On the regularity of a free boundary for a nonlinear obstacle problem arising in superconductor modelling.

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Résumé

Nous étudions les frontières libres asociées à des solutions d'une classe de problèmes de l'obstacle non linéaires. Cette classe de problèmes contient un modèle particulier dérivé des équations de Ginzburg-Landau de la supraconductivité. Nous considérons des solutions dans un ouvert borné Ω à bord Lipschitz, et nous prouvons que la frontière libre est régulière lorsque celle-ci est suffisamment proche du bord fixe $\partial\Omega$. Nous prouvons aussi un résultat de stabilité de la frontière libre et donnons une borne a priori sur la mesure de Hausdorff de cette frontière libre.

Abstract

We study the free boundary of solutions to a class of nonlinear obstacle problems. This class of problems contains a particular model derived from the Ginzburg-Landau equation of superconductivity. We consider solutions in a Lipschitz bounded open set Ω and prove the regularity of the free boundary when it is close enough to the fixed boundary $\partial\Omega$. We also give a result of stability of the free boundary and give a bound on the Hausdorff measure of the free boundary.

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1 Introduction

In this article we are interested in solutions to a nonlinear obstacle problem. This problem is motivated by a work of Chapman, Rubinstein, Schatzman [13] where a model is formally derived from the Ginzburg-Landau theory for a superconductor with a density of vortices in an interior region whose boundary is a free boundary. A rigorous derivation of this model has been done by Sandier, Serfaty [23]. Se also [4, 12, 25, 24] for some related works on the mathematical analysis of superconductivity. Here we will prove rigorous results on the regularity of the free boundary contained in a Lipschitz domain. The core of the technical part of this article is an adaptation in the framework of the nonlinear obstacle problem on non-smooth domains of Caffarelli-type techniques [8, 9] originally developed for linear obstacle problems on smooth domains.

The model that we consider in this paper is a nonlinear obstacle problem in a Lipschitz bounded open set $\Omega \subset \mathbf{R}^n$. We are interested in the minimization of the energy

$$E(u) = \int_\Omega F(|\nabla u|^2) + u^2$$

on the convex set

$$K_\lambda = \left\{ u \in H^1(\Omega), \quad u \geq \lambda \quad \text{on} \quad \Omega, \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega \right\}$$

where $0 \leq \lambda \leq \lambda_0$ are two constants. We make the following assumption (which implies that the energy E is strictly convex)

(A0) F is a C^∞ convex function satisfying $F'(0) = 1$ and $\lim_{q \to +\infty} F'(q) < +\infty$.

It is classical that for each λ there exists a unique minimizer u_λ of the energy E on K_λ. For such a minimizer the coincidence set is

$$\{u = \lambda\}$$

and the free boundary is

$$\partial\{u = \lambda\}$$

When the free boundary $\partial\{u = \lambda\}$ is smooth, the solution u satisfies the following Euler-Lagrange equation

$$\begin{cases} div\left(F'(|\nabla u|^2)\nabla u\right) = u \quad \text{on} \quad \Omega\backslash\{u = \lambda\} \\ u = \lambda_0 \quad \text{on} \quad \partial\Omega \\ \left.\begin{array}{l} u = \lambda \\ \frac{\partial u}{\partial n} = 0 \end{array}\right| \quad \text{on} \quad \partial\{u = \lambda\} \end{cases}$$

Although there are two boundary conditions on the free boundary, the problem is not overdeterminated. These two boundary conditions allow to characterize the free boundary $\partial\{u = \lambda\}$ which is an unknown in this problem.
We refer the reader to the monographs [17, 14, 22] for a presentation of the classical results on the free boundary of the obstacle problem.

