Equations aux dérivées partielles / Partial differential equations

Convergence of Meissner minimisers of the Ginzburg-Landau energy as $\kappa \to +\infty$.

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Abstract. We consider Meissner solutions, critical points of the Ginzburg-Landau energy $\mathcal{E}_\kappa$ for an infinite cylinder of smooth bounded section $\Omega \subset \mathbf{R}^2$. These Meissner solutions do not contain vortices, satisfy a convexity condition, and are local minimisers. For an exterior magnetic field H_0 less than a critical field H_0^*, we prove the existence and uniqueness of such a Meissner solution $u_\kappa^{H_0}$ for κ large enough. Moreover we prove the convergence of $u_\kappa^{H_0}$ towards $u_\infty^{H_0}$ as $\kappa \to +\infty$, where $u_\infty^{H_0}$ is a local minimiser of some energy $\mathcal{E}_\infty$, and can be understood as a Meissner solution for $\kappa = +\infty$. On another hand we prove that $\mathcal{E}_\infty$ has no global minimisers.

Convergence des solutions de Meissner, minimiseurs de l'énergie de Ginzburg-Landau lorsque $\kappa \to +\infty$.

Résumé. Nous considérons les solutions de Meissner, points critiques de l'énergie de Ginzburg-Landau $\mathcal{E}_\kappa$ pour un cylindre infini de section $\Omega \subset \mathbf{R}^2$ ouvert borné régulier. Ces solutions de Meissner ne contiennent pas de vortex, satisfont une condition de convexité, et sont des minima locaux. Pour un champ magnétique extérieur H_0 plus petit qu'un champ critique H_0^*, nous prouvons l'existence et l'unicité de telles solutions de Meissner $u_\kappa^{H_0}$ pour κ suffisamment grand. De plus nous prouvons la convergence de $u_\kappa^{H_0}$ vers $u_\infty^{H_0}$ lorsque $\kappa \to +\infty$, où $u_\infty^{H_0}$ est un minimum local d'une certaine énergie $\mathcal{E}_\infty$, et peut être vu comme une solution de Meissner pour $\kappa = +\infty$. D'un autre côté nous prouvons que $\mathcal{E}_\infty$ n'a pas de minimum global.

Version française abrégée.

Soit $\Omega \subset \mathbf{R}^2$ un ouvert connexe et simplement connexe tel que $\partial\Omega \in C^{2,\alpha}$ pour un $\alpha \in (0,1)$. Pour $\kappa \in (0,+\infty]$, nous considérons l'énergie de Ginzburg-Landau avec $f > 0$:

$$\mathcal{E}_\kappa(f,Q) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\ \mathrm{curl}\ Q - H_0|^2 + G(f,Q)$$

où

$$G(f,Q) = f^2 Q^2 + \frac{(f^2-1)^2}{2}$$

et pour $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, $\mathrm{curl}\ Q = \partial_1 Q_2 - \partial_2 Q_1$.

Remarquons que $\frac{1}{2}G''(f,Q)\cdot(g,q)^2 = (fq + 2gQ)^2 + 3g^2(f^2 - Q^2 - \frac{1}{3})$. Pour se restreindre à des sous-ensembles sur lesquels $\mathcal{E}_\kappa$ est convexe, nous introduisons l'espace

$$V = \{(f,Q) \in L^\infty \times L^\infty,\ \nabla f \in L^2,\ \ \mathrm{curl}\ Q \in L^2\},$$

et les sous-ensembles convexes fermés de V,

$$K_{\delta_0} = \{(f,Q) \in V,\ f^2 - Q^2 \geq \frac{1}{3} + \delta_0\}, \delta_0 \geq 0.$$

Alors $Int(K_0) = \cup_{\delta_0>0} K_{\delta_0}$, et pour $\kappa \in (0,+\infty]$, $\mathcal{E}_\kappa$ est strictement convexe sur $Int(K_0)$.

Définition 1 *Le couple (f,Q) est dit minimum local de $\mathcal{E}_\kappa$ si $\exists \epsilon > 0,\ \forall (\overline{f},\overline{Q}) \in V,\ |(\overline{f},\overline{Q})|_V < \epsilon,\ \mathcal{E}_\kappa(f+\overline{f}, Q+\overline{Q}) \geq \mathcal{E}_\kappa(f,Q)$. Et le minimum local est dit strict si de plus l'inégalité est stricte pour $(\overline{f},\overline{Q}) \neq 0$.*

