follows from earlier results of Bonnet and Hamel [9], and had already been used in several papers (*see e.g.* Lewis, Von Elbe [28], Sivashinsky [38], Williams [40]), is very natural. Indeed, any solution ϕ of (1.1-1.2) gives rise to a solution $u(t, x, y) = \phi(x, y + ct)$ of the evolution problem (1.3) with $u_0 = \phi$. The planar speed c_0 is now nothing else than the projection on the directions $(\pm\cos\alpha, -\sin\alpha)$ of the vertical speed c of the curved front $\phi(x, y + ct)$ moving downwards. The speed c_0 is the speed of two planar waves moving in the directions $(\pm\cos\alpha, -\sin\alpha)$ perpendicular to the half-lines making an angle α with the vertical axis.

Remark 1.2 *1. The dimension 2 is quite different from other dimensions since, as soon as $N \geq 3$, there is no solution of problem (1.1) in $I\!R^N$, with $\alpha < \pi/2$ and conical conditions of the type (1.2) (*see *[19]). But the possible existence of solutions of (1.1) in $I\!R^N$ under some weaker conical conditions is still open in dimensions $N \geq 3$.*

2. It was also proved in [19] that no solution of (1.1-1.2) exists whenever $\alpha \in (\pi/2, \pi)$, in dimensions 2 and higher.

Whereas there are many papers dealing with the stability of the travelling fronts for one-dimensional equations of the type (1.5) with various types of nonlinearities f (*see e.g.* [2], [10], [17], [25], [36], [37]), or for wrinkled travelling fronts of multidimensional equations in infinite cylinders (*see* [4] and [8] for the existence and uniqueness results, and[5], [29], [33], [34], [35] for the stability results), or lastly for planar fronts in the whole space (*see* [27], [41]), nothing seems to be known about the stability of the solutions of two-dimensional

problem (1.1) under conical conditions of the type (1.2), for $\alpha < \pi/2$. As already emphasized, the travelling fronts $\phi(x, y + ct)$ are special time-global solutions of (1.3) satisfying, at each time, the conical conditions (1.2) in the frame moving downwards with speed $c = c_0/\sin\alpha$. Therefore, the question of the global stability of these travelling waves and the question of the asymptotic behaviour for large time of the solutions of the Cauchy problem (1.3) starts from the study of the global attractor of equation (1.3) under conical conditions of the type (1.2) in a frame moving downwards with speed c.

The next theorem states that the travelling waves are the only time-global solutions of (1.3) satisfying such conical conditions.

Theorem 1.3 (Liouville type result) *Let $\alpha \in (0, \pi/2]$ and $0 \le u(t, x, y) \le 1$ solve the equation*

$$u_t = \Delta u + f(u), \quad (x, y) \in I\!R^2 \tag{1.6}$$

with $t \in (-\infty, +\infty)$ and f satisfying (1.4), and assume that

$$\begin{cases} \lim\limits_{y_0 \to -\infty} \sup\limits_{t \in I\!R,\ y \le y_0 - |x| \cot\alpha} u(t, x, y - ct) = 0 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{t \in I\!R,\ y \ge y_0 - |x| \cot\alpha} u(t, x, y - ct) = 1. \end{cases} \tag{1.7}$$

Then there exists a couple $(h, k) \in I\!R^2$ such that $u(t, x, y) = \tau_{h,k}\phi(x, y + ct)$ for all $(t, x, y) \in I\!R \times I\!R^2$, where ϕ is given by Theorem 1.1.

Since $\phi(x, y) \to 0$ (resp. $\to 1$) uniformly as $y + |x| \cot\alpha \to -\infty$ (resp. $y + |x| \cot\alpha \to +\infty$), the following corollary holds :

Corollary 1.4 *Let $0 \le u(t, x, y) \le 1$ be a solution of (1.6); assume the existence of two couples (a_1, b_1) and $(a_2, b_2) \in I\!R^2$ for which $\tau_{a_1,b_1}\phi(x, y + ct) \le u(t, x, y) \le \tau_{a_2,b_2}\phi(x, y + ct)$ for all $(t, x, y) \in I\!R^3$. Then the conclusion of Theorem 1.3 holds.*

The idea for proving Theorem 1.3 is based on a sliding method (*see* [7]) in the variable t and some versions of the maximum principle for parabolic equations in unbounded domains. Similar methods were used in [35] and [3] to get some monotonicity results for the solutions of some semilinear parabolic equations in various domains.

Theorem 1.3 especially implies the following

Theorem 1.5 (Convergence of a subsequence to a travelling wave) *Let ϕ be a solution of (1.1-1.2) for $\alpha \in (0, \pi/2]$ with assumptions (1.4) on f. Let $u(t, x, y)$ be a solution of the Cauchy problem (1.3) such that*

$$\begin{cases} u_0 \le \phi \quad \text{in } I\!R^2 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{y \ge y_0 - |x| \cot\alpha} u_0(x, y) > \theta. \end{cases} \tag{1.8}$$

Then, for every sequence $t_n \to +\infty$, there exist a subsequence $t_{n'} \to +\infty$ and $(a,b) \in I\!R^2$ such that

$$u(t_{n'} + t, x, y - ct_{n'} - ct) \to \phi(x+a, y+b) \quad \text{as } n' \to +\infty$$

locally uniformly in $(t,x,y) \in I\!R^3$.

A consequence of this result is that, if u_0 satisfies (1.8) and if $\omega(u_0)$ is the ω-limit set of u_0 for the semi-group $\mathcal{S}(t)$ given by (1.3), then $\omega(u_0)$ is made up of travelling waves. Condition (1.8) is especially satisfied when u_0 lies between two translates of a solution ϕ of (1.1-1.2). But, even under condition (1.8), the ω-limit set $\omega(u_0)$ of u_0 may well be a continuum, and one may ask for sufficient conditions for $\omega(u_0)$ to be a singleton. This is the goal of Theorem 1.6 below.

Before stating this result, let us first introduce some notations. Denote by $UC(I\!R^2)$ the space of all bounded uniformly continuous functions from $I\!R^2$ to $I\!R$. We fix a C^∞ function $g : I\!R \to I\!R$ such that $g(x) = |x|$ for $|x|$ large enough. For $\rho > 0$, we set

$$q(x,y) = e^{-\rho(g(x)\sin\alpha - y\cos\alpha)} \tag{1.9}$$

and

$$G_\rho = \{w \in UC(I\!R^2),\ \limsup_{|(x,y)|\to+\infty} |w(x,y)| = 0,\ w/q \in L^\infty(I\!R^2)\}.$$

The space G_ρ is a Banach space with the norm

$$\|w\|_{G_\rho} = \|w\|_{L^\infty(I\!R^2)} + \|w/q\|_{L^\infty(I\!R^2)}.$$

Theorem 1.6 (Stability result) *Choose $\alpha \in (0, \pi/2)$ and let f satisfy (1.4). Let $u(t,x,y)$ be a solution of the Cauchy problem (1.3) with initial datum $u_0 \in UC(I\!R^2)$ such that $0 \le u_0 \le 1$. Assume the existence of ρ_0, $C_0 > 0$ and of a solution ϕ of (1.1-1.2) such that $|u_0(x,y) - \phi(x,y)| \le C_0 e^{-\rho_0\sqrt{x^2+y^2}}$ in $I\!R^2$. Also assume that there exists $(a,b) \in I\!R^2$ such that $u_0 \le \tau_{a,b}\phi$ in $I\!R^2$.*

Then there are four constants $T \ge 0$, $K \ge 0$, $\omega > 0$ and $\rho > 0$, such that

$$\forall t \ge T, \quad \|u(t,x,y-ct) - \phi(x,y)\|_{G_\rho} \le Ke^{-\omega t}.$$

Under the above assumptions, it especially follows that $u(t,\cdot,\cdot - ct)$ converges to ϕ uniformly in $I\!R^2$, and exponentially in time. Notice also that if $|u_0 - \phi| \le C_0 e^{-\rho_0\sqrt{x^2+y^2}}$ in $I\!R^2$ for some solution ϕ of (1.1-1.2), then u_0 and ϕ have the same limits along the lines $y = -|x|\cot\alpha$ as $x \to \pm\infty$, whence such a ϕ, if any, is unique.

Notice that Theorem 1.6 holds especially if $u_0 \in UC(I\!R^2)$ is such that, say, $0 \le u_0 < 1$ and if there exists a solution ϕ of (1.1-1.2) such that $u_0 - \phi$ has compact support.

Lastly, the following theorem holds :

Theorem 1.7 *Let $\alpha \in (0, \pi/2)$, and f satisfy (1.4). Let $0 \le u(t,x,y) \le 1$ be a solution of the Cauchy problem (1.3) with u_0 bounded in $C^1(I\!R^2)$ and $0 \le u_0 \le 1$. Assume that $\lim_{y\to+\infty} \inf_{\mathcal{C}_+(y,\pi-\alpha)} u_0 > \theta$ and that there exists a solution ϕ of (1.1-1.2) such that $u_0 \le \phi$ in $I\!R^2$. Also assume that for some $\rho_0 > 0$*

$$|\partial_{e_\alpha} u_0(x,y)| \le C e^{\rho_0(y\sin\alpha - x\cos\alpha)}, \quad |\partial_{e'_\alpha} u_0(x,y)| \le C e^{\rho_0(y\sin\alpha + x\cos\alpha)}$$

for all $(x,y) \in I\!R^2$, where

$$e_\alpha = (\sin\alpha, -\cos\alpha) \quad \text{and } e'_\alpha = (-\sin\alpha, -\cos\alpha). \tag{1.10}$$

Then the function $u(t,\cdot,\cdot - ct)$ converges uniformly in $I\!R^2$, as $t \to +\infty$, to a solution ϕ' of (1.1-1.2).

Remark 1.8 *The convergence phenomenon is really governed by the behaviour of the initial datum when the space variable becomes infinite along the directions e_α and e'_α. In that sense, the situation is similar to the KPP situation ;* see *[29]. It may well happen that, if the initial datum u_0 has no limit in the e_α and e'_α directions, its ω-limit is made up of a continuum of waves (*see *[15]).*

Let us mention here similar stability results were obtained by Ninomiya and Taniguchi [32] for curved fronts in singular limits for Allen-Cahn bistable equations. Existence of smooth solutions of problem (1.1-1.2) with bistable nonlinearity f was obtained by Fife [16] for angles $\alpha < \pi/2$ close to $\pi/2$. Conical-shaped and more general curved fronts also exist for the Fisher-KPP equation, with concave nonlinearity f (*see* [11], [21]). Other stability results were also obtained by Michelson [31] for Bunsen fronts solving the Kuramoto-Sivashinsky equation, in some asymptotic regimes. Formal stability results in the nearly equidiffusional case were also given in [30].

The plan of the paper is the following. Section 2 is devoted to the proof of the existence of travelling waves with the conical conditions at infinity. In Section 3, we prove that global solutions - i.e. defined for all $t \in I\!R$ - are travelling wave solutions. In order to prove Theorem 1.6, we present a local stability result in Section 4 ; combined to Section 3, this implies the global stability : this last item will be treated in Section 5.

2 Existence of travelling waves solutions

2.1 Proof of Theorem 1.1

Let $\alpha \in (0, \pi/2]$ be given. We are looking for a solution ϕ of (1.1), i.e.

$$\Delta\phi - c\partial_y\phi + f(\phi) = 0, \quad 0 < \phi < 1 \quad \text{in} \quad I\!R^2$$

with $c = c_0/\sin\alpha$, satisfying the conditions (1.2) at infinity, i.e.

$$\begin{cases} \lim\limits_{y_0\to+\infty} \inf\limits_{y\geq y_0-|x|\cot\alpha} \phi(x,y) = 1 \\ \lim\limits_{y_0\to-\infty} \sup\limits_{y\leq y_0-|x|\cot\alpha} \phi(x,y) = 0 \end{cases}$$

The strategy to prove Theorem 1.1 is to build a solution ϕ between a sub- and a supersolution in the whole plane $I\!R^2$.

We perform the proof in three steps.

Step 1 : *Construction of a subsolution.* A natural candidate for a subsolution is the following function :

$$\underline{\phi}(x,y) = \phi_0((y-\gamma_0(x))\sin\alpha),$$

where

$$\gamma_0(x) = -\frac{1}{c_0\sin\alpha}\ln(\cosh(x\ c_0\cos\alpha)),$$

and ϕ_0 is the solution of the one-dimensional problem (1.5) satisfying $\phi_0(0) = \theta$. It can easily be checked (*see* also [19] where such subsolutions were used) that $\underline{\phi}$ is a classical subsolution of

$$\Delta\underline{\phi} - c\partial_y\underline{\phi} + f(\underline{\phi}) = \frac{\cos^2\alpha}{\cosh^2(x\ c_0\cos\alpha)} f(\phi_0((y-\gamma_0(x))\sin\alpha)) \geq 0 \ \text{ in } I\!R^2.$$

Furthermore, $\underline{\phi}$ is a solution of $\Delta\underline{\phi} - c\partial_y\underline{\phi} = 0$ in $\{y \leq \gamma_0(x)\}$. Notice that since $\underline{\phi}$ is of class C^2, it is also a subsolution of $\Delta\underline{\phi} - c\partial_y\underline{\phi} + f(\underline{\phi}) \geq 0$ in the viscosity sense.

Moreover, the function γ_0 satisfies $\sup_{x\in I\!R}|\gamma_0(x) + |x|\cot\alpha| < +\infty$. This implies in particular

$$\lim_{y_0\to-\infty} \sup_{\{y\leq y_0-|x|\cot\alpha\}} \underline{\phi}(x,y) = 0 \tag{2.1}$$

and

$$\lim_{y_0\to+\infty} \inf_{\{y\geq y_0-|x|\cot\alpha\}} \underline{\phi}(x,y) = 1. \tag{2.2}$$

Step 2 : *Construction of a supersolution.* On the contrary, the construction of a supersolution which is above the subsolution is a nontrivial fact, and requires the use of the solution ψ to an associated free boundary problem.

We define the candidate for the supersolution as :

$$\overline{\phi}(x,y) = \begin{cases} \theta\ \psi(x,y) \ \text{ in } \Omega := \{\psi < 1\} \\ \phi_0(\mathrm{dist}((x,y),\Omega)) \ \text{ in } I\!R^2\backslash\Omega \end{cases}$$

where dist denotes the euclidean distance function and ψ is the unique (up to shift) solution to the following free boundary problem (*see* [20]) :

Theorem 2.1 (A free boundary problem,[1] [20]) *For $\alpha \in (0, \frac{\pi}{2}]$, $c_0 > 0$ and $c = c_0/\sin\alpha$, there exists a function ψ satisfying*

$$
(2.3) \qquad \begin{cases} \Delta\psi - c\partial_y\psi = 0 \quad \text{in } \Omega := \{\psi < 1\}, \\ 0 < \psi \le 1 \quad \text{in } I\!R^2, \\ \dfrac{\partial\psi}{\partial n} = c_0 \quad \text{on } \Gamma := \partial\Omega, \\ \lim\limits_{y\to-\infty} \sup\limits_{\mathcal{C}^-(y,\alpha)} \psi = 0, \\ \psi = 1 \quad \text{in } \mathcal{C}^+(y_0, \pi-\alpha) \quad \text{for some } y_0 \in I\!R, \end{cases}
$$

where $\frac{\partial\psi}{\partial n}$ stands for the normal derivative on Γ of the restriction of ψ to $\overline{\Omega}$. Furthermore, ψ is continuous in $I\!R^2$, the set $\Gamma = \partial\Omega$ is a C^∞ graph $\Gamma = \{y = \varphi(x),\ x \in I\!R\}$ such that

$$
\sup_{x\in I\!R} |\varphi(x) + |x|\cot\alpha| < +\infty,
$$

Ω is the subgraph $\Omega = \{y < \varphi(x)\}$, the restriction of ψ is C^∞ in $\overline{\Omega}$, and $|\varphi'(x)| \le \cot\alpha$ in $I\!R$. Lastly, ψ is nondecreasing in y, even in x and satisfies

$$
\partial_x\psi(x,y) \ge 0 \quad \text{for } x \ge 0,\ y < \varphi(x).
$$

From Theorem 2.1 and from the definition of γ_0, it is easy to see that there exist two positive constants r_0 and C such that

$$
\forall r \ge r_0, \quad \underline{\phi}^r(x,y) := \underline{\phi}(x, y-r) \le \theta \ \text{ in } \overline{\Omega}
$$

and

$$
\mathrm{dist}((x,y),\Omega)) \ge -C + (y - \gamma_0(x))\sin\alpha \ \text{ in } I\!R^2\backslash\Omega = \{\psi = 1\}.
$$

Because of (2.1), and from the comparison principles proved in [19], it follows that $\underline{\phi}^r \le \overline{\phi}$ in Ω for all $r \ge r_0$ and then, by construction of $\overline{\phi}$, we get that

$$
\underline{\phi}^r \le \overline{\phi} \ \text{ in } I\!R^2
$$

as soon as $r \ge \max(r_0, C/\sin\alpha)$.

