Convergence of Meissner minimisers of the Ginzburg-Landau energy of superconductivity as $\kappa \to +\infty$

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Abstract

The Meissner solution of a smooth cylindrical superconducting domain subject to a uniform applied axial magnetic field is examined. Under an additional convexity condition the uniqueness of the Meissner solution is proved. It is then shown that it is a local minimiser of the Ginzburg-Landau energy $\mathcal{E}_\kappa$. For applied fields less than a critical value the existence of the Meissner solution is proved for large enough Ginzburg-Landau parameter κ. Moreover it is proved that the Meissner solution converges to a local minimiser of a certain energy $\mathcal{E}_\infty$ in the limit as $\kappa \to \infty$. Finally, it is proved that for κ large enough the Meissner solution is not a global minimiser of $\mathcal{E}_\kappa$.

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1 Introduction

At temperatures below their critical temperature, type II superconductors can exist in one of three different states, depending on the strength of the applied magnetic field. For low magnetic fields they are in what is known as the Meissner state, in which the magnetic field is excluded from the interior except in thin boundary layers whose thickness is known as the penetration depth, denoted by λ. At intermediate applied magnetic fields the magnetic

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field penetrates the sample in the form of quantised flux tubes, usually known as vortices, since they are each circled by a vortex of superconducting current. The cores of these tubes comprise non-superconducting (normal) material, and are of a radius ξ, which is known as the coherence length. An important physical parameter is the ratio of these two lengthscales, $\kappa = \lambda/\xi$, which is known as the Ginzburg-Landau parameter. For Type II superconductors $\kappa > 1/\sqrt{2}$, but κ is typically much larger than this, especially in superconducting alloys. We note that all the high-temperature superconducting materials discovered to date have very large (typically 50 to 100) values of κ.

As the applied magnetic field increases further the density of vortices increases. For large applied magnetic fields the material becomes completely normal.

The transition from the vortex state to the normal state occurs at the upper critical field, H_{c_2}, as is second order and therefore reversible. On the other hand, the transition from the Meissner state to the vortex state is first order, and exhibits hysteresis. The magnetic field at which the vortex state becomes theoretically energetically favourable is known as the lower critical field, H_{c_1}, but there is both "superheating" and "supercooling" around this field. The superheating of the Meissner state is the subject of this paper.

We study the Meissner state for finite κ and in the limit as $\kappa \to \infty$, in which case the model is substantially simplified. The layout of the paper is as follows. In the remainder of the introduction we introduce the Ginzburg-Landau theory of superconductivity and state our main results. We also establish the convergence of the Meissner solution to the solution of the simplified problem as $\kappa \to \infty$.

In section 3 we prove the uniqueness of the Meissner solution, while in section 4 we establish its existence for magnetic fields less than a critical value, and for large enough κ. In section 5 we consider global and local minimisers of the GL free energy. We show that the Meissner solution is a local minimiser for magnetic fields below a (second) critical value, but that for large κ and nonzero applied magnetic field it is not the global minimiser. Finally, we present our conclusions.

1.1 The Ginzburg-Landau model

We consider a superconductor material occupying a domain $\Sigma \subset I\!R^3$, in a uniform exterior magnetic field $H_0\boldsymbol{e_3}$. The state of the superconductor is characterised by a complex order parameter Ψ (defined on Σ) such that $|\Psi|^2$ represents the number density of superconducting electrons, which may be thought of as a kind of "macroscopic wavefunction", and the magnetic vector potential $\boldsymbol{A}$, which is such that the magnetic field is given by

$$\boldsymbol{H} = \text{curl } \boldsymbol{A}.$$

In the theory introduced by Ginzburg and Landau [9] the equilibrium state of the superconductor is given by the minimiser of the Ginzburg-Landau energy (for a review of the theory the reader may consult [4, 6, 7]):

$$\mathcal{E} = \int_\Sigma \left| \left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right) \Psi \right|^2 + \frac{(|\Psi|^2 - 1)^2}{2} + | \textbf{curl } \boldsymbol{A} - H_0 \boldsymbol{e_3}|^2 \, dV + \int_{I\!R^3 \setminus \Sigma} | \textbf{curl } \boldsymbol{A} - H_0 \boldsymbol{e_3}|^2 \, dV \tag{1.1}$$

Here κ is the Ginzburg-Landau parameter of the introduction. The energy (1.1) is in the usual nondimensional form in which $|\Psi| = 1$ represents wholly superconducting material, $|\Psi| = 0$ represents wholly normal material, and lengths have been scaled with the penetration depth λ, which is the typical lengthscale for variations in the magnetic potential. The vortex core radius is then given by $1/\kappa$.

The Euler-Lagrange equations corresponding to (1.1) are the celebrated Ginzburg-Landau equations:

$$\begin{aligned}
\left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right)^2 \Psi &= \left(|\Psi|^2 - 1 \right) \Psi \qquad \text{in } \Sigma, && (1.2)\\
(\text{curl})^2 \boldsymbol{A} &= \frac{i}{2\kappa} \left(\Psi \nabla \Psi^* - \Psi^* \nabla \Psi \right) - |\Psi|^2 \boldsymbol{A} \qquad \text{in } \Sigma, && (1.3)\\
(\text{curl})^2 \boldsymbol{A} &= \mathbf{0} \qquad \text{in } I\!R^3 \setminus \Sigma, && (1.4)\\
\boldsymbol{n} \cdot \left(\frac{1}{\kappa} \nabla - i\boldsymbol{A} \right) \Psi &= 0 \qquad \text{on } \partial\Sigma, && (1.5)\\
[\boldsymbol{n} \wedge \boldsymbol{A}] &= \mathbf{0}, && (1.6)\\
[\boldsymbol{n} \wedge \text{curl } \boldsymbol{A}] &= \mathbf{0}, && (1.7)\\
\text{curl } \boldsymbol{A} &\to H_0 \boldsymbol{e_3} \qquad \text{as } |\boldsymbol{x}| \to \infty, && (1.8)
\end{aligned}$$

where $[\,]$ represents the jump in the enclosed quantity across $\partial\Sigma$, and $\boldsymbol{n}$ is the unit outward normal to Σ. We consider the case where the superconductor has the shape of a long cylinder, mathematically idealised by $\Sigma = \Omega \times I\!R$ where the section of the cylinder $\Omega \subset I\!R^2$ is a bounded smooth open set, and the axis of the cylinder is parallel to $\boldsymbol{e_3}$. Then the energy (1.1) makes no sense, but assuming that the problem is invariant by translation in the direction $\boldsymbol{e_3}$ of the axis of the cylinder, we can redefine simply the energy as an integral over the cross-section $I\!R^2$. If we assume also that the magnetic field lies only in the $\boldsymbol{e}_3$ direction, then (1.4) and (1.8) imply that the magnetic field is constant and equal to $H_0\boldsymbol{e}_3$ in $I\!R^2 \setminus \Omega$. Then the continuity conditions on $\partial\Omega$ can be replaced by $\boldsymbol{H} = H_0 \boldsymbol{e}_3$ there.

Vortices correspond to zeros of the order parameter Ψ, while the Meissner solution is a solution in which $|\Psi| > 0$ on $\overline{\Omega}$. In this case we can make the change of gauge

$$\begin{cases} \Psi = f e^{i\chi}, f > 0 \\ \mathbf{Q} = \boldsymbol{A} - \frac{1}{\kappa} \nabla \chi \end{cases}$$

where χ is uniquely defined (up to $2k\pi$, $k \in \mathbf{Z}$) if for example Ψ is smooth. Then we define the new energy for $\kappa \in (0, +\infty]$:

$$\mathcal{E}_\kappa(f, \mathbf{Q}) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\text{curl } \mathbf{Q} - H_0|^2 + G(f, \mathbf{Q})$$

where

$$G(f,\mathbf{Q}) = f^2\mathbf{Q}^2 + \frac{(f^2-1)^2}{2}$$

and curl $\mathbf{Q} = \partial_1 Q_2 - \partial_2 Q_1$ with $\mathbf{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$. For $\kappa = +\infty$, we take

$$\mathcal{E}_\infty(f,\mathbf{Q}) = \int_\Omega |\text{curl } \mathbf{Q} - H_0|^2 + G(f,\mathbf{Q})$$

Let us remark that $\frac{1}{2}G''(f,\mathbf{Q})\cdot(g,\boldsymbol{q})^2 = (f\boldsymbol{q}+2g\mathbf{Q})^2 + 3g^2(f^2-\mathbf{Q}^2-\frac{1}{3})$, so that G is convex if $f^2-\mathbf{Q}^2 \geq \frac{1}{3}$.

1.2 A classical difficulty and the mathematical framework

Let V be the space (to be defined) on which we want to study the energy $\mathcal{E}_\kappa$. Let us remark that the functional $\mathcal{E}_\kappa$ is convex on the convex set $K_0 = \{(f,\mathbf{Q}) \in V,\ f^2-\mathbf{Q}^2 \geq \frac{1}{3}\}$. We are interested in the solutions $(f_\kappa, \mathbf{Q}_\kappa)$ such that

$$\inf_{(f,\mathbf{Q})\in K_0} \mathcal{E}_\kappa(f,\mathbf{Q}) \tag{1.9}$$

is attained in $(f_\kappa,\mathbf{Q}_\kappa)$ and $(f_\kappa,\mathbf{Q}_\kappa)$ belongs to $\text{Int}(K_0)$, the interior of K_0. For the value H_0 of the exterior magnetic field, we will denote by $u_\kappa^{H_0}$ such a minimising point $(f_\kappa,\mathbf{Q}_\kappa)$.
Then at least two choices seem natural:
(a) $V = V_4$, where $V_4 = \{(f,\mathbf{Q}) \in L^4 \times L^4,\ \nabla f \in L^2,\ \text{curl } \mathbf{Q} \in L^2\}$, space on which the energy $\mathcal{E}_\kappa$ is naturally defined.
(b) The constraint which defines K_0 requires an L^∞ control on f and $\mathbf{Q}$. Then it is natural to define $V = V_\infty$ where $V_\infty = \{(f,\mathbf{Q}) \in L^\infty \times L^\infty,\ \nabla f \in L^2,\ \text{curl } \mathbf{Q} \in L^2\}$.

