Quasilinear Elliptic System Arising in a Three-dimensional Type-II Superconductor for Infinite κ.

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Abstract

We study a quasilinear elliptic system arising in a three-dimensional superconductor Ω. This model is formally derived from the Ginzburg-Landau energy at $\kappa = +\infty$ for a Meissner solution. If the tangential trace of the magnetic field $\mathbf{H}$ is given on $\partial\Omega$, we prove the existence of a unique solution for small data, and nonexistence for large data. On the other hand we prove that the current $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ is such that $\mathbf{J}^2$ is maximum on the boundary $\partial\Omega$.

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1 Introduction

In this paper we study a three-dimensional superconductor submitted to an exterior magnetic field. The underlying model for the superconductor is the Ginzburg-Landau model which involves a parameter κ called the Ginburg-Landau parameter in the litterature [8, 9, 12]. When $\kappa > \frac{1}{\sqrt{2}}$ the superconductor is called of type-II, and may contain some vortices. As a first approach, we restrict our study to a solution without vortices, called the Meissner solution. Outside the superconductor, the magnetic field is assumed to satisfy Maxwell equations. In this paper we derive formally a model when $\kappa = +\infty$, for a three-dimensional superconductor submitted to an exterior magnetic field, and we study rigorously this particular model. We prove some existence, non-existence and uniqueness results. Our approach is the same in its spirit (but different from the technical point of view) as this of Berestycki, Bonnet and Chapman [3]. In their paper, these authors derive a two-dimensional model for $\kappa = +\infty$ posed on the cross-section of an infinite

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cylindrical superconductor. From this point of view our work can be seen as an extension of their work where in place of a quasilinear scalar equation, we study a quasilinear elliptic system.

1.1 The model studied in this paper

To simplify the presentation, we assume that the superconductor is represented by a smooth open set $\Omega \subset \mathbf{R}^3$, homeomorphic to a ball. We study the following model, which describes the magnetic field $\mathbf{H}$ in a three-dimensional type-II superconductor Ω for a Meissner solution at $\kappa = +\infty$. We search for the solutions $\mathbf{H}$ satisfying:

$$-\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ on } \Omega \tag{1.1}$$

$$\mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \tag{1.2}$$

where the scalar function F is given by the relation:

$$h = (1-u^2)u \Longleftrightarrow u = F(h^2)h$$

The function F is uniquely defined for $u^2 < \frac{1}{3}$ or equivalently for $h^2 < \frac{4}{27}$ and $F(0) = 1$. Here $\mathbf{H}_T$ denotes the tangential trace of the magnetic field $\mathbf{H}$: $\mathbf{H}_T = \mathbf{H} - (\mathbf{H} \cdot n)n$, where n is the exterior unit normal to Ω.
We will prove results for system (1.1)-(1.2).

Remark 1.1 *For some reasons which will be clear in section 2, the physical solutions* $\mathbf{H}$ *satisfy moreover the boundary condition*

$$\mathbf{curl}\ \mathbf{H} \cdot n = 0 \text{ on } \partial\Omega \tag{1.3}$$

It can be seen that $\mathbf{curl}\ \mathbf{H} \cdot n$ *only depends on* $\mathbf{H}_T^{ext}$ *on* $\partial\Omega$ *and then (1.3) is a consequence of (1.2) for an appropriate choice of* $\mathbf{H}_T^{ext}$*. As a consequence, results for system (1.1)-(1.2)-(1.3) are particular subcases of the results we prove on system (1.1)-(1.2).*

1.2 Main results

We assume a sufficient regularity, let say $\partial\Omega \in C^{2+\alpha}$. Our main results are

Theorem 1.2 (Existence, Uniqueness, Non-Existence of the Magnetic Field)
i) There exists a constant $\epsilon > 0$*, such that if* $|\mathbf{H}_T^{ext}|_{C^{2+\alpha}(\partial\Omega)} < \epsilon$*, then there exists a unique* $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega})$ *solution to (1.1) with the boundary condition (1.2).*
ii) Let $\mathbf{H}_T^{ext} \in C^{2+\alpha}(\partial\Omega)$*. Then there is no solution* $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega})$ *to (1.1) with the boundary condition* $\mathbf{H}_T = \mu\mathbf{H}_T^{ext}$ *on* $\partial\Omega$ *for* $\mu > 0$ *large enough.*

Theorem 1.3 (Properties of the Current)
Let us consider a solution $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega})$ *to (1.1) and let us define its current by*

$$\mathbf{J} = \mathbf{curl\ H}$$

i) Then by definition $\mathbf{J}^2 < \frac{4}{27}$ *on* $\overline{\Omega}$. *Moreover* $\mathbf{J}^2$ *is maximal on* $\partial\Omega$.
ii) Let $\mathbf{J}^{ext} \in C^{2+\alpha}(\partial\Omega)$. *Then there is at most one magnetic field* $\mathbf{H}$ *solution to (1.1) such that* $\mathbf{curl\ H} = \mathbf{J}^{ext}$ *on* $\partial\Omega$.

Remark 1.4 *From theorem 1.3, we see that the value of the current on the boundary of the superconductor determines uniquely the magnetic field inside the superconductor.*

1.3 Organization of the article

In section 2 we explain the formal derivation of the model presented in the introduction. Next we give in section 3 the proofs of the main theorems. In section 4 we complete this study by the heuristical derivation (from the Ginzburg-Landau model) of a model for a torus superconductor which formally shows the existence of a maximal magnetic capacity. Finally, we indicate in the appendix, the variational method for quasilinear elliptic systems as an alternative method which gives less information than our proofs in section 3.