1.1 Main results

Our main result (for a smooth open set and in the linear case) is the following :

Theorem 1.1 (Regularity transfer from the fixed boundary to the free boundary)
Let us assume that the open set Ω is smooth, and that $F(q) = q$, then the energy E has a unique minimizer u_λ on K_λ for all $\lambda \in [0, \lambda_0]$. Moreover there exists $\delta > 0$ such that for all $\lambda \in (\lambda_0 - \delta, \lambda_0)$, the free boundary $\partial\{u_\lambda = \lambda\}$ is a C^∞ $(n-1)$-dimensional manifold homeomorphic to $\partial\Omega$.

Although this result seems very natural, it was an open problem (even in this linear case), that we solve here applying the approach of blow-ups developed by Caffarelli [8] for the regularity of the free boundary of the obstacle problem. Under the assumption that $\partial\Omega \in C^\infty$, a nonlinear variant of theorem 1.1 was proved in [5] by A. Bonnet and the author, using the Nash-Moser inverse function theorem in dimension 2. This Nash-Moser approach could work in fact in any dimensions, but it can not be applied to a fixed boundary $\partial\Omega$ less regular than C^∞. On the contrary the approach of Caffarelli [8] allows to deal with non-smooth fixed boundaries $\partial\Omega$.

We extend theorem 1.1 to Lipschitz open set Ω and for general convex functions F satisfying assumption (A0). More precisely we make the following two assumptions on the regularity of Ω:

(A1) *Exterior sphere condition*:
There exists $r_0 > 0$ such that for every point X_0 of the boundary $\partial\Omega$, there exists a point $X_1 \in \mathbf{R}^n$, such that the ball $B_{r_0}(X_1)$ is included in $\mathbf{R}^n\backslash\Omega$ and is tangent to $\partial\Omega$ at X_0.

(A2) *Interior cone condition*:
There exist $r_0 > 0$ and an angle $\alpha_0 \in (0, \frac{\pi}{2})$ such that for every point X_0 of the boundary $\partial\Omega$, there exists a unit vector $\nu \in \mathbf{S}^{n-1}$, such that Ω contains the cone

$$\left\{X \in B_{r_0}(X_0), \quad < \frac{X - X_0}{|X - X_0|}, \nu > \geq \ \cos\alpha_0\right\}$$

where $< \cdot, \cdot >$ is the usual scalar product. Theorem 1.1 is a corollary of the following more general result:

Theorem 1.2 (Regularity transfer from a Lipschitz fixed boundary)
Under assumptions (A0)-(A1)-(A2), the energy E has a unique minimizer u_λ on K_λ for all $\lambda \in [0, \lambda_0]$. Moreover there exists $\delta > 0$ such that for all $\lambda \in (\lambda_0 - \delta, \lambda_0)$, the free boundary $\partial\{u_\lambda = \lambda\}$ is a C^∞ $(n-1)$-dimensional manifold homeomorphic to $\partial\Omega$.

In the application that we have in mind, namely a nonlinear free boundary problem arising in the description of superconductors in dimension two (see Bonnet, Monneau [5], Berestycki, Bonnet, Chapman [2]), the function F_0 is analytic convex but only defined on $[0, \frac{4}{27})$ by $F_0'(0) = 1$ and

$$h = (1 - v^2)v \Longleftrightarrow v = F_0'(h^2)h$$

Using a L^∞ control on the gradient of the solution we deduce the following result in this particular case:

Corollary 1.3 (Application to a superconducting model)
Under assumption (A1)-(A2), with $F = F_0$, there exists $\delta > 0$ such that $\forall \lambda \in (\lambda_0 - \delta, \lambda_0)$, there exists a unique solution u_λ minimizer of E on K_λ satisfying $\sup_{\overline{\Omega}} |\nabla u_\lambda|^2 < \frac{4}{27}$; moreover the free boundary $\partial\{u_\lambda = \lambda\}$ is a C^∞ $(n-1)$-dimensional manifold homeomorphic to $\partial\Omega$.

Let us mention that part of the methods of [20] could be adapted to this model of superconductivity to get informations on the singularities of the free boundary when $\lambda < \lambda_0 - \delta$.