Let us remark that in each case (a) or (b) the energy $\mathcal{E}_\kappa$ is defined and continuous on V. Generally the existence of a minimiser $u_\kappa^{H_0}$ is a consequence of the fact that $\mathcal{E}_\kappa(f,\mathbf{Q}) \to +\infty$ as $|(f,\mathbf{Q})|_V \to +\infty$; but here in both cases (a) and (b) this property is not clear. A second difficulty is that if $u_\kappa^{H_0}$ is a minimiser on K_0, how can we know that $u_\kappa^{H_0} \notin \partial K_0$ (and then $u_\kappa^{H_0} \in \text{Int}(K_0)$). A great difficulty in the case (a) is that if $u_\kappa^{H_0} \in \text{Int}(K_0)$, then for a perturbation of applied fields h in a neighbourhood of H_0, it is not possible to claim that u_κ^h stays in $\text{Int}(K_0)$; on the contrary this is possible in the case (b). This difficulty is associated to the fact that K_0 is not closed when $V = V_4$, but is closed if $V = V_\infty$.
Then we chose $V = V_\infty$ and we define the convex closed sets

$$K_{\delta_0} = \{(f,\mathbf{Q}) \in V,\ f^2-\mathbf{Q}^2 \geq \frac{1}{3}+\delta_0\},\ \delta_0 \geq 0$$

Then $\text{Int}(K_0) = \cup_{\delta_0>0} K_{\delta_0}$. In particular $\mathcal{E}_\kappa$ is strictly convex on K_{δ_0}. Moreover there exists $c = c(\delta_0) > 0$ such that if $(f,\mathbf{Q})$ is a critical point of $\mathcal{E}_\kappa$ and $(f,\mathbf{Q}), (f+\overline{f}, \mathbf{Q}+\overline{\mathbf{Q}}) \in K_{\delta_0}$, then

$$\mathcal{E}_\kappa(f+\overline{f},\mathbf{Q}+\overline{\mathbf{Q}}) \geq \mathcal{E}_\kappa(f,\mathbf{Q}) + \int_\Omega \frac{|\nabla\overline{f}|^2}{\kappa^2} + |\text{curl } \overline{\mathbf{Q}}|^2 + c(|\overline{f}|^2 + |\overline{\mathbf{Q}}|^2)$$

1.3 Main results

DEFINITION 1.1 *We say that* $(f, \mathbf{Q}) \in V$ *is a local minimiser of* $\mathcal{E}_\kappa$ *on* V *if* $\exists \epsilon = \epsilon(f, \mathbf{Q}) > 0$ *such that if* $|(\overline{f}, \overline{\mathbf{Q}})|_V < \epsilon$*, then* $\mathcal{E}_\kappa(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) \geq \mathcal{E}_\kappa(f, \mathbf{Q})$*. The function* $(f, \mathbf{Q})$ *is said to be a strict local minimiser if moreover* $\mathcal{E}_\kappa(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) > \mathcal{E}_\kappa(f, \mathbf{Q})$ *for* $(\overline{f}, \overline{\mathbf{Q}}) \neq 0$*. Finally,* $(f, \mathbf{Q})$ *is a global minimiser if we can take* $\epsilon = +\infty$ *in the definition of a local minimiser.*

An element $(f, \mathbf{Q}) \in V$ is said to be a critical point of $\mathcal{E}_\kappa$ on V if it satisfies the Euler-Lagrange equations:

$$\left\{ \begin{array}{l} \frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f \\ -\,\mathbf{curl}\,(\mathrm{curl}\,\mathbf{Q}) = \frac{1}{2}G'_{\mathbf{Q}} \end{array} \right| \text{ on } \Omega \tag{1.10}$$

with the boundary conditions (that we obtain formally)

$$\left\{ \begin{array}{l} \frac{\partial_n f}{\kappa^2} = 0 \\ \mathrm{curl}\,\mathbf{Q} = H_0 \end{array} \right| \text{ on } \partial\Omega \tag{1.11}$$

We assume that $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0,1)$. Let us remark that we can deduce (1.11) rigorously as weak boundary conditions (see equation (2.2)).
Then we have (see [12]-[13] for similar uniqueness results for the Meissner solution)

THEOREM 1.2 *If* $(f, \mathbf{Q})$ *is a critical point of* $\mathcal{E}_\kappa$ *on* V *(a weak solution of (1.10)-(1.11)), and* $(f, \mathbf{Q}) \in Int(K_0)$*, then* $(f, \mathbf{Q})$ *is unique. Moreover* $(f, \mathbf{Q})$ *is a strict local minimiser of* $\mathcal{E}_\kappa$*.*

In what follows we denote by $(f_\kappa, \mathbf{Q}_\kappa)$ this local minimiser when it exists. We will use the more precise and compact notation $u_\kappa^{H_0}$ to denote this solution under an exterior magnetic field H_0.
Let us recall that in [1], Berestycki Bonnet and Chapman have proved with help of an inverse function theorem that there exists a critical magnetic field H_0^* which only depends on Ω such that $\forall H_0 \in [0, H_0^*)$, $u_\infty^{H_0}$ exists; $u_\infty^{H_0^*} \in \partial K_0$; and $\forall H_0 > H_0^*$, $u_\infty^{H_0}$ does not exist. Then we have

THEOREM 1.3 *There exists a positive countinuous and nondecreasing function* γ *defined on* $(0, +\infty]$ *such that* $\gamma(+\infty) = H_0^*$ *and such that* $\forall (\kappa, H_0) \in \mathcal{P} := \{(\kappa, H_0),\ \kappa \in (0, +\infty],\ 0 \leq H_0 < \gamma(\kappa)\}$*,* $u_\kappa^{H_0} \in Int(K_0)$ *exists. Moreover* $(\kappa, H_0) \to u_\kappa^{H_0} \in V$ *is continuous on* $\mathcal{P}$*. In particular this implies that for* $0 \leq H_0 < H_0^*$*:* $u_\kappa^{H_0} \to u_\infty^{H_0}$ *in* V *as* $\kappa \to +\infty$*.*

Now we give a result which proves that for κ large enough, $u_\kappa^{H_0}$ is only a local minimiser and not a global minimiser.

THEOREM 1.4 $\forall H_0 \geq 0, \exists C = C(H_0) > 0$ *such that*

$$\inf_{(f, \mathbf{Q}) \in V} \mathcal{E}_\kappa(f, \mathbf{Q}) \leq \frac{C}{\sqrt{\kappa}}$$

In particular for $\kappa = +\infty$*,* $\mathcal{E}_\infty$ *has no global minimiser if* $H_0 > 0$*.*

1.4 Proof of theorem 1.3

We give here the proof of theorem 1.3, because it is very short and shows the main idea of this article. The proof of theorem 1.3 uses the following two propositions:

PROPOSITION 1.5 *(case $\kappa \in (0,+\infty)$)*
For $\kappa_0 \in (0,+\infty)$ and $H_0 \geq 0$, if $u_{\kappa_0}^{H_0} \in Int(K_0)$, then $\exists \epsilon = \epsilon(\kappa_0, H_0) > 0$, such that if $|h - H_0|, |\kappa - \kappa_0| < \epsilon$, then u_κ^h exists and $u_\kappa^h \in Int(K_0)$. Moreover the map $(\kappa, h) \longmapsto u_\kappa^h \in V$ is continuous.

If we denote curl $\mathbf{Q}_\infty$ by H_∞, and if $(f_\infty, \mathbf{Q}_\infty)$ is the solution $u_\infty^{H_0}$, let us remark that from [1], the map $H_0 \longmapsto \sup_{\overline{\Omega}} |\nabla H_\infty|$ is nondecreasing. Then for all $\delta_0 > 0$, there exists a unique $H_{\delta_0}^* \in (0, H_0^*)$ such that $\forall h \in [0, H_{\delta_0}^*]$, $u_\infty^h \in K_{\delta_0}$, and $\forall h > H_{\delta_0}^*$, $u_\infty^h \notin K_{\delta_0}$.
We need to have an a priori L^∞-control on f and $\mathbf{Q}$, so we define for $M > 1$ the set

$$K_{\delta_0}^M = \{(f, \mathbf{Q}) \in K_{\delta_0},\ f, |\mathbf{Q}| \leq M \text{ and } f \geq \frac{1}{M}\}$$

Then we have

PROPOSITION 1.6 *(case κ close to $+\infty$)*
For $\delta_0 > 0$, $\exists \kappa_0 = \kappa_0(\delta_0) > 0$, $\forall M > 1$, $\forall \theta \in (0,1)$, $\exists C_\theta = C_\theta(\delta_0, M) > 0$ such that $\forall H_0 \in [0, H_{\delta_0}^*]$, $\forall \kappa \geq \kappa_0$, then if $u_\kappa^{H_0}$ exists and $u_\kappa^{H_0} \in K_{\delta_0}^M$, then

$$|u_\kappa^{H_0} - u_\infty^{H_0}|_V \leq \frac{C_\theta(\delta_0, M)}{\kappa^{\frac{1}{2}-\theta}}$$

Given these two propositions, the proof of theorem 1.3 can be made in three steps: (with some $\theta \in (0, \frac{1}{2})$ fixed):

Step 1: Let $\delta_0 > 0$ be given. Choose $M_0 = M_0(\delta_0) > 1$ such that $\forall H_0 \in [0, H_{\delta_0}^*]$, $u_\infty^{H_0} \in K_{\delta_0}^{M_0}$. Then choose $\kappa_1 = \kappa_1(\delta_0) > 0$, large enough such that for all $H_0 \in [0, H_{\delta_0}^*]$, then

$$\forall u \in V,\ \left(|u - u_\infty^{H_0}|_V \leq \frac{C_\theta(\frac{\delta_0}{4}, 4M_0)}{\kappa_1^{\frac{1}{2}-\theta}}\right) \Longrightarrow \left(u \in K_{\frac{\delta_0}{2}}^{2M_0}\right) \tag{1.12}$$

Step 2: Let $\kappa \geq \kappa_1$. We have $u_\kappa^0 = (1,0) \in Int(K_0)$ and proposition 1.5 allows us to build a u_κ^h for h in a neighbourhood of 0, and with help of the control given by proposition 1.6 and (1.12), u_κ^h stays in $K_{\frac{\delta_0}{2}}^{2M_0}$, and then we have existence of u_κ^h until $h = H_{\delta_0}^*$.

Step 3: For $H_0 \in [0, H_0^*)$, from proposition 1.5, the map $\kappa \longmapsto u_\kappa^{H_0} \in V$ is continuous, and from proposition 1.6, $u_\kappa^{H_0}$ admits the limit $u_\infty^{H_0}$ as $\kappa \to +\infty$. Consequently we can always chose a positive continuous and nondecreasing function γ as described in theorem 1.3.
The theorem 1.3 is then proved.

2 Uniqueness of Meissner solutions in infinite cylinder under a convexity condition.

Here we are interested in the uniqueness of Meissner solutions of the Ginzburg-Landau equations in infinite cylinder $\Sigma_\infty = \Omega \times I\!R$. The problem is invariant by translation, and we usually restrict the study to solutions which have this symmetry (which is the case in theorem 1.2):

$$f = f(x_1, x_2), \qquad \mathbf{Q} = \begin{pmatrix} Q_1(x_1, x_2) \\ Q_2(x_1, x_2) \\ 0 \end{pmatrix}, \qquad (x_1, x_2) \in \partial\Omega.$$

Here we relax these conditions somewhat, but not completely. We still suppose that the magnetic field outside the superconductor lies in the $\boldsymbol{e}_3$ direction, so that it is constant and equal to $H_0\boldsymbol{e}_3$, and then energy density in $I\!R^3 \backslash \Sigma_\infty$ vanishes. However, we relax the symmetry assumption inside the superconductor, so that we prove the uniqueness (under a certain convexity condition) of solutions $(f, \mathbf{Q})$ defined in all the cylinder Σ_∞ of the form:

$$f = f(x_1, x_2, x_3), \qquad \mathbf{Q} = \begin{pmatrix} Q_1(x_1, x_2, x_3) \\ Q_2(x_1, x_2, x_3) \\ Q_3(x_1, x_2, x_3) \end{pmatrix};$$

theorem 1.2 will be a particular case of theorem 2.1. As we have already mentioned, we can not derive the Ginzburg-Landau equations on an infinite cylinder from an energy. Let us then take as a model the case of a periodic cylinder, and derive rigorously the Euler-Lagrange equations. In this way we will find the minimal regularity assumption to impose to our solution on the infinite cylinder.
Then for $\Lambda > 0$, let the periodic cylinder $\Sigma_\Lambda = \Omega \times \Lambda\mathbf{S}^1$ where $\mathbf{S}^1 = \mathbf{Z}/(2\pi\mathbf{Z})$, and let f, $\mathbf{Q}$ be functions defined on $\overline{\Sigma}_\Lambda$. Then for $\kappa \in (0, +\infty]$, the Ginzburg-Landau energy is:

$$\mathcal{E}_\kappa^\Lambda(f, \mathbf{Q}) = \int_{\Sigma_\Lambda} \frac{|\nabla f|^2}{\kappa^2} + |\ \mathbf{curl}\ \mathbf{Q} - H_0\boldsymbol{e}_\mathbf{3}|^2 + G(f, \mathbf{Q})$$