2 Formal derivation of model (1.1)-(1.2)

The Ginzburg-Landau model
The superconductor material is represented by the open set Ω, and in the complementar $\mathbf{R}^3\backslash\Omega$ we would assume (in a realistic case) the existence of an exterior magnetic field which is given at infinity. As a consequence of this exterior magnetic field, the superconductor will create its own magnetic field in order to minimize the total magnetic field inside the superconductor. This is the reason why we can only impose the exterior magnetic field at infinity, in a real experimentation.
Let us recall that if $\mathbf{A}$ is the magnetic potential vector inside the superconductor, and Ψ the complex wave function which describes the superconducting state, then the Ginzburg-Landau energy (without exterior forces) is

$$\mathcal{E}_\kappa(\mathbf{A}, \psi) = \int_\Omega \left| \left(\frac{1}{\kappa}\nabla - i\mathbf{A} \right) \Psi \right|^2 + \frac{1}{2}(|\Psi|^2 - 1)^2 + |\mathbf{curl\ A}|^2 \tag{2.1}$$

In the case where there are no vortices (the Meissner solution), we have $|\Psi| > 0$, and we can take (see [3]) the new gauge $\Psi = fe^{i\chi}$, $\mathbf{Q} = A - \frac{1}{\kappa}\nabla\chi$, where $f > 0$ and $\chi \in \mathbf{R}$.

Then we get

$$\mathcal{E}_\kappa(f,\mathbf{Q}) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\textbf{curl } \mathbf{Q}|^2 + \frac{(f^2-1)^2}{2} + f^2\mathbf{Q}^2$$

with Euler-Lagrange equations in Ω:

$$\begin{cases} \frac{1}{\kappa^2}\Delta f = f(f^2+\mathbf{Q}^2-1) \\ -(\textbf{curl })^2\mathbf{Q} = f^2\mathbf{Q} \end{cases} \tag{2.2}$$

and the interior magnetic field is

$$\mathbf{H}^{int} = \textbf{curl } \mathbf{Q} \tag{2.3}$$

Outside the superconductor material we have the Maxwell equations for the exterior magnetic field $\mathbf{H}^{ext}$ in $\mathbf{R}^3\backslash\overline{\Omega}$:

$$\begin{cases} \text{div } \mathbf{H}^{ext} = 0 \\ \textbf{curl } \mathbf{H}^{ext} = 0 \end{cases} \tag{2.4}$$

Moreover on the interface $\partial\Omega$, we have the continuity of the magnetic field:

$$\mathbf{H}^{int} = \mathbf{H}^{ext} \tag{2.5}$$

and formally from the variation of the Ginzburg-Landau energy we find the boundary conditions:

$$\frac{\partial_n f}{\kappa^2} = 0 \tag{2.6}$$

$$\mathbf{Q}\cdot n = 0 \tag{2.7}$$

Moreover we impose in some sense the exterior magnetic field at infinity:

$$\mathbf{H}^{ext} \simeq \mathbf{H}_0 \tag{2.8}$$

where $\mathbf{H}_0$ is the (constant) magnetic field that we would have if there was no superconducting material in the space.

Remark 2.1 *Let us remark that the tangential trace of the interior magnetic field satisfies* $\mathbf{H}_T^{int} = \mathbf{H}_T^{ext}$ *and then is not arbitrar. It can be seen that* $\textbf{curl } \mathbf{H}^{int}\cdot n$ *only depends on* $\mathbf{H}_T^{int}$*, but on the other hand* $\textbf{curl } \mathbf{H}^{ext} = 0$*, then*

$$\textbf{curl } \mathbf{H}^{int}\cdot n = 0$$

This can also be seen from (2.7),(2.2). In particular it means that the current $\mathbf{J} = \textbf{curl } \mathbf{H}$ *is tangential to the boundary of the superconductor* $\partial\Omega$*.*

Previous models

The case $\mathbf{H}_0 = H_0 e_3$ along the axis of the cylinder $\Omega = \omega \times \mathbf{R}$ with $\omega \subset \mathbf{R}^2$ was studied in [3] in the case $\kappa = +\infty$, and in [4] for finite κ. See also [1], [2].

A model in $\mathbf{R}^3$

Let us recall that we consider the case where Ω is a bounded open set, homeomorphic to a ball. For infinite κ, boundary condition (2.6) has no sense, and then we just forget it. Moreover equation (2.2) gives formally

$$\left\{ \begin{array}{l} f^2 = 1 - \mathbf{Q}^2 \\ -\mathbf{curl}\ \mathbf{H} = (1 - \mathbf{Q}^2)\mathbf{Q} \end{array} \right. \tag{2.9}$$

where we have set $\mathbf{H} = \mathbf{curl}\ \mathbf{Q}$. Now we get with the function F defined in the introduction: $\mathbf{Q} = -F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}$, and if we take the **curl** , we get the new system:

$$\left\{ \begin{array}{l} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ in } \Omega \\ \mathbf{curl}\ \mathbf{H} = 0 \text{ in } \mathbf{R}^3 \backslash \overline{\Omega} \\ \mathbf{H} \text{ continuous on } \partial\Omega \\ \text{div } \mathbf{H} = 0 \text{ in } \mathbf{R}^3 \\ \mathbf{H} = \mathbf{H}_0 \text{ at infinity} \end{array} \right. \tag{2.10}$$

The model that we consider in this paper.

At a first glance system (2.10) seems a little bit too complicated to be analysed. That is why we only consider a particular submodel. In fact we are only interested in the magnetic field in the superconductor, so we will consider that the tangential trace of the exterior magnetic field is given on the surface of the superconductor. Then we consider the following model which was introduced in subsection 1.1:

$$\left\{ \begin{array}{l} -\mathbf{curl}\ (F(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ in } \Omega \\ \mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \end{array} \right. \tag{2.11}$$

where $\mathbf{H}_T^{ext}$ is physically chosen such that $\mathbf{curl}\ \mathbf{H}_T^{ext} \cdot n = 0$.