We also prove a result on the perturbation (locally in space) of the free boundary.

Theorem 1.4 (Local stability of the free boundary)
We assume (A0)-(A1)-(A2). Let $\lambda^* \in (0, \lambda_0)$ be such that there exists a minimizer u_{λ^*} of the energy E on K_{λ^*} with a free boundary $\partial\{u_{\lambda^*} = \lambda^*\}$ which is C^∞ in a compact set $\mathcal{K}^*$ of Ω. Then for every smaller compact set $\mathcal{K} \subset\subset \mathcal{K}^*$ there exists $\varepsilon > 0$ such that for every λ satisfying $|\lambda - \lambda^*| < \varepsilon$, the unique solution u_λ has a free boundary $\partial\{u_\lambda = \lambda\}$ which is C^∞ in $\mathcal{K}$.

The proof of this result is based on a geometric criterion for the regularity of the free boundary given by Caffarelli in [8] and on the continuity of the map $\lambda \mapsto u_\lambda$. We also refer to the book of Rodrigues [22] for classical results on the global stability of the free boundary.

Finally we give a bound on the Hausdorff measure of the free boundary, generalizing to non-smooth fixed boundaries $\partial\Omega$, a result of Brezis, Kinderlehrer [6] based on fine estimates for variational inequalities. Here the proof is an adaptation of the work of Caffarelli [9], developed for linear equations.

Theorem 1.5 (Bound on the Hausdorff measure of the free boundary)
Under assumptions (A0)-(A1)-(A2), there exists a constant $C > 0$ only depending on Ω, λ_0, F such that for any minimizer u_λ of E on K_λ with $\lambda \in [0, \lambda_0]$, we have

$$\mathcal{H}^{n-1}\left(\partial\left\{u_\lambda = \lambda\right\}\right) \quad \leq \quad C$$

2 Some known results on blow-up limits

2.1 The simple blow-up limit

To prove regularity results on the free boundary, the main tool (first introduced for the obstacle problem by Caffarelli in [10]) is the notion of blow-up.

Let us consider a solution u to

$$\left\{\begin{array}{l} \Delta u = f \geq 1 \quad \text{on} \quad \{u > 0\} \cap \Omega \\ \\ u \geq 0 \quad \text{on} \quad \Omega \quad \text{and} \quad |D^2 u|_{L^\infty(\Omega)} \leq M \end{array}\right. \tag{2.1}$$

with $f \in C^{0,\alpha}(\Omega)$ and $f(0) = 1$. We assume that X_0 is a point of the free boundary $\partial\{u = 0\}$. Let us consider the following blow-up sequence of functions

$$u^\varepsilon(X) = \frac{u(X_0 + \varepsilon X)}{\varepsilon^2}$$

By assumptions, $u^\varepsilon(0) = \nabla u^\varepsilon(0) = 0$ and the second derivatives $|D^2 u^\varepsilon|$ are bounded by a constant independent on $\varepsilon > 0$. By Ascoli-Arzela theorem, up to extraction of a convergent subsequence (ε'), we get

$$u^{\varepsilon'} \longrightarrow u^0 \quad \text{uniformly on compact sets of} \quad \mathbf{R}^n$$

This function u^0 is called a blow-up limit of the function u at the point X_0.

In any dimensions, the main result for blow-up limits is the following

Theorem 2.1 (Caffarelli [10, 8, 11], Weiss [26]; Characterization of a Simple Blow-up Limit)
The blow-up limit u^0 is unique and only depends on the point X_0 on the free boundary. Moreover either X_0 is a **singular** *point and then u^0 is a quadratic form, i.e.*

$$u^0(X) = \frac{1}{2} \quad {}^tX \cdot Q_{X_0} \cdot X \quad \geq 0$$

where Q_{X_0} is a symmetric matrix $n \times n$ such that tr $Q_{X_0} = 1$.
Or X_0 is a **regular** *point and then there exists a unit vector $\nu_{X_0} \in \mathbf{S}^{n-1}$ such that*

$$u^0(X) = \frac{1}{2} \left(\max\left(< X, \nu_{X_0} >, 0\right)\right)^2$$

and the free boundary is a C^1 $(n-1)$-dimensional manifold in a neighbourhood of X_0.