Let us recall that $V = \{(f, \mathbf{Q}) \in L^\infty \times L^\infty,\ \nabla f \in L^2,\ \ \mathbf{curl}\ \mathbf{Q} \in L^2\}$. Then $\mathcal{E}_\kappa$ is defined on V. In particular if $(f, \mathbf{Q})$ is an extremum of $\mathcal{E}_\kappa^\Lambda$, then we get the Euler-Lagrange equations in the distributional sense inside Σ_λ:

$$\left\{ \begin{array}{l} \frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f \in L^\infty \\ -(\ \mathbf{curl}\ )^2\mathbf{Q} = \frac{1}{2}G'_\mathbf{Q} \in L^\infty \end{array} \right| \text{ in } \mathcal{D}'(\Sigma_\Lambda) \tag{2.1}$$

and the boundary conditions in $H^{-\frac{1}{2}}(\partial\Sigma_\Lambda)$:

$$\left\{ \begin{array}{l} \partial_n f = 0 \\ (\ \mathbf{curl}\ \mathbf{Q} - H_0\boldsymbol{e}_\mathbf{3}) \wedge n = 0 \end{array} \right. \tag{2.2}$$

The only difficulty is to justify the weak sense of the boundary conditions. It is possible to define these conditions by duality (for example $< \partial_n f, \phi >_{H^{-\frac{1}{2}}(\partial\Sigma_\Lambda)\times H^{\frac{1}{2}}(\partial\Sigma_\Lambda)} = \int_{\Sigma_\Lambda} \Delta f \phi + \nabla f \cdot \nabla \phi$ for $\phi \in H^1(\Sigma_\Lambda)$). In particular see [8] for the definition of the tangential trace of a vector field $\boldsymbol{A} \in L^2$ with $\mathbf{curl}\ \boldsymbol{A} \in L^2$.

Minimal regularity assumption
In the following we assume at least the minimal regularity $f, \mathbf{Q} \in L^\infty(\Sigma_\infty)$, ∇f, $\mathbf{curl}\ \mathbf{Q} \in L^2_{loc}(\Sigma_\infty)$ (and $\mathbf{curl}\ ^2\mathbf{Q} \in L^2_{loc}(\Sigma_\infty)$ which is a consequence of the Ginzburg-Landau equations).

Then we have in the case of the infinite cylinder:

THEOREM 2.1 *For $\kappa \in (0, +\infty]$, if $(f, \mathbf{Q})$ satisfies (2.1)-(2.2) on $\Sigma_\infty = \Omega \times I\!R$, with the minimal regularity assumption and $(f, \mathbf{Q}) \in K_{\delta_0} = \{(f, \mathbf{Q}) \in V,\ f^2 - \mathbf{Q}^2 \geq \frac{1}{3} + \delta_0\}, \delta_0 > 0$, then $(f, \mathbf{Q})$ is unique and*

$$\begin{cases} Q_3 \equiv 0 \\ f \text{ and } \mathbf{Q} \text{ are independent on } x_3 \end{cases}$$

In particular $\boldsymbol{H} = H(x_1, x_2)\boldsymbol{e_3}$ where $\boldsymbol{H} = \mathbf{curl}\ \mathbf{Q}$, $(x_1, x_2) \in \Omega$.

Now we can identify the solutions $(f, \mathbf{Q})$ on Σ_∞ as functions defined on Ω, and we have:

PROPOSITION 2.2 *The solutions $(f, \mathbf{Q})$ of theorem 2.1 are analytic in Ω. On $\overline{\Omega}$, the solutions $(f, \mathbf{Q})$ have at least the regularity $C^{1+m,\alpha}$ if $\partial\Omega \in C^{2+m,\alpha}$, $m \geq 0$.*

Proof of proposition 2.2
If $f, \mathbf{Q} \in L^\infty(\Omega)$, ∇f, $\mathbf{curl}\ \mathbf{Q} \in L^2(\Omega)$, then for finite κ:

$$\begin{cases} \frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f \in L^\infty \text{ in } \mathcal{D}'(\Omega) \\ \partial_n f = 0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega). \end{cases}$$

Then (see [2] for the regularity of weak solutions) $f \in H^2(\Omega)$, and by the standard L^p-elliptic theory, $f \in W^{2,p}$ and therefore $f \in C^{1,\theta}$ (Schauder theory). Let us remark that the equation on $\mathbf{Q}$ is elliptic in dimension 1, but is not elliptic in dimension $n \geq 2$. So it is interesting to obtain an elliptic formulation, and it was done in [1] introducing the magnetic field $H = \text{curl}\ \mathbf{Q} = \partial_1 Q_2 - \partial_2 Q_1$. Using the explicit form of G it is easy to obtain an elliptic equation satisfied by H. Writing $-\ \mathbf{curl}\ (\text{curl}\ \mathbf{Q}) = \frac{1}{2}G'_{\mathbf{Q}}$ as $-\ \mathbf{curl}\ H = f^2\mathbf{Q}$ where $\mathbf{curl}\ H = \begin{pmatrix} \partial_2 H \\ -\partial_1 H \end{pmatrix}$, we see that $H = \text{curl}\ \mathbf{Q} = \text{curl}\ (-\frac{\mathbf{curl}\ H}{f^2}) = \nabla \cdot (\frac{\nabla H}{f^2})$. Thus

$$\begin{cases} \nabla \cdot (\frac{\nabla H}{f^2}) - H = 0 \text{ in } \mathcal{D}'(\Omega) \\ H = H_0 \text{ in } H^{-\frac{1}{2}}(\partial\Omega); \end{cases} \tag{2.3}$$

$\mathbf{Q}$ may be found by the inverse formula $\mathbf{Q} = -\frac{\textbf{curl } H}{f^2}$. Then $H \in C^{2,\theta}$ (see [2]). Consider the elliptic system:

$$\left\{\begin{array}{l} \frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f(f, -\frac{\textbf{curl } H}{f^2}) \\ \partial_n f = 0 \\ \nabla(\frac{\nabla H}{f^2}) - H = 0 \\ H = H_0 \end{array}\right. \tag{2.4}$$

A classical bootstrap argument permits us to see that if $(f, H) \in C^{m,\alpha} \times C^{m+1,\alpha}$ with $m \geq 1$, then $(f, H) \in C^{m+1,\alpha} \times C^{m+2,\alpha}$, and then $f, H \in C^\infty(\Omega)$. Moreover because the elliptic (non-linear) system is analytic, the solutions (f, H) are analytic on Ω (result of Morrey, see [10]). In particular if $\partial\Omega$ is analytic, then (f, H) is analytic on $\overline{\Omega}$. Finally we find the regularity on $\mathbf{Q}$, taking $\mathbf{Q} = -\frac{\textbf{curl } H}{f^2}$. For infinite κ, the proof can be found in [1]. This ends the proof.

Proof of theorem 2.1

Let $(f_j, \mathbf{Q}_j)$, $j = 1, 2$ be two solutions in K_{δ_0} of (2.1)-(2.2) on $\Sigma_\infty = \Omega \times I\!R$. Let $\psi(x_3)$ a function with compact support. We multiply the equation $\frac{1}{\kappa^2}\Delta(f_2 - f_1) = \frac{1}{2}[G'_f]^{f_2,\mathbf{Q}_2}_{f_1,\mathbf{Q}_1}$ by $\psi(f_2 - f_1)$, and the equation $-(\textbf{ curl })^2(\mathbf{Q}_2 - \mathbf{Q}_1) = \frac{1}{2}[G'_\mathbf{Q}]^{f_2,\mathbf{Q}_2}_{f_1,\mathbf{Q}_1}$ by $\psi(\mathbf{Q}_2 - \mathbf{Q}_1)$. Then by integration on $\Sigma = \Sigma_\infty$ we obtain:

$$\int_\Sigma \Big(\frac{1}{\kappa^2}\nabla(\psi(f_2 - f_1)) \cdot \nabla(f_2 - f_1) + \textbf{ curl } (\psi(\mathbf{Q}_2 - \mathbf{Q}_1)) \cdot \textbf{ curl } (\mathbf{Q}_2 - \mathbf{Q}_1) \\ + \frac{1}{2}[G']^{f_2,\mathbf{Q}_2}_{f_1,\mathbf{Q}_1} \cdot \psi \begin{pmatrix} f_2 - f_1 \\ \mathbf{Q}_2 - \mathbf{Q}_1 \end{pmatrix}\Big) = 0$$

Here we have integrated by parts one time in f, and one time in $\mathbf{Q}$, using the equality $\int_\Sigma(\textbf{ curl } \boldsymbol{A}) \cdot \boldsymbol{B} = \int_\Sigma \boldsymbol{A} \cdot (\textbf{ curl } \boldsymbol{B}) + \int_{\partial\Sigma}(\boldsymbol{A} \wedge \boldsymbol{B}) \cdot \boldsymbol{n}$. Moreover the boundary terms obtained for f and for $\mathbf{Q}$ are zero because of (2.2). Then we obtain (using $\textbf{curl } (f\boldsymbol{A}) = \nabla f \wedge \boldsymbol{A} + f \textbf{ curl } \boldsymbol{A}$):

$$\int_\Sigma \psi \left\{\frac{1}{\kappa^2}|\nabla(f_2 - f_1)|^2 + |\textbf{ curl } (\mathbf{Q}_2 - \mathbf{Q}_1)|^2 + \frac{1}{2}[G']^{f_2,\mathbf{Q}_2}_{f_1,\mathbf{Q}_1} \cdot \begin{pmatrix} f_2 - f_1 \\ \mathbf{Q}_2 - \mathbf{Q}_1 \end{pmatrix}\right\} = \\ \int_\Sigma \left(-\frac{f_2 - f_1}{\kappa}\right) \cdot \left(\frac{\nabla\psi}{\sqrt{\psi}}\right) \cdot \left(\sqrt{\psi}\frac{\nabla(f_2 - f_1)}{\kappa}\right) \\ - \left(\frac{\nabla\psi}{\sqrt{\psi}} \wedge (\mathbf{Q}_2 - \mathbf{Q}_1)\right) \cdot \left(\sqrt{\psi} \textbf{ curl } (\mathbf{Q}_2 - \mathbf{Q}_1)\right) \tag{2.5}$$

But G is strictly convex on K_{δ_0}, and from the inequality of convexity we deduce that

$$\exists c = c(\delta_0) > 0 \text{ such that } \frac{1}{2}[G']^{f_2,\mathbf{Q}_2}_{f_1,\mathbf{Q}_1} \cdot \begin{pmatrix} f_2 - f_1 \\ \mathbf{Q}_2 - \mathbf{Q}_1 \end{pmatrix} \geq c((f_2 - f_1)^2 + (\mathbf{Q}_2 - \mathbf{Q}_1)^2).$$

On another hand we have the L^∞-estimates on the right-hand side of (2.5): $|f_2 - f_1| \leq C$, $|\mathbf{Q}_2 - \mathbf{Q}_1| \leq C$. Then from the Cauchy-Schwarz inequality we have:

$$\int_\Sigma \psi \left\{ \frac{|\nabla (f_2 - f_1)|^2}{\kappa^2} + |\ \mathbf{curl}\ (\mathbf{Q}_2 - \mathbf{Q}_1)|^2 + \delta((f_2 - f_1)^2 + (\mathbf{Q}_2 - \mathbf{Q}_1)^2) \right\}$$
$$\leq C \left(\int_\Sigma \frac{|\nabla \psi|^2}{\psi} \right)^{\frac{1}{2}} \left\{ \left(\int_\Sigma \psi \frac{|\nabla (f_2 - f_2)|^2}{\kappa^2} \right)^{\frac{1}{2}} + \left(\int_\Sigma \psi |\ \mathbf{curl}\ (\mathbf{Q}_2 - \mathbf{Q}_1)|^2 \right)^{\frac{1}{2}} \right\}$$