Moreover, notice that the construction of $\overline{\phi}$ implies that

$$
(2.4) \qquad \lim_{y_0\to-\infty} \sup_{\{y\le y_0-|x|\cot\alpha\}} \overline{\phi}(x,y) = 0.
$$

We shall prove in Section 2.2 the following result

[1] This problem arises in models of equidiffusional premixed Bunsen flames in the limit of high activation energy. The existence of a solution ψ of problem (2.3) can be obtained by regularizing approximations, starting from solutions of problems of the type (1.1) with nonlinearities f_ε approximating a Dirac mass at 1.

Proposition 2.2 *The function $\overline{\phi}$ is a supersolution of (1.1) in the vicosity sense.*

Step 3 : *Existence of a solution.* Choose a real number r such that $r \geq \max(r_0, C/\sin\alpha)$. By using the Perron method for viscosity solutions (*see* [14] and H. Ishii [23], Theorem 7.2 page 41), we get the existence of a vicosity solution ϕ of $\Delta\phi - c\partial_y\phi + f(\phi) = 0$, which satisfies :

$$0 \leq \underline{\phi}^r \leq \phi \leq \overline{\phi} \leq 1 \quad \text{in } I\!R^2.$$

Now by the regularity theory for viscosity solutions (see [13]), it follows that ϕ is $C^{2+\beta}$ (with $\beta > 0$), and then ϕ is a classical solution of (1.1). Finally ϕ satisfies the conditions at infinity (1.2) because of (2.2) and (2.4). This completes the proof of Theorem 1.1. □

2.2 Proof of Proposition 2.2

The proof of Proposition 2.2 is based on the following result :

Lemma 2.3 *Let ξ be the function defined by*

$$\xi(x,y) = \phi_0^{-1}(\theta\ \psi(x,y)) \quad \textit{in } \overline{\Omega} = \{y \leq \varphi(x)\},$$

where ψ is the solution to the free boundary problem given by Theorem 2.1. Then

$$|\nabla\xi| \leq 1 \quad \textit{in } \overline{\Omega}.$$

Proof. We have

$$\begin{cases} \Delta\xi + c_0\left(|\nabla\xi|^2 - \dfrac{\partial_y\xi}{\sin\alpha}\right) = 0 \ \text{ in } \Omega = \{\xi < 0\}, \\ \xi = 0 \ \text{ and } \dfrac{\partial\xi}{\partial n} = 1 \ \text{ on } \Gamma = \partial\{\xi < 0\} \end{cases}$$

since $\phi_0(s) = \theta e^{c_0 s}$ for all $s \leq 0$. A straightforward computation gives, for $v = |\nabla\xi|^2$:

$$\Delta v + b \cdot \nabla v = 2|D^2\xi|^2,$$

where $b = 2c_0\nabla\xi - c_0/\sin\alpha\ e_y$ and $e_y = (0,1)$.

Let us define $M = \sup v$. We want to prove that $M \leq 1$. Let us assume that $M > 1$. We know that $v = 1$ on Γ and $v(x,y) \to 1$ as $|x| \to +\infty$ and $d((x,y),\Gamma)$ stays bounded. From the maximum principle we conclude that there exists a sequence of points (x_n, y_n) such that $v(x_n, y_n) \to M$, $d((x_n, y_n), \Gamma) \to +\infty$, and the sequence of functions $v_n(x,y) = v(x_n + x, y_n + y)$ converges to the function $v_\infty(x,y)$ which from the strong maximum principle satisfies $v_\infty(x,y) \equiv M$. Moreover $\xi_{0,n}(x,y) = \xi_{0,n}(x_n + x, y_n + y) - \xi_{0,n}(x_n, y_n)$ converges to a function

$\xi_{0,\infty}(x,y)$ such that $v_\infty = |\nabla \xi_{0,\infty}|^2$, and $D^2\xi_{0,\infty} \equiv 0$.
On the other hand, the following function

$$w(x,y) = e^{c_0(y\sin\alpha + x\cos\alpha)} + e^{c_0(y\sin\alpha - x\cos\alpha)} = e^{c_0\sin\alpha(y-g(x)))}$$

is a solution of the equation $\Delta w - \frac{c_0}{\sin\alpha}\partial_y w = 0$ on the whole space. Using the comparison principle on the Lipschitz subgraph $\Omega = \{y < \varphi(x)\}$, we can deduce that there exist two constants $y_1 > y_2$ such that

$$e^{c_0\sin\alpha(y-g(x)-y_1))} \le \psi \le e^{c_0\sin\alpha(y-g(x)-y_2))} \quad \text{on} \quad \Omega$$

and then a simple computation implies

$$|\xi_{0,\infty}(x,y)| \le |y_1 - y_2|\sin\alpha + \sqrt{x^2+y^2}.$$

Because $M > 1$, this is in contradiction with $\nabla\xi_{0,\infty} \equiv \nu\sqrt{M}$ for a constant vector ν satisfying $||\nu|| = 1$. This ends the proof of the lemma. □

Let us now turn to the
Proof of Proposition 2.2. Let us define

$$I[u] := \Delta u - c\partial_y u + f(u).$$

By construction $\overline{\phi}$ is a classical solution of $I[\overline{\phi}] = 0$ in the open set $\Omega = \{\overline{\phi} < \theta\}$. Moreover the gradient of $\overline{\phi}$ is continuous across $\Gamma = \partial\{\overline{\phi} < \theta\}$, which is smooth.

Let us now consider the function $\xi(x,y) = \phi_0^{-1}(\overline{\phi}(x,y))$, now defined in the whole plane $I\!R^2$. We have

$$J[\xi] := \frac{I[\overline{\phi}]}{\phi_0'(\xi)} = \Delta\xi + c_0\left(|\nabla\xi|^2 - \frac{\partial_y\xi}{\sin\alpha}\right) + G(\xi)\left(1 - |\nabla\xi|^2\right) \tag{2.5}$$

in the viscosity sense in $I\!R^2$, where $G(\xi) = f(\phi_0(\xi))/\phi_0'(\xi) \ge 0$. Because $\xi(x,y) = d((x,y),\Gamma)$ in $I\!R^2\backslash\Omega = \{\xi \ge 0\}$, the following inequality holds in the *viscosity sense* :

$$J[\xi] \le H[\xi] := -\frac{K}{1-K\xi} + c_0\left(1 - \frac{n\cdot e_y}{\sin\alpha}\right) \quad \text{in } \{y > \varphi(x)\}, \tag{2.6}$$

and equality holds where ξ is smooth (*see* Gilbarg, Trudinger [18]). Here $K = K(Y)$ and $n = n(Y)$ are respectively the curvature[2] and the exterior normal to the set $\overline{\Omega}$ at a point $Y = Y(x,y) \in \Gamma$ where the ball $B_{\xi(x,y)}((x,y))$ is tangent to Γ.

On the other hand, on the level set Γ we have $|\nabla\xi| = 1$ and because of Lemma 2.3 we get $D^2_{nn}\xi \ge 0$. Therefore, since $I[\overline{\phi}] = 0$ in $\overline{\Omega}$, we deduce from (2.5) that

$$-K(Y) + c_0\left(1 - \frac{n(Y)\cdot e_y}{\sin\alpha}\right) \le 0 \quad \text{for all } Y \in \Gamma.$$

[2]under the convention that the curvature of a disk is negative

Furthermore, observe that the inequality

$$\frac{-K(Y)}{1-K(Y)\xi(x,y)} \leq -K(Y)$$

holds for all $(x,y) \in I\!R^2 \backslash \Omega = \{\xi \geq 0\}$, whatever the sign of K is, under the same notations as above for Y.

Therefore, $H[\xi] \leq 0$ in $I\!R^2 \backslash \Omega$ and finally $J[\xi] \leq 0$ in $\{y > \varphi(x)\} = \{\xi > 0\}$ in the viscosity sense. Hence, $I[\overline{\phi}] \leq 0$ in $I\!R^2$ in the viscosity sense, which ends the proof of Proposition 2.2. □

3 Global solutions are travelling waves

This section is devoted to the proof of Theorems 1.3 and Theorem 1.5 below, the latter being a consequence of the former.

One of the main tools in the proof of Theorem 1.3 is the following comparison principle :

Proposition 3.1 (Comparison principle) *Let $\delta \in I\!R$ and $g : I\!R \to I\!R$ be a Lipschitz-continuous function which is nonincreasing in $(-\infty, \delta]$. Let $\psi : I\!R \to I\!R$ be a Lipschitz-continuous function. Let $\underline{v} : (t,x,y) \mapsto \underline{v}(t,x,y)$ and $\overline{v} : (t,x,y) \mapsto \overline{v}(t,x,y)$ be two bounded and Lipschitz-continuous functions defined on $I\!R \times \overline{\Omega}$, where $\Omega = \{y < \psi(x)\}$. Let $\kappa \in I\!R$. Assume that*

$$\begin{cases} \underline{v}_t \leq \Delta \underline{v} + \kappa \partial_y \underline{v} + g(\underline{v}) & \text{in } \mathcal{D}'(I\!R \times \Omega) \\ \overline{v}_t \geq \Delta \overline{v} + \kappa \partial_y \overline{v} + g(\overline{v}) & \text{in } \mathcal{D}'(I\!R \times \Omega), \end{cases}$$

$\underline{v} \leq \delta$ in $I\!R \times \overline{\Omega}$, $\underline{v}(t,x,\psi(x)) \leq \overline{v}(t,x,\psi(x))$ for all $(t,x) \in I\!R^2$, and

$$\lim_{y_0 \to -\infty} \sup_{t \in I\!R,\ y < \psi(x) + y_0} (\underline{v}(t,x,y) - \overline{v}(t,x,y)) \leq 0.$$

Then $\underline{v}(t,x,y) \leq \overline{v}(t,x,y)$ for all $(t,x,y) \in I\!R \times \overline{\Omega}$.

Proof. Since $\underline{v}$ and $\overline{v}$ are bounded in $I\!R \times \overline{\Omega}$, one has $\underline{v} - \varepsilon \leq \overline{v}$ in $I\!R \times \overline{\Omega}$ for $\varepsilon > 0$ large enough. Let us now define

$$\varepsilon^* = \inf \{\varepsilon > 0,\ \underline{v} - \varepsilon' \leq \overline{v} \ \text{ in } I\!R \times \overline{\Omega} \ \text{ for all } \varepsilon' \geq \varepsilon\}.$$

By continuity, one can immediately say that $\underline{v} - \varepsilon^* \leq \overline{v}$ in $I\!R \times \overline{\Omega}$.

Let us now assume that $\varepsilon^* > 0$. There exists then a sequence $\varepsilon_n \overset{<}{\to} \varepsilon^*$ and a sequence of points (t_n, x_n, y_n) in $I\!R \times \overline{\Omega}$ such that

$$\underline{v}(t_n, x_n, y_n) - \varepsilon_n > \overline{v}(t_n, x_n, y_n).$$

Since $\varepsilon_n \geq \varepsilon^*/2 > 0$ for n large enough, it follows from the assumptions of Proposition 3.1 that there exist two real numbers $0 < A \leq B$ such that

$$\psi(x_n) - B \leq y_n \leq \psi(x_n) - A \tag{3.1}$$

for n large enough.

Call $\psi_n(x) = \psi(x + x_n) - y_n$ and let $\underline{v}_n$ and $\overline{v}_n$ the functions defined in $I\!R \times \{y \leq \psi_n(x)\}$ by

$$\underline{v}_n(t, x, y) = \underline{v}(t + t_n, x + x_n, y + y_n) \text{ and } \overline{v}_n(t, x, y) = \overline{v}(t + t_n, x + x_n, y + y_n).$$

Since the functions ψ_n are uniformly Lipschitz-continuous, they locally converge, up to extraction of some subsequence, to a globally Lipschitz-continuous function ψ_∞. Similarly, up to extraction of another subsequence, the functions $\underline{v}_n$ and $\overline{v}_n$ converge locally uniformly in $I\!R \times \{y < \psi_\infty(x)\}$ to two globally Lipschitz-continuous functions $\underline{v}_\infty$ and $\overline{v}_\infty$, which can be extended by continuity on $I\!R \times \{y = \psi_\infty(x)\}$. Call

$$\Omega_\infty = \{(x, y) \in I\!R^2,\ y < \psi_\infty(x)\}.$$

Since $\underline{v}(t, x, \psi(x)) \leq \overline{v}(t, x, \psi(x))$ for all $(t, x) \in I\!R^2$ and since $\underline{v}$ and $\overline{v}$ are globally Lipschitz-continuous in $I\!R \times \overline{\Omega}$, it follows that

$$\underline{v}_\infty(t, x, \psi_\infty(x)) \leq \overline{v}_\infty(t, x, \psi_\infty(x))$$

for all $(t, x) \in I\!R^2$.

By passage to the limit, the functions $\underline{v}_\infty$ and $\overline{v}_\infty$ satisfy

$$\begin{cases} (\underline{v}_\infty)_t \leq \Delta \underline{v}_\infty + \kappa \partial_y \underline{v}_\infty + g(\underline{v}_\infty) & \text{in } \mathcal{D}'(I\!R \times \Omega_\infty) \\ (\overline{v}_\infty)_t \geq \Delta \overline{v}_\infty + \kappa \partial_y \overline{v}_\infty + g(\overline{v}_\infty) & \text{in } \mathcal{D}'(I\!R \times \Omega_\infty) \end{cases}$$

and $\underline{v}_\infty - \varepsilon^* \leq \underline{v}_\infty$ in $I\!R \times \overline{\Omega_\infty}$. On the other hand, $\psi_\infty(0) \geq A > 0$ from (3.1), and $\underline{v}_\infty(0, 0, 0) - \varepsilon^* = \underline{v}_\infty(0, 0, 0)$. Lastly, $\underline{v}_\infty - \varepsilon^* \leq \underline{v}_\infty \leq \delta$ in $I\!R \times \overline{\Omega_\infty}$ and the function g was assumed to be nonincreasing in $(-\infty, \delta]$. Hence, $g(\underline{v}_\infty - \varepsilon^*) \geq g(\underline{v}_\infty)$ in $I\!R \times \overline{\Omega_\infty}$.

Therefore, the function $w := \underline{v}_\infty - \varepsilon^* - \overline{v}_\infty$ is a bounded, globally Lipschitz-continuous and nonpositive function in $I\!R \times \overline{\Omega_\infty}$, vanishing at the point $(0, 0, 0)$ and satisfying

$$w_t \leq \Delta w + \kappa \partial_y w + \gamma(t, x, y) w \quad \text{in } \mathcal{D}'(I\!R \times \Omega_\infty),$$

where γ is globally bounded function (here we use the fact that g is globally Lipschitz-continuous). The strong parabolic maximum principle then implies that $w(t, x, y) = 0$, *i.e.* $\underline{v}_\infty(t, x, y) - \varepsilon^* = \overline{v}_\infty(t, x, y)$, for all $t \leq 0$ and $(x, y) \in \overline{\Omega_\infty}$. But the positivity of ε^* contradicts the fact that $\underline{v}_\infty \leq \overline{v}_\infty$ on $I\!R \times \partial\Omega_\infty$.

As a conclusion, $\varepsilon^* = 0$ and $\underline{v} \leq \overline{v}$ in $I\!R \times \overline{\Omega}$. □

Remark 3.2 *The above comparison principle is a version of a parabolic maximum principle for time-global solutions in an unbounded space-domain. This comparison principle actually holds the same way in any space-dimension for more general second-order parabolic operators with smooth coefficients depending on time and space and a non-linearity $g(t, x_1, \cdots, x_N, u)$ satisfying the same monotonicity assumption with respect to u as in Proposition 3.1.*

Let us now turn to the

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3, the function v defined in $I\!R^3$ by $v(t,x,y) := u(t,x,y-ct)$ is such that $0 \le v \le 1$ and it solves

$$v_t = \Delta v - c\partial_y v + f(v). \tag{3.2}$$

From standard parabolic estimates, the function v is globally Lipschitz-continuous with respect to all variables (t,x,y). Furthermore, v satisfies

$$\left\{ \begin{array}{l} \displaystyle\lim_{y_0\to-\infty} \sup_{t\in I\!R,\ y\le y_0-|x|\cot\alpha} v(t,x,y) = 0 \\ \displaystyle\lim_{y_0\to+\infty} \inf_{t\in I\!R,\ y\ge y_0-|x|\cot\alpha} v(t,x,y) = 1. \end{array} \right. \tag{3.3}$$

We shall now prove that v is actually independent of t. That will imply that $v = v(x,y)$ is a solution of (1.1-1.2) (notice that from the strong maximum principle, one then has $0 < v < 1$). From Theorem 1.1 and from the uniqueness results in [19], it will follow that $v = \tau_{a,b}\phi$ in $I\!R^2$, for some pair $(a,b) \in I\!R^2$.