3 Proof of theorems 1.2 and 1.3.

3.1 Preliminaries

We define the primitive $G(s) = \int^s f$ for $s \in [0, \frac{4}{27} - \delta]$ for some arbitrarily small $\delta > 0$, which is extended on $\mathbf{R}^+$ as a C^∞-convex function, which is affine on $[\frac{4}{27}, +\infty[$.

In all what follows we consider the new system:

$$-\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ on } \Omega \tag{3.1}$$

$$\mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \tag{3.2}$$

Important Remark

From now on, we will prove results on system (3.1)-(3.2), which will imply the announced results for system (1.1)-(1.2).

3.2 An approach by the inverse function theorem

We assume that $\partial\Omega \in C^{2+\alpha}$. Let us define the spaces:

$$X^{2+\alpha}_{0T,div} = \{\mathbf{H} \in C^{2+\alpha}(\overline{\Omega}), \mathbf{H}_T = 0, \text{div } \mathbf{H} = 0\}$$

$$X^{m+\alpha}_{div} = \{\mathbf{H} \in C^{m+\alpha}(\overline{\Omega}), \text{div } \mathbf{H} = 0\},\ m = 0, 1, 2$$

Lemma 3.1 *If* $\mathbf{H}^{ext}_T \in C^{2+\alpha}(\partial\Omega)$*, then there exists* $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ *such that div* $\mathbf{H}^p = 0$ *and* $\mathbf{H}^p_T = \mathbf{H}^{ext}_T$*.*

Proof of lemma 3.1

It is quite easy to build a particular interior magnetic field $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ such that div $\mathbf{H}^p = 0$ and $\mathbf{H}^p_T = \mathbf{H}^{ext}_T$. To do this, we take a field $\mathbf{H}^1 \in C^{2+\alpha}(\overline{\Omega})$, such that $\mathbf{H}^1_T = \mathbf{H}^{ext}_T$. We then consider the function $\phi \in C^{2+\alpha}(\overline{\Omega})$ such that $\Delta\phi = \text{div } \mathbf{H}^1$, $\phi = 1$ on $\partial\Omega$, and take $\mathbf{H}^p = \mathbf{H}^1 - \nabla\phi \in C^{1+\alpha}(\overline{\Omega})$. Then

$$\begin{cases} \Delta\mathbf{H}^p = -(\mathbf{curl}\ )^2\mathbf{H}^1 \in C^{\alpha}(\Omega) \\ \mathbf{H}^p_T = \mathbf{H}^{ext}_T \text{ in } C^{2+\alpha}(\partial\Omega) \\ \text{div } \mathbf{H}^p = 0 \text{ in } C^{1+\alpha}(\partial\Omega) \end{cases}$$

Then from the Schauder theory for elliptic system (see [15]) we get $\mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$, which proves lemma 3.1.

From lemma 3.1, there exists $\mathbf{H}^p \in X^{2+\alpha}_{div}$ such that $\mathbf{H}^p_T = \mathbf{H}^{ext}_T$. Let

$$X = \mathbf{H}^p + X^{2+\alpha}_{0T,div}$$

and

$$\begin{array}{rccl} \mathcal{A}: & X & \longrightarrow & X^{\alpha}_{div} \\ & \mathbf{H} & \longmapsto & \mathcal{A}(\mathbf{H}) = -\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} \end{array}$$

Then we have

Proposition 3.2 *For all* $\mathbf{H}^{ext}_T \in C^{2+\alpha}(\partial\Omega)$*, there exists a unique solution* $\mathbf{H} \in X^{2+\alpha}_{div}$ *solution to (3.1)-(3.2).*

This proposition proves in particular the i) of theorem 1.2.

Proof of proposition 3.2

Let us compute $\mathcal{A}(\mathbf{H}) = G'(\mathbf{J}^2)(-(\mathbf{curl}\ )^2\mathbf{H}) - \nabla(G'(\mathbf{J}^2)) \wedge \mathbf{curl}\ \mathbf{H} - \mathbf{H}$, where $\mathbf{J} = \mathbf{curl}\ \mathbf{H}$ is the current. But

$$-(\mathbf{curl}\ )^2\mathbf{H} = \Delta\mathbf{H} - \nabla(\mathrm{div}\ \mathbf{H})$$

and $\mathrm{div}\ \mathbf{H} = 0$, $\mathbf{J}_i = \epsilon_{ijk}\nabla_j\mathbf{H}_k$, where ϵ_{ijk} is the complelety antisymetric tensor such that $\epsilon_{123} = 1$. Then we get

$$(\mathcal{A}(\mathbf{H}))_i = A_{ijmn}(\mathbf{J})D^2_{jm}\mathbf{H}_n - \mathbf{H}_i$$

with

$$A_{ijmn}(\mathbf{J}) = G'(\mathbf{J}^2)\delta_{jm}\delta_{in} - 2G''(\mathbf{J}^2)\mathbf{J}_k\epsilon_{kij}\mathbf{J}_l\epsilon_{lmn}$$

where we have used for the intermediate computation: $(\nabla(G'(\mathbf{J}^2))\wedge\mathbf{J})_i = \epsilon_{ijk}\nabla_j(G'(\mathbf{J}^2))\mathbf{J}_k$, $\nabla_j(G'(\mathbf{J}^2)) = 2G''\mathbf{J}_l\epsilon_{lmn}\nabla_j\nabla_m\mathbf{H}_n$.