But if we take $\psi(x) = \psi_0(\lambda x_3)$, where $\psi_0 \geq 0$ has a support in $[-2, 2]$, is equal to 1 on $[-1, 1]$, then using the fact that $\int_\Sigma \frac{|\nabla \psi|^2}{\psi} = \lambda \int_\Sigma \frac{|\nabla \psi_0|^2}{\psi_0}$, we deduce as $\lambda \to 0$, that:

$$\int_\Sigma \frac{|\nabla (f_1 - f_2)|^2}{\kappa^2} + |\ \mathbf{curl}\ (\mathbf{Q}_2 - \mathbf{Q}_1)|^2 + c((f_2 - f_1)^2 + (\mathbf{Q}_2 - \mathbf{Q}_1)^2) = 0.$$

Consequently $f_2 = f_1$ and $\mathbf{Q}_2 = \mathbf{Q}_1$.
Now if $(f, \mathbf{Q})$ is a solution, for $t \in I\!R$ let $f^t(x_1, x_2, x_3) = f(x_1, x_2, x_3 + t)$, and $\mathbf{Q}^t(x_1, x_2, x_3) = \mathbf{Q}(x_1, x_2, x_3 + t)$. It is clear that $(f^t, \mathbf{Q}^t)$ is a solution, and from the uniqueness result we deduce that f and $\mathbf{Q}$ do not depend on the coordinate x_3.
Now let $\tilde{\mathbf{Q}} = \begin{pmatrix} Q_1(x_1, x_2) \\ Q_2(x_1, x_2) \\ 0 \end{pmatrix}$. Then (for every fixed Λ) we have by construction

$$\mathcal{E}_\kappa^\Lambda(f, \mathbf{Q} + \overline{\mathbf{Q}}) \leq \mathcal{E}_\kappa^\Lambda(f, \mathbf{Q}) \text{ where } \overline{\mathbf{Q}} = \tilde{\mathbf{Q}} - \mathbf{Q}. \tag{2.6}$$

We conclude with the help of the following lemma:

LEMMA 2.3 *Assume that $(f, \mathbf{Q}) \in V$ is a solution of (2.1)-(2.2) on Σ_Λ. Consider $(\overline{f}, \overline{\mathbf{Q}}) \in V$, and assume that $(f, \mathbf{Q})$ and $(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}})$ are in K_{δ_0}. Then we have*

$$\mathcal{E}_\kappa^\Lambda(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) \geq \mathcal{E}_\kappa^\Lambda(f, \mathbf{Q}) + \int_{\Sigma_\Lambda} \frac{|\nabla \overline{f}|^2}{\kappa^2} + |\ \mathbf{curl}\ \overline{\mathbf{Q}}|^2 + c(|\overline{f}|^2 + |\overline{\mathbf{Q}}|^2). \tag{2.7}$$

From (2.6), this lemma implies (with $\overline{f} = 0$) that $\int_{\Sigma_\Lambda} |\ \mathbf{curl}\ \overline{\mathbf{Q}}|^2 + c|\overline{\mathbf{Q}}|^2 = 0$, so that $\overline{\mathbf{Q}} = 0$. Hence $Q_3 = 0$. Then $\boldsymbol{H} = \ \mathbf{curl}\ \mathbf{Q} = H(x_1, x_2)\boldsymbol{e_3}$, and the theorem 2.1 is proved.

Proof of lemma 2.3
We have

$$\mathcal{E}_\kappa(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) = \mathcal{E}_\kappa(f, \mathbf{Q}) + \int_{\Sigma_\Lambda} \left\{ \frac{2}{\kappa^2} \nabla f \cdot \nabla \overline{f} + \frac{1}{\kappa^2} |\nabla \overline{f}|^2 \right.$$
$$\left. + 2(\ \mathbf{curl}\ \overline{\mathbf{Q}}) \cdot (\ \mathbf{curl}\ \mathbf{Q} - H_0 \boldsymbol{e_3}) + |\ \mathbf{curl}\ \overline{\mathbf{Q}}|^2 + G(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) - G(f, \mathbf{Q}) \right\}.$$

Moreover, from the Taylor formula with integral rest, we obtain:

$$G(f + \overline{f}, \mathbf{Q} + \overline{\mathbf{Q}}) - G(f, \mathbf{Q}) = G'_{(f, \mathbf{Q})}(f, \mathbf{Q}) \cdot (\overline{f}, \overline{\mathbf{Q}}) + \int_0^1 (1 - t)(\overline{f}, \overline{\mathbf{Q}}) \cdot G''(f, \mathbf{Q} + t\overline{\mathbf{Q}}) \cdot (\overline{f}, \overline{\mathbf{Q}}) dt,$$

which gives, after an integration by parts,

$$\begin{aligned}
\mathcal{E}_\kappa(f+\overline{f},\mathbf{Q}+\overline{\mathbf{Q}}) &= \mathcal{E}_\kappa(f,\mathbf{Q}) + \int_{\Sigma_\Lambda} \frac{1}{\kappa^2}|\nabla\overline{f}|^2 + |\,\mathbf{curl}\;\overline{\mathbf{Q}}|^2 \\
&+ \int_0^1 dt(1-t)\left(\int_{\Sigma_\Lambda} (\overline{f},\overline{\mathbf{Q}})\cdot G''(f+t\overline{f},\mathbf{Q}+t\overline{\mathbf{Q}})\cdot(\overline{f},\overline{\mathbf{Q}})\right) \\
&+ 2\left\{\int_{\Sigma_\Lambda} \overline{f}\cdot\left(-\frac{1}{\kappa^2}\Delta f + \frac{1}{2}G'_f(f,\mathbf{Q})\right) + \overline{\mathbf{Q}}\cdot\left((\,\mathbf{curl}\,)^2\mathbf{Q} + \frac{1}{2}G'_{\mathbf{Q}}(f,\mathbf{Q})\right)\right. \\
&\qquad \left. + \int_{\partial\Sigma_\Lambda} \frac{1}{\kappa^2}\overline{f}\cdot\frac{\partial f}{\partial n} - (\,\mathbf{curl}\;\mathbf{Q} - H_0\boldsymbol{e_3})\wedge\overline{\mathbf{Q}}\cdot n\right\}.
\end{aligned}$$

The last term is zero, since $(f,\mathbf{Q})$ satisfies (2.1)-(2.2). On the other hand $(f,\tilde{\mathbf{Q}})\in K_{\delta_0}$, so from the inequality of convexity, we have

$$\frac{1}{2}(\overline{f},\overline{\mathbf{Q}})\cdot G''(f+t\overline{f},\mathbf{Q}+t\overline{\mathbf{Q}})\cdot(\overline{f},\overline{\mathbf{Q}}) \geq c(|\overline{f}|^2+|\overline{\mathbf{Q}}|^2).$$

This implies the lemma.

3 Existence of Meissner solutions

3.1 A priori estimates as $\kappa\to+\infty$

In the following we consider only the functions f and $\mathbf{Q}=(Q_1,Q_2)$ defined on Ω, with curl $\mathbf{Q}=\partial_1 Q_2-\partial_2 Q_1$, and the associated energy defined for $(f,\mathbf{Q})\in V$, $\kappa\in(0,+\infty]$:

$$\mathcal{E}_\kappa(f,\mathbf{Q}) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\text{curl }\mathbf{Q}-H_0|^2 + G(f,\mathbf{Q}).$$

We recall that a critical point $(f_\kappa,\mathbf{Q}_\kappa)$ of $\mathcal{E}_\kappa$ satisfies

$$\left\{\begin{array}{l}
\left.\begin{array}{l}
\frac{1}{\kappa^2}\Delta f_\kappa = \frac{1}{2}G'_f(f_\kappa,\mathbf{Q}_\kappa) \\
-\,\mathbf{curl}\;(\text{curl }\mathbf{Q}_\kappa) = \frac{1}{2}G'_{\mathbf{Q}}(f_\kappa,\mathbf{Q}_\kappa)
\end{array}\right| \text{ on } \Omega \\
\left.\begin{array}{l}
\frac{\partial_n f_\kappa}{\kappa^2} = 0 \\
\text{curl }\mathbf{Q}_\kappa = H_0
\end{array}\right| \text{ on } \partial\Omega,
\end{array}\right. \tag{3.1}$$

which implies $H_\kappa=$ curl $\mathbf{Q}_\kappa$ satisfies

$$\left\{\begin{array}{l}
\nabla\cdot(\frac{\nabla H_\kappa}{f_\kappa^2}) - H_\kappa = 0 \\
H_\kappa = H_0
\end{array}\right. \tag{3.2}$$

with $\mathbf{Q}_\kappa$ given by the inverse formula

$$\mathbf{Q}_\kappa = -\frac{\mathbf{curl}\;H_\kappa}{f_\kappa^2}.$$

We are interested in the solutions of (3.1) in $Int(K_{\delta_0})$. Theorem 2.1 claims the uniqueness of these solutions (when they exist). Then we have the following a priori estimates as $\kappa\to+\infty$ (which imply proposition 1.6):

PROPOSITION 3.1 *$\forall \delta_0 > 0$, $\exists \kappa_0 = \kappa_0(\delta_0) > 0$ such that $\forall M > 1$, $\forall \theta \in (0,1)$, $\exists C_\theta = C_\theta(\delta_0, M) > 0$ such that $\forall H_0 \in [0, H^*_{\delta_0}]$, $\forall \kappa \geq \kappa_0$, for all solutions $(f_\kappa, \mathbf{Q}_\kappa) \in K^M_{\delta_0}$ of (3.1), we have*

$$\left.\begin{array}{l} |f_\kappa - f_\infty|_{C^{0,\theta}(\overline{\Omega})} \\ |\mathbf{Q}_\kappa - \mathbf{Q}_\infty|_{C^{0,\theta}(\overline{\Omega})} \\ |H_\kappa - H_\infty|_{C^{1,\theta}(\overline{\Omega})} \end{array}\right\} \leq \frac{C_\theta}{\kappa^{\frac{1}{2}-\theta}}$$

and

$$|\nabla(f_\kappa - f_\infty)|_{L^2} = O(\frac{1}{\kappa^{\frac{1}{2}}}).$$

REMARK 3.2 *Here C_θ only depends on θ, Ω and on $\max(M, \frac{1}{\delta_0})$.*

Motivation

The essential difficulty as $\kappa \to +\infty$ is the singular perturbation of f resulting from the highest order derivatives in the equation $\frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f$ vanishing in the limit. The limiting solution, $f = f_\infty$, does not satisfy the boundary condition ($\partial_n f_\infty \neq 0$), and there is a boundary layer of width $1/\kappa$. If (s,n) are coordinates tangential and normal to $\partial\Omega$ respectively, then rescaling $n = \bar{n}/\kappa$ in the boundary layer we find that

$$f \sim f_\infty(s,0) + \frac{1}{\kappa}\left(\partial_n f_\infty(s,0)\bar{n} + \frac{\partial_n f_\infty(s,0)}{\sqrt{2} f_\infty(s,0)} e^{-\sqrt{2} f_\infty(s,0)\bar{n}}\right)$$

there. Thus a good approximation to f both in the bulk and in the boundary layer region is

$$f_\infty(x) + \frac{1}{\kappa}\frac{\partial_n f_\infty(y)}{\sqrt{2} f_\infty(y)} e^{-\sqrt{2} f_\infty(y)\kappa d(x,\partial\Omega)} \tag{3.3}$$

where $y \in \partial\Omega$ is such that $d(x, \partial\Omega) = |x - y|$.