The regularity C^1 can then be improved by Kinderlehrer, Nirenberg results [16], and gives C^∞ regularity for an obstacle problem where the elliptic operator has C^∞ coefficients. It is also possible to get similar results with analyticity of the solutions when the coefficients are analytic.

2.2 More general blow-up limits

We now recall a result which characterizes the limits of some more general blow-up sequences where the origin moves with the scaling.

Lemma 2.2 (General Blow-up Limits, [8])
Let

$$u^\varepsilon(X) = \frac{u_\varepsilon(X_\varepsilon + \varepsilon X)}{\varepsilon^2}$$

where u_ε is a sequence of solutions to

$$\begin{cases} \Delta u_\varepsilon = f_\varepsilon \geq 1 \quad \text{on} \quad \{u_\varepsilon > 0\} \cap \Omega_\varepsilon \\ \\ u_\varepsilon \geq 0 \quad \text{on} \quad \Omega_\varepsilon \quad \text{and} \quad |D^2 u_\varepsilon|_{L^\infty(\Omega_\varepsilon)} \leq M \end{cases}$$

with $|f_\varepsilon|_{C^{0,\alpha}(\Omega_\varepsilon)} \leq M$*. We assume that* $u_\varepsilon(X_\varepsilon) = 0$ *and that* $\frac{1}{\varepsilon} d(X_\varepsilon, \partial\Omega_\varepsilon) \geq r > 0$ *as* $\varepsilon \to 0$*. Then up to extraction of a convergent subsequence* (ε')*, we get*

$$u^{\varepsilon'} \longrightarrow u^0 \quad \textit{uniformly on compact sets of} \quad \Omega^0$$

for some open set Ω^0 *and where* u^0 *is convex and satisfies*

$$\begin{cases} \Delta u^0 = f_0(0) \geq 1 \quad \textit{on} \quad \{u^0 > 0\} \cap \Omega^0 \\ \\ u^0 \geq 0 \quad \textit{on} \quad \Omega^0 \quad \textit{and} \quad |D^2 u^0|_{L^\infty(\Omega^0)} \leq M \end{cases}$$

Moreover either

i) the interior of the coincidence set of the blow-up limit is empty:

$$\{u^0 = 0\}^0 = \emptyset$$

Or

ii) the interior of the coincidence set of the blow-up limit satisfies

$$\{u^0 = 0\}^0 \neq \emptyset$$

and 0 *is a regular point for* u^0 *and also for all* $u^{\varepsilon'}$ *with* ε' *small enough.*

Another useful result is the following nondegeneracy property of the solution:

Lemma 2.3 (Nondegeneracy, [8])

Let u *be a solution to problem (2.1) and* $0 \in \overline{\{u > 0\}}$*. If* $B_r(0) \subset \Omega$*, then*

$$\sup_{B_r(0)} (u(X) - u(0)) \geq \frac{r^2}{2n}$$

Proof of lemma 2.3. Apply the maximum principle to $w(X) = u(X) - u(0) - \frac{1}{2n}|X|^2$ in $B_r(0) \cap \{u > 0\}$.