Fix now any real number t_0. For $s \in I\!R$, call w^s the function defined in $I\!R^3$ by

$$w^s(t,x,y) = v(t+t_0, x, y+s).$$

The function w^s is a solution of (3.2) as well.

From the assumptions on f, there exists $\rho > 0$ such that $\theta \le 1-\rho$ and f is nonincreasing on the interval $[1-\rho, +\infty)$. Remember also that f is identically equal to 0 on $(-\infty, \theta]$. From (3.3), there exists $A > 0$ such that

$$\left\{ \begin{array}{ll} v(t,x,y) \ge 1-\rho & \text{for all } t\in I\!R,\ y \ge A - |x|\cot\alpha \\ v(t,x,y) \le \theta & \text{for all } t\in I\!R,\ y \le -A - |x|\cot\alpha. \end{array} \right.$$

Choose any $s \ge 2A$ and observe that

$$\begin{array}{rcl} w^s(t,x,-A-|x|\cot\alpha) & = & v(t+t_0, x, s-A-|x|\cot\alpha) \\ & \ge & 1-\rho \ \ge\ \theta \ \ge\ v(t,x,-A-|x|\cot\alpha) \end{array}$$

for all $(t,x) \in I\!R^2$. It is then immediate to check that all the assumptions of Proposition 3.1 are satisfied with $g = f$, $\delta = \theta$, $\psi(x) = -A - |x|\cot\alpha$, $\kappa = -c$, $\underline{v} = v$, $\overline{v} = w^s$. Therefore,

$$w^s(t,x,y) \ge v(t,x,y) \ \text{ for all } t \in I\!R \text{ and } y \le -A - |x|\cot\alpha.$$

Similarly, the assumptions of Proposition 3.1 are also satisfied with the choices $g(\tau) = -f(1-\tau)$, $\delta = \rho$, $\psi(x) = A + |x|\cot\alpha$, $\kappa = c$, $\underline{v}(t,x,y) = 1 - w^s(t,x,-y)$ and $\overline{v}(t,x,y) = 1 - v(t,x,-y)$. Therefore, $\underline{v}(t,x,y) \le \overline{v}(t,x,y)$ for all $t \in I\!R$ and $y \le A + |x|\cot\alpha$, which means that

$$v(t,x,y) \le w^s(t,x,y) \ \text{ for all } t \in I\!R \text{ and } y \ge -A - |x|\cot\alpha.$$

As a consequence, one has $v \le w^s$ in $I\!R^3$ for all $s \ge 2A$. Let us now define

$$s^* = \inf \{s > 0,\ v \le w^\tau \text{ in } I\!R^3 \text{ for all } \tau \ge s\}.$$

By continuity, one has $v \le w^{s^*}$. Let us assume by contradiction that $s^* > 0$. One shall consider two cases, namely whether the infimum of $w^{s^*} - v$ is positive or zero on the strip

$$S = \{(t, x, y) \in I\!R^3,\ |y + |x| \cot \alpha| \le A\}.$$

Case 1 : $\inf_S (w^{s^*} - v) > 0$. Since the function v, as well as w^{s^*}, is globally Lipschitz-continuous, there exists $\eta_0 \in (0, s^*)$ such that $w^{s^*-\eta} \ge v$ in S for all $\eta \in [0, \eta_0]$. Choose any $\eta \in [0, \eta_0]$. Since $s^* - \eta \ge 0$, one has $w^{s^*-\eta}(t, x, y) \ge 1 - \rho$ for all $t \in I\!R$ and $y \ge A - |x| \cot \alpha$. It also follows from the choice of η that

$$w^{s^*-\eta}(t, x, A - |x| \cot \alpha) \ge v(t, x, A - |x| \cot \alpha)$$

for all $(t, x) \in I\!R^2$ (since $(t, x, A - |x| \cot \alpha) \in \partial S$ for all $(t, x) \in I\!R^2$). As above, it is straightforward to check that Proposition 3.1 implies that

$$w^{s^*-\eta}(t, x, y) \ge v(t, x, y) \ \text{ for all } t \in I\!R \text{ and } y \ge A - |x| \cot \alpha.$$

Similarly, it can also be deduced that

$$w^{s^*-\eta}(t, x, y) \ge v(t, x, y) \ \text{ for all } t \in I\!R \text{ and } y \le -A - |x| \cot \alpha.$$

Putting all the preceeding facts together, one concludes that $w^{s^*-\eta} \ge v$ in $I\!R^3$ for all $\eta \in [0, \eta_0]$. This is contradiction with the minimality of s^*, since $\eta_0 > 0$. Therefore, case 1 is ruled out.

Case 2 : $\inf_S (w^{s^*} - v) = 0$. There exists then a sequence (t_n, x_n, y_n) such that $t_n \in I\!R$, $-A - |x_n| \cot \alpha \le y_n \le A - |x_n| \cot \alpha$ and

$$w^{s^*}(t_n, x_n, y_n) - v(t_n, x_n, y_n) \to 0 \text{ as } n \to +\infty.$$

Call $v_n(t, x, y) = v(t + t_n, x + x_n, y + y_n)$. Each function v_n is a solution of (3.2) and ranges in $[0, 1]$. From standard parabolic estimates, the functions v_n converge locally uniformly, up to extraction of some subsequence, to a global solution v_∞ of (3.2) such that $0 \le v_\infty \le 1$. Furthermore, $v_\infty(t_0, 0, s^*) = v_\infty(0, 0, 0)$. Therefore, the function $z(t, x, y) := v_\infty(t+t_0, x, y+s^*) - v_\infty(t, x, y)$, which is nonnegative since $v \le w^{s^*}$, vanishes at $(0, 0, 0)$ and is a global bounded solution of

$$z_t = \Delta z - c\partial_y z + \gamma(t, x, y) z$$

for some bounded function γ (here we use the fact that f is globally Lipschitz-continuous). The strong maximum principle for $t \le 0$ and the uniqueness of the Cauchy problem for the above equation then imply that $z(t, x, y) = 0$ for all $(t, x, y) \in I\!R^3$. As a consequence,

$$v_\infty(t + nt_0, x, y + ns^*) = v_\infty(t, x, y) \tag{3.4}$$

for all $(t,x,y) \in I\!R^3$ and $n \in \mathbf{Z}$.

Furthermore, from the definitions of (t_n, x_n, y_n), one of the following three cases occur up to extraction of some subsequence : (i) the sequence (x_n, y_n) is bounded, (ii) $x_n \to -\infty$, or (iii) $x_n \to +\infty$.

If case (i) occurs, then (3.3) holds for v_∞. If case (ii) occurs, then the function v_∞ satisfies

$$\begin{cases} \lim\limits_{y_0 \to -\infty} \sup\limits_{t \in I\!R,\ y \le y_0 + x\cot\alpha} v_\infty(t,x,y) = 0 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{t \in I\!R,\ y \ge y_0 + x\cot\alpha} v_\infty(t,x,y) = 1. \end{cases}$$

Lastly, if case (iii) occurs, then v_∞ satisfies

$$\begin{cases} \lim\limits_{y_0 \to -\infty} \sup\limits_{t \in I\!R,\ y \le y_0 - x\cot\alpha} v_\infty(t,x,y) = 0 \\ \lim\limits_{y_0 \to +\infty} \inf\limits_{t \in I\!R,\ y \ge y_0 - x\cot\alpha} v_\infty(t,x,y) = 1. \end{cases}$$

In each of the three cases (i), (ii) or (iii), one gets a contradiction with property (3.4). Therefore case 2 is ruled out too.

As a conclusion, the assumption $s^* > 0$ is impossible, whence

$$v(t,x,y) \le w^0(t,x,y) = v(t+t_0,x,y)$$

for all $(t,x,y) \in I\!R^3$. Since t_0 is arbitrary in $I\!R$, one concludes that v does not depend on the variable t. As already emphasized, that completes the proof of Theorem 1.3. □

Let us now turn to the

Proof of Theorem 1.5. The functions $v_n(t,x,y) = u(t_n+t, x, y-ct_n-ct)$ solve

$$\partial_t v_n = \Delta v_n - c\partial_y v_n + f(v_n) \tag{3.5}$$

for $t > -t_n$. Furthermore, since ϕ is a solution of (1.1), the maximum principle implies that

$$v_n(t,x,y) \le \phi(x,y)$$

for all $(x,y) \in I\!R^2$ and for all $t \ge -t_n$. On the other hand, because of the second inequality in (1.8) and because u_0 is nonnegative, there exist $\eta \in (\theta, 1]$ and $s_0 \in I\!R$ such that

$$\forall (x,y) \in I\!R^2, \quad u_0(x,y) \ge \max(H(\pm x\cos\alpha + y\sin\alpha + s_0)),$$

where $H(s) = 0$ if $s < 0$ and $H(s) = \eta$ if $s \ge 0$. Therefore,

$$\forall t \ge -t_n,\ \forall (x,y) \in I\!R^2, \quad v_n(t,x,y) \ge \max(v^+(t_n+t,x,y), v^-(t_n+t,x,y)),$$

where the functions $v^\pm$ solve equation (3.5) with initial conditions $v^\pm(0,x,y) = H(\pm x\cos\alpha + y\sin\alpha + s_0)$. Consider the function v^+. Since equation (3.5) is

invariant up to translation and since $v^+(0,\cdot,\cdot)$ only depends on the variable $s = x\cos\alpha + y\sin\alpha$, so does $v^+(t,\cdot,\cdot)$ at any time $t \geq 0$. Therefore, $v^+(t,x,y)$ can be written as $v^+(t,x,y) = V^+(t,s)$ where V^+ solves

$$\left\{ \begin{array}{rcl} \partial_t V^+ & = & \partial_s^2 V^+ - c_0 \partial_s V^+ + f(V^+) \\ V^+(0,s) & = & H(s+s_0). \end{array} \right.$$

A result of Kanel' [25], [26] (*see* also Roquejoffre [35]) yields the convergence of $V^+(t,s)$ to $\phi_0(s+s_1)$ uniformly in $s \in I\!R$ as $t \to +\infty$, for some $s_1 \in I\!R$, where ϕ_0 is the solution of (1.5) such that, say, $\phi_0(0) = \theta$. By symmetry in the x-variable, it follows that $v^-(t,x,y) \to \phi_0(s'+s_1)$ uniformly in $(x,y) \in I\!R^2$ as $t \to +\infty$, where $s' = -x\cos\alpha + y\sin\alpha$. Consequently,

$$\forall (t,x,y) \in I\!R^3, \quad \liminf_{n\to+\infty} v_n(t,x,y) \geq \max(\phi_0(\pm x\cos\alpha + y\sin\alpha + s_1)).$$

Eventually, from standard parabolic estimates, there exists a subsequence $n' \to +\infty$ such that the functions $v_{n'}$ converge locally uniformly in $I\!R \times I\!R^2$ to a classical solution $v(t,x,y)$ of $v_t = \Delta v - c v_y + f(v)$ such that

$$\max(\phi_0(\pm x\cos\alpha + y\sin\alpha + s_1)) \leq v(t,x,y) \leq \phi(x,y)$$

for all $(t,x,y) \in I\!R^3$.

The function $u(t,x,y) = v(t,x,y+ct)$ then satisfies (1.6) and (1.7). Theorem 1.3 yields that $u(t,x,y) = \phi(x+a, y+b+ct)$ for some $(a,b) \in I\!R^2$ and for all $(t,x,y) \in I\!R \times I\!R^2$. Therefore, $v(t,x,y) = \phi(x+a, y+b)$ and the conclusion of Theorem 1.5 follows. □

4 Local stability

The goal of this section is to prove the following stability result :

Theorem 4.1 (Local Stability) *Let $\alpha \in (0,\pi/2)$ and f satisfy (1.4). Let $u(t,x,y)$ be a solution of the Cauchy problem (1.3). There exists $\overline{\rho} > 0$ (one may choose $\overline{\rho} = c_0 \cot\alpha$) such that the following holds : for any $\rho \in (0,\overline{\rho})$, there is $\varepsilon > 0$ such that if $0 \leq u_0 \leq 1$, $u_0 \in UC(I\!R^2)$, and $\|u_0 - \phi\|_{G_\rho} \leq \varepsilon$ for some solution ϕ of (1.1-1.2), then there are two constants $K \geq 0$ and $\omega > 0$ such that*

$$\forall t \geq 0, \qquad \|u(t,\cdot,\cdot - ct) - \phi(\cdot,\cdot)\|_{G_\rho} \leq K e^{-\omega t}.$$

The object to study is the linearized operator around a wave solution ϕ :

$$Lv = -\Delta v + c\partial_y v - f'(\phi) v.$$

In the whole section, we choose the (unique) wave ϕ solving (1.1-1.2) such that :

$$\phi(x,y) = \phi(-x,y), \quad \phi(0,0) = \theta.$$

Proposition 4.2 (No eigenvalue with negative real part) *Let* $\rho > 0$ *and* $u \in C(I\!R^2, \mathbf{C})$ *be a classical solution of* $Lu = \lambda u$ *such that* $\mathrm{Re}(u),\ \mathrm{Im}(u) \in G_\rho$.

- *If* $\mathrm{Re}(\lambda) < 0$, *then* $u = 0$.
- *If* $\mathrm{Re}(\lambda) = 0$, *then there is* $C > 0$ *such that*

$$|u| \leq C\phi_y \quad \text{in } I\!R^2.$$

Proof. We wish to follow the idea in [5]. The result is obtained by proving first that u decays faster than any derivative of the wave, then to conclude with the aid of the parabolic equation

$$U_t + LU = 0, \qquad U(0) = \mathrm{Re}(u). \tag{4.1}$$

This first part of the programme does not seem to be done as easily as in [5], due to the lack of precise boundaries where to apply an exact boundary condition - hence an evolution equation approach -.

In order to circumvent the difficulty we directly use equation (4.1) and construct a Fife-McLeod type super-solution (*see* [17]) : set

$$w_0(x, y) = \min(e^{\tilde{\rho}(y \sin\alpha - x\cos\alpha)}, e^{\tilde{\rho}(y\sin\alpha + x\cos\alpha)}),$$

where $\tilde{\rho} \in (0, c_0)$ shall be chosen later. We also set

$$\bar{U}(t, x, y) = a_0(t)\phi_y(x, y) + a_1(t)\gamma_1(x, y) + a_2(t)w_0(x, y)\gamma_2(x, y).$$

Define $y_1 > 0$ and $k > 0$ such that

$$\forall (x, y) \in \mathcal{C}_+(y_1, \pi - \alpha), \quad f'(\phi(x, y)) \leq -k \tag{4.2}$$

and choose $y_2 < 0$ such that

$$\forall (x, y) \in \mathcal{C}_-(y_2, \alpha), \quad f'(\phi(x, y)) = 0. \tag{4.3}$$

Actually, any negative y_2 works since $\phi(0, 0) = \theta$ and it is known ([9], [19]) that ϕ is nonincreasing in any direction $\tau = (\cos\beta, \sin\beta)$ such that $-\pi/2 - \alpha \leq \beta \leq -\pi/2 + \alpha$. The functions γ_1 and γ_2 are required to be in $C^2(I\!R^2)$ and to satisfy

- $0 \leq \gamma_1,\ \gamma_2 \leq 1$ in $I\!R^2$;
- $\gamma_1 \equiv 1$ in $\mathcal{C}_+(2y_1, \pi - \alpha)$ and $\gamma_1 \equiv 0$ in $\mathcal{C}_-(y_1, \alpha)$;
- $\gamma_2 \equiv 1$ in $\mathcal{C}_-(2y_2, \alpha)$ and $\gamma_2 \equiv 0$ in $\mathcal{C}_+(y_2, \pi - \alpha)$;
- $\partial_x\gamma_1(0, y) = \partial_x\gamma_2(0, y) \equiv 0$.

Then set

$$\mathcal{L}\bar{U} = \bar{U}_t + L\bar{U};$$

as is now classical we anticipate that $\dot{a}_0(t)$ will be nonnegative, and break the evaluation of $\mathcal{L}\bar{U}$ in three parts.