Now let f^κ_∞ be any approximation to f_∞ which takes into account the boundary layer, i.e. is such that

$$\partial_n f^\kappa_\infty = 0 \text{ on } \partial\Omega. \tag{3.4}$$

Write $\overline{f} = f_\kappa - f^\kappa_\infty$, $\overline{\mathbf{Q}} = \mathbf{Q}_\kappa - \mathbf{Q}_\infty$, $\overline{H} = H_\kappa - H_\infty$. Recall that $\forall H_0 \in [0, H^*_{\delta_0}]$, $(f_\infty, \mathbf{Q}_\infty) \in K_{\delta_0}$. Then under the assumption of proposition 3.1 and (3.4) we have

LEMMA 3.3 *$\forall H_0 \in [0, H^*_{\delta_0}]$, $\exists C = C(\delta_0, M) > 0$ such that if $(f^\kappa_\infty, \mathbf{Q}_\infty) \in K_{\frac{\delta_0}{2}}$ then*

$$\left.\begin{array}{l} |\overline{f}|_{L^2} \\ \frac{1}{\kappa}|\nabla\overline{f}|_{L^2} \\ |\overline{\mathbf{Q}}|_{L^2} \\ |\mathrm{curl}\overline{\mathbf{Q}}|_{L^2} \end{array}\right\} \leq C\left\{\left|\frac{\Delta f^\kappa_\infty}{\kappa^2} - \frac{1}{2}[G'_f]^{f^\kappa_\infty,\mathbf{Q}_\infty}_{f_\infty,\mathbf{Q}_\infty}\right|_{L^2} + \left|\frac{1}{2}[G'_{\mathbf{Q}}]^{f^\kappa_\infty,\mathbf{Q}_\infty}_{f_\infty,\mathbf{Q}_\infty}\right|_{L^2}\right\} \tag{3.5}$$

$$\frac{\left|\Delta\overline{f}\right|_{L^2}}{\kappa^2} \leq \left|\frac{\Delta f^\kappa_\infty}{\kappa^2} - \frac{1}{2}[G'_f]^{f^\kappa_\infty,\mathbf{Q}_\infty}_{f_\infty,\mathbf{Q}_\infty}\right|_{L^2} + \left|\frac{1}{2}[G'_f]^{f_\kappa,\mathbf{Q}_\kappa}_{f^\kappa_\infty,\mathbf{Q}_\infty}\right|_{L^2} \tag{3.6}$$

$$\left.\begin{array}{l}\left|D^2\overline{H}\right|_{L^p}\\ \left|\nabla\overline{H}\right|_{L^p}\\ |\overline{H}|_{L^p}\end{array}\right\} \leq C(1+|\overline{H}|_{L^\infty}+|\nabla\overline{H}|_{L^\infty})\{|\overline{f}|_{L^p}+|\nabla\overline{f}|_{L^p}+|f_\infty^\kappa-f_\infty|_{L^p}+|\nabla(f_\infty^\kappa-f_\infty)|_{L^p}\} \quad (3.7)$$

REMARK 3.4 *We note that under the assumption of proposition 3.1 the quantities* $|\frac{1}{2}[G']_{f_\infty,\mathbf{Q}_\infty}^{f_\infty^\kappa,\mathbf{Q}_\infty}|_{L^2}$ *are bounded by a constant times* $|f_\infty^\kappa - f_\infty|_{L^2}$. *For*

$$f_\infty^\kappa - f_\infty = \frac{1}{\kappa}\frac{\partial_n f_\infty(y)}{\sqrt{2}f_\infty(y)}e^{-\sqrt{2}f_\infty(y)\kappa d(x,\partial\Omega)}$$

we have $|f_\infty^\kappa - f_\infty|_{L^2} \sim C/\kappa^{\frac{3}{2}}$ *which is therefore heuristically the best estimate of* $|f_\kappa - f_\infty|_{L^2}$ *that we can hope for.*

Rather than using the exponential correction (3.3) we choose a linear function times a cut off, which makes it easier to obtain the necessary estimates. Precisely our choice is:

$$f_\infty^\kappa = f_\infty - \chi_\kappa\psi_{f_\infty}$$

where $\psi_{f_\infty}(x) = d(x,\partial\Omega)\cdot\partial_n f_\infty(y)$ where $y\in\partial\Omega$ satisfies $d(x,\partial\Omega) = |x-y|$, and where $\chi_\kappa(x) = \chi_0(\kappa d(x,\partial\Omega))$, with χ_0 a smooth nonincreasing function which satisfies $\chi_0 \geq 0$ on $[0,+\infty[$, $\chi_0 = 1$ on $[0,\frac{1}{2}]$, $\chi_0 = 0$ on $[1,+\infty[$. We assume that $\partial\Omega\in C^{2,\alpha}$; then in particular $\chi_\kappa\psi_{f_\infty}\in C^{1,\alpha}$. With this definition we have $\partial_n f_\infty^\kappa = 0$ and (compare with remark 3.4):

$$|f_\infty^\kappa - f_\infty|_{L^2} = O\left(\frac{1}{\kappa^{\frac{3}{2}}}\right).$$

More general estimates on $f_\infty^\kappa - f_\infty$ are put together in the following lemma

LEMMA 3.5 *For* $1\leq p<+\infty$, $\theta\in(0,1)$, *we have:*

$$\begin{aligned}|f_\infty^\kappa - f_\infty|_{L^p} &= O\left(\frac{1}{\kappa^{1+\frac{1}{p}}}\right),\\ |\nabla(f_\infty^\kappa - f_\infty)|_{L^p} &= O\left(\frac{1}{\kappa^{\frac{1}{p}}}\right),\\ |f_\infty^\kappa - f_\infty|_{C^{0,\theta}} &= O\left(\frac{1}{\kappa^{1-\theta}}\right).\end{aligned}$$

Proof. The calculus is straightforward.

REMARK 3.6 *If* $\partial\Omega\in C^{3,\alpha}$, *then* $\chi_\kappa\psi_{f_\infty}\in C^{2,\alpha}$ *in which case* $\Delta(f_\infty^\kappa - f_\infty)$ *makes sense and we have* $|\Delta(f_\infty^\kappa - f_\infty)|_{L^2} = O(\sqrt{\kappa})$.

Then we deduce

LEMMA 3.7 $\exists \kappa_0 = \kappa_0(\delta_0) > 0$ *such that* $\forall \kappa \geq \kappa_0$

$$|\overline{f}|_{L^2}, |\overline{\mathbf{Q}}|_{L^2}, |curl\ \overline{\mathbf{Q}}|_{L^2} = O\left(\frac{1}{\kappa^{\frac{3}{2}}}\right), \qquad |\nabla \overline{f}|_{L^2} = O\left(\frac{1}{\kappa^{\frac{1}{2}}}\right) \tag{3.8}$$

$$|\Delta \overline{f}|_{L^2} = O(\sqrt{\kappa}) \tag{3.9}$$

$$|D^2\overline{H}|_{L^p}, |\nabla \overline{H}|_{L^p}, |\overline{H}|_{L^p} = O\left(\frac{1}{\kappa^{\frac{1}{p}}} + |\overline{f}|_{L^p} + |\nabla \overline{f}|_{L^p}\right) \tag{3.10}$$

Proof of lemma 3.7
For κ larger than some κ_0, for every $H_0 \in [0, H^*_{\delta_0}]$, we obtain $(f^\kappa_\infty, Q_\infty) \in K_{\frac{\delta_0}{2}}$, and then lemma 3.3 can be used. The estimates (3.8) and (3.9) are directly deduced from lemmas 3.3 and 3.5. To deduce 3.10 it is sufficient to remark that $\mathbf{Q}_\kappa = -\frac{\mathbf{curl}\ H_\kappa}{f^2_\kappa}$ and then $|\nabla H_\kappa|_{L^\infty} \leq |\mathbf{Q}_\kappa|_{L^\infty} |f^2_\kappa|_{L^\infty} \leq C$ because $(f_\kappa, \mathbf{Q}_\kappa) \in K^M_{\delta_0}$. Moreover $|\overline{H}|_{L^2} = |\text{curl}\ \overline{\mathbf{Q}}|_{L^2} = O(1/\kappa^{\frac{3}{2}})$, and therefore $|H|_{L^\infty} \leq C$, and $|\overline{H}|_{L^\infty}, |\nabla \overline{H}|_{L^\infty} \leq C$, and we deduce (3.10), which ends the proof of the lemma.

As we have already mentioned in the introduction, we have an L^∞ constraint on the solution, while the natural spaces and estimates are L^p. Thus we need to use Gagliardo-Nirenberg inequalities (see [11]). The proof of proposition 3.1 uses these inequalities many times, and for the convenience of the reader we recall them here for a bounded open set $\Omega \subset I\!R^n$.

Gagliardo-Nirenberg inequalities
Let $1 \leq q, r \leq +\infty$, $0 \leq j < m$ and $q' > 0$. Then

$$|D^j u|_p \leq C\{|D^m u|^a_r\, |u|^{1-a}_q + |u|_{q'}\}$$

where

$$\frac{1}{p} = \frac{j}{n} + a\left(\frac{1}{r} - \frac{m}{n}\right) + (1-a)\frac{1}{q}$$

for $\frac{j}{m} \leq a \leq 1$, and only for $\frac{j}{m} \leq a < 1$ if $m - j - \frac{n}{r} \in I\!N$ and $1 < r < +\infty$. Here $D^s u$ symbolises the set of all partial derivatives of total order s, and for $-\infty < 1/p < +\infty$, we use the definition

$$|v|_p = \begin{cases} (\int_\Omega |v|^p)^{\frac{1}{p}} \text{ if } 0 < \frac{1}{p} < +\infty \\ \sup_{x\in\Omega} |v(x)| \text{ if } \frac{1}{p} = 0 \\ [D^s v]_\alpha = \sup_{x,x'\in\Omega, x\neq x'} \frac{|D^s v(x') - D^s v(x)|}{|x'-x|^\alpha} \text{ if } -\infty < \frac{1}{p} < 0, s = [-\frac{n}{p}], \alpha = (-\frac{n}{p}) - s \end{cases}$$

End of the proof of proposition 3.1
We use various inequalities of Gagliardo-Nirenberg.
From the inequality

$$|\overline{f}|_{L^\infty} \leq C\{|D^2\overline{f}|_{L^2}^{\frac{1}{2}}|\overline{f}|_{L^2}^{\frac{1}{2}} + |\overline{f}|_{L^2}\};\ n = 2$$

we deduce that

$$|\overline{f}|_{L^\infty} = O\left(\kappa^{-\frac{1}{2}}\right) \tag{3.11}$$

From the inequality

$$[\overline{f}]_{2a-1} \leq C\{|D^2\overline{f}|_{L^2}^{a}|\overline{f}|_{L^2}^{1-a} + |\overline{f}|_{L^2}\};\ n = 2, \frac{1}{2} < a < 1$$

we deduce $[\overline{f}]_{2a-1} = O(\kappa^{2(a-\frac{3}{4})})$. With $\theta = 2a-1$ this implies that $\forall\theta \in (0,1), |\overline{f}|_{C^{0,\theta}} = O(1/\kappa^{\frac{1}{2}-\theta})$, and from lemma 3.5 we deduce that

$$\forall\theta \in (0,1), |f_\kappa - f_\infty|_{C^{0,\theta}} = O\left(\frac{1}{\kappa^{\frac{1}{2}-\theta}}\right).$$

From the inequality

$$|\nabla\overline{f}|_{L^p} \leq C\{|D^2\overline{f}|_{L^2}^{a}|\overline{f}|_{L^2}^{1-a} + |\overline{f}|_{L^2}\};\ n = 2,\ \frac{1}{p} = 1-a,\ \frac{1}{2} \leq a < 1$$