Nous noterons $u_\kappa^{H_0}$ le couple (f_κ, Q_κ), si pour la valeur H_0 du champ magnétique extérieur, $(f_\kappa, Q_\kappa) \in Int(K_0)$ est un minimum local de $\mathcal{E}_\kappa$ sur V.
Dans [1] Berestycki, Bonnet et Chapman ont prouvé qu'il existe un champ critique H_0^* (qui ne dépend que de Ω), tel qu'il existe un minimum local $u_\infty^{H_0} \in Int(K_0)$ de $\mathcal{E}_\infty$ sur V pour tout $H_0 \in [0, H_0^*)$, et il n'y a pas de points critiques de $\mathcal{E}_\infty$ dans $Int(K_0)$ si $H_0 \geq H_0^*$. Nous prouvons:

Théorème 1 *(Unicité de la solution de Meissner) Pour tout $\kappa \in (0,+\infty]$, si (f_k, Q_κ) est un point critique de $\mathcal{E}_\kappa$ sur V qui est dans $Int(K_0)$, alors (f_κ, Q_κ) est unique et est un minimum local strict.*

Théorème 2 *(Existence d'un minimiseur de Meissner pour $\mathcal{E}_\kappa$ et convergence lorsque $\kappa \to +\infty$) Dans l'espace des paramètres $(\kappa, H_0) \in (0,+\infty] \times [0, H_0^*)$, il existe un voisinage $\mathcal{P}$ de $\{+\infty\} \times [0, H_0^*)$ tel que $\forall (\kappa, H_0) \in \mathcal{P}$, $u_\kappa^{H_0} \in Int(K_0)$ existe. De plus $(\kappa, H_0) \mapsto u_\kappa^{H_0} \in V$ est continue sur $\mathcal{P}$. En particulier cela implique que pour $0 \leq H_0 < H_0^*$: $u_\kappa^{H_0} \to u_\infty^{H_0}$ dans V lorsque $\kappa \to +\infty$.*

Théorème 3 *(Non-existence de minimiseurs globaux) Pour $H_0 \geq 0$, $\exists C = C(H_0) > 0$, $\forall \kappa \in (0,+\infty]$: $\inf_{(f,Q)\in V} \mathcal{E}_\kappa(f,Q) \leq \frac{C}{\sqrt{\kappa}}$. En particulier pour $\kappa = +\infty$, $\mathcal{E}_\infty$ n'a pas de minimum global si $H_0 \neq 0$.*

I. INTRODUCTION.
We consider the Ginzburg-Landau energy of a superconductor Σ under an exterior magnetic field $H_0 e_3$:

$$\mathcal{E}_\kappa(\psi, A) = \int_\Sigma |(\frac{1}{\kappa}\nabla - iA)\psi|^2 + \frac{(|\psi|^2-1)^2}{2} + |\ \mathbf{curl}\ A - H_0 e_3|^2, \tag{1}$$

where $\psi \in \mathbf{C}$ is the superconducting wave function, A is the magnetic vector potential in the superconductor, and κ is the Ginzburg-Landau parameter; in particular $\kappa \geq 1/\sqrt{2}$ for superconductors of type II. Here we consider a cylinder $\Sigma = \Omega \times \mathbf{R}$ whose section $\Omega \subset \mathbf{R}^2$ is a smooth open set connected and simply connected; precisely we assume that $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0,1)$. Then the energy (1) is not defined. If we assume that the problem is invariant by translation in the direction of the axis e_3 of the cylinder, then we can define a new energy as an integral on the section Ω of the cylinder.
We are interested in the Meissner solutions, it means solutions such that $|\psi| > 0$ on Ω. Then

we can set: $\psi = fe^{i\chi}, f > 0, Q = A - \nabla\chi$, where χ is uniquely defined (up to $2k\pi$, $k \in \mathbf{Z}$) for example if ψ is smooth. Then we consider for $\kappa \in (0, +\infty]$:

$$\mathcal{E}_\kappa(f, Q) = \int_\Omega \frac{|\nabla f|^2}{\kappa^2} + |\text{ curl } Q - H_0|^2 + G(f, Q),$$

where

$$G(f, Q) = f^2 Q^2 + \frac{(f^2 - 1)^2}{2}$$

and for $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, curl $Q = \partial_1 Q_2 - \partial_2 Q_1$.
Let us remark that $\frac{1}{2} G''(f, Q) \cdot (g, q)^2 = (fq + 2gQ)^2 + 3g^2(f^2 - Q^2 - \frac{1}{3})$. To restrict our study to subsets on which $\mathcal{E}_\kappa$ is convex, we introduce the space

$$V = \{(f, Q) \in L^\infty \times L^\infty,\ \nabla f \in L^2,\ \text{ curl } Q \in L^2\},$$

and the closed convex subsets of V,

$$K_{\delta_0} = \{(f, Q) \in V,\ f^2 - Q^2 \geq \frac{1}{3} + \delta_0\}, \delta_0 \geq 0.$$

Then $Int(K_0) = \cup_{\delta_0 > 0} K_{\delta_0}$, and for $\kappa \in (0, +\infty]$, $\mathcal{E}_\kappa$ is strictly convex on $Int(K_0)$.