3 A bound on the second derivatives

In this section we will prove the following result

Proposition 3.1 (Control near the fixed boundary $\partial\Omega$)
Under the assumptions of theorem 1.2, let us define $\varepsilon = \sqrt{2\left(\frac{\lambda_0-\lambda}{\lambda}\right)}$. *Then there exist constants* $C, c > 0$ *such that for all* $\lambda \in [0, \lambda_0]$ *we have*

$$u_\lambda(X) - \lambda \quad \geq \quad c\varepsilon^2 \quad on \quad \{X \in \Omega, \quad dist(X, \partial\Omega) \leq c\varepsilon\} \tag{3.1}$$

$$|\nabla u_\lambda(X)| \quad \leq \quad C\varepsilon \quad on \quad \Omega \tag{3.2}$$

and for all $\delta \in (0, 1]$

$$\left|D^2 u_\lambda(X)\right| \quad \leq \quad C/\delta^2 \quad on \quad \{X \in \Omega, \quad dist(X, \partial\Omega) \geq c\varepsilon\delta\} \tag{3.3}$$

Moreover we have

$$div\left(F'(|\nabla u_\lambda|^2)\nabla u_\lambda\right) = u_\lambda \ 1_{\{u_\lambda > \lambda\}} \quad on \quad \Omega$$

where for the function $u_\lambda \geq \lambda$ *we define*

$$1_{\{u_\lambda>\lambda\}}(X) = \begin{cases} 1 & if \quad u_\lambda(X) > \lambda \\ 0 & if \quad u_\lambda(X) = \lambda \end{cases}$$

Remark 3.2 *For a smooth* Ω*, some* L^∞ *bounds on the second derivatives are given in [6] for fixed* λ*. Here we need to precise the dependence of the constants as* λ *goes to* λ_0*. The exterior sphere condition gives a control (3.1) from below on* u_λ*, and with the help of Harnack inequality we get the* L^∞ *bounds (3.3) on the second derivatives up to the case* $\lambda = \lambda_0$*. Because the fixed boundary* $\partial\Omega$ *is not smooth here, the bound (3.3) on the second derivatives goes to infinity when the point reaches the fixed boundary* $\partial\Omega$ *(case* $\delta = 0$*).*

We consider the minimizer u_λ of the convex energy

$$E(u) = \int_\Omega F(|\nabla u|^2) + u^2$$

on the convex set

$$K_\lambda = \big\{u \in H^1(\Omega), \quad u \geq \lambda \quad \text{on} \quad \Omega, \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega\big\}$$

We first prove that the minimizer u_λ satisfies the following Euler-Lagrange equation

Lemma 3.3 (Euler-Lagrange equation)

$$div\left(F'(|\nabla u_\lambda|^2)\nabla u_\lambda\right) = u_\lambda \ 1_{\{u_\lambda > \lambda\}} \quad on \quad \Omega$$

Although this result seems natural, we do not know any references where it is proved (except in the linear case). We give a complete proof below.

Proof of lemma 3.3

Let

$$(s)^+ = \begin{cases} s & \text{if} \quad s > 0 \\ \\ 0 & \text{if} \quad s \leq 0 \end{cases}$$

Then the minimization of E on K_λ is equivalent to the minimization of the convex energy

$$E_\lambda(u) = \int_\Omega F(|\nabla u|^2) + \Big((u-\lambda)^+ + \lambda\Big)^2$$

on the convex set

$$K = \big\{u \in H^1(\Omega), \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega\big\}$$

Because u_λ is the minimizer of E_λ on K, we have for every $\varphi \in C_0^\infty(\Omega)$ and $t \in [0,1]$:

$$E_\lambda(u_\lambda + t\varphi) \geq E_\lambda(u_\lambda)$$

Then Lebesgue's dominated convergence theorem gives

$$\begin{aligned} 0 \quad &\leq \quad \lim_{t\to 0} \quad \left(\frac{E_\lambda(u_\lambda + t\varphi) - E_\lambda(u_\lambda)}{t}\right) \\ &= \quad \int_\Omega 2F'\left(|\nabla u_\lambda|^2\right) \nabla u_\lambda \nabla\varphi + 2\ u_\lambda\left(\varphi\ \text{sgn}^+(u_\lambda - \lambda) + \varphi^+\left(1 - \text{sgn}^+(u_\lambda - \lambda)\right)\right) \end{aligned}$$

where

$$\text{sgn}^+(s) = \begin{cases} 1 & \text{if} \quad s > 0 \\ \\ 0 & \text{if} \quad s \leq 0 \end{cases}$$