1. $(x, y) \in \mathcal{C}_-(2y_2, \alpha)$. Then we have, because of (4.3) and because $\phi_y \geq 0$:

$$\mathcal{L}\bar{U} \geq \left(\dot{a}_2(t) + (c\tilde{\rho}\sin\alpha - \tilde{\rho}^2)a_2(t)\right) w_0,$$

provided that $a_2(t)$ is nonnegative. Remember that $c\sin\alpha = c_0$ and set

$$a_2(t) = \alpha_2 e^{-t\tilde{\rho}(c_0-\tilde{\rho})}$$

with $\alpha_2 > 0$ to be chosen later. Observe here that $\tilde{\rho}(c_0 - \tilde{\rho}) > 0$ since $\tilde{\rho}$ is in $(0, c_0)$.

2. $(x, y) \in \mathcal{C}_+(2y_1, \pi - \alpha)$. Then we have, because of (4.2) and provided that $a_1(t)$ is nonnegative :

$$\mathcal{L}\bar{U} \geq \dot{a}_1(t) + ka_1(t),$$

and we define

$$a_1(t) = \alpha_1 e^{-kt}$$

with $\alpha_1 > 0$ to be chosen later.

3. $(x, y) \in \mathcal{C}_-(2y_1, \alpha) \cap \mathcal{C}_+(2y_2, \pi - \alpha)$. There is a large constant $C_1 > 0$ and a small positive constant ω such that

$$\mathcal{L}\bar{U} \geq \dot{a}_0(t)\phi_y(x, y) - C_1 e^{-\omega t}.$$

Then, because ϕ_y is positive and bounded away from 0 in the region under consideration, there is a large constant C_2 such that we may take

$$a_0(t) = \alpha_0 - C_2 e^{-\omega t}, \quad \alpha_0 > 0,$$

and

$$\mathcal{L}\bar{U} \geq 0 \ \text{ in } \mathcal{C}_-(2y_1, \alpha) \cap \mathcal{C}_+(2y_2, \pi - \alpha).$$

Combining the above steps, and since $\mathrm{Re}(u) \in G_\rho$, one can choose α_0, α_1, α_2 large enough and $\tilde{\rho} > 0$ small enough so that $\bar{U}$ satisfies $\mathcal{L}\bar{U} \geq 0$ in $I\!R_+ \times I\!R^2$ and $\mathrm{Re}(u) \leq \bar{U}(0, .)$. Then we have :

$$\mathrm{Re}(e^{-\lambda t}u)(t, x, y) \leq \bar{U}(t, x, y).$$

We may repeat the argument with $-\mathrm{Re}(u)$ so as to obtain a similar lower bound for $\mathrm{Re}(e^{-\lambda t}u)$.

We can now conclude :

- If $\mathrm{Re}(\lambda) < 0$, assuming $u \neq 0$ contradicts the unboundedness of $\mathrm{Re}(e^{-\lambda t}u)$.

- If $\mathrm{Re}(\lambda) = 0$, then we argue similarly with $\mathrm{Im}(u)$ and $\mathrm{Im}(e^{-\lambda t}u)$ and we get an upper bound of the type $|\mathrm{Im}(e^{-\lambda t}u)| \leq \bar{V}(t,x,y)$, where $\bar{V}$ is of the same type as $\bar{U}$. We then only have to let $t \to +\infty$ to get that $|u| \leq C\phi_y$ in $I\!R^2$.

This ends the proof of Proposition 4.2. □

The next step is to show that 0 is NOT an eigenvalue of L when L is restricted to G_ρ. We first observe that

$$L(\phi_x) = L(\phi_y) = 0,$$

but neither ϕ_x nor ϕ_y belongs to G_ρ since $\liminf_{|x|\to+\infty} \phi_x(x, -|x|\cot\alpha)$ and $\liminf_{|x|\to+\infty} \phi_y(x, -|x|\cot\alpha)$ are positive (indeed, $\phi(x+\xi, y-|\xi|\cot\alpha) \to \phi_0(\pm x\cos\alpha + y\sin\alpha + t_\pm)$ as $\xi \to \pm\infty$ in $C^2_{loc}(I\!R^2)$, for some $t_\pm \in I\!R$).[3]

Then remark that a function $u(x,y)$ may be decomposed in an even and odd part (with respect to x) : $u = u_1 + u_2$ with

$$u_1(x,y) = \frac{u(x,y)+u(-x,y)}{2}, \quad u_2(x,y) = \frac{u(x,y)-u(-x,y)}{2}.$$

Notice also that $\partial_x u_1(0,y) = 0$ - provided u_1 is smooth enough - and that $u_2(0,y) = 0$. This trivial remark implies in fact boundary conditions for u_1 and u_2 on the y-axis if u_1 and u_2 are considered as functions from the right half-space that we denote $I\!R^2_+ = \{x > 0\}$. Notice finally that ϕ_x is odd and ϕ_y is even (with respect to the x-variable).

On the other hand, the operator L commutes with the reflections with respect to the y axis. Hence, if a function u in G_ρ solves $Lu = 0$, then both functions u_1 and u_2 are in G_ρ and solve $Lu = 0$. On the basis of all the above remarks we have the

Proposition 4.3 (No eigenfunctions in the null space of L) *(i). Let $\rho > 0$ and $u \in C^2(\overline{I\!R^2_+}) \cap G_\rho$ solve*

$$Lu = 0 \quad \text{in } I\!R^2_+, \quad u(0,y) = 0. \tag{4.4}$$

Then $u = 0$.

(ii). Let $\rho > 0$ and $u \in C^2(\overline{I\!R^2_+}) \cap G_\rho$ solve

$$Lu = 0 \quad \text{in } I\!R^2_+, \quad u_x(0,y) = 0.$$

Then $u = 0$.

We will only prove part (i) of Proposition 4.3, part (ii) being completely similar and being actually included in the proof of Proposition 4.5 below. The proof of part (i) will be based on the following

[3] We have of course $t_- = t_+$ as soon as ϕ is symmetric with respect to the x-variable.

Lemma 4.4 *Let $\rho > 0$ and $u \in C^2(\overline{I\!R^2_+}) \cap G_\rho$ satisfy (4.4). Then there is a constant $C > 0$ such that*

$$|u| \leq C\phi_x \quad \text{in } I\!R^2_+.$$

Proof. Argue as in Proposition 4.2, but this time u vanishes at the boundary $\{x = 0\}$, as well as ϕ_x. To circumvent this we define the supersolution $\bar{U}$ as

$$\bar{U}(t,x,y) = a_0(t)\phi_x(x,y) + \Big(a_1(t)\gamma_1(x,y) + a_2(t)w_0(x,y)\gamma_2(x,y)\Big)w_1(x),$$

where w_1 is bounded, increasing and concave, and satisfies moreover $w_1(0) = w_1''(0) = 0$ and $w_1(+\infty) = 1$. With similar choices for the functions a_0, a_1, a_2, w_0 as in Proposition 4.2, one has $\mathcal{L}\bar{U} \geq 0$ for all $t \geq 0$ and $(x,y) \in I\!R^2_+$. Furthermore, since $L\phi_x = 0$, $\phi_x > 0$ in $I\!R^2_+$ and $\phi_x = 0$ on $\{x = 0\}$, it follows from Hopf lemma that $\phi_{xx}(0,y) > 0$ for all $y \in I\!R$. On the other hand, the standard elliptic estimates up to the boundary imply that, say, $\|\nabla u\|_{L^\infty(\{0\leq x\leq 1,\ y_0\leq y\leq y_0+1\})} \leq C_0\|u\|_{L^\infty(\{0\leq x\leq 2,\ y_0-1\leq y\leq y_0+2\})}$ for some constant C_0 independant of $y_0 \in I\!R$. Therefore, suitable choices of a_0, a_1, a_2 and $\tilde{\rho}$ guarantee that $u(x,y) \leq \bar{U}(0,x,y)$ in $I\!R^2_+$. Hence, $u(x,y) \leq \bar{U}(t,x,y)$ for all $t \geq 0$ and $(x,y) \in I\!R^2_+$. Passing to the limit $t \to +\infty$ as in Proposition 4.2 leads to $u \leq C\phi_x$ in $I\!R^2_+$.

The same reasoning with $-u$ completes the proof of Lemma 4.4. □

Proof of Proposition 4.3. As already emphasized, we will only prove part (i). Under the assumptions of part (i), and from Lemma 4.4, let us denote by C_0 the biggest (maybe negative) constant C such that $u \geq C\phi_x$ in $I\!R^2_+$. We would like to prove that $C_0 \geq 0$. To see this, we assume the contrary and try to prove that $u \geq (C_0 + \delta)\phi_x$, for all δ in a small range.

First of all, since $u \in G_\rho$, $\phi_x \notin G_\rho$ and $C_0 \neq 0$, one gets that $u \not\equiv C_0\phi_x$. Therefore, $u > C_0\phi_x$ in $I\!R^2_+$ and $u_x(0,y) > C_0\phi_{xx}(0,y)$ for all $y \in I\!R$, due to the strong maximum principle and the Hopf Lemma. Consequently, for all subdomain Ω of $I\!R^2_+$ such that $I\!R^2_+\backslash\Omega$ is bounded, there exists $\delta_0(\Omega) > 0$ such that

$$\forall \delta \in [0, \delta_0(\Omega)],\ \forall (x,y) \in I\!R^2_+\backslash\Omega, \quad u(x,y) \geq (C_0 + \delta)\phi_x(x,y). \tag{4.5}$$

Rotate the coordinates (x,y) so as to bring the vector $(1,0)$ to the vector e_α defined by (1.10); let

$$(X,Y) = (x\sin\alpha - y\cos\alpha, x\cos\alpha + y\sin\alpha)$$

be the new coordinates.

In this new system the operator L reads

$$L = -\Delta - c\cos\alpha\frac{\partial}{\partial X} + c_0\frac{\partial}{\partial Y} - f'(\phi).$$

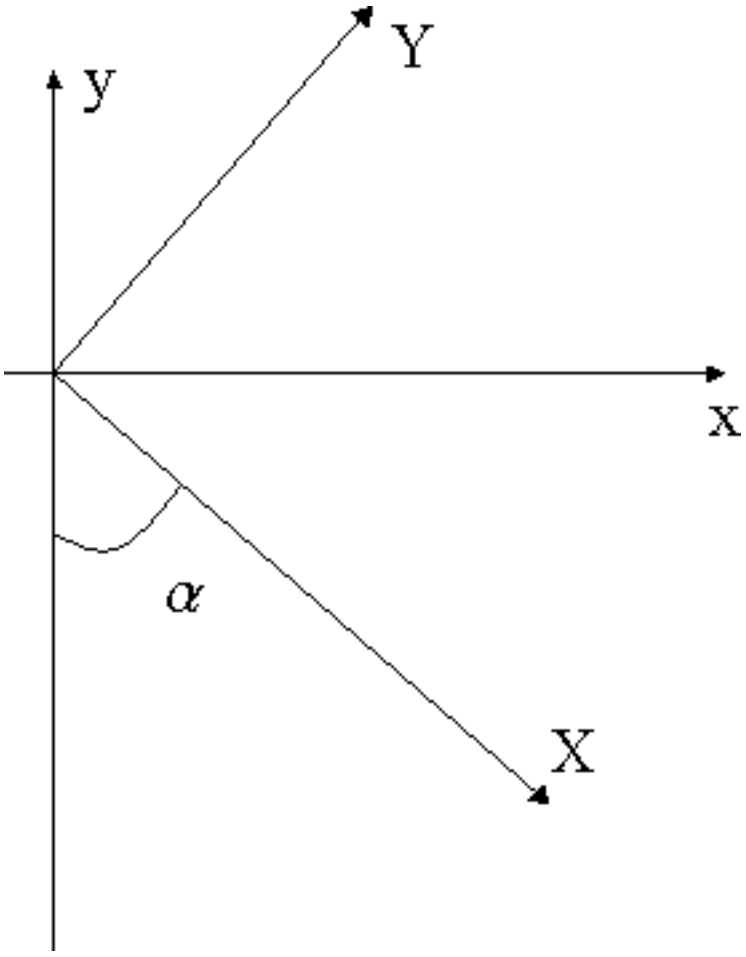

Figure 2: Rotated axes

To describe some portions of the plane, we will indifferently use the (x, y) or (X, Y) coordinate system, and we make a slight abuse of notations identifying a point in Ω with its coordinates in the rotated frame.

Since $f'(1^-) < 0$ and $\phi \to 1^-$ uniformly in $\mathcal{C}_+(y, \pi - \alpha)$ as $y \to +\infty$, one can then choose $Y_1 > 0$ such that :

$$\exists k > 0,\ \forall Y \geq Y_1, \quad f'(\phi(X, Y)) \leq -k.$$

Let Ω and S be the subsets of $I\!R^2_+$ defined by

$$\Omega = \{x > 0 \text{ and } (Y > Y_1 \text{ or } X > 1)\}, \tag{4.6}$$

$$S = \{X_1 \geq 1,\ 0 \leq Y \leq Y_1\}, \tag{4.7}$$

and let $\delta_0(\Omega) > 0$ satisfy (4.5).

Since $u \geq C_0\phi_x$ in $I\!R^2_+$, two cases may occur :

case 1 : $\inf_S(u - C_0\phi_x) = 0$. In that case, there exists a sequence (X_n, Y_n) (in the (X, Y)-frame) such that $u(X_n, Y_n) - C_0\phi_x(X_n, Y_n) \to 0$ as $n \to +\infty$.

Since the distance between S and $\partial I\!R^2_+ = \{x = 0\}$ is positive and $u > C_0\phi_x$ in $I\!R^2_+$, one concludes that $X_n \to +\infty$. Furthermore, since the sequence (Y_n) ranges in $[0, Y_1]$, one can assume, up to extraction of some subsequence, that $Y_n \to Y_\infty$ as $n \to +\infty$.

On the one hand, one has already mentionned the existence of $t_+ \in I\!R$ such that $\phi(x+\xi, y-|\xi|\cot\alpha) \to \phi_0(x\cos\alpha + y\sin\alpha + t_+)$ as $\xi \to +\infty$ in $C^2_{loc}(I\!R^2)$. Therefore, the functions $(X,Y) \mapsto \phi_x(X+X_n, Y+Y_n)$ locally converge to the function $\cos\alpha\ \phi_0'(Y+Y_\infty+t_+)$ as $n \to +\infty$.

On the other hand, from standard elliptic estimates, the functions $(X,Y) \mapsto u_n(X+X_n, Y+Y_n)$ locally converge, up to extraction of some subsequence, to a solution $u_\infty(X,Y)$ of

$$-\Delta u_\infty - c\cos\alpha\partial_X u_\infty + c_0\partial_Y u_\infty - f'(\phi_0(Y+Y_\infty+t_+))u_\infty = 0 \ \text{ in } I\!R^2.$$

Both functions u_∞ and $C_0\cos\alpha\ \phi_0'(Y+Y_\infty+t_+)$ satisfy the above equation, and

$$u_\infty(X,Y) \geq C_0\cos\alpha\ \phi_0'(Y+Y_\infty+t_+) \text{ in } I\!R^2$$

with equality at $(0,0)$. The strong maximum principle implies that $u_\infty(X,Y) \equiv C_0\cos\alpha\ \phi_0'(Y+Y_\infty+t_+)$ in $I\!R^2$.

But, since $u \in G_\rho$, one has, say, $u(X,0) \to 0$ as $X \to +\infty$. Hence, $u_\infty(0,-Y_\infty) = 0$, and $\phi_0'(t_+) = 0$ since $C_0\cos\alpha \neq 0$. But $\phi_0' > 0$ in $I\!R$. Therefore, case 1 is ruled out.

case 2 : $\inf_S(u - C_0\phi_x) > 0$. In that case, since ϕ_x is globally bounded, there exists $\eta_0 > 0$ such that $u \geq (C_0+\eta)\phi_x$ in S for all $\eta \in [0,\eta_0]$.

Choose now any δ such that $0 < \delta \leq \min(\delta_0(\Omega), \eta_0)$. One then has $u \geq (C_0+\delta)\phi_x$ in $S \cup (I\!R^2_+ \backslash \Omega)$. Let us now prove that the latter also holds in the two other parts of $I\!R^2_+$, namely in $\Omega_1 = \{x > 0,\ Y < 0,\ X > 1\}$ and $\Omega_2 = \{x > 0,\ Y > Y_1\}$.