We deduce $|\nabla\overline{f}|_{L^p} = O(1/\kappa^{\frac{2}{p}-\frac{1}{2}});\ 2 \leq p < +\infty$. In particular from lemma 3.5 we deduce

$$|\nabla(f_\kappa - f_\infty)|_{L^2} = O(\frac{1}{\kappa^{\frac{1}{2}}})$$

Moreover from (3.11), we get obviously $|\overline{f}|_{L^p} = O(1/\kappa^{\frac{1}{2}})$ for $2 \leq p < +\infty$; and (3.10) gives $|D^2\overline{H}|_{L^p}, |\nabla\overline{H}|_{L^p}, |\overline{H}|_{L^p} = O(1/\kappa^{\frac{2}{p}-\frac{1}{2}});\ 2 \leq p < +\infty$. From the inequality

$$|\nabla\overline{H}|_{L^{\overline{p}}} \leq C\{|D^2\overline{H}|_{L^p}^{a}|\overline{H}|_{L^2}^{1-a} + |\overline{H}|_{L^2}\};\ n = 2,\quad -\frac{n}{\overline{p}} = a(3-\frac{2}{p}) - 2$$

$$\frac{1}{2} \leq a < 1 \text{ if } p = 2;\ \frac{1}{2} \leq a \leq 1 \text{ if } 2 < p < +\infty$$

we deduce with $\theta = 1-\frac{2}{p},\ a = 1,\ 2 < p < +\infty$, that $\forall\theta \in (0,1),\ [\nabla\overline{H}]_\theta = O(1/\kappa^{\frac{1}{2}-\theta})$. In the limit case $\theta = 0$, we have $|\nabla\overline{H}|_{L^\infty} \leq \frac{C_\eta}{\kappa^{\frac{1}{2}-\eta}}$ where η is arbitrarily small. Therefore $\forall\theta \in (0,1), |\nabla\overline{H}|_{C^{0,\theta}} = O(1/\kappa^{\frac{1}{2}-\theta})$. In particular because $|\overline{H}|_{L^2} = O(1/\kappa^{\frac{3}{2}})$ we deduce that $|\overline{H}|_{L^\infty} = O(1/\kappa^{\frac{1}{2}-\theta})$, thus

$$\forall\theta \in (0,1),\ |\overline{H}|_{C^{1,\theta}} = O\left(\frac{1}{\kappa^{\frac{1}{2}-\theta}}\right).$$

Moreover

$$\overline{\mathbf{Q}} = \mathbf{Q}_\kappa - \mathbf{Q}_\infty = -\frac{\mathbf{curl}\ \overline{H}}{f^2} + (\ \mathbf{curl}\ H_\infty)\left(\frac{1}{f_\infty^2} - \frac{1}{f_\kappa^2}\right).$$

Thus using $|f_\infty^\kappa - f_\infty|_{C^{0,\theta}} = O(1/\kappa^{1-\theta})$ from lemma 3.5, we deduce that

$$\forall \theta \in (0,1), \quad |\overline{\mathbf{Q}}|_{C^{0,\theta}} = O\left(\frac{1}{\kappa^{\frac{1}{2}-\theta}}\right)$$

which ends the proof of the proposition 3.1.

Proof of lemma 3.3

From (3.1) we get:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \frac{1}{\kappa^2}\Delta\overline{f} = \frac{1}{2}\{[G'_f]^{f_\kappa,\mathbf{Q}_\kappa}_{f^\kappa_\infty,\mathbf{Q}_\infty} + [G'_f]^{f^\kappa_\infty,\mathbf{Q}_\infty}_{f_\infty,\mathbf{Q}_\infty}\} - \frac{\Delta f^\kappa_\infty}{\kappa^2} \\ -\ \mathbf{curl}\ (\mathrm{curl}\ \overline{\mathbf{Q}}) = \frac{1}{2}\{[G'_{\mathbf{Q}}]^{f_\kappa,\mathbf{Q}_\kappa}_{f^\kappa_\infty,\mathbf{Q}_\infty} + [G'_{\mathbf{Q}}]^{f^\kappa_\infty,\mathbf{Q}_\infty}_{f_\infty,\mathbf{Q}_\infty}\} \end{array}\right| \text{ on } \Omega \\ \left.\begin{array}{l} \partial_n\overline{f} = 0 \\ \mathrm{curl}\ \overline{\mathbf{Q}} = 0 \end{array}\right| \text{ on } \partial\Omega \end{array}\right. \tag{3.12}$$

The first equation gives the inequality (3.6). To obtain the inequality (3.5), we multiply the first equation of (3.12) (resp. the second equation of (3.12)) by $\overline{f}$ (resp. $\overline{\mathbf{Q}}$) and integrate by parts. Then we obtain

$$\int_\Omega \frac{|\nabla\overline{f}|^2}{\kappa^2} + |\mathrm{curl}\ \overline{\mathbf{Q}}|^2 + \frac{1}{2}[G']^{f,\mathbf{Q}}_{f^\kappa,\mathbf{Q}^*} \cdot \begin{pmatrix} \overline{f} \\ \overline{\mathbf{Q}} \end{pmatrix} = \int_\Omega \left(\frac{\Delta f^\kappa}{\kappa^2} - \frac{1}{2}[G'_f]^{f^\kappa,\mathbf{Q}^*}_{f^*,\mathbf{Q}^*}\right)\overline{f} - \frac{1}{2}[G'_{\mathbf{Q}}]^{f^\kappa,\mathbf{Q}^*}_{f^*,\mathbf{Q}^*} \cdot \overline{\mathbf{Q}}.$$

Using the inequality of convexity because $(f^\kappa_\infty, \mathbf{Q}_\infty) \in K_{\frac{\delta_0}{2}}$, we get for $c = c(\frac{\delta_0}{2})$

$$\frac{1}{2}[G']^{f_\kappa,\mathbf{Q}_\kappa}_{f^\kappa_\infty,\mathbf{Q}_\infty} \cdot \begin{pmatrix} \overline{f} \\ \overline{\mathbf{Q}} \end{pmatrix} \geq c(\overline{f}^2 + \overline{\mathbf{Q}}^2)$$

and we find the inequality (3.5).

From (3.2) we have

$$\left\{\begin{array}{l} \nabla\cdot(\frac{\nabla\overline{H}}{f^2_\infty}) - \overline{H} = -g \\ \overline{H} = 0 \end{array}\right.$$

where

$$g = \nabla\cdot((\nabla H_\kappa)(\frac{1}{f^2_\kappa} - \frac{1}{(f_\infty)^2})) = (\Delta H_\kappa)(\frac{1}{f^2_\kappa} - \frac{1}{(f_\infty)^2}) - 2(\nabla H_\kappa)(\frac{\nabla f_\kappa}{f^3_\kappa} - \frac{\nabla f_\infty}{(f_\infty)^3}),$$

but from (3.2) we obtain $\Delta H_\kappa = f^2_\kappa H_\kappa + 2\nabla H_\kappa \cdot \frac{\nabla f_\kappa}{f_\kappa}$ and then

$$g = \frac{H_\kappa}{(f_\infty)^2}((f_\infty)^2 - f^2_\kappa) + 2\frac{\nabla H_\kappa}{(f_\infty)^2}(\frac{\nabla f_\infty}{f_\infty} - \frac{\nabla f_\kappa}{f_\kappa}).$$

Then $|g|_{L^p} \leq C(|H|_{L^\infty} + |\nabla H|_{L^\infty})(|f_\kappa - f_\infty|_{L^p} + |\nabla(f_\kappa - f_\infty)|_{L^p})$. On another hand we have the standard elliptic L^p-estimate, $|\overline{H}|_{W^{2,p}} \leq C|g|_{L^p}$. Consequently we obtain the inequality (3.7), and the lemma is proved.

3.2 Local existence of the Meissner solution

We aim to find by perturbation from a known solution some solutions $(f,\mathbf{Q})$ of (3.1), that is, solutions $(f,\tilde{H})$ of the system:

$$\left\{\begin{array}{l} \frac{1}{\kappa^2}\Delta f = \frac{1}{2}G'_f(f, -\frac{\mathbf{curl}\ \tilde{H}}{f^2}) \\ \partial_n f = 0 \\ \nabla\cdot(\frac{\nabla \tilde{H}}{f^2}) - \tilde{H} = H_0 \\ \tilde{H} = 0 \end{array}\right. \tag{3.13}$$

where we have introduced the notation $\tilde{H} = H - H_0$ to obtain homogeneous boundary conditions. We will perturb using H_0. On another hand we are only interested in the solutions $(f,\mathbf{Q}) \in Int(K_0)$, i.e. $(f,\tilde{H})$ in the open set:

$$\mathcal{U} = \{(f,\tilde{H}) : \text{ for } \mathbf{Q} = -\frac{\mathbf{curl}\ \tilde{H}}{f^2},\ (f,\mathbf{Q}) \in Int(K_0)\}$$

Then it is natural to introduce the operator:

$$\mathcal{A}(f,\tilde{H}) = \left\{\begin{array}{l} \frac{1}{\kappa^2}\Delta f - \frac{1}{2}G'_f(f, -\frac{\mathbf{curl}\ \tilde{H}}{f^2}) \\ \nabla\cdot(\frac{\nabla \tilde{H}}{f^2}) - \tilde{H} \end{array}\right. .$$

We want to apply an inverse function theorem in Hölder spaces, which is possible because the solutions $(f,\tilde{H})$ are in fact regular (see proposition 2.2). Define the spaces

$$X_0^{m,\alpha} = \{f,\tilde{H} \in C^{m,\alpha}(\overline{\Omega}),\ \partial_n f = 0, \tilde{H} = 0 \text{ on } \partial\Omega\},$$

$$X^{m,\alpha} = \{f,\tilde{H} \in C^{m,\alpha}(\overline{\Omega})\}.$$

We will work in the space of minimal regularity $X_0^{2,\alpha}$. In particular we assume that $\partial\Omega \in C^{2,\alpha}$. Then $\mathcal{A} : \mathcal{U}_0 = \mathcal{U} \cap X_0^{2,\alpha} \to X^{0,\alpha}$ is a regular map. We want to solve

$$\mathcal{A}(f,\tilde{H}) = \begin{pmatrix} 0 \\ H_0 \end{pmatrix}.$$

We start with the particular solution:

$$\mathcal{A}(1,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \text{where } (1,0) \in \mathcal{U}_0$$

Proof of proposition 1.5
We have

$$D_{(f,\tilde{H})}\mathcal{A}(f,\tilde{H})\cdot(\phi,\tilde{h}) = \left\{\begin{array}{l} \frac{1}{\kappa^2}\Delta\phi - 3(f^2 - \frac{|\nabla\tilde{H}|^2}{f^4} - \frac{1}{3})\phi - 2\frac{(\ \mathbf{curl}\ \tilde{H})\cdot(\ \mathbf{curl}\ \tilde{h})}{f^3} \\ \nabla\cdot(\frac{\nabla\tilde{h}}{f^2} - 2\frac{\nabla\tilde{H}}{f^3}\phi) - \tilde{h} \end{array}\right. .$$

We denote $\chi = f^2 - \frac{|\nabla \tilde{H}|^2}{f^4} - \frac{1}{3}$, and we consider the equation

$$D_{(f,\tilde{H})}\mathcal{A}(f,\tilde{H}) \cdot (\phi, \tilde{h}) = \left\{ \begin{array}{l} \phi_0 \\ \tilde{h}_0 \end{array} \in X^{0,\alpha}. \right. \tag{3.14}$$