Definition 1 *The couple (f, Q) is said to be a local minimiser of $\mathcal{E}_\kappa$ if $\exists \epsilon > 0$, $\forall (\overline{f}, \overline{Q}) \in V$, $|(\overline{f}, \overline{Q})|_V < \epsilon$, $\mathcal{E}_\kappa(f + \overline{f}, Q + \overline{Q}) \geq \mathcal{E}_\kappa(f, Q)$. And the local minimiser is said strict if moreover the inequality is strict for $(\overline{f}, \overline{Q}) \neq 0$.*

II. MAIN RESULTS.
We will denote by $u_\kappa^{H_0}$ the couple (f_κ, Q_κ), if for the value H_0 of the exterior magnetic field, $(f_\kappa, Q_\kappa) \in Int(K_0)$ is a local minimiser of $\mathcal{E}_\kappa$ on V.
In [1] Berestycki, Bonnet and Chapman have proved that there exists a critical field H_0^* (which only depends on Ω), such that there exists a local minimiser $u_\infty^{H_0} \in Int(K_0)$ of $\mathcal{E}_\infty$ on V for every $H_0 \in [0, H_0^*)$, and there is no critical points of $\mathcal{E}_\infty$ in $Int(K_0)$ if $H_0 \geq H_0^*$. We prove

Theorem 1 *(Uniqueness of Meissner solution) For all $\kappa \in (0, +\infty]$, if (f_k, Q_κ) is a critical point of $\mathcal{E}_\kappa$ on V which is in $Int(K_0)$, then (f_κ, Q_κ) is unique and is a strict local minimiser.*

Theorem 2 *(Existence of a Meissner minimiser for $\mathcal{E}_\kappa$ and convergence as $\kappa \to +\infty$) In the space of parameters $(\kappa, H_0) \in (0, +\infty] \times [0, H_0^*)$ there exists a neighbourhood $\mathcal{P}$ of $\{+\infty\} \times [0, H_0^*)$ such that $\forall (\kappa, H_0) \in \mathcal{P}$, $u_\kappa^{H_0} \in Int(K_0)$ exists. Moreover $(\kappa, H_0) \mapsto u_\kappa^{H_0} \in V$ is continuous on $\mathcal{P}$. In particular this implies that for $0 \leq H_0 < H_0^*$, $u_\kappa^{H_0} \to u_\infty^{H_0}$ in V as $\kappa \to +\infty$.*

Theorem 3 *(Nonexistence of global minimisers) For $H_0 \geq 0$, $\exists C = C(H_0) > 0$ such that $\forall \kappa \in (0, +\infty], \inf_{(f,Q) \in V} \mathcal{E}_\kappa(f, Q) \leq \frac{C}{\sqrt{\kappa}}$. In particular for $\kappa = +\infty$, $\mathcal{E}_\infty$ has no global minimisers if $H_0 \neq 0$.*

III. ELEMENTS OF DEMONSTRATION.

Idea of proof of theorem 1
The uniqueness is a consequence of the

Lemma 1 *Let $\delta_0 > 0$, then $\exists c = c(\delta_0) > 0$ such that if $(f_\kappa, Q_\kappa) \in K_{\delta_0}$ is a critical point of $\mathcal{E}_\kappa$ on V, and if $(f, Q) \in K_{\delta_0}$, then*

$$\mathcal{E}_\kappa(f,Q) \geq \mathcal{E}_\kappa(f_\kappa, Q_\kappa) + \int_\Omega \frac{|\nabla(f - f_\kappa)|^2}{\kappa^2} + |\ curl\ (Q - Q_\kappa)|^2 + c(|f - f_\kappa|^2 + |Q - Q_\kappa|^2).$$

Idea of proof of theorem 2
Let us introduce for $1 > \delta_0 > 0$ the set

$$\mathcal{K}_{\delta_0} = \{(f,Q) \in K_{\delta_0},\ f, |Q| \leq \frac{1}{\delta_0}\}.$$