Let us first deal with Ω_1. Notice that $f(\phi) = f'(\phi) = 0$ in $\mathcal{C}_-(0,\alpha)$ since $\phi(0,0) = \theta$ and ϕ is nonincreasing in any direction of this cone $\mathcal{C}_-(0,\alpha)$. Hence, $f'(\phi(X,Y)) = 0$ in Ω_1 and both u and $(C_0+\delta)\phi_x$ satisfy

$$-\Delta v - c\cos\alpha v_X + c_0 v_Y = 0 \text{ in } \Omega_1.$$

Furthermore, $u \geq (C_0+\delta)\phi_x$ on $\partial\Omega_1$. Lastly, remember that $|u| \leq C\phi_x$ in $I\!R^2$ from Lemma 4.4. Because of (1.2) and standard elliptic estimates, one can say that $\lim_{y\to-\infty}\sup_{\mathcal{C}_-(y,\alpha)}|\phi_x| = 0$. Hence, as ϕ_x does, $u(X,Y) \to 0$ uniformly as $Y \to -\infty$ with $(X,Y) \in \Omega_1$. Therefore, with a method similar to the proof of Proposition 3.1 (*see* also Lemma 5.1 in [19]), one can prove that $u \geq (C_0+\delta)\phi_x - \varepsilon$ in Ω_1 for all $\varepsilon > 0$, whence $u \geq (C_0+\delta)\phi_x$ in Ω_1.

Similarly, both u and $(C_0+\delta)\phi_x$ satisfy

$$-\Delta v - c\cos\alpha v_X + c_0 v_Y - f'(\phi)v = 0 \text{ in } \Omega_2, \tag{4.8}$$

with $f'(\phi(X,Y)) \leq 0$ in Ω_2. Furthermore, $u \geq (C_0+\delta)\phi_x$ on $\partial\Omega_2$. Lastly, $\lim_{y\to+\infty}\sup_{\mathcal{C}_+(y,\pi-\alpha)}|\phi_x| = 0$, whence $u(X,Y)$ and $\phi_x(X,Y) \to 0$ uniformly

as $Y \to +\infty$ with $(X,Y) \in \Omega_2$. Since $(C_0+\delta)\phi_x - \varepsilon$ is a subsolution of (4.8) for all $\varepsilon > 0$, it then follows similarly that $u \geq (C_0+\delta)\phi_x - \varepsilon$ in Ω_2 for all $\varepsilon > 0$, whence $u \geq (C_0+\delta)\phi_x$ in Ω_2.

As a conclusion, $u \geq (C_0+\delta)\phi_x$ in $I\!R^2_+$ for all $\delta \in [0, \min(\delta_0(\Omega), \eta_0)]$. This contradicts the definition of C_0.

Therefore, $C_0 \geq 0$ and $u \geq 0$ in $I\!R^2_+$. However we would prove in the same way that $u \leq 0$ in $I\!R^2_+$. This proves $u = 0$ in $I\!R^2_+$. □

Proposition 4.5 (No eigenfunctions with pure imaginary eigenvalue) *Let* $\rho > 0$ *and* $u \in C^2(I\!R^2, \mathbf{C})$ *such that* $\mathrm{Re}(u)$, $\mathrm{Im}(u) \in G_\rho$. *Assume that* $Lu = \lambda u$ *with* $\mathrm{Re}(\lambda) = 0$ *and* $\mathrm{Im}(\lambda) \neq 0$. *Then* $u = 0$.

Proof. The proof is a generalisation of the above proposition, combined with the parabolic maximum principle. Once again, we may assume that u is either odd or even; suppose it is even. If u is as described above, Proposition 4.2 applies, and we may define the infimum (maybe nonpositive) of all C such that $\mathrm{Re}(u) \leq C\phi_y$ in $\overline{I\!R^2_+}$ (or equivalently in $I\!R^2$ by evenness). Denote it by C_0.

We wish to prove that $C_0 \leq 0$, as is now usual. Assume by contradiction that $C_0 > 0$.

Set $\lambda = i\omega$, with $\omega \neq 0$. The function

$$U(t) = \mathrm{Re}(e^{-\lambda t}u) = \mathrm{Re}(u)\cos\omega t + \mathrm{Im}(u)\sin\omega t$$

solves (4.1) in $\overline{I\!R^2_+}$ together with Neumann boundary conditions on $\partial I\!R^2_+$ for all $t \in I\!R$, as does ϕ_y. Therefore, $U(t)(x,y) \leq C_0\phi_y(x,y)$ for all $(x,y) \in \overline{I\!R^2_+}$ and for all $t \geq 0$, whence for all $t \in I\!R$ since $U(t)$ is $2\pi/\omega$-periodic in t.

If there exists $(x_0,y_0) \in I\!R^2_+$ such that $\mathrm{Re}(u)(x_0,y_0) = C_0\phi_y(x_0,y_0)$, then $U(t) - C_0\phi_y$ has an interior minimum at $t = 0$ and (x_0,y_0). Hence, $U(t) \equiv C_0\phi_y$ in $I\!R^2_+$ for all $t \leq 0$, and thus $\mathrm{Re}(u) \equiv C_0\phi_y$. The latter is impossible since $\mathrm{Re}(u) \in G_\rho$ and $\phi_y \notin G_\rho$. Therefore, $\mathrm{Re}(u) < C_0\phi_y$ in $I\!R^2_+$. Similarly, the parabolic Hopf lemma then implies that $\mathrm{Re}(u) < C_0\phi_y$ in $\partial I\!R^2_+$.

Under the notations in the proof of Proposition 4.3, let S' be the strip

$$S' = \{x \geq 0,\ 0 \leq Y \leq Y_1\}.$$

Since $\mathrm{Re}(u) \leq C_0\phi_y$ (in $I\!R^2$), two cases may occur :

case 1 : $\sup_{S'}(\mathrm{Re}(u) - C_0\phi_y) = 0$. Since $\mathrm{Re}(u) < C_0\phi_y$ in $\overline{I\!R^2_+}$, there exists then a sequence of points $(X_n,Y_n) \in S'$ such that $X_n \to +\infty$, $Y_n \to Y_\infty \in I\!R$ and $\mathrm{Re}(u)(X_n,Y_n) - C_0\phi_y(X_n,Y_n) \to 0$ as $n \to +\infty$. From standard elliptic estimates, the functions $(X,Y) \mapsto \mathrm{Re}(u)(X+X_n, Y+Y_n)$ and $(X,Y) \mapsto \mathrm{Im}(u)(X+X_n, Y+Y_n)$ converge, up to extraction of some subsequence, to two real-valued bounded functions $v_\infty(X,Y)$ and $w_\infty(X,Y)$ solving

$$L_\infty v_\infty = -\omega w_\infty \text{ and } L_\infty w_\infty = \omega v_\infty \ \text{ in } I\!R^2,$$

where $L_\infty = -\Delta - c\cos\alpha\partial_X + c_0\partial_Y - f'(\phi_0(Y+Y_\infty+t_+))$. Therefore, the function $u_\infty = v_\infty + iw_\infty$ solves $L_\infty u_\infty = \lambda u_\infty$.

On the other hand, one recalls that the functions $(X,Y) \mapsto \phi_y(X+X_n, Y+Y_n)$ locally converge to the function $\sin\alpha\ \phi_0'(Y+Y_\infty+t_+)$ as $n \to +\infty$.

Furthermore, $\mathrm{Re}(u_\infty) = v_\infty \le C_0 \sin\alpha\ \phi_0'(Y+Y_\infty+t_+)$ in $I\!R^2$ with equality at $(0,0)$, and both functions $\mathrm{Re}(e^{-\lambda t}u_\infty)$ and $C_0\sin\alpha\ \phi_0'(Y+Y_\infty+t_+)$ solve (4.1) with the operator L_∞ instead of L. As done several lines above, one then concludes that $v_\infty = \mathrm{Re}(u_\infty) \equiv C_0\sin\alpha\ \phi_0'(Y+Y_\infty+t_+)$. But $\mathrm{Re}(u) \in G_\rho$, whence, say, $\mathrm{Re}(u)(X,0) \to 0$ as $X \to +\infty$ and $v_\infty(0,-Y_\infty) = 0$. Thus $\phi_0'(t_+) = 0$ since $C_0\sin\alpha \neq 0$. One gets a contradition and case 1 is ruled out.

case 2 : $\sup_{S'}(\mathrm{Re}(u) - C_0\phi_y) < 0$. In that case, there exists $\eta_0 > 0$ such that $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ in S' for all $\delta \in [0,\eta_0]$. Choose any δ such that $0 \le \delta \le \eta_0$. Let $\Omega_1' = \{x \ge 0,\ Y \le 0\}$ and $\Omega_2' = \{x \ge 0,\ Y \ge Y_1\}$, and let us prove that $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ in $\Omega_1' \cup \Omega_2'$, which would yield that $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ in $I\!R^2_+$ and would contradict the minimality of C_0.

Let us first deal with Ω_1'. Since $f'(\phi) = 0$ in $\mathcal{C}_-(0,\alpha)$, both even (in x) functions $U(t)(x,y)$ and $(C_0-\delta)\phi_y(x,y)$ satisfy

$$v_t - \Delta v + c\partial_y v = 0 \text{ for all } (t,x,y) \in I\!R \times \mathcal{C}_-(0,\alpha),$$

and $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ on $\partial\mathcal{C}_-(0,\alpha)$. Let ε^* be the smallest nonnegative ε such that $\mathrm{Re}(u) \le (C_0-\delta)\phi_y + \varepsilon$ in $\mathcal{C}_-(0,\alpha)$. Assume $\varepsilon^* > 0$. Since $|u| \le C\phi_y$, one knows that $\lim_{y\to-\infty}\sup_{\mathcal{C}_-(y,\alpha)}|\mathrm{Re}(u) - (C_0-\delta)\phi_y| = 0$. Therefore, we may assume the existence of some sequence $\varepsilon_n \to \varepsilon^*$ and $(X_n,Y_n) \in \mathcal{C}_-(0,\alpha)$ such that $X_n \to +\infty$ and $Y_n \to Y_\infty < 0$ such that

$$\mathrm{Re}(u)(X_n,Y_n) - (C_0-\delta)\phi_y(X_n,Y_n) - \varepsilon_n \to 0 \ \text{ as } n \to +\infty.$$

Arguing as in case 1 above and in the proof of Proposition 3.1, one then gets a contradiction with the positivity of ε^*.

Therefore, $\varepsilon^* = 0$ and $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ in $\mathcal{C}_-(0,\alpha)$.

Similarly, using the fact that $f'(\phi) \le 0$ in $\{(x,y),\ (x,y) \in \Omega_2' \text{ or } (-x,y) \in \Omega_2'\} = \mathcal{C}_+(Y_1/\sin\alpha, \pi-\alpha)$, one can prove that

$$\mathrm{Re}(u) \le (C_0-\delta)\phi_y \ \text{ in } \mathcal{C}_+(Y_1/\sin\alpha, \pi-\alpha).$$

Eventually, $\mathrm{Re}(u) \le (C_0-\delta)\phi_y$ in $I\!R^2$ for all $\delta > 0$ small enough. This contradicts the definition of C_0.

Therefore, $C_0 \le 0$ and $\mathrm{Re}(u) \le 0$. We may prove that $\mathrm{Re}(u) \ge 0$ in the same fashion, which implies $\mathrm{Re}(u) = 0$, and then $u = 0$ since $Lu = i\omega u$. □

Proof of Theorem 4.1 It remains to prove that L is a Fredholm operator; namely

$$L = T + K, \quad \mathrm{Re}(\sigma(T)) \ge \beta \text{ for some } \beta > 0,\ KT^{-1} \text{ compact.}$$

To do so, we wish to find a weight function $p(x,y)$ such that the operator M, defined by

$$L = pM(pI)$$

is a second order elliptic operator whose zero-order coefficient is positive and bounded away from 0 outside a compact subset. A natural choice would be - at least in the right half-space -

$$p(x,y) = e^{-\rho X}\phi_0'(Y)$$

where $\rho > 0$ is small and X, Y are the rotated coordinates. Such a choice would almost work, up to the fact that we are here asking too much decay at infinity. Therefore the weight will have to be slightly modified, in order to keep only the decay that is asked to functions belonging to G_ρ.

In the sequel of the proof of Theorem 4.1, we fix $\rho \in (0, c_0 \cot\alpha)$, and call

$$\lambda = \frac{\rho(c_0\cot\alpha - \rho)}{4} > 0.$$

Step 1 (an auxiliary function). Let ϕ_0 be the 1D wave, and let the real number t_+ be chosen so that $\phi(x+x_n, y-|x_n|\cot\alpha) \to \phi_0(\pm x\cos\alpha + y\sin\alpha + t_+)$ as $x_n \to \pm\infty$ (we recall that ϕ is assumed to be even in the variable x). Set

$$L_0 = -\partial_Y^2 + c_0\partial_Y - f'(\phi_0(.+t_+)).$$

It is known that $\phi_0' > 0$ in $I\!R$, $\phi_0''(s)/\phi_0'(s) \to c_0$ (resp. $\to -\mu$) as $s \to -\infty$ (resp. as $s \to +\infty$), and $\phi_0'''(s)/\phi_0'(s) \to c_0^2$ (resp. $\to \mu^2$) as $s \to -\infty$ (resp. as $s \to +\infty$), where $\mu = (c_0 + \sqrt{c_0^2 - 4f'(1^-)})/2$.

Lastly, one has that $f'(\phi_0(s)) = 0$ for $-s$ large enough, and $f'(\phi_0(s)) \to f'(1^-)$ as $s \to +\infty$. Therefore, there exist $A > 0$ and a function ψ of class C^2 such that

$$\begin{cases} \forall |Y| \le A, & \psi(Y) = \phi_0'(Y+t_+), \\ \forall |Y| \ge 2A, & \psi'(Y) = 0, \\ \forall -2A \le Y \le -A, & -\dfrac{\psi''(Y)}{\psi(Y)} + c_0\dfrac{\psi'(Y)}{\psi(Y)} - f'(\phi_0(Y+t_+)) \ge -\lambda, \\ \forall A \le Y \le 2A, & -\dfrac{\psi''(Y)}{\psi(Y)} + c_0\dfrac{\psi'(Y)}{\psi(Y)} - f'(\phi_0(Y+t_+)) \ge -\lambda, \\ \min_{I\!R}\psi > 0. \end{cases}$$

The existence of such a function ψ can be obtained through a slight perturbation of the exponential tails of ϕ_0'. We call C_0 a positive constant such that

$$\|\frac{\psi'}{\psi}\|_{L^\infty(I\!R)} + \|\frac{\psi''}{\psi}\|_{L^\infty(I\!R)} \le C_0.$$

We also choose two C^∞ functions k_1 and $k_2 : I\!R \to I\!R$ such that $0 \le k_1, k_2 \le 1$, $k_1 = k_2 = 1$ on $[-A, A]$ and $k_1 = k_2 = 0$ outside $[-2A, 2A]$.

Step 2 (construction of T). We next choose a C^∞ convex function $h : I\!R \to I\!R$ such that $h(x) = |x|$ for $|x|$ large enough, and

$$0 \le h'' \le \lambda/(C_0\cos\alpha), \tag{4.9}$$

where C_0 is as above. Notice that the above properties especially imply that $|h'| \leq 1$.

We set, *only in this particular step 2:*

$$\begin{aligned} X = &\ h(x)\sin\alpha - y\cos\alpha \\ Y = &\ h(x)\cos\alpha + y\sin\alpha. \end{aligned}$$

Since $f'(\phi_0(Y+t_+)) - f'(\phi(x,y)) \to 0$ uniformly as $|(x,y)| \to +\infty$, and

$$\lim_{y_0\to+\infty} \sup_{y\geq y_0-|x|\cot\alpha} |f'(\phi(x,y)) - f'(1^-)| = \lim_{y_0\to-\infty} \sup_{y\leq y_0-|x|\cot\alpha} |f'(\phi(x,y))| = 0,$$

one may, without loss of generality, choose A large enough so that

$$[f'(\phi_0(Y+t_+))(1-k_1(x)) - f'(\phi(x,y))](1-k_1(x)k_2(Y)) \geq -\lambda \tag{4.10}$$

for all $(x,y) \in I\!R^2$.

We finally set

$$u(x,y) = p(x,y)v(x,y),$$

where

$$p(x,y) = e^{-\rho X}\psi(Y).$$

Let us write, for every C^2 function u:

$$Lu(x,y) = p(x,y)Mv(x,y).$$

The operator M has the form

$$M = -\Delta + B(x,y).\nabla + \frac{Lp}{p},$$

where $B = -2\nabla p/p + (0,c)$ is a C^1 bounded vector-valued function.