Now we can consider

$$\begin{array}{rccc} D_{\mathbf{H}}\mathcal{A}(\mathbf{H}): & X^{2+\alpha}_{0T,div} & \longrightarrow & X^{\alpha}_{div} \\ & \mathbf{h} & \longmapsto & D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h} \end{array}$$

and

$$(D_{\mathbf{H}}\mathcal{A}(\mathbf{H})\cdot\mathbf{h})_i = A_{ijmn}(\mathbf{J})D^2_{jm}\mathbf{h}_n + 2\frac{dA_{ilmn}}{d(\mathbf{J})}(D^2_{jm}\mathbf{H}_n)\mathbf{J}_r\epsilon_{rpq}\nabla_p\mathbf{h}_q - \mathbf{h}_i \qquad (3.3)$$

It is easy to compute $det(L_{in}(x,\lambda))$ where $\lambda = (\lambda_1,\lambda_2,\lambda_3)$, and $L_{in}(x,\lambda) = \lambda_j\lambda_m A_{ijmn}(\mathbf{J}) = G'\lambda^2\delta_{in} + 2G''(\lambda\wedge\mathbf{J})_i(\lambda\wedge\mathbf{J})_n$. Thus, with $\mathbf{J} = \mathbf{J}(x)$:

$$det(L_{in}(x,\lambda)) = G'^2(\mathbf{J}^2)\lambda^4(G'(\mathbf{J}^2)\lambda^2 + 2G''(\mathbf{J}^2)|\lambda\wedge\mathbf{J}|^2) \neq 0 \text{ if } \lambda \neq 0$$

Then $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$ is elliptic on $X^{2+\alpha}_{0T,div}$, and we want to prove that it satisfies a Schauder estimate, which will be used later. To do this, we first extend the operator $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$, as an elliptic operator $\mathcal{L}(\mathbf{H})$ which has expression (3.3), and is defined from $X^{2+\alpha}_{0T,0div} = \{\mathbf{H}\in C^{2+\alpha}, \mathbf{H}_T = 0, (\mathrm{div}\ \mathbf{H})_{|\partial\Omega} = 0\}$ into C^α. Now using the theory of elliptic systems developped in [15], it is easy to verify that the conditions $\mathbf{H}_T = 0$, $(\mathrm{div}\ \mathbf{H})_{|\partial\Omega} = 0$ are complementar conditions, which in particular permit us to get the Schauder estimate:

$$|\mathbf{h}|_{C^{2+\alpha}(\overline{\Omega})} \leq C(\mathbf{H})\left\{|\mathcal{L}(\mathbf{H})\cdot\mathbf{h}|_{C^\alpha(\overline{\Omega})} + |\mathbf{h}_T|_{C^{2+\alpha}(\partial\Omega)} + |\mathrm{div}\ \mathbf{h}|_{C^{1+\alpha}(\partial\Omega)} + |\mathbf{h}|_{L^1(\Omega)}\right\} \qquad (3.4)$$

<u>Injectivity</u>

Let us first prove that $KerD_{\mathbf{H}}\mathcal{A}(\mathbf{H}) = \{0\}$. To do this, we remark that for the energy

$\mathcal{E}(\mathbf{H}) = \int_\Omega \frac{G(|\mathbf{curl\ H}|^2)}{2} + \frac{\mathbf{H}^2}{2}$, we have $\mathcal{E}'(\mathbf{H}) \cdot \mathbf{h} = \int_\Omega G'(\mathbf{J}^2)\mathbf{curl\ H} \cdot \mathbf{curl\ h} + \mathbf{H} \cdot \mathbf{h} = \int_\Omega -\mathcal{A}(\mathbf{H}) \cdot \mathbf{h}$, where we have used the general equality

$$\int_\Omega \mathbf{curl\ A} \cdot \mathbf{B} = \int_\Omega \mathbf{curl\ B} \cdot \mathbf{A} + \int_{\partial\Omega} (\mathbf{A} \wedge \mathbf{B}) \cdot n$$

Consequently

$$\mathcal{E}''(\mathbf{H}) \cdot (\mathbf{h}, \mathbf{h}) = -\int_\Omega (D_{\mathbf{H}}\mathcal{A}(\mathbf{H}) \cdot \mathbf{h}) \cdot \mathbf{h} \tag{3.5}$$

But on the other hand we compute explicitly

$$\mathcal{E}''(\mathbf{H}) \cdot (\mathbf{h}, \mathbf{h}) = \int_\Omega G'(\mathbf{J}^2)|\mathbf{curl\ h}|^2 + 2G''(\mathbf{J}^2)(\mathbf{curl\ H} \cdot \mathbf{curl\ h})^2 + \mathbf{h}^2 \tag{3.6}$$

then $\mathbf{h} = 0$ if $D_{\mathbf{H}}\mathcal{A}(\mathbf{H}) \cdot \mathbf{h} = 0$, which proves the injectivity of $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$.

Surjectivity

One way to prove the surjectivity of $D_{\mathbf{H}}\mathcal{A}(\mathbf{H})$, is to prove it for a weak version using Lax-Milgram theorem. We introduce the following symmetric bilinear form defined on $Y^1_{0T,div} = \{\mathbf{H} \in \mathcal{H}^1(\Omega),\ \mathbf{H}_T = 0,\ \text{div } \mathbf{H} = 0\}$:

$$a(\mathbf{h}, \mathbf{v}) = \mathcal{E}''(\mathbf{H}) \cdot (\mathbf{h}, \mathbf{v}),\ \forall \mathbf{h}, \mathbf{v} \in Y^1_{0T,div}$$

and we want to solve

$$a(\mathbf{h}, \mathbf{v}) = < -\mathbf{f}, \mathbf{v} >,\ \forall \mathbf{v} \in Y^1_{0T,div} \tag{3.7}$$

where $\mathbf{f} \in (Y^1_{0T,div})'$.