This set allows us to have an a priori L^∞-control on f and Q.
As $\kappa \to +\infty$, there is creation of a boundary layer for the function f_κ, because the Newman boundary condition disappears at $\kappa = +\infty$:

$$f_\kappa(x) = f_\infty(x) + \frac{1}{\kappa}\frac{\partial_n f_\infty(y)}{\sqrt{2} f_\infty(y)} e^{\sqrt{2} f_\infty(y) \kappa d(x,\partial\Omega)} + \cdots$$

where $y \in \partial\Omega$ satisfies $d(x, \partial\Omega) = |x - y|$. Taking into account this boundary layer, and with the help of Euler-Lagrange equations of the problem, we obtain several energy estimates, and from Gagliardo-Nirenberg inequalities, we deduce

Proposition 1 *(case κ close to $+\infty$)*
There exists a continuous and increasing function η with $\eta(0) = 0$ such that for $\delta_0 > 0$ small enough, we have $u_\infty^{H_0} \in \mathcal{K}_{\delta_0}$ for every $H_0 \in [0, H_0^* - \eta(\delta_0)]$. Then there exists $\kappa_0 = \kappa_0(\delta_0) > 0$ and a constant $C(\delta_0) > 0$ such that $\forall H_0 \in [0, H_0^* - \eta(\delta_0)]$, $\forall \kappa \geq \kappa_0$, if $u_\kappa^{H_0}$ exists, then

$$\left(u_\kappa^{H_0} \in \mathcal{K}_{\frac{\delta_0}{2}}\right) \Longrightarrow \left(|u_\kappa^{H_0} - u_\infty^{H_0}|_V \leq \frac{C(\delta_0)}{\kappa^\gamma}\right) \tag{2}$$

for some fixed $\gamma \in (0, \frac{1}{2})$.

With help of an inverse function theorem in Hölder spaces applied to functions $(f, H =$ curl $Q)$, we prove

Proposition 2 *(case $\kappa \in (0, +\infty)$)*
For $\kappa_0 \in (0, +\infty)$ and $H_0 \geq 0$, if $u_{\kappa_0}^{H_0} \in Int(K_0)$, then $\exists \epsilon = \epsilon(\kappa_0, H_0) > 0$, such that if $|h - H_0|, |\kappa - \kappa_0| < \epsilon$, then u_κ^h exists with $u_\kappa^h \in Int(K_0)$. Moreover the map $(\kappa, h) \longmapsto u_\kappa^h \in V$ is continuous.

We finish the proof of theorem 2 in three steps (with some $\theta \in (0, \frac{1}{2})$ fixed).

Step 1

Let us chose $\delta_0 > 0$ small enough, and let $\kappa_1 = \kappa_1(\delta_0) > 0$, large enough such that for all $H_0 \in [0, H_0^* - \eta(\delta_0)]$ and $\kappa \geq \kappa_1$:

$$\forall u \in V, \quad \left(|u - u_\infty^{H_0}|_V \leq \frac{C(\delta_0)}{\kappa^\gamma} \right) \Longrightarrow (u \in \mathcal{K}_{\delta_0}) \tag{3}$$

Step 2

Let $\kappa \geq \kappa_1$. We have $u_\kappa^0 = (1,0) \in \mathcal{K}_{\delta_0} \subset Int(K_0)$ and proposition 2 allows us to build u_κ^h for h in a neighbourhood of 0, and u_κ^h stays in $\mathcal{K}_{\frac{\delta_0}{2}} \supset\supset \mathcal{K}_{\delta_0}$, and from (2) and (3) u_κ^h stays in $\mathcal{K}_{\delta_0} \subset Int(K_{\delta_0})$, and then we have existence of u_κ^h until $h = H_0^* - \eta(\delta_0)$.

Step 3

For $H_0 \in [0, H_0^*)$, from proposition 2, the map $\kappa \longmapsto u_\kappa^{H_0} \in V$ is continuous, and from proposition 1, $u_\kappa^{H_0}$ admits the limit $u_\infty^{H_0}$ as $\kappa \to +\infty$. Consequently it proves the existence of a neighbourhood $\mathcal{P}$ as described in theorem 2.

Idea of proof of theorem 3

We build for $\kappa = +\infty$ a minimising sequence which gives us the estimate for finite κ. Let us take coordinates $(x, y) \in \Omega$. This sequence is a regularization of $f_n = 1$ outside the lines $x = \frac{k}{n}$, $Q_n = H_0(-\frac{y}{n} \sum_{k \in \mathbf{Z}} \delta_{\frac{k}{n}}(x), \frac{nx - [nx]}{n})$. Thus curl $Q_n = H_0$, and $|Q_n|^2 \simeq \frac{1}{n^2}$ except on the lines $x = \frac{k}{n}$ (vortex sheet) where we force the function f_n to take the value 0 such that we assure $|Q_n|^2 |f_n|^2 \simeq 0$. Then for $\kappa = \infty$ and $n \to +\infty$, we get $\inf_V \mathcal{E}_\infty = 0$.

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