Let us now evaluate Lp. Using (1.5) and the fact that $c = c_0/\sin\alpha$, a straigthforward calculation gives :

$$\frac{Lp}{p} = a(x,y) + b(x,y),$$

where

$$\left\{ \begin{aligned} a(x,y) &= c_0\rho\cot\alpha - \rho^2\sin^2\alpha\ h'^2(x) \\ &\quad + \left[-\frac{\psi''(Y)}{\psi(Y)} + c_0\frac{\psi''(Y)}{\psi(Y)} - f'(\phi_0(Y+t_+))\right](1-k_1(x)) \\ &\quad + [f'(\phi_0(Y+t_+))(1-k_1(x)) - f'(\phi(x,y))](1-k_1(x)k_2(Y)) \\ &\quad + \left(\rho\sin\alpha - \cos\alpha\frac{\psi'(Y)}{\psi(Y)}\right)h''(x), \\ b(x,y) &= \left[-\frac{\psi''(Y)}{\psi(Y)} + c_0\frac{\psi''(Y)}{\psi(Y)}\right]k_1(x) \\ &\quad + \cos\alpha\left[\cos\alpha\frac{\psi''(Y)}{\psi(Y)} - 2\rho\sin\alpha\frac{\psi'(Y)}{\psi(Y)}\right](1-h'^2(x)) \\ &\quad + [f'(\phi_0(Y+t_+))(1-k_1(x)) - f'(\phi(x,y))]k_1(x)k_2(Y). \end{aligned} \right.$$

The function a is clearly continuous and bounded in $I\!R^2$. Let us now estimate it from below. Assume that $A > 0$ is large enough so that

$$f'(\phi(s+t_+)) = 0 \text{ for } s \leq -2A, \quad -f'(\phi(s+t_+)) \geq -f'(1^-)/2 \geq 0 \text{ for } s \geq 2A.$$

It follows then from the choice of ψ that

$$\left[-\frac{\psi''(Y)}{\psi(Y)} + c_0\frac{\psi''(Y)}{\psi(Y)} - f'(\phi_0(Y+t_+))\right](1-k_1(x)) \geq -\lambda(1-k_1(x)) \geq -\lambda$$

for all $(x,y) \in I\!R^2$. Putting the above estimate together with (4.10), (4.9) and the fact that $|h'| \leq 1$, it follows that

$$\forall (x,y) \in I\!R^2, \quad a(x,y) \geq \rho(c_0 \cot\alpha - \rho) - 3\lambda = \lambda > 0.$$

On the other hand, it follows from the choices of h, k_1 and k_2 that the function b is continuous with compact support in $I\!R^2$.

Set

$$\tilde{M} = -\Delta + B(x,y).\nabla + a(x,y)$$

and

$$Tu = p\tilde{M}(\frac{u}{p}); \quad Ku = Lu - Tu = b(x,y)u.$$

<u>Step 3</u>. Let us prove the existence of $\beta > 0$ such that $\mathrm{Re}(\sigma(T)) \geq \beta$. For this we estimate $\|e^{-tT}\|_{\mathcal{L}(G_\rho)}$. Let $u_0 \in G_\rho$. We have

$$e^{-tT}u_0 = pe^{-t\tilde{M}}(\frac{u_0}{p})$$

and the maximum principle yields

$$\|p^{-1}e^{-tT}u_0\|_\infty \leq e^{-\lambda t}\|p^{-1}u_0\|_\infty \leq Ce^{-\lambda t}\|q^{-1}u_0\|_\infty, \tag{4.11}$$

where q was defined in (1.9) and C is a constant which does not depend on u_0. We here use the definition of G_ρ, and the fact that the function ψ is bounded from below by a positive constant. Furthermore, since ψ is bounded from above, one infers that

$$\|q^{-1}e^{-tT}u_0\|_\infty \leq C'\|p^{-1}e^{-tT}u_0\|_\infty$$

for some constant $C' > 0$. Hence,

$$\|q^{-1}e^{-tT}u_0\|_\infty \leq CC'e^{-\lambda t}\|q^{-1}u_0\|_\infty.$$

On the other hand, let us choose $B > 0$ large enough so that $b = 0$ and $f'(\phi) \leq f'(1^-)/2 < 0$ in $\mathcal{C}_+(B, \pi - \alpha)$. Observe now that the function q is bounded in $\mathcal{C}_-(B,\alpha)$. Therefore, there exists a constant C'' (independent of u_0) such that

$$\|e^{-tT}u_0\|_{L^\infty(\mathcal{C}_-(B,\alpha))} \leq C''\|q^{-1}e^{-tT}u_0\|_{L^\infty(\mathcal{C}_-(B,\alpha))} \leq CC'C''e^{-\lambda t}\|q^{-1}u_0\|_\infty.$$

Lastly, the function $u(t) := e^{-tT}u_0$ satisfies

$$\begin{cases} u_t - \Delta u + cu_y - f'(\phi)u &= 0 & \text{in } \mathcal{C}_+(B, \pi-\alpha) \\ |u(t,x,y)| &\leq CC'C''e^{-\lambda t}\|q^{-1}u_0\|_\infty & \text{on } \partial\mathcal{C}_+(B,\pi-\alpha). \end{cases}$$

Hence we have

$$\|e^{-tT}u_0\|_{L^\infty(\mathcal{C}_+(B,\pi-\alpha))} \leq (1+CC'C'')e^{-\beta t}(\|u_0\|_{L^\infty(\mathcal{C}_+(B,\pi-\alpha))} + \|q^{-1}u_0\|_\infty),$$

where $\beta = \min(\lambda, -f'(1^-)/2) > 0$.

Summing up, one gets that

$$\|e^{-tT}u_0\|_{G_\rho} \leq \tilde{C}e^{-\beta t}\|u_0\|_{G_\rho}$$

for some constant $\tilde{C}$. Therefore - by a standard Laplace transform argument - the spectrum of L satisfies $\mathrm{Re}(\sigma(T)) \geq \beta$.

Step 4 (conclusion). For every λ such that $\mathrm{Re}(\lambda) \in (-\infty, \beta)$, the operator $T - \lambda I$ is an isomorphism of a dense subspace of G_ρ onto G_ρ, and $K(T-\lambda I)^{-1}$ is compact. Moreover, L is sectorial in G_ρ (Stewart [39]).

Combining these considerations with Propositions 4.2-4.5, we obtain the existence of a cone with aperture less than $\pi/2$ and positive vertex containing the spectrum of L - see [29] for more details. Classical stability results [22] apply subsequently. □

5 Convergence to a single wave

The aim of this section is to prove Theorem 1.6 and 1.7. In this section we keep the notations of the preceding section. In particular, we use the rotated coordinate system (X, Y). We will have to investigate the behaviour of different functions as the space variable becomes infinite along the directions $e_\alpha = (\sin\alpha, -\cos\alpha)$ and $e'_\alpha = (-\sin\alpha, -\cos\alpha)$. Only the direction e_α will be investigated, the case of e'_α being similar.

The first result that we need is another Liouville type property.

Proposition 5.1 *Let $v(t,X,Y)$ ranging in $[0,1]$ be a classical solution of*

$$\begin{array}{rcl} v_t - \Delta v - c_0\cot\alpha\ v_X + c_0 v_Y &=& f(v), \quad (t,X,Y) \in I\!R^3 \\ \limsup\limits_{Y\to-\infty,\ (t,X)\in I\!R^2} v(t,X,Y) &=& 0 \\ \liminf\limits_{Y\to+\infty,\ (t,X)\in I\!R^2} v(t,X,Y) &>& \theta. \end{array} \tag{5.1}$$

Then there exists $Y_0 \in I\!R$ such that $v(t,X,Y) = \phi_0(Y+Y_0)$ for all $(t,X,Y) \in I\!R^3$.

Proof. The first part of the proof consists in observing that there exists $Y_1 \in I\!R$ and $\eta \in (\theta, 1]$ such that $v(t, X, Y) \geq H(Y + Y_1)$ for all $(t, X, Y) \in I\!R^3$, where $H(s) = 0$ if $s < 0$ and $H(s) = \eta$ if $s \geq 0$. Therefore, arguing as in the proof of Theorem 1.5, one gets the existence of $Y_2 \in I\!R$ such that $v(t, X, Y) \geq \phi_0(Y + Y_2)$ for all $(t, X, Y) \in I\!R^3$.

The second part of the proof is identical to the proof of Theorem 1.3. □

Let u_0 and ϕ satisfy the assumptions of Theorem 1.6, and let $u(t, x, y)$ be the solution of (1.3). Up to a same shift of both u_0 and ϕ, we may assume without loss of generality that $\phi(0, 0) = \theta$ and ϕ is even in x. A standard argument from local existence theory for nonlinear parabolic equations - *see* [22], Chapter 3, and [1] - would yield the exponential spatial convergence of the solution u under investigation to a 1D wave in the X direction, locally in Y. Notice especially that, for the function ϕ, there exists $t_+ \in I\!R$ such that, for all $K > 0$, there are $C_K > 0$ and $\lambda_K > 0$ such that

$$\forall X \geq 0,\ \forall |Y| \leq K,\ \ |\phi(X, Y) - \phi_0(Y + t_+)| \leq C_K e^{-\lambda_K X}. \tag{5.2}$$

The same type of property holds in the left plane $\{x < 0\}$.

As far as the function $u(t, x, y)$ is concerned, such an exponential decay is *a priori* not uniform in time, and our point is that this convergence is indeed uniform in time. This is the goal of the next proposition.

Proposition 5.2 *Under the assumptions of Theorem 1.6, there are constants $C > 0$, $t_0 > 0$ and $\rho_1 > 0$, such that*

$$|\partial_{e_\alpha} u(t, x, y - ct)| \leq C e^{-\rho_1 X + c_0 Y/2}, \quad |\partial_{e_\alpha} u(t, -x, y - ct)| \leq C e^{-\rho_1 X + c_0 Y/2}$$

for all $t \geq t_0$ and $(x, y) \in I\!R^2_+$, where $X = x \sin\alpha - y \cos\alpha$, $Y = x \cos\alpha + y \sin\alpha$.

Proof. It is divided into several steps.

Step 1 (estimates for u). Set $\overline{u}(t, x, y) = u(t, x, y - ct)$. The function $\overline{u}$ satisfies $\overline{u}_t = \Delta\overline{u} - c\overline{u}_y + f(\overline{u})$ and $0 \leq \overline{u}(t, x, y) \leq \tau_{a,b}\phi(x, y)$ for all $t \geq 0$ and $(x, y) \in I\!R^2$. Since both functions ϕ and

$$w(x, y) = e^{c_0(x \cos\alpha + y \sin\alpha)} + e^{c_0(-x \cos\alpha + y \sin\alpha)}$$

satisfy $\Delta v - c v_y = 0$ in $\mathcal{C}_-(0, \alpha)$, together with $\lim_{y \to -\infty} \sup_{\mathcal{C}_-(y,\alpha)} \phi$ (resp. w) $= 0$ and $\phi \leq \theta \leq w$ on $\partial\mathcal{C}_-(0, \alpha)$, it follows from Lemma 5.1 in [19] that

$$\phi \leq w \ \text{ in } \mathcal{C}_-(0, \alpha).$$

On the other hand, $\nabla_{x,y}\overline{u}$, as well as $\partial_t \nabla_{x,y}\overline{u}$ and the spatial derivatives of $\nabla_{x,y}\overline{u}$ up to the second order, are globally bounded for all $t \geq 1$ and $(x, y) \in I\!R^2$. Furthermore, the function w is bounded from below by a positive constant

in any strip of the type $\mathcal{C}_+(B,\pi-\alpha)\backslash\mathcal{C}_-(A,\alpha)$ for each $A<B$. Standard parabolic estimates then imply that there exist some constants C_1 and $C_1'(y_0)$ such that

$$(5.3)\quad \begin{array}{l} |\nabla_{x,y}\overline{u}(t,x,y)|+|\partial_t\nabla_{x,y}\overline{u}(t,x,y)|+|D^2\overline{u}(t,x,y)|+|D^3\overline{u}(t,x,y)| \\ \leq \left\{\begin{array}{ll} C_1'(y_0)\left(e^{c_0(x\cos\alpha+y\sin\alpha)}+e^{c_0(-x\cos\alpha+y\sin\alpha)}\right) & \text{in } \mathcal{C}_-(y_0,\alpha) \\ C_1 & \text{in } I\!R^2 \end{array}\right. \end{array}$$

and for all $t\geq 1$, where $|D^2\overline{u}|$ and $|D^3\overline{u}|$ respectively mean the maximum of the absolute values of the second order (resp. third order) spatial derivatives of $\overline{u}$.

Step 2 (estimates for ϕ_X). First of all, it follows from (5.2) and standard elliptic estimates that

$$|\phi_X|\leq C_2e^{-\lambda X}\ \text{ on } \{y=-x\cot\alpha,\ x\geq 0\}=\{X\geq 0,\ Y=0\}$$

for some $\lambda>0$. Similar estimates as (5.3) obviously hold for the derivatives of ϕ in $\mathcal{C}_-(0,\alpha)$. Therefore, even if it means increasing $C_2>0$, decreasing $\lambda>0$, one can assume that

$$|\phi_X|\leq C_2e^{-\lambda X+\frac{3c_0}{4}(x\cos\alpha+y\sin\alpha)}=:v(x,y)\ \text{ on } \partial\left(\mathcal{C}_-(0,\alpha)\cap I\!R^2_+\right).$$

A direct calculation shows that v satisfies $\Delta v-cv_y\leq 0$ in $\mathcal{C}_-(0,\alpha)\cap I\!R^2_+$, as soon as $\lambda^2-c_0\lambda\cot\alpha\leq 3c_0^2/16$, which can always be assumed even if it means decreasing λ. Since $\limsup_{dist((x,y),\partial(\mathcal{C}_-(0,\alpha)\cap I\!R^2_+))\to+\infty,\ (x,y)\in\mathcal{C}_-(0,\alpha)\cap I\!R^2_+}|\phi_X(x,y)|=0$ and $v\geq 0$, it follows therefore from the proof of Lemma 5.1 in [19][4] that $|\phi_X|\leq v$ in $\mathcal{C}_-(0,\alpha)\cap I\!R^2_+$. In other words,

$$(5.4)\qquad |\phi_X(x,y)|\leq C_2e^{-\lambda X+\frac{3c_0}{4}Y}\ \text{ in } \mathcal{C}_-(0,\alpha)\cap I\!R^2_+.$$

On the other hand, because of (1.2) and since $f'(1^-)<0$, there exists y_1 such that $f'(\phi)\leq f'(1^-)/2<0$ in $\mathcal{C}_+(y_1,\pi-\alpha)$. Even if it means decreasing $\lambda>0$, the function $\zeta:=e^{-\lambda X+\lambda Y}$ satisfies

$$\Delta\zeta-c\partial_y\zeta+f'(\phi)\zeta\leq 0\ \text{ in } \mathcal{C}_+(y_1,\pi-\alpha),$$

while $|\phi_X|\leq C_3\zeta$ on $\partial(\mathcal{C}_+(y_1,\pi-\alpha)\cap I\!R^2_+)$ for some constant C_3. Since $\lim_{y\to+\infty}\sup_{\mathcal{C}_+(y,\pi-\alpha)}|\phi_X|=0$, it follows from the proof of Lemma 5.1 in [19] that

$$(5.5)\qquad |\phi_X|\leq C_3\zeta=C_3e^{-\lambda X+\lambda Y}\ \text{ in } \mathcal{C}_+(y_1,\pi-\alpha)\cap I\!R^2_+.$$

Step 3 (estimates for $\overline{u}_X-\phi_X$). The function $z(t,x,y):=\overline{u}(t,x,y)-\phi(x,y)$ satisfies an equation of the type

$$\partial_tz-\Delta z+c\partial_yz+\gamma(t,x,y)z=0,$$

[4]The proof can easily be adapted to our situation, the boundary of $\mathcal{C}_-(0,\alpha)\cap I\!R^2_+$ being a Lipschitz graph in a rotated frame.

where γ is bounded and $\|\gamma\|_{L^\infty((0,+\infty)\times I\!R^2)} \leq \|f\|_{Lip}$ ($\|f\|_{Lip}$ denotes the Lipschitz norm of f). Choose now any direction ν of $I\!R^2$ such that $|\nu| = 1$. It follows from the assumptions of Theorem 1.6 that $|z(0,x,y)| \leq C_0 e^{-\rho_0 \nu\cdot(x,y)}$ in $I\!R^2$. Let $\omega_0 = \rho_0^2 + c\rho_0 + \|f\|_{Lip}$. It is easy to check that the function $\kappa(t,x,y) := C_0 e^{\omega_0 t - \rho_0 \nu\cdot(x,y)}$ satisfies

$$\partial_t \kappa - \Delta\kappa + c\partial_y \kappa - \|f\|_{Lip}\ \kappa \geq 0,$$

together with $\kappa(0,x,y) \geq |z(0,x,y)|$ in $I\!R^2$. The maximum principle then yields that $|z(t,x,y)| \leq \kappa(t,x,y)$, whence

$$|\overline{u}(t,x,y) - \phi(x,y)| \leq C_0 e^{\omega_0 t - \rho_0 \nu\cdot(x,y)}$$

for all $t \geq 0$ and $(x,y) \in I\!R^2$. Since the above estimate holds for all $\nu \in I\!R^2$ with $|\nu| = 1$, one concludes that

$$|\overline{u}(t,x,y) - \phi(x,y)| \leq C_0 e^{\omega_0 t - \rho_0 \sqrt{x^2+y^2}} \tag{5.6}$$

for all $t \geq 0$ and $(x,y) \in I\!R^2$.