Let us recall here the classical result (see [7] page 247): For Ω at least of class C^2, there exists a constant $C = C(\Omega) > 0$ such that

$$\forall \mathbf{v} \in \mathcal{H}^1(\Omega), |\mathbf{v}|_{\mathcal{H}^1} \leq C\{|\mathbf{v}|_{L^2} + |\mathbf{curl\ v}|_{L^2} + |\text{div } \mathbf{v}|_{L^2} + |\mathbf{v}_T|_{\mathcal{H}^{\frac{1}{2}}(\partial\Omega)}\} \tag{3.8}$$

From (3.6) and (3.8) we see that there exists a constant $C > 0$ such that $a(\mathbf{h}, \mathbf{h}) \geq C|\mathbf{h}|_{\mathcal{H}^1(\Omega)}$ holds for $h \in Y^1_{0T,div}$. Then Lax-Milgram theorem applies. In particular for all $\mathbf{f} \in X^\alpha_{div} \subset (Y^1_{0T,div})'$, we deduce from (3.5) that there exists a unique $h \in Y^1_{0T,div}$ such that

$$\int_\Omega (\mathbf{curl}\ (\frac{d}{d\mathbf{H}}(G'(|\mathbf{curl\ H}|^2)\mathbf{curl\ H}) \cdot \mathbf{h}) + \mathbf{h} + \mathbf{f}) \cdot \mathbf{v} = 0$$

Now we use the following lemma (see theorem 3.4 in [13]):

Lemma 3.3 *If* $\mathbf{v} \in L^2(\Omega),\ div\ \mathbf{v} = 0,$ *then* $\exists \phi \in \mathcal{H}^1(\Omega)$ *such that* $\mathbf{v} = \mathbf{curl}\ \phi$.

We apply this lemma to $\mathbf{h}$ and $\mathbf{f}$. We get the existence of $\mathbf{k}, \mathbf{g} \in \mathcal{H}^1(\Omega)$ such that $\mathbf{h} = \mathbf{curl\ k}$, $\mathbf{f} = \mathbf{curl\ g}$. Then using the general equality

$$\int_\Omega \mathbf{v} \cdot \mathbf{curl\ B} = \int_\Omega \mathbf{curl\ v} \cdot \mathbf{B} + \int_{\partial\Omega} \mathbf{B} \cdot (\mathbf{v} \wedge n)$$

and $\mathbf{v} \wedge n = 0$ on $\partial\Omega$ because $\mathbf{v} \in Y^1_{0T,div}$, we get

$$\int_\Omega \mathbf{curl}\ \mathbf{v} \cdot \mathbf{A} = 0$$

where $\mathbf{A} = \frac{d}{d\mathbf{H}}(G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) \cdot \mathbf{h} + \mathbf{k} + \mathbf{g} \in L^2(\Omega)$. But from the proof of proposition 4 page 260 in [7], we get $\mathbf{curl}\ Y^1_{0T,div} = \mathbf{curl}\ \{\mathbf{H} \in \mathcal{H}^1(\Omega), \mathbf{H}_T = 0\}$, and the orthogonal of this space for the scalar product in $L^2(\Omega)$ is the space $\{\mathbf{V} \in L^2(\Omega), \mathbf{curl}\ \mathbf{V} = 0\}$. Consequently, we get $\mathbf{curl}\ \mathbf{A} = 0$, which means that the unique solution $\mathbf{h} \in Y^1_{0T,div}$ to (3.7) for $\mathbf{f} \in X^\alpha_{div}$ satisfies the Euler-Lagrange equation :

$$D_\mathbf{H}\mathcal{A}(\mathbf{H}) \cdot \mathbf{h} = \mathbf{f}$$

Then the Schauder estimate (3.4) applies and proves that $\mathbf{h} \in X^{2+\alpha}_{0T,div}$, which proves that $D_\mathbf{H}\mathcal{A}(\mathbf{H})$ is surjectif.
Then $D_\mathbf{H}\mathcal{A}(\mathbf{H})$ is an isomorphism, and the inverse function theorem applies and proves proposition 3.2.

From the results of Morrey [15], we deduce that $\mathbf{H} \in C^\infty(\overline{\Omega})$, and $\mathbf{H}$ is analytic on $\{|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27} - \delta\}$.

Proof of ii) of theorem 1.2

From lemma 3.1, for a given $\mathbf{H}^{ext}_T \in C^{2+\alpha}(\partial\Omega)$, $\exists \mathbf{H}^p \in C^{2+\alpha}(\overline{\Omega})$ such that div $\mathbf{H}^p = 0$, $\mathbf{H}^p_T = \mathbf{H}^{ext}_T$. We then minimize

$$\min_{\mathbf{H} \in Y^1_{0T,div}} E(\mathbf{H} + \mathbf{H}^p)$$

where $E(\mathbf{H}) = \int_\Omega \mathbf{H}^2 + |\mathbf{curl}\ \mathbf{H}|^2$ and $Y^1_{0T,div} = \{\mathbf{H} \in \mathcal{H}^1(\Omega),\ \text{div}\ \mathbf{H} = 0,\ \mathbf{H}_T = 0\}$. Because E is continuous, strictly convex and infinite at infinity on $Y^1_{0T,div}$, there exists a unique minimizer $\mathbf{H}^* \in Y^1_{0T,div}$. Moreover $\mathbf{H}^* + \mathbf{H}^p \not\equiv 0$ (because $(\mathbf{H}^* + \mathbf{H}^p)_T \not\equiv 0$ and $\mathbf{H}^* + \mathbf{H}^p \in \mathcal{H}^1(\Omega)$), then $E(\mathbf{H}^* + \mathbf{H}^p) > 0$.
We now consider a solution $\mathbf{H}$ to (3.1)-(3.2) such that $\mathbf{H}_T = \mu\mathbf{H}^{ext}_T$ for some $\mu > 0$. If such a solution exists, then

$$0 < \mu^2 E(\mathbf{H}^* + \mathbf{H}^p) \leq \int_\Omega \mathbf{H}^2 + \mathbf{J}^2 G'(\mathbf{J}^2) = \int_{\partial\Omega} G'(\mathbf{J}^2_T)(\mathbf{J}_T \wedge \mathbf{H}_T) \cdot n \leq \mu C |\mathbf{H}^{ext}_T|_{L^1(\partial\Omega)}$$

Then for μ large enough there is no solution $\mathbf{H}$ to (3.1)-(3.2).