Considering φ and $-\varphi$ we get that div $\left(F'(|\nabla u_\lambda|^2)\nabla u_\lambda\right) \in L^\infty(\Omega)$. Using the regularity theory for elliptic equations (see [21]) we deduce that $u \in C^{1,\alpha}_{loc}(\Omega)$. Consequently $\{u_\lambda > \lambda\}$ is an open set and the Euler-Lagrange equation is satisfied on this open set. Furthermore a classical argument using the nondegeneracy lemma 2.3 proves that the Lebesgue measure of the free boundary $\partial\{u_\lambda = \lambda\}$ is zero. This implies the full Euler-Lagrange equation. This ends the proof of lemma 3.3.

Let us recall that when Ω is smooth, there exists a constant $C_0 > 0$ such that for each $\lambda \in [0, \lambda_0]$ we have the following properties (see Brézis, Kinderlehrer [6]):

(H1)

$$|\nabla u_\lambda(X)| \le C_0 \quad \text{on} \quad \Omega$$

(H2)

$$u \in C^{1,1}_{loc}(\Omega)$$

In a first case we will prove proposition 3.1 assuming (H1)-(H2), and in a second case we will justify these assumptions.

Case A: we assume (H1)-(H2) and that $\partial\Omega$ is smooth.

Step 1: proof of (3.1)

We will build a subsolution u_0 such that (for some point X_ε which will be precised below)

$$\frac{u_\lambda(X) - \lambda}{\lambda} \quad \ge \quad \varepsilon^2 u_0\left(\frac{|X - X_\varepsilon|}{\varepsilon}\right) \quad \text{for} \quad \frac{|X - X_\varepsilon|}{\varepsilon} \in [r_0, r_0 + \tau_0] \qquad (3.4)$$

with $\varepsilon = \sqrt{2\left(\frac{\lambda_0 - \lambda}{\lambda}\right)}$.

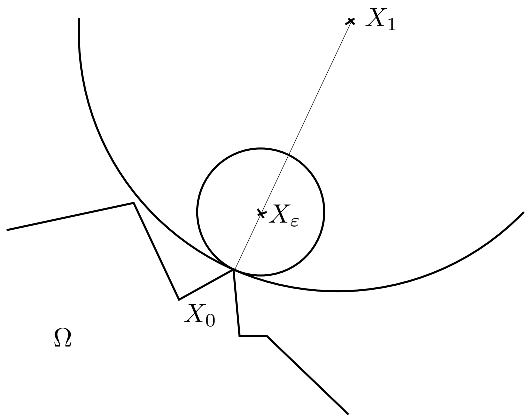

Figure 1: Construction of a subsolution outside the ball $B_{|X_0-X_\varepsilon|}(X_\varepsilon)$

For some $\tau_0 > 0$, we consider a solution u_0 of

$$\begin{cases} \Delta u_0 = \mu > 1 \quad \text{on} \quad B_{r_0+\tau_0}(0) \backslash B_{r_0}(0) \\ \\ u_0 = \frac{1}{2} \quad \text{on} \quad \partial B_{r_0}(0) \\ \\ u_0 = 0 \quad \text{on} \quad \partial B_{r_0+\tau_0}(0) \end{cases}$$