Standard parabolic estimates then imply that

$$|\overline{u}_X(t,x,y) - \phi_X(x,y)| \leq C_4 e^{\omega_0 t - \rho_0 \sqrt{x^2+y^2}} \tag{5.7}$$

for all $t \geq 1$ and $(x,y) \in I\!R^2$, for some constant C_4.

Furthermore, estimates of the type (5.3) also hold by replacing $\overline{u}$ with ϕ (take $u_0 = \phi$ as the initial condition). Therefore,

$$|\overline{u}_X(t,x,y) - \phi_X(x,y)| \leq C_4'(e^{c_0(x\cos\alpha + y\sin\alpha)} + e^{c_0(-x\cos\alpha + y\sin\alpha)}) \tag{5.8}$$

in $\mathcal{C}_-(0,\alpha)$, for all $t \geq 1$, and for some constant C_4'.

Step 4 (auxiliary functions and definition of a set Ω'). Choose now some positive coefficients ρ_1, ρ_2 and c_1 such that $0 < \rho_2 < \rho_1$, $0 < c_1 < c_0$ and $2(\rho_1 + \rho_2) < (c_0 - c_1)\tan\alpha$. Consider the function

$$v_1(t,x,y) = e^{\rho_1 X - c_0 Y/2}\overline{u}_X(t,x,y)$$

defined for all $t > 0$ and $(x,y) \in I\!R^2$, and let Ω' be the set defined by

$$\Omega' = \{(x,y) \in I\!R^2,\ x > -1 \text{ and } (X > X_1 \text{ or } Y > Y_1)\},$$

where $X_1 > 0$ and $Y_1 > 0$ shall be chosen below.

From the above upper bounds for $|\nabla_{x,y}\overline{u}|$ given in Step 1, it is straightforward to check that there is a constant $C_5 = C_5(X_1, Y_1) > 0$ such that

$$\begin{aligned} |v_1(t,x,y)| + |\partial_t v_1(t,x,y)| + |\nabla_{x,y} v_1(t,x,y)| & \\ + |D^2 v_1(t,x,y)| \quad &\leq \quad C_5 e^{-\rho_2|X| - c_1|Y|/2} \end{aligned} \tag{5.9}$$

for all $t \geq 1$ and $(x,y) \in \partial\Omega'$ (remember that the quantity x is bounded on $\partial\Omega'$). Note that (5.9) is not optimal when Y (or y) becomes positive and large; all we need, however, is an integrability condition for v_1.

Set $\psi(t,x,y) = v_1(t,x,y)$ for all $t \geq 1$ and $(x,y) \in \partial\Omega'$ and extend ψ in $[1,+\infty) \times \overline{\Omega'}$ by a C^2 function, still denoted by ψ, such that ψ, as well as ψ_t and the space derivatives of ψ up to the second order, are bounded by $C_6 e^{-\rho_2|X|-c_1|Y|/2}$ in $[1,+\infty) \times \overline{\Omega'}$ for some constant C_6. Finally set, for all $t \geq 1$ and $(x,y) \in \overline{\Omega'}$,

$$v(t,x,y) = v_1(t,x,y) - \psi(t,x,y).$$

Step 5 ($v(t,\cdot,\cdot) \in L^2(\Omega')$ for each $t \geq 1$). First, the function $\psi(t,\cdot,\cdot)$ is in $L^2(\Omega')$ by construction. Write now

$$v_1(t,x,y) = v_2(x,y) + v_3(t,x,y),$$

where $v_2(x,y) = e^{\rho_1 X - c_0 Y/2}\phi_X(x,y)$ and $v_3(t,x,y) = e^{\rho_1 X - c_0 Y/2}(\overline{u}_X(t,x,y) - \phi_X(x,y))$.

The function v_2 is in $L^2(\Omega' \cap \mathcal{C}_-(0,\alpha))$ because of (5.4), even if it means decreasing ρ_1 so that

$$0 < \rho_1 < \lambda. \tag{5.10}$$

One has $v_2 \in L^2(\Omega' \cap \{0 \leq Y \leq y_1/\sin\alpha\})$ because of (5.2) and (5.10). On the other hand, $v_2 \in L^2(\Omega' \cap \{Y \geq y_1/\sin\alpha,\ X \geq 0\})$ because of (5.5), (5.10), and even if it means decreasing λ so that $0 < \lambda < c_0/2$. Lastly, $v_2 \in L^2(\Omega' \cap \{X \leq 0\})$ because ϕ_X is globally bounded. Therefore, $v_2 \in L^2(\Omega')$.

Fix now a real number $\beta > 0$ such that $\rho_0 > c_0\beta/2$ and $\beta < \tan\alpha$. Let $t \geq 1$. The function $v_3(t,\cdot,\cdot)$ is in $L^2(\Omega' \cap \{Y \leq -\beta X\})$ because of (5.8), even if it means decreasing ρ_1 so that $0 < \rho_1 < c_0\beta/2$. The function $v_3(t,\cdot,\cdot)$ is in $L^2(\Omega' \cap \{|Y| \leq \beta X\})$ because of (5.7), even if it means decreasing ρ_1 so that $0 < \rho_1 < \rho_0 - c_0\beta/2$. On the other hand, $v_3(t,\cdot,\cdot) \in L^2(\Omega' \cap \{Y \geq \beta X,\ X \geq 0\})$ because $u_X - \phi_X$ is globally bounded in $L^\infty([1,+\infty) \times I\!R^2)$ and because $0 < \rho_1 < c_0\beta/2$. Lastly, $v_3(t,\cdot,\cdot) \in L^2(\Omega' \cap \{X \leq 0\})$ because $u_X - \phi_X$ is globally bounded in $L^\infty([1,+\infty) \times I\!R^2)$. Therefore, $v_3(t,\cdot,\cdot) \in L^2(\Omega')$ for each $t \geq 1$.

One concludes that $v(t,\cdot,\cdot) \in L^2(\Omega')$ for each $t \geq 1$.

Step 6 (integration by parts over Ω'). Multiply the equation for v by v; integrate by parts over Ω'. We get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\Omega'} v^2 \;=\; & \underbrace{-\int_{\Omega'}\Big(v_X^2 + (c_0\cot\alpha\ \rho_1 - \rho_1^2)v^2\Big)}_{I}\ \underbrace{-\int_{\Omega'}\Big(v_Y^2 - (f'(\overline{u}) + \frac{c_0^2}{4})v^2\Big)}_{II} \\ & \underbrace{-\int_{\Omega'}(P\psi)v}_{III}\end{aligned}$$

where P is a parabolic operator with bounded coefficients. Let us analyse these three terms.

The term I is the one that will control the estimate of v. We may obviously estimate it by

$$I \leq -(c_0 \cot\alpha \ \rho_1 - \rho_1^2) \int_{\Omega'} v^2, \tag{5.11}$$

and, even if it means decreasing both ρ_1 and ρ_2, we may assume that

$$0 < \rho_1 < c_0 \cot\alpha.$$

By assumption, $|u_0(x,y) - \phi(x,y)| = O(e^{-\rho_0\sqrt{x^2+y^2}})$ as $x^2+y^2 \to +\infty$, whence $\lim_{y\to+\infty} \inf_{\mathcal{C}_-(y,\pi-\alpha)} u_0 = 1 > \theta$. From the proof of Theorem 1.5, there exist two functions $V^{\pm}(t,s)$ such that

$$\overline{u}(t,x,y) \geq \max(V^{\pm}(t, \pm x\cos\alpha + y\sin\alpha))$$

for all $t \geq 0$ and $(x,y) \in I\!R^2$, where $V^{\pm}(t,s) - \phi_0(s+s_1) \to 0$ uniformly in $s \in I\!R$ as $t \to +\infty$. In particular, it also follows that, for any $\varepsilon > 0$, one has $\overline{u}(t,x,y) \geq 1-\varepsilon$ as soon as Y and t are large enough, uniformly in $X \in I\!R$. On the other hand, $u_0 \leq \tau_{a,b}\phi$ implies that $\overline{u}(t,x,y) \leq \tau_{a,b}\phi(x,y)$ for all $t \geq 0$ and $(x,y) \in I\!R^2$. From Proposition 5.1, there is then a bounded function $t \mapsto Y_t$, defined for t large enough, such that

$$\lim_{X\to+\infty} \lim_{t\to+\infty} |\overline{u}(t,X,Y) - \phi_0(Y+Y_t)| = 0$$

uniformly in $Y \in I\!R$ (under the restriction that $x > -1$) - argue by contradiction.

As a consequence, there exist $X_1 > 0$ and $Y_1 > 0$ in the definition of Ω' so that

$$|f'(\overline{u}(t,x,y)) - f'(\phi_0(Y+Y_t))| \leq \frac{1}{3}(c_0\cot\alpha \ \rho_1 - \rho_1^2) \text{ in } \Omega'$$

for t large enough (remember that $0 < \rho_1 < c_0\cot\alpha$).

On the other hand we have, for all $a \in I\!R$, for all $V \in H_0^1(a,+\infty)$, and as long as Y_t is defined :

$$\int_a^{+\infty} (V_Y^2 - (f'(\phi_0(Y+Y_t)) + \frac{c_0^2}{4})V^2) \ dY \geq 0.$$

This is due to the linear stability of the 1D wave ϕ_0.

Hence, integral II can be estimated by

$$II \leq \frac{1}{3}(c_0\cot\alpha \ \rho_1 - \rho_1^2) \int_{\Omega'} v^2 \tag{5.12}$$

for t is large enough.

Because of the choice of ψ, the spatial $L^2(\Omega')$ norm of ψ_t, as well as that of the spatial derivatives of ψ up to second order, is uniformly bounded in t. Hence we may, as is classical, estimate III by

$$III \leq \frac{1}{3}(c_0\cot\alpha \ \rho_1 - \rho_1^2) \int_{\Omega'} v^2 \ + \ C_7 \tag{5.13}$$

for some constant C_7 independent of $t \geq 1$.

Summing up (5.11), (5.12) and (5.13), we obtain a uniform control of the L^2-norm $\|v(t,.,.)\|_{L^2(\Omega')}$, and thus a uniform control of $\|v_1(t,.,.)\|_{L^2(\Omega')}$, for t large enough. Therefore, standard parabolic estimates imply that the function $e^{\rho_1 X - c_0 Y/2}\partial_{e_\alpha}\overline{u} = e^{\rho_1 X - c_0 Y/2}\overline{u}_X$ is bounded in $L^\infty((t_0,+\infty)\times\tilde{\Omega}')$ for some $t_0 > 0$, where, say, $\tilde{\Omega}' = \{(x,y)\in I\!R^2,\ (x-1,y)\in\Omega'\}$. Eventually, since $|\nabla_{x,y}\overline{u}|$ is globally bounded in $(x,y)\in I\!R^2$ independently of $t\geq t_0$, one concludes that $e^{\rho_1 X - c_0 Y/2}\overline{u}_X$ is bounded in $L^\infty((t_0,+\infty)\times I\!R^2_+)$.

Similar estimates can be proven for $\partial_{e_\alpha}u(t,-x,y-ct)$. That completes the proof of Proposition 5.2. □

Proof of Theorem 1.6. Step 1. Even if it means shifting both u_0 and ϕ, with the same shift, one can assume without loss of generality that ϕ is even in x and that

$$\phi(x+x_n, y-|x_n|\cot\alpha) \to \phi_0(\pm x\cos\alpha + y\sin\alpha) \tag{5.14}$$

locally in (x,y) for any sequence $x_n\to\pm\infty$. It then follows from (5.6) that

$$\begin{cases} u(t,x+r\sin\alpha, y-r\cos\alpha-ct)\to\phi_0(x\cos\alpha+y\sin\alpha) \\ u(t,x-r\sin\alpha, y-r\cos\alpha-ct)\to\phi_0(-x\cos\alpha+y\sin\alpha)\end{cases} \text{ as } r\to+\infty,$$

for all $(x,y)\in I\!R^2$ and $t\geq 0$ (and also for $t=0$ by assumption on u_0).

Therefore, integrating in e_α the bounds given in Proposition 5.2 yields the existence of $C>0$, $t_0>0$ such that

$$\begin{aligned} |u(t,x,y-ct)-\phi_0(x\cos\alpha+y\sin\alpha)| &\leq Ce^{-\rho_1 X + c_0 Y/2} \\ |u(t,-x,y-ct)-\phi_0(-x\cos\alpha+y\sin\alpha)| &\leq Ce^{-\rho_1 X + c_0 Y/2}\end{aligned} \tag{5.15}$$

for all $t\geq t_0$ and $(x,y)\in I\!R^2_+$.

Since the initial datum $u_0 := \phi$ obviously falls within the assumptions of Theorem 1.6, and since $\phi(x,y+ct)$ is the solution of (1.3) with initial condition ϕ, one concludes that similar estimates as (5.15) also hold with $u(t,\pm x,y-ct)$ replaced with $\phi(x,y)$. Summing (5.15) with these estimates for ϕ implies that

$$\begin{aligned} |u(t,x,y-ct)-\phi(x,y)| &\leq C'e^{-\rho_1' X + c_0 Y/2} \\ |u(t,-x,y-ct)-\phi(-x,y)| &\leq C'e^{-\rho_1' X + c_0 Y/2}\end{aligned} \tag{5.16}$$

for all $t\geq t_1$ and $(x,y)\in I\!R^2_+$, where C', t_1 and ρ_1' are positive constants.

Therefore, there exists $\rho>0$ (depending only on ρ_1', c_0 and α), which we may choose less than $\overline{\rho}$ as in Theorem 4.1, such that : for all $\varepsilon>0$ and $y_1\in I\!R$, there is $r\geq 0$ such that

$$\begin{aligned} &\|u(t,\cdot,\cdot-ct)-\phi\|_{L^\infty(\mathcal{C}_-(y_1,\alpha)\setminus B_r)} \\ &+\|q^{-1}(u(t,\cdot,\cdot-ct)-\phi)\|_{L^\infty(\mathcal{C}_-(y_1,\alpha)\setminus B_r)} \leq \varepsilon/2\end{aligned} \tag{5.17}$$

for all $t\geq t_1$, where q has been defined in (1.9) and B_r denotes the euclidean open ball of center 0 and radius r.

Step 2. Let us now prove that $u(t,x,y-ct) \to \phi(x,y)$ as $t \to +\infty$ locally uniformly in $(x,y) \in I\!R^2$. If not, there exists a sequence $(x_n,y_n) \to (x_\infty,y_\infty) \in I\!R^2$ such that

$$\liminf_{n\to+\infty} |u(t_n,x_n,y_n-ct_n) - \phi(x_n,y_n)| > 0.$$

From Theorem 1.5, the functions $u(t_n+t,x,y-ct_n-ct)$ converge, up to extraction of some subsequence, locally uniformly in $(t,x,y) \in I\!R^3$ to a translate $\tau_{h,k}\phi$ as $n \to +\infty$. Owing to the definition of (x_n,y_n), one has

$$|\tau_{h,k}\phi(x_\infty,y_\infty) - \phi(x_\infty,y_\infty)| > 0. \tag{5.18}$$

On the other hand, the inequalities (5.16) imply, after passage to the limit $t_n \to +\infty$, that

$$|\tau_{h,k}\phi(x,y) - \phi(x,y)| \le C' e^{-\rho_1' X + c_0 Y/2}$$

and

$$|\tau_{h,k}\phi(-x,y) - \phi(-x,y)| \le C' e^{-\rho_1' X + c_0 Y/2}$$

for all $(x,y) \in I\!R^2_+$. It especially follows that $\tau_{h,k}\phi$ and ϕ have the same limits along the direction e_α and e'_α. Hence $\tau_{h,k}\phi = \phi$, which contradicts (5.18).

Therefore,

$$u(t,x,y-ct) \to \phi(x,y) \quad \text{as } t \to +\infty$$

locally uniformly in $(x,y) \in I\!R^2$.