3.3 Further results

We have

Proposition 3.4 *If* $\mathbf{H} \in C^{2+\alpha}(\overline{\Omega})$ *is a solution to (3.1), then the current* $\mathbf{J} = \mathbf{curl\ H}$ *is such that* $\mathbf{J}^2$ *is maximal on* $\partial\Omega$.

It implies the i) of theorem 1.3.

Proof of proposition 3.4
Taking the **curl** of equation $-\mathbf{curl}\,(G'(|\mathbf{curl\ H}|^2)\mathbf{curl\ H}) - \mathbf{H} = 0$ in Ω, we get an equation satisfied by the current $\mathbf{J}$:

$$-(\mathbf{curl}\,)^2(G'(\mathbf{J}^2)\mathbf{J}) - \mathbf{J} = 0 \text{ in } \Omega \tag{3.9}$$

But the general relation

$$-(\mathbf{curl}\,)^2\mathbf{A} = \Delta\mathbf{A} - \nabla(\text{div }\mathbf{A})$$

gives

$$\Delta(G'\mathbf{J}) - \nabla(G''\mathbf{J}\cdot\nabla(\mathbf{J}^2)) - \mathbf{J} = 0$$

where we have used div $\mathbf{J} = 0$. With the notation $u = \mathbf{J}^2$, the computation of the first term gives: $\Delta(G'\mathbf{J}) = G'(u)\Delta\mathbf{J} + G''(u)(\Delta u)\mathbf{J} + b_1(\mathbf{J},\nabla\mathbf{J})\nabla u$. The second term gives $\nabla_i(G''\mathbf{J}\cdot\nabla(\mathbf{J}^2)) = G''(u)\mathbf{J}_j D^2_{ij}u + (b_2(\mathbf{J},\nabla\mathbf{J})\nabla u)_i$. Then multiplying by $\mathbf{J}$ and using the equality $\Delta\frac{\mathbf{J}^2}{2} = \mathbf{J}\Delta\mathbf{J} + |\nabla\mathbf{J}|^2$, we get:

$$\frac{G'(u)}{2}\Delta u + G''(u)u\sum_{j\neq\frac{\mathbf{J}}{|\mathbf{J}|}} D^2_{jj}u + b(\mathbf{J},\nabla\mathbf{J})\nabla u - u = G'(u)|\nabla\mathbf{J}|^2 \geq 0$$

Then from the maximum principle we deduce that u can not be maximum inside Ω, except in the trivial case where $\mathbf{J}\equiv 0$. This proves the proposition.

Moreover, we have

Proposition 3.5 *If* $\mathbf{J}^{ext}$ *is the trace of* $\mathbf{J}$ *on* $\partial\Omega$, *then there is at most one current* $\mathbf{J} = \mathbf{curl\ H}$ *and one magnetic field* $\mathbf{H}$ *solution to (3.1) such that* $\mathbf{curl\ H} = \mathbf{J}^{ext}$ *on* $\partial\Omega$.

Proof of proposition 3.5
An easy proof consists to write the difference of equation (3.9) satisfied by two currents $\mathbf{J}^{(1)}$ and $\mathbf{J}^{(2)}$, to multiply it by $\mathbf{v} = G'((\mathbf{J}^{(2)})^2)\mathbf{J}^{(2)} - G'((\mathbf{J}^{(1)})^2)\mathbf{J}^{(1)}$ and to integrate by part. The boundary term is nul because $\mathbf{J}^{(1)} = \mathbf{J}^{(2)}$ on $\partial\Omega$, and we obtain:

$$\int_\Omega |\mathbf{curl\ v}|^2 + (\mathbf{J}^{(2)} - \mathbf{J}^{(1)})\cdot\mathbf{v} = 0$$

and by strict convexity of the map $\mathbf{J} \longmapsto G(\mathbf{J}^2)$, there exists $c = c(\delta) > 0$, such that $(\mathbf{J}^{(2)} - \mathbf{J}^{(1)}) \cdot \mathbf{v} \geq c(\mathbf{J}^{(2)} - \mathbf{J}^{(1)})^2$, which implies that $\mathbf{J}^{(2)} - \mathbf{J}^{(1)} = 0$, and proves the uniqueness of the current. Consequently the magnetic field is unique, because $\mathbf{H} = -\mathbf{curl}\ (G'(\mathbf{J}^2)\mathbf{J})$. This ends the proof.

4 The magnetic capacity of a superconducting torus: some heuristical remarks.

In this section we want to relax the assumption that the superconductor is homeomorphic to a ball. To simplify the presentation, we assume that the superconductor Ω is a torus. We will examine the consequences of the multiconnectivity property of the superconducting torus, and in particular the existence of a phase parameter. This phase parameter will give the existence of permanent currents in the superconductor.
Let us note the cylindrical basis by $e_r = (\cos\phi, \sin\phi, 0)$, $e_\phi = (-\sin\phi, \cos\phi, 0)$ and $e_z = (0, 0, 1)$. Let $\omega \subset \mathbf{R}^2$ a bounded smooth open set which will be the section of the torus. We assume that $\sup_{x\in\overline{\omega}} |x| < R$. Then we define the torus by

$$\Omega = \{M = (R + \rho\cos\theta)e_r + \rho\sin\theta e_z, \phi \in [0, 2\pi[, (\rho\cos\theta, \rho\sin\theta) \in \omega\} \tag{4.1}$$