We remark that if $(\phi, \tilde{h}) \in KerD\mathcal{A}(f,\tilde{H})$, then multiplying the first equation by ϕ and the second by $\tilde{h}$, and integrating by parts we obtain (with the cancellation of two terms $\textbf{curl}\ \tilde{H} \cdot\ \textbf{curl}\ \tilde{h} - \nabla\tilde{H} \cdot \nabla\tilde{h} = 0$)

$$\int_\Omega \frac{|\nabla\phi|^2}{\kappa^2} + 3\chi\phi^2 + \frac{|\nabla\tilde{h}|}{f^2} + \tilde{h}^2 = 0.$$

But $(f,\tilde{H}) \in \mathcal{U}_0$ then there exists some $\eta_0 > 0$ such that $\chi > \eta_0 > 0$. Therefore $\phi = \tilde{h} = 0$ and $KerD\mathcal{A}(f,\tilde{H}) = \{0\}$.
Incidentally, it is possible to see the surjectivity of $D\mathcal{A}(f,\tilde{H})$, with the help of the Fredholm theory. In fact (3.14) can be written:

$$\left\{ \begin{array}{l} \frac{1}{\kappa^2}\Delta\phi - 3\chi\phi - 2\frac{\nabla\tilde{H}\cdot\nabla\tilde{h}}{f^3} = \phi_0 \\ \Delta\tilde{h} - 2\frac{\nabla f\cdot\nabla\tilde{H}}{f} - 2f^2\nabla\cdot(\frac{\nabla\tilde{H}\phi}{f^3}) - f^2\tilde{h} = f^2\tilde{h}_0 \end{array} \right.$$

Here the terms of second derivatives are decoupled in ϕ and $\tilde{h}$, and we can write these equations as a compact perturbation of the identity. If we introduce (for example) the two well known isomorphisms

$$\begin{array}{rll} A_1 : & \{\phi \in C^{2,\alpha},\ \partial_n\phi = 0\} & \to C^{0,\alpha} \\ & \phi \longmapsto \Delta\phi - \phi & \end{array},$$

$$\begin{array}{rll} A_2 : & \{\tilde{h} \in C^{2,\alpha},\ \tilde{h} = 0\} & \to C^{0,\alpha} \\ & \tilde{h} \longmapsto \Delta\tilde{h} - \tilde{h} & \end{array},$$

then (3.14) can be written

$$(Id + T)(\phi, \tilde{h}) = (\phi_1, \tilde{h}_1),$$

where

$$\begin{array}{rcl} \phi_1 & = & A_1^{-1}(\kappa^2\phi_0), \\ \tilde{h}_1 & = & A_2^{-1}(f^2\tilde{h}_0), \end{array}$$

and

$$T(\phi,\tilde{h}) = \left\{ \begin{array}{l} A_1^{-1}((1-3\kappa^2\chi)\phi - 2\kappa^2\frac{\nabla\tilde{H}\cdot\nabla\tilde{h}}{f^3}) \\ A_2^{-1}((1-f^2)\tilde{h} - 2\frac{\nabla\cdot\nabla\tilde{h}}{f} - 2f^2\nabla\cdot(\frac{\phi\nabla\tilde{H}}{f^3})) \end{array} \right.$$

Then $T : X_0^{0,\alpha} \to X_0^{0,\alpha}$ is clearly compact, and then

$$Ind(Id + T) = Ind(Id).$$

However, $Ind(Id{+}T) = dimKer(Id{+}T) - codimIm(Id{+}T)$ and $Ind(Id) = 0$, $dimKer(Id{+}T) = 0$, so that $Id{+}T$ is surjective and $D\mathcal{A}(f,\tilde{H})$ is surjective. Therefore $D\mathcal{A}(f,\tilde{H}) : X_0^{2,\alpha} \to X^{0,\alpha}$ is invertible and the inverse function theorem applies, which proves proposition 1.5.

4 Global and local minimisers

4.1 A minimising sequence

We start with the existence of a minimising sequence for $\mathcal{E}_\infty$. Recall that

$$V = \{(f, \mathbf{Q}) \in L^\infty \times L^\infty,\ \nabla f \in L^2,\ \textbf{curl}\ \mathbf{Q} \in L^2\}.$$

PROPOSITION 4.1 *(i) For any $H_0 \geq 0$,*

$$\inf_{(f,\mathbf{Q})\in V} \mathcal{E}_\infty(f, \boldsymbol{Q}) = 0.$$

The minimising sequence can be chosen in C^∞.

(ii) For $H_0 > 0$ there is no global minimiser of $\mathcal{E}_\infty$

Proof of proposition 4.1
We remark that zero is the absolute minimum of $\mathcal{E}_\infty$, whose density is non-negative. Assuming that the first part of the theorem holds, a global minimiser should satisfy almost everywhere

$$f^2 |\boldsymbol{Q}|^2 + \frac{(f^2-1)^2}{2} + (\operatorname{curl} \boldsymbol{Q} - H_0)^2 = 0.$$

Then we have a.e. in Ω: $f = 1$, $\boldsymbol{Q} = 0$ and curl $\boldsymbol{Q} = H_0$. Then the distributional derivative of $\mathbf{Q}$ is zero and therefore $H_0 = 0$.

The minimising sequence The problem is somewhat analogous to the problem of minimising the functional

$$\int_{-1}^{1} ((F')^2 - 1)^2 + F^2\, dx.$$

There it is well known that a minimising sequence is furnished by the sawtooth function

$$F_n = \begin{cases} x - \frac{2j+1/2}{n} & x \in \left[\frac{2j}{n}, \frac{2j+1}{n}\right) \\ -x + \frac{2j+3/2}{n} & x \in \left(\frac{2j+1}{n}, \frac{2j+2}{n}\right] \end{cases} \quad j \in \mathbf{Z}.$$

Thus we have that $(F')^2 - 1$ is identically zero, while the derivative F'_n switches between the values -1 and $+1$ increasingly rapidly as $n \to \infty$ so that F itself stays small.

In the present case we need to satisfy curl $\boldsymbol{Q} = H_0$ while keeping $\boldsymbol{Q}$ small. If we take the second component of $\boldsymbol{Q}$ to be a sawtooth function

$$Q_2^{(n)} = H_0 \left(x - \frac{j+1/2}{n}\right) \qquad x \in \left[\frac{j}{n}, \frac{j+1}{n}\right), \quad j \in \mathbf{Z},$$

then the jumps in $Q_2^{(n)}$ will create δ-function spikes in first component at the points $x = j/n$. However, we can neutralise the effect on the energy of these spikes by simply dropping f down to zero locally.

We now construct a C^∞ minimising sequence which is a smooth version of that described above. Let ψ be a real C^∞ function, $0 \le \psi \le 1$ such that

$$\psi(x) \equiv 0 \text{ for } |x| \ge 2,$$
$$\psi(x) \equiv 1 \text{ for } |x| \le 1.$$

We introduce

$$g_n(x,y) = 1 - \sum_{k=-\infty}^{\infty} \psi\left(n^2\left(x - \frac{k}{n}\right)\right).$$

The function g_n is a sum of C^∞ functions with compact supports $\{\frac{k}{n} - \frac{2}{n^2} \le x \le \frac{k}{n} + \frac{2}{n^2}\}$. Now let χ_ϵ be an approximation of the identity in $I\!R$ (a nonnegative real C^∞ function with support in $(-\epsilon, \epsilon)$ and mass $\int \chi_\epsilon = 1$). Let $[x]$ denote the integral part of x (i.e. $[x] = \max\{n \in I\!N, n \le x\}$). We define h_n as the convolution:

$$h_n(x) = \left(x - \frac{[nx]}{n}\right) * \chi_{\frac{1}{2n^2}}.$$

We define then the functional

$$\boldsymbol{P_n}(x,y) = (y(h_n'(x) - 1), h_n(x)),$$

and compute curl $\boldsymbol{P_n} = h_n'(x) - (h_n'(x) - 1) = 1$. We construct then a sequence $(f_n, \boldsymbol{Q_n})$ as

$$\begin{cases} f_n = g_{n|\Omega} \\ \boldsymbol{Q_n} = H_0 \boldsymbol{P_n}_{|\Omega} \end{cases} \tag{4.1}$$

Let us consider the sets $B_j^n = \Omega \bigcap \bigcup_{k=-\infty}^{+\infty} \{(x,y), \{\frac{k}{n} - \frac{j}{n^2} \le x \le \frac{k}{n} + \frac{j}{n^2}\}$ for $j = 1, 2$. The sequence $(f_n, \boldsymbol{Q_n})$ is such that :

$$\begin{cases} \text{curl } \boldsymbol{Q_n} = H_0 & \text{in } \Omega, \\ 0 \le f_n \le 1 & \text{in } \Omega, \\ f_n = 1 & \text{in } \Omega \backslash B_2^n, \\ |\boldsymbol{Q_n}| = H_0 h_n(x) \le \frac{H_0}{n} & \text{in } \Omega \backslash B_1^n, \\ f_n = 0 & \text{in } B_1^n. \end{cases}$$

We remark that $\text{meas}(B_j^n) \le 2\frac{j}{n}(\text{diam}(\Omega))^2$. Therefore, if we estimate the energy on $\Omega \backslash B_2^n$, $B_2^n \backslash B_1^n$ and B_1^n, we obtain

$$\mathcal{E}_\infty(f_n, \boldsymbol{Q_n}) \le \frac{C}{n}.$$

Consequently, $\inf \mathcal{E}_\infty = 0$ and $(f_n, \boldsymbol{Q_n})$ is a minimising sequence. This completes the proof of proposition 4.1.

From the construction above, we notice that

$$\mathcal{E}_\kappa(f_n, \boldsymbol{Q_n}) \le \frac{C}{n} + C' \frac{n^3}{\kappa^2}$$

Therefore, for $n = \sqrt{\kappa}$, we deduce theorem 1.4.

4.2 Properties of local minimisers

Here we present some properties of local minimisers (see definition 1.1).

PROPOSITION 4.2 *For $\kappa \in (0, +\infty)$, if $(f, \mathbf{Q})$ is a local minimiser in V of $\mathcal{E}_\kappa$, then $\exists \epsilon > 0$ such that either $\epsilon \le f \le 1 - \epsilon$ or $f \equiv 0$ or $f \equiv 1$.*

REMARK 4.3 *If $f \equiv 1$, then $\mathbf{Q} \equiv 0$, but curl $\mathbf{Q} = H_0$ on $\partial\Omega$, then $H_0 = 0$.*

Proof of proposition 4.2
Let us first remark that if $(f, \mathbf{Q})$ is a global minimiser, then $\mathcal{E}_\kappa(\min(f, 1), \mathbf{Q}) \le \mathcal{E}_\kappa(f, \mathbf{Q})$ and therefore $0 \le f \le 1$. If $(f, \mathbf{Q})$ is a local minimiser, then let $\overline{f} = (f - (\sup_\Omega f - \epsilon))^+$. If $\sup_\Omega f > 1$, then for ϵ small enough, $\mathcal{E}_\kappa(f - \overline{f}, \mathbf{Q}) < \mathcal{E}_\kappa(f, \mathbf{Q})$, which is impossible, therefore $0 \le f \le 1$.
Now

$$\begin{cases} \frac{1}{\kappa^2}\Delta f = f(f^2 + \mathbf{Q}^2 - 1) \\ f \ge 0 \end{cases}$$

Then by the strong maximum principle either $f \equiv 0$, either $f > 0$ on Ω. Since $\partial_n f = 0$, by the Hopf lemma we deduce that $\inf_\Omega f > 0$, and then $\inf_{\overline{\Omega}} f > 0$.
Now let $v = 1 - f$, then

$$\begin{cases} \frac{1}{\kappa^2}\Delta v - f(f+1)v = -f\mathbf{Q}^2 \le 0 \\ v \ge 0 \end{cases}$$

Then, as previously, we deduce that either $v = 0$ or $\inf_{\overline{\Omega}} v > 0$, and therefore $\sup_{\overline{\Omega}} f < 1$, and the proposition is proved.