By symmetry we have $u_0(X) = u_0(|X|)$. Let us recall that for each point $X_0 \in \partial\Omega$, there exists $X_1 \in \mathbf{R}^n$, such that $B_{r_0}(X_1)$ is included in $\mathbf{R}^n \backslash \Omega$ and is tangent to $\partial\Omega$ at X_0. Now considering the function u_λ at a scale close to the fixed boundary $\partial\Omega$ we introduce the point $X_\varepsilon = X_0 + \varepsilon\,(X_1 - X_0)$ and the following function (see figure 1)

$$w^\varepsilon(X) = \frac{u_\lambda(X_\varepsilon + \varepsilon X) - \lambda}{\lambda \varepsilon^2}$$

which satisfies on $\frac{\Omega - X_\varepsilon}{\varepsilon}$:

$$\begin{cases} A_\varepsilon(w^\varepsilon) \leq 1 \\ \\ 0 \leq w^\varepsilon \leq \frac{1}{2} \end{cases}$$

where the quasilinear elliptic partial differential operator A_ε is defined in (4.1). Moreover for a good choice of $\mu > 1, \tau_0 > 0$, we have on $B_{r_0+\tau_0}(0) \backslash B_{r_0}(0)$:

$$\begin{cases} A_\varepsilon(u_0) \geq 1 \\ \\ 0 \leq u_0 \leq \frac{1}{2} \end{cases}$$

Then by the Maximum Principle (see Berestycki, Nirenberg [3]), we can slide u_0 below w^ε and we get

$$w^\varepsilon \geq u_0 \quad \text{on} \quad B_{r_0+\tau_0}(0) \backslash B_{r_0}(0)$$

This is equivalent to (3.4) whose we deduce (3.1). This ends the proof of step 1.

Step 2: proof of (3.2): estimate on the gradient : $|\nabla u_\lambda| \leq \lambda\varepsilon |u_0'(r_0)|$

We first remark that a straightforward consequence of step 1 is that

$$\limsup_{X \to \partial\Omega} \left(\frac{\lambda_0 - u_\lambda}{\operatorname{dist}(X, \partial\Omega)} \right) \quad \leq \quad \lambda\varepsilon |u_0'(r_0)|$$

From the fact that $u = \mathit{constant} = \lambda_0$ on $\partial\Omega$, we deduce that $|\nabla u| \leq \lambda\varepsilon |u_0'(r_0)|$ on $\partial\Omega$. Now the estimate on the gradient comes from the fact that the gradient is maximal on the boundary $\partial\Omega$. For the convenience of the reader we recall this classical argument.
For $u = u_\lambda$, we have

$$a_{ij}(\nabla u)\, u_{ij} = u \quad \text{on} \quad \Omega \backslash \{u = \lambda\}$$

where $a_{ij}(p) = F'(|p|^2)\delta_{ij} + 2F''(|p|^2)p_i p_j$. Let us take $v = \partial_\xi u$ where $\xi \in \mathbf{S}^{n-1}$. Then

$$a_{ij} v_{ij} + b_k v_k = v \quad \text{on} \quad \Omega \backslash \{u = \lambda\}$$

where $b_k = (a_{ij})'_{p_k} \cdot u_{ij}$. The Maximum Principle implies that $v = \partial_\xi u$ is maximal on $\partial\Omega \cup \partial\{u = \lambda\}$. Taking all directions $\xi \in \mathbf{S}^{n-1}$ we deduce that $|\nabla u|$ is maximal on $\partial\Omega$, because $\nabla u = 0$ on $\partial\{u = \lambda\}$.
This ends the proof of step 2.

Step 3: proof of (3.3)

Let

$$w(X) = \frac{u_\lambda(\varepsilon X) - \lambda}{\lambda\varepsilon^2}$$

Then

$$\begin{cases} A_\varepsilon(w) = 1 \quad \text{on} \quad \{w > 0\} \\ \\ 0 \leq w \leq \frac{1}{2} \end{cases}$$

where the operator A_ε is defined in (4.1). Let $Y_0 \in \frac{\Omega}{\varepsilon}$ such that $\operatorname{dist}\left(Y_0, \frac{\partial\Omega}{\varepsilon}\right) \geq c$. We will prove a bound on $|D^2 w(Y_0)|$. To this end we will apply the method of Alt and Phillips [1], using the following Harnack inequality of Krylov, Safonov for non-divergence operator (a similar Harnack inequality for divergence operator is also applicable, see Gilbarg, Trudinger [15]):