Let us recall that the penetration length of the magnetic field is a quantity λ very small in comparison to the macroscopic scale. This λ is usually called the London wave length. Now we recall the Ginzburg-Landau energy with the explicit dependence on λ:

$$\mathcal{E}_\kappa^\lambda(\mathbf{A}, \psi) = \int_\Omega \left| \left(\frac{\lambda}{\kappa}\nabla - i\mathbf{A} \right) \Psi \right|^2 + \frac{1}{2}(|\Psi|^2 - 1)^2 + \lambda^2 |\mathbf{curl}\ \mathbf{A}|^2 \tag{4.2}$$

In (2.1) we have considered a superconductor whose size is of order λ, such that it was possible to take $\lambda = 1$ in new coordinates. Here, on the contrary we are interested in macroscopic superconductors Ω, and we want to take the double singular limit $\lambda \to 0$, $\kappa \to +\infty$.

After a gauge transformation, we can assume that for some $N \in \mathbf{Z}$:

$$\left\{ \begin{array}{l} \psi = e^{iN\phi} f,\ f > 0 \\ \mathbf{A} = \mathbf{Q} \text{ is real} \end{array} \right.$$

Then

$$\mathcal{E}_\kappa^\lambda(\mathbf{A}, \psi) = \mathcal{E}_\kappa^\lambda(N, f, \mathbf{Q}) = \int_\Omega \lambda^2 \frac{|\nabla f|}{\kappa^2} + \left(\frac{\lambda N}{\kappa} e_\phi - \mathbf{Q} \right)^2 f^2 + \frac{1}{2}(f^2 - 1)^2 + \lambda^2 |\mathbf{curl}\ \mathbf{Q}|^2$$

In particular for $\kappa = +\infty$, $N = +\infty$ we obtain formally for some $C \in \mathbf{R}$:

$$\mathcal{E}^\lambda_\infty(C, f, \mathbf{Q}) = \int_\Omega (\lambda C e_\phi - \mathbf{Q})^2 f^2 + \frac{1}{2}(f^2 - 1)^2 + \lambda^2 |\mathbf{curl}\ \mathbf{Q}|^2$$

with the Euler-Lagrange equations

$$\begin{cases} (\lambda C e_\phi - \mathbf{Q})^2 + f^2 = 1 \\ \lambda^2 (\mathbf{curl}\ )^2 \mathbf{Q} + (\mathbf{Q} - \lambda C e_\phi) f^2 = 0 \end{cases}$$

For $\mathbf{H} = \mathbf{curl}\ \mathbf{Q}$, $\mathbf{B} = \mathbf{Q} - \lambda C e_\phi$, and while $\mathbf{B}^2 < \frac{1}{3}$, or equivalently $|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27}$, we get

$$\begin{cases} -\lambda^2 \mathbf{curl}\ (F(\lambda^2 |\mathbf{curl}\ \mathbf{H}|^2) \mathbf{curl}\ \mathbf{H}) - \mathbf{H} = -\frac{\lambda C}{r} e_z \text{ in } \Omega \\ \mathbf{H}_T = \mathbf{H}^{ext}_T \text{ on } \partial\Omega \end{cases}$$

We note $r = R + \rho \cos\theta$ and we have used the fact that for $\mathbf{B} = \mathbf{B}_r e_r + \mathbf{B}_\phi e_\phi + \mathbf{B}_z e_z$ we have

$$\mathbf{curl}\ \mathbf{B} = \begin{pmatrix} \frac{1}{r}\partial_\phi \mathbf{B}_z - \partial_z \mathbf{B}_\phi, \\ \partial_z \mathbf{B}_r - \partial_r \mathbf{B}_z, \\ \frac{1}{r}\partial_r (r\mathbf{B}_\phi) - \frac{1}{r}\partial_\phi \mathbf{B}_r \end{pmatrix}$$

in the direct basis (e_r, e_ϕ, e_z).

If we take

$$C = \frac{C_1}{\lambda}$$

at the limit $\lambda \to 0$, we get formally

$$\mathbf{H} = \frac{C_1}{r} e_z \text{ on } \Omega$$

and outside Ω we have the Maxwell equations

$$\begin{cases} \text{div } \mathbf{H} = 0 \text{ on } \Omega^c \\ \mathbf{curl}\ \mathbf{H} = 0 \text{ on } \Omega^c \end{cases} \tag{4.3}$$

and on $\partial\Omega$, $\mathbf{H}_T$ is discontinuous (because there are some surface currents for $\lambda = 0$), and

$$[\mathbf{H} \cdot n]^{ext}_{int} = 0 \text{ on } \partial\Omega \tag{4.4}$$

We now focus on the behaviour on the boundary $\partial\Omega$ of the superconductor in the limit $\lambda \to 0$. Let a point on the boundary $x^* \in \partial\Omega$. We introduce a change of coordinates

$$\tilde{x} = \frac{x - x^*}{\lambda}$$

and

$$\tilde{\mathbf{H}}(\tilde{x}) = \mathbf{H}(x)$$

Let $n^* = n(x^*)$ the exterior unit normal vector to Ω in x^* and $r^* = r(x^*)$. Then formally at the limit $\lambda \to 0$, we get

$$\left\{ \begin{array}{l} -\mathbf{curl}\ (F(|\mathbf{curl}\ \tilde{\mathbf{H}}|^2)\mathbf{curl}\ \tilde{\mathbf{H}}) - \tilde{\mathbf{H}} = -\frac{C_1}{r^*} e_z \text{ in } \Pi^* \\ \tilde{\mathbf{H}}_T = \mathbf{H}_T^{ext}(x^*) \text{ on } \partial\Pi^* \end{array} \right.$$

where $\Pi^* = \{\tilde{x},\ \tilde{x} \cdot n^* < 0\}$. Now we can formally assume by symmetry that