Let now $\rho \in (0,\overline{\rho})$ be as in Step 1 above. Let $\varepsilon > 0$ be any positive number. As already underlined in the proofs of Proposition 5.2 and Theorem 1.5, there exist $y_2 \ge 0$ and $t_2 > 0$ such that

$$\forall t \ge t_2,\ \forall (x,y) \in \mathcal{C}_+(y_2,\pi-\alpha), \quad \left\{ \begin{array}{rcl} \phi(x,y) & \ge & 1-\varepsilon/8 \\ u(t,x,y-ct) & \ge & 1-\varepsilon/8. \end{array} \right. \tag{5.19}$$

Therefore, for all $t \ge t_2$,

$$\|u(t,\cdot,\cdot-ct) - \phi\|_{L^\infty(\mathcal{C}_+(y_2,\pi-\alpha))} \le \varepsilon/4.$$

The function $z(t,x,y) = u(t,x,y-ct) - \phi(x,y)$ satisfies the equation

$$\partial_t z - \Delta z + c\partial_y z + \gamma(t,x,y) z = 0$$

for some globally bounded function γ. Without loss of generality, one may also assume that y_2 and t_2 are such that

$$\forall t \ge t_2,\ \forall (x,y) \in \mathcal{C}_+(y_2,\pi-\alpha), \quad \gamma(t,x,y) \ge -\frac{f'(1^-)}{2} > 0.$$

The inequalities (5.17) applied to, say, $\varepsilon/8$ and y_2, yield the existence of $r > 0$ such that

$$\forall (x,y) \in \partial\mathcal{C}_+(y_2,\pi-\alpha)\backslash B_r, \quad |z(t,x,y)| \le \varepsilon q(x,y)/8$$

for all $t \geq t_2'$ large enough (we may choose $t_2' \geq t_2$). Furthermore, one has proved that z converges to 0 locally uniformly as $t \to +\infty$. Since q is bounded from below in B_r, one may then assume that $|z(t,x,y)| \leq \varepsilon q(x,y)/8$ for all $t \geq t_2'$ large enough and for all $(x,y) \in B_r$. Therefore,

$$\forall t \geq t_2',\ \forall (x,y) \in \partial\mathcal{C}_+(y_2, \pi-\alpha), \quad |z(t,x,y)| \leq \varepsilon q(x,y)/8.$$

On the other hand, because of (5.6), even if it means decreasing ρ (depending only on ρ_0, ρ_1' and α), there is a constant $C > 0$ such that

$$\forall (x,y) \in \mathcal{C}_+(y_2, \pi-\alpha), \quad |z(t_2',x,y)| = |u(t_2',x,y-ct_2') - \phi(x,y)| \leq Cq(x,y).$$

Next, the function $h(t,x,y) = \varepsilon q(x,y)/8 + Ce^{-\delta(t-t_2')}q(x,y)$ is such that

$$\partial_t h - \Delta h + c\partial_y h - \frac{f'(1^-)}{2}h \geq 0$$

for $\delta > 0$ and $\rho > 0$ small enough (ρ depending only on c, $f'(1^-)$, α, $\|g'\|_\infty$ and $\|g''\|_\infty$). Furthermore, $|z(t_2',x,y)| \leq h(t_2',x,y)$ in $\mathcal{C}_+(y_2,\pi-\alpha)$, and $|z(t,x,y)| \leq h(t,x,y)$ on $\partial\mathcal{C}_+(y_2,\pi-\alpha)$ for all $t \geq t_2'$. The maximum principle yields

$$\forall t \geq t_2',\ \forall (x,y) \in \mathcal{C}_+(y_2, \pi-\alpha), \quad |z(t,x,y)| \leq h(t,x,y).$$

As a consequence,

$$\|q^{-1}(u(t,\cdot,\cdot-ct)-\phi)\|_{L^\infty(\mathcal{C}_+(y_2,\pi-\alpha))} \leq \varepsilon/4$$

for all $t \geq t_2'' \geq t_2'$ large enough.

As a conclusion of this step 2, one has

$$\begin{array}{rcl} \|u(t,\cdot,\cdot-ct)-\phi\|_{L^\infty(\mathcal{C}_+(y_2,\pi-\alpha))} & & \\ +\|q^{-1}(u(t,\cdot,\cdot-ct)-\phi)\|_{L^\infty(\mathcal{C}_+(y_2,\pi-\alpha))} & \leq & \varepsilon/2 \end{array} \tag{5.20}$$

for all $t \geq t_2''$.

Step 3 (conclusion). Once $\rho > 0$ has been defined in Steps 1 and 2, let now $\varepsilon > 0$ be as in Theorem 4.1. Let y_2 be as in Step 2, let $y_1 = y_2$ and let $r > 0$ be such that (5.17) holds for t large enough. Remember that (5.20) holds for t large enough. Lastly, $u(t,x,y-ct) \to \phi(x,y)$ locally in (x,y) as $t \to +\infty$. Since q is bounded from below in B_r, one gets

$$\|u(t,\cdot,\cdot-ct)-\phi\|_{L^\infty(B_r)} + \|q^{-1}(u(t,\cdot,\cdot-ct)-\phi)\|_{L^\infty(B_r)} \leq \varepsilon/2$$

for t large enough.

Eventually, there exists $t_3 \geq 0$ such that the function $\tilde{u}_0(x,y) := u(t_3,x,y-ct_3)$ satisfies : $\tilde{u}_0 - \phi \in G_\rho$ and

$$\|\tilde{u}_0 - \phi\|_{G_\rho} \leq \varepsilon.$$

From Theorem 4.1, one concludes that

$$\forall t \geq 0, \quad \|u(t+t_3, \cdot, \cdot - ct - ct_3) - \phi\|_{G_\rho} \leq K' e^{-\omega t}$$

for some constants $K' \geq 0$ and $\omega > 0$. The conclusion of Theorem 1.6 follows.□

Proof of Theorem 1.7. In $I\!R^2_+$ denote, as above, the rotated coordinates by (X, Y). According to the assumptions of the theorem we may define

$$u_{0,+\infty}(Y) = \lim_{X \to +\infty} u_0(X, Y). \tag{5.21}$$

The function $u_{0,+\infty}$ is such that $\liminf_{Y \to +\infty} u_{0,+\infty}(Y) \in (\theta, 1]$, and $u_{0,+\infty}(\cdot) \leq \phi_0(\cdot + Y_0)$ for some $Y_0 \in I\!R$, because of the assumptions on u_0. The solution $u_{+\infty}(t, Y)$ of the Cauchy problem

$$\begin{aligned} u_t - u_{YY} + c_0 u_Y = & \quad f(u) \quad (t > 0, Y \in I\!R) \\ u(0, Y) = & \quad u_{0,+\infty}(Y) \end{aligned}$$

converges exponentially in time and uniformly in $Y \in I\!R$ to a steady 1D solution of the above problem (*see* [26], [34]), which is a 1D wave that we denote by $\phi_0(Y + Y_{+\infty})$, where $Y_{+\infty} \in I\!R$.

Fix any sequence $x_n \to +\infty$. The functions

$$u_n(t, x, y) = u(t, x + x_n, y - |x_n| \cot \alpha - ct)$$

are bounded in $C^{1,\delta}_t((0, +\infty) \times I\!R^2)$ and $C^{2,\delta}_{(x,y)}((0, +\infty) \times I\!R^2)$ locally in $(t, x, y) \in (0, +\infty) \times I\!R^2$, for some $\delta > 0$. Up to extraction of some subsequence, these functions u_n converge locally uniformly in $(0, +\infty) \times I\!R^2$ to a solution u_∞ of $\partial_t u_\infty = \Delta u_\infty - c\partial_y u_\infty + f(u_\infty)$ in $(0, +\infty) \times I\!R^2$.

Fix now any $\varepsilon > 0$. Let v_0 be a function bounded in $C^3(I\!R^2)$ such that $u_0 - \varepsilon \leq v_0 \leq u_0 + \varepsilon$ in $I\!R^2$ (remember that $u_0 \in UC(I\!R^2)$), and let $v(t, x, y)$ be the solution of (1.3) with initial condition v_0. It follows that $\|u(t, \cdot, \cdot) - v(t, \cdot, \cdot)\|_{L^\infty(I\!R^2)} \leq \varepsilon e^{\|f\|_{Lip} t}$ for all $t \geq 0$. The functions

$$v_n(t, x, y) = v(t, x + x_n, y - |x_n| \cot \alpha - ct)$$

converge locally uniformly in $[0, +\infty) \times I\!R^2$ to a solution v_∞ of the same equation as u_∞, and such that $\|u_\infty(t, \cdot, \cdot) - v_\infty(t, \cdot, \cdot)\|_{L^\infty(I\!R^2)} \leq \varepsilon e^{\|f\|_{Lip} t}$ for all $t > 0$. Furthermore, one can say from (5.21) that

$$u_{0,+\infty}(x \cos \alpha + y \sin \alpha) - \varepsilon \leq v_\infty(0, x, y) \leq u_{0,+\infty}(x \cos \alpha + y \sin \alpha) + \varepsilon$$

for all $(x, y) \in I\!R^2$. Since the function $u_{+\infty}(t, x \cos \alpha + y \sin \alpha)$ is a solution of the equation satisfied by v_∞, one then has $|v_\infty(t, x, y) - u_{+\infty}(t, x \cos \alpha + y \sin \alpha)| \leq \varepsilon e^{\|f\|_{Lip} t}$ for all $t \geq 0$ and $(x, y) \in I\!R^2$. It follows that $|u_\infty(t, x, y) - u_{+\infty}(t, x \cos \alpha + y \sin \alpha)| \leq 2\varepsilon e^{\|f\|_{Lip} t}$ for all $t > 0$ and $(x, y) \in I\!R^2$. Since $\varepsilon > 0$

was arbitrary, one then has that $u_\infty(t,x,y) \equiv u_{+\infty}(t, x\cos\alpha + y\sin\alpha)$. By uniqueness of the limit, one concludes that

$$u(t, x + r\sin\alpha, y - r\cos\alpha) \to u_{+\infty}(t, Y) \quad \text{as } r \to +\infty \tag{5.22}$$

for all $t \geq 0$ and $(x,y) \in I\!R^2$.

In a similar fashion, we may define $u_{-\infty}(t, Y')$ and $\phi_0(Y' + Y_{-\infty})$ for the left side, so that $u_{-\infty}(t, Y') \to \phi_0(Y' + Y_{-\infty})$ uniformly in $Y' \in I\!R$ as $t \to +\infty$, and $u(t, x - r\sin\alpha, y - r\cos\alpha) \to u_{-\infty}(t, Y')$ as $r \to +\infty$ for all $t \geq 0$ and $(x,y) \in I\!R^2$, where $Y' = -x\cos\alpha + y\sin\alpha$.

Under the notations of the proof of Proposition 5.2, the function $\overline{u}_X$ satisfies a parabolic linear equation with bounded coefficients. Because of the assumptions on u_0, it then easily follows that there exists $\omega_0 \in I\!R$ such that

$$|\overline{u}_X(t,x,y)| \leq Ce^{\rho_0(y\sin\alpha - x\cos\alpha)}e^{\omega_0 t} \tag{5.23}$$

for all $t \geq 0$ and $(x,y) \in I\!R^2$. Furthermore, Step 1 of Proposition 5.2 can be reproduced word by word and it gives some estimates of $\overline{u}_X$ in lower cones $\mathcal{C}_-(y_0, \alpha)$ for $t \geq 1$. Therefore, there exists $\rho_1 > 0$ small enough such that (5.9) holds and the function

$$v_1(t,x,y) = e^{\rho_1 X - c_0 Y/2}\overline{u}_X(t,x,y)$$

is in $L^2(\Omega')$ for each $t \geq 1$. To see it, divide Ω' into the following four regions : $\Omega' \cap \{X \leq 0\}$ (use here the fact that $\overline{u}_X$ is globally bounded for $t \geq 1$), $\Omega' \cap \{X \geq 0,\ Y \geq \beta X\}$ (use the fact that $\overline{u}_X$ is globally bounded in this region for $t \geq 1$, and choose $0 < \rho_1 < c_0\beta/2$), $\Omega' \cap \{X \geq 0,\ |Y| \leq \beta X\}$ (use (5.23) and choose $\beta(c_0/2 + \rho_0|\sin^2\alpha - \cos^2\alpha|) + \rho_1 < 2\rho_0\sin\alpha\cos\alpha$), and $\Omega' \cap \{X \geq 0,\ Y \leq -\beta X\}$ (use the estimates in Step 1 of Proposition 5.2 and choose again $0 < \rho_1 < c_0\beta/2$). Step 6 of Proposition 5.2 can be reproduced and the conclusion of Proposition 5.2 still holds.

Let now (x_∞, y_∞) be the unique couple of real numbers such that $\tau_{x_\infty,y_\infty}\phi$ converges to $\phi_0(Y + Y_{+\infty})$ (resp. $\phi_0(Y + Y_{-\infty})$) along the direction e_α (resp. e'_α).

Let us fix any $\varepsilon > 0$ and let us prove that

$$\|u(t,\cdot,\cdot - ct) - \tau_{x_\infty,y_\infty}\phi\|_{L^\infty(I\!R^2)} \leq \varepsilon$$

for t large enough.

First of all, as already emphasized, there exists $A \in I\!R$ such that $1 - \varepsilon/2 \leq u(t,x,y-ct) \leq 1$ in $\mathcal{C}_+(A, \pi - \alpha)$ for t large enough, and $1 - \varepsilon/2 \leq \tau_{x_\infty,y_\infty}\phi \leq 1$ in $\mathcal{C}_+(A, \pi - \alpha)$, whence

$$\|u(t,\cdot,\cdot - ct) - \tau_{x_\infty,y_\infty}\phi\|_{L^\infty(\mathcal{C}_+(A,\pi-\alpha))} \leq \varepsilon$$

for t large enough. Similarly, since $0 \leq u(t,x,y-ct) \leq \phi(x,y)$ (because $0 \leq u_0 \leq \phi$), there exists $B \leq A$ such that

$$\|u(t,\cdot,\cdot - ct) - \tau_{x_\infty,y_\infty}\phi\|_{L^\infty(\mathcal{C}_-(B,\alpha))} \leq \varepsilon,$$

for all $t \geq 0$.

Let $S = \mathcal{C}_+(A, \pi - \alpha) \backslash \mathcal{C}_-(B, \alpha)$. Because of the estimates for $\overline{u}_X$ as in the conclusion of Proposition 5.2, and because of (5.22), there exists $t_+ \geq 0$ and $x_+ \geq 0$ such that

$$|u(t, x, y - ct) - u_{+\infty}(t, Y)| \leq \varepsilon/3$$

for all $t \geq t_+$ and for all $(x, y) \in S \cap \{x \geq x_+\}$. On the other hand,

$$\|u_{+\infty}(t, \cdot) - \phi_0(\cdot + Y_{+\infty})\|_{L^\infty(B/\sin\alpha, A/\sin\alpha)} \leq \varepsilon/3$$

for t large enough. Lastly, even if it means increasing x_+, one can assume that $|\tau_{x_\infty, y_\infty}\phi(x, y) - \phi_0(Y + Y_{+\infty})| \leq \varepsilon/3$ for all $(x, y) \in S \cap \{x \geq x_+\}$. Therefore,

$$\|u(t, \cdot, \cdot - ct) - \tau_{x_\infty, y_\infty}\phi\|_{L^\infty(S \cap \{x \geq x_+\})} \leq \varepsilon$$

for t large enough.

Similarly, there exists $x_- \leq 0$ such that

$$\|u(t, \cdot, \cdot - ct) - \tau_{x_\infty, y_\infty}\phi\|_{L^\infty(S \cap \{x \leq x_-\})} \leq \varepsilon$$

for t large enough.

Lastly, from Theorem 1.5, there exists a sequence $t_n \to +\infty$ and $(h, k) \in I\!R^2$ such that

$$u(t_n, x, y - ct_n) \to \tau_{h,k}\phi(x, y)$$

locally uniformly in $(x, y) \in I\!R^2$ as $n \to +\infty$. The arguments above prove that for each $\varepsilon' > 0$, there exists $R = R_{\varepsilon'} \geq 0$ such that $\|\tau_{h,k}\phi - \tau_{x_\infty, y_\infty}\phi\|_{L^\infty(I\!R^2 \backslash B_R)} \leq \varepsilon'$. As a consequence, $\tau_{h,k}\phi$ and $\tau_{x_\infty, y_\infty}\phi$ have the same limits along the directions e_α and e'_α, whence $\tau_{h,k}\phi = \tau_{x_\infty, y_\infty}\phi$. Hence, by uniqueness of the limit, one can say that the whole family $u(t, x, y - ct)$ converges to $\tau_{x_\infty, y_\infty}\phi$ as $t \to +\infty$, locally uniformly in $(x, y) \in I\!R^2$.

Eventually, one concludes that

$$\|u(t, \cdot, \cdot - ct) - \tau_{x_\infty, y_\infty}\phi\|_{L^\infty(S \cap \{x_- \leq x \leq x_+\})} \leq \varepsilon$$

for t large enough.

As a conclusion,

$$\|u(t, \cdot, \cdot - ct) - \tau_{x_\infty, y_\infty}\phi\|_{L^\infty(I\!R^2)} \leq \varepsilon$$

for t large enough. That completes the proof of Theorem 1.7. □

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