PROPOSITION 4.4 *If $(f, \mathbf{Q})$ is a critical point and $f^2 - \mathbf{Q}^2 - \frac{1}{3} > \delta_0$, then $(f, \mathbf{Q})$ is a local minimiser of $\mathcal{E}_\kappa$ (for finite or infinite κ).*

Proof of proposition 4.4
This follows immediately from lemma 2.3.

COROLLARY 4.5 *For $\kappa = +\infty$, if $(\mathbf{Q}_\infty)^2 < \frac{1}{3} - \delta_0$, then $(f_\infty, \mathbf{Q}_\infty)$ is a local minimiser of $\mathcal{E}_\infty$.*

PROPOSITION 4.6 *For $\kappa = +\infty$, if $(f, \mathbf{Q}) \in V$ is a critical point of $\mathcal{E}_\infty$ such that*

$$\exists x_0 \in \Omega,\ |\mathbf{Q}(x_0)|^2 > \frac{1}{3} \text{ and } f(x_0) > 0$$

and $(f, \mathbf{Q})$ is continuous at $x_0 \in \Omega$, then $(f, \mathbf{Q})$ is not a local minimiser of $\mathcal{E}_\infty$.

Proof of proposition 4.6
Let us first consider the general case $\kappa \in (0,+\infty]$. Without loss of generality we take x_0 to be the origin, and we choose the coordinates (x,y) in $I\!R^2$, such that $\mathbf{Q}(0) = |\mathbf{Q}(0)|e_x$. We consider the following function:

$$\phi(x,y) = \chi(y)\cos(\alpha x)1_{\{\alpha x \in [-\frac{\pi}{2},\frac{\pi}{2}]\}},$$

where for $\lambda, \mu > 0$,

$$\chi(y) = \begin{cases} 0 \text{ if } |y| > \frac{\lambda+\mu}{\alpha} \\ 1 \text{ if } |y| \le \frac{\lambda}{\alpha} \\ \text{ continuous and linear on } [-\frac{\lambda+\mu}{\alpha}, -\frac{\lambda}{\alpha}] \cup [\frac{\lambda}{\alpha}, \frac{\lambda+\mu}{\alpha}] \end{cases}.$$

Consider

$$\begin{cases} \overline{\mathbf{Q}} = \nabla\phi \\ \overline{f} = -\Lambda\partial_x\phi \end{cases}.$$

Then

$$\mathcal{E}_\kappa(f+t\overline{f},\mathbf{Q}+t\overline{\mathbf{Q}}) = \mathcal{E}_\kappa(f,\mathbf{Q}) + t\mathcal{E}_\kappa'(f,\mathbf{Q})\cdot(\overline{f},\overline{\mathbf{Q}}) + \frac{t^2}{2}\mathcal{E}_\kappa''(f,\mathbf{Q})\cdot(\overline{f},\overline{\mathbf{Q}})^2 + O(t^3).$$

Suppose that $(f,\mathbf{Q})$ is a local minimiser. Then $\mathcal{E}_\kappa'(f,\mathbf{Q})\cdot(\overline{f},\overline{\mathbf{Q}}) = 0$, and it remains only to examine the sign of the quantity

$$\begin{aligned} J &= \tfrac{1}{2}\mathcal{E}_\kappa''(f,\mathbf{Q})\cdot(\overline{f},\overline{\mathbf{Q}})^2 \\ &= \textstyle\int_\Omega \frac{2}{\kappa^2}|\nabla\overline{f}|^2 + 2|\,\mathbf{curl}\ \overline{\mathbf{Q}}|^2 + (f\overline{\mathbf{Q}} + 2\overline{f}\mathbf{Q})^2 + 3\overline{f}^2(f^2 - \mathbf{Q}^2 - \frac{1}{3}) \\ &= \textstyle\int_\Omega \frac{2\Lambda^2}{\kappa^2}|\nabla\partial_x\phi|^2 - 3\Lambda^2(\partial_x\phi)^2(\frac{1}{3} - f^2 + \mathbf{Q}^2) + (\partial_x\phi)^2(f - 2\Lambda\mathbf{Q}_x)^2 + (f\partial_y\phi - 2\Lambda\mathbf{Q}_y\partial_x\phi)^2 \end{aligned}.$$

Now, if we choose

$$\Lambda = \frac{f(0)}{2|\mathbf{Q}_x(0)|},$$

let $\omega = \mathit{supp}\ \phi$, $\delta = \inf_\omega(\frac{1}{3} - f^2 + \mathbf{Q}^2)$, and $R = \sup_\omega(|\mathbf{Q}_y|\Lambda, |f - 2\Lambda\mathbf{Q}_x|)$, we find (using the fact that $0 \le f \le 1$)

$$J \le \frac{2\Lambda^2}{\kappa^2}|\nabla\partial_x\phi|_{L^2}^2 + 2|\partial_y\phi|_{L^2}^2 - |\partial_x\phi|_{L^2}^2\{3\Lambda^2\delta - C'R^2\}.$$

Moreover

$$\begin{aligned} |\partial_x\phi|_{L^2}^2 &= \pi(\lambda + \frac{\mu}{3}), \\ |\partial_y\phi|_{L^2}^2 &= \frac{\pi}{\mu}, \\ |\nabla\partial_x\phi|_{L^2}^2 &= \alpha^2(|\partial_x\phi|_{L^2}^2 + |\partial_y\phi|_{L^2}^2) = \alpha^2\pi(\lambda + \frac{\mu}{3} + \frac{1}{\mu}). \end{aligned}$$

Then if we take $\lambda = 1$, $\mu = \sqrt{\alpha}$, then $|\partial_y \phi|_{L^2} \to 0$, $|\partial_x \phi|_{L^2} \to +\infty$ and $R \to 0$ as $\alpha \to +\infty$, and in particular for $\kappa = +\infty$ we find $\delta > 0$ (because $f^2 + \mathbf{Q}^2 = 1$). Then we obtain $J < 0$ which gives a contradiction. Consequently $(f, \mathbf{Q})$ is not a local minimiser, and proposition 4.6 is proved.

REMARK 4.7 Large but finite κ

For large but finite κ we can see heuristically the condition for $(f, \mathbf{Q})$ not to be a local minimiser should be

$$\left|(f^2 - \mathbf{Q}^2 - \frac{1}{3})^-\right|_{L^2(\omega)} \geq O(\frac{1}{\kappa}). \tag{4.2}$$

The argument goes as follows. We first remark that for finite κ if $(f, \mathbf{Q}) \in V$ is a local minimiser with $f \not\equiv 0$ and $\mathbf{Q} \not\equiv 0$, then $(f, \mathbf{Q})$ is analytic, and then if $f(x_0) = 0$ or $\mathbf{Q}(x_0) = 0$, we can chose another point x_0 arbitrarily closed to the first one such that $f(x_0) \neq 0$ and $\mathbf{Q}(x_0) \neq 0$.
Then, in the previous calculation let $L = |(f, \mathbf{Q})|_{C^1(\omega)}$ and $r^2 = (\frac{\pi}{2\alpha})^2 + (\frac{\lambda+\mu}{\alpha})^2$. Then $R^2 \leq CL^2 r^2 (1 + \Lambda^2)$. Then, for $\lambda = 1$ we obtain

$$\frac{J}{\pi} \leq \frac{2\Lambda^2}{\kappa^2}\alpha^2(1 + \frac{\mu}{3} + \frac{1}{\mu}) + \frac{2}{\mu} - (1 + \frac{\mu}{3})\{3\Lambda^2\delta - C''\frac{L^2}{\alpha^2}(1 + \mu^2)(1 + \Lambda^2)\}.$$

Heuristically, if $\delta \simeq 0$, then (for κ large enough) $\Lambda \simeq \Lambda_\infty = f_\infty/(2|\mathbf{Q}_\infty|) = 1/\sqrt{2}$ (since $f_\infty^2 + \mathbf{Q}_\infty^2 = 1$). Then $\Lambda^2\delta \leq 1$. Let us chose $\mu = O(1/\delta)$, $\alpha = O(\kappa\sqrt{\delta})$ and $\kappa \geq L/\delta^{\frac{5}{2}}$. Then $J < 0$, so that $(f, \mathbf{Q})$ is not a local minimiser (and (4.2) is satisfied).

4.3 On the critical field at $\kappa = +\infty$

We consider here the field H_0^* at which (for $\kappa = +\infty$) the solution $|\boldsymbol{Q}_\infty|$ first reaches $1/\sqrt{3}$. In one dimension there is an explicit solution for the infinite-κ Ginzburg-Landau equations on a halfspace $]-\infty, a]$ (see [3]), namely $H(x) = -\sqrt{2}\sinh x/\cosh^2 x$, $H(a) = H_0$, and $H' = (1 - Q^2)Q$. Then $Q(a) = 1/\sqrt{3}$ when $H_0 = \sqrt{5/18} = H_0^*$, and $Q(a) = 1$ when $H_0 = 1/\sqrt{2}$. For more general domains, we have the following comparison theorem.

PROPOSITION 4.8 *For $1/R \in I\!R$ let $B_R = \{|x| < R\}$ if $R > 0$, $B_\infty = \{(x_1, x_2), x_1 > 0\}$ if $1/R = 0$, $B_R = \{|x| > |R|\}$ if $R < 0$. Let $K_+ = \max_{x \in \partial\Omega} curv(x)$ (with $K_+ > 0$ for a disk), $K_- = \min_{x \in \partial\Omega} curv(x)$. Then*

$$H_0^*\left(B_{\frac{1}{K_-}}\right) \leq H_0^*(\Omega) \leq H_0^*\left(B_{\frac{1}{K_+}}\right)$$

In particular if Ω is convex then $H_0^*(\Omega) \geq \sqrt{5/18}$.

Proof of proposition 4.8
This is straightforward using sub and supersolutions.

5 Conclusion

We have been concerned with the Meissner solution of the Ginzburg-Landau model, that is, the superconducting solution for which there are no vortices and the modulus of the order parameter is strictly positive. We have demonstrated the existence of this solution for values of the applied field H_0 less that a critical value H_0^*, and (under an additional convexity assumption) its uniqueness. We have shown that in the limit as the Ginzburg-Landau parameter $\kappa \to \infty$ the Meissner solution approaches the solution of the limiting problem formulated in [3, 1], which is a local minimiser of the limiting Ginzburg-Landau energy $\mathcal{E}_\infty$. Moreover, we have shown that the Meissner solution is only a local minimiser of the Ginzburg-Landau energy $\mathcal{E}_\kappa$, and not a global minimiser, for large enough κ.

The minimising sequence we constructed for the energy $\mathcal{E}_\infty$ corresponds to filling the material with vortex sheets, and is in some sense comparable to the solution of the vortex density model in [5] in which the domain contains a uniform vortex density equal to the applied magnetic field. Indeed, since

$$\boldsymbol{Q} \sim \frac{1}{r}\boldsymbol{e}_\theta$$

at a vortex, a vortex may be thought of as generating a δ-function in curl $\boldsymbol{Q}$, and thus a natural way to define the Ginzburg-Landau energy for $\kappa = +\infty$ in the presence of vortices is

$$\mathcal{E}_\infty(f, \mathbf{Q}) = \int_\Omega |\text{curl } \mathbf{Q} + \omega - H_0|^2 + G(f, \mathbf{Q}),$$

where ω is the vortex density. In this case the minimiser is clearly $\boldsymbol{Q} \equiv \mathbf{0}$, $f \equiv 1$, $\omega \equiv H_0$. It is interesting that this solution is somehow found by the minimising sequence, even when the details of the vortex cores have been ignored completely.

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