$$\tilde{\mathbf{H}} = \tilde{\mathbf{H}}(\tilde{x} \cdot n^*)$$

Then a solution is

$$n^* \cdot \tilde{\mathbf{H}}(\tilde{x} \cdot n^*) = \frac{C_1}{r^*} e_z \cdot n^*$$

and the new function

$$\overline{\mathbf{H}}_T := \tilde{\mathbf{H}} - \frac{C_1}{r^*} e_z$$

satisfies

$$\left\{ \begin{array}{l} -\mathbf{curl}\ (F(|\mathbf{curl}\ \overline{\mathbf{H}}_T|^2)\mathbf{curl}\ \overline{\mathbf{H}}_T) - \overline{\mathbf{H}}_T = 0 \text{ in } \Pi^* \\ \overline{\mathbf{H}}_T = \mathbf{H}_T^{ext}(x^*) - (\frac{C_1}{r^*} e_z)_T \text{ on } \partial\Pi^* \end{array} \right.$$

and then

$$\overline{\mathbf{H}}_T(\tilde{x} \cdot n^*) = \phi(\tilde{x} \cdot n^*) e_T \text{ on } \Pi^*$$

where $\phi(y)$ satisfies an ordinary differential equation and e_T is a constant unit vector parallel to $\partial\Pi^*$. It is possible to show (see [6]) that

$$\phi(y) = \sqrt{2}\frac{\sinh(y+a)}{\cosh^2(y+a)} \text{ for some } a \in \mathbf{R}$$

In particular we can see that $|\mathbf{curl}\ \overline{\mathbf{H}}|^2 < \frac{4}{27}$ in Π^* if and only if $|\overline{\mathbf{H}}_T| < \sqrt{\frac{5}{18}}$ in Π^*. Consequently as $\lambda \to 0$, we deduce that the limit $\mathbf{H}$ satisfies

$$\left\{ \begin{array}{l} \mathbf{H}^{int} = \frac{C_1}{r} e_z \text{ on } \Omega \\ |\mathbf{H}_T^{ext} - \mathbf{H}_T^{int}| \leq \sqrt{\frac{5}{18}} \quad \text{on} \quad \partial\Omega \end{array} \right. \tag{4.5}$$

Conclusion

Formally system (4.3)-(4.4)-(4.5) has solutions $\mathbf{H}$ for C_1 small enough. And the maximum value of C_1 such that there exists a solution $\mathbf{H}$, can be interpretated as the magnetic capacity of the superconducting torus Ω.

5 Appendix: Direct study using a variational formulation

We introduce a variational formulation for model (3.1)-(3.2). This approach has the advantage to give us directly a weak solution, but the disadvantage that we have a very poor information on the solution.

It is natural to introduce the following energy

$$\mathcal{E}(\mathbf{H}) = \int_\Omega \frac{G(|\mathbf{curl}\ \mathbf{H}|^2)}{2} + \frac{\mathbf{H}^2}{2}$$

which is defined on the space

$$Y = \{\mathbf{H} \in \mathcal{H}^1(\Omega), \text{div } \mathbf{H} = 0, \mathbf{H}_T = \mathbf{H}_T^{ext}\}$$

where $\mathbf{H}_T^{ext} \in \mathcal{H}^{\frac{1}{2}}(\partial\Omega)$ is a given boundary condition. Let us remark that from lemma 2.2 in [13], there exists a (not unique) particular vector field $\mathbf{H}^p \in \mathcal{H}^1(\Omega)$, such that div $\mathbf{H}^p = 0$ and $\mathbf{H}_T^p = \mathbf{H}_T^{ext}$ on $\partial\Omega$. Then $Y \neq \emptyset$, and $Y = \mathbf{H}^p + Y^1_{0T,div}$, where $Y^1_{0T,div} = \{\mathbf{H} \in \mathcal{H}^1(\Omega), \mathbf{H}_T = 0, \text{div } \mathbf{H} = 0\}$. In particular we expect that the minimizer satisfies:

$$\begin{cases} -\mathbf{curl}\ (G'(|\mathbf{curl}\ \mathbf{H}|^2)\mathbf{curl}\ \mathbf{H}) - \mathbf{H} = 0 \text{ on } \Omega \\ \mathbf{H}_T = \mathbf{H}_T^{ext} \text{ on } \partial\Omega \end{cases} \tag{5.1}$$

Then we have the

Theorem 5.1 *There exists a unique minimmizer* $\mathbf{H}$ *of* $\mathcal{E}$ *on* Y. *Moreover* $\mathbf{H}$ *is a solution to (5.1).*

Proof of theorem 5.1
The existence and uniqueness of the solution comes from the fact that $\mathcal{E}$ is continuous, striclty convex and infinite at infinity on the space Y, because of inequality (3.8). On the contrary the Euler-Lagrange equation (5.1) is not immediate. But with the same argument as in the proof of proposition 3.2, we can justify this Euler-Lagrange equation. This ends the proof of the theorem 5.1.

Let us remark that if **curl H** is continuous on Ω, then the results of Morrey (see [15]) permit us to deduce that the magnetic field $\mathbf{H}$ is C^∞ on Ω, and even analytic where $|\mathbf{curl}\ \mathbf{H}|^2 < \frac{4}{27} - \delta$. Nevertheless this variational formulation seems not so easy to use to deduce some qualitative properties of the solutions. In particular we have not found in the literature some results which permit us directly to deduce that **curl H** is continuous, and then that $\mathbf{H}$ is a classical solution if $\mathbf{H}_T^{ext}$ is smooth enough. This approach would involve more work in this direction. That is why we simply have used the inverse function theorem in section 3.2, in the framework of solutions in $C^{2+\alpha}(\overline{\Omega})$.

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