Theorem 3.4 (Harnack inequality for non-divergence operators; [7])

If

$$\begin{cases} a_{ij}v_{ij} = f \quad on \quad B_1 \subset \mathbf{R}^n \\ \\ v \geq 0 \quad on \quad B_1 \end{cases}$$

and for the matrix $a = (a_{ij})$

$$0 < c_0 \leq a \leq C_0$$

then there exists a constant $C = C(n, C_0, c_0) > 0$ such that

$$\sup_{B_{\frac{1}{2}}} v \leq C \left(\inf_{B_{\frac{1}{2}}} v + |f|_{L^n(B_1)} \right)$$

We will also use the following interior estimate:

Theorem 3.5 (Interior estimate,[15])

Let us assume that

$$a_{ij}v_{ij} + cv = f \quad on \quad B_r \subset \mathbf{R}^n$$

and for the matrix $a = (a_{ij})$

$$0 < c_0 \leq a$$

If for some $\alpha \in (0,1)$ there exists a constant $C_0 > 0$ such that

$$|a_{ij}|_{L^\infty(B_r)} + r^\alpha [a_{ij}]_{\alpha;B_r} + r^2 |c|_{L^\infty(B_r)} + r^{2+\alpha} [c]_{\alpha;B_r} \leq C_0$$

where $[\cdot]_{\alpha;B_r}$ is defined by

$$[g]_{\alpha;B_r} = \sup_{x,y \in B_r, x \neq y} \left(\frac{|g(x) - g(y)|}{|x-y|^\alpha} \right)$$

Then

$$r^2|D^2v|_{L^\infty(B_{\frac{r}{2}})} \le C\left(|v|_{L^\infty(B_r)} + r^2|f|_{L^\infty(B_r)} + r^{2+\alpha}[f]_{\alpha;B_r}\right)$$

for some constant $C = C(n, \alpha, C_0, c_0) > 0$.

Let $w_r(X) = w(Y_0 + rX)$. Applying Harnack inequality theorem 3.4 to w_r we get

$$\sup_{B_{\frac{r}{2}}(Y_0)} w \le C\left(\inf_{B_{\frac{r}{2}}(Y_0)} w + r^2\right) \tag{3.5}$$

Let

$$\rho = \sqrt{\frac{w(Y_0)}{2C}}$$

i) Case $\rho < c\delta$.

Then Y_0 is close to $\{w = 0\}$ and ρ can be arbitrarily small. We apply Harnack inequality (3.5) with $r = \rho$ and we get

$$0 < w(Y_0) \le \sup_{B_{\frac{\rho}{2}}(Y_0)} w \le 2C \inf_{B_{\frac{\rho}{2}}(Y_0)} w$$

Let us remark that we have (see theorem 6.1, p. 281 of Ladyshenskaya, Ural'tseva [18])

$$[w]_{\alpha;B_1} \le C$$

where the constant C has the following dependence $C = C(n, \alpha, |w|_{L^\infty(B_2)}, F, \lambda_0, r_0) > 0$. Then applying theorem 3.5, we deduce that

$$r^2|D^2w|_{L^\infty\left(B_{\frac{r}{2}(Y_0)}\right)} \le C\left(|w|_{L^\infty(B_r(Y_0))} + r^2\right)$$

With the choice $r = \rho$, this implies

$$|D^2w(Y_0)| \le C$$

ii) case $\rho \ge c\delta$.

We apply the previous interior estimate with $r = c\delta$. Using the fact that $|w| \le \frac{1}{2}$, we find

$$|D^2w(Y_0)| \le C/\delta^2$$

iii) Conclusion :

$$|D^2 u_\lambda| \le C/\delta^2 \quad \text{on} \quad \{X \in \Omega, \quad \text{dist}\,(X, \partial\Omega) \ge c\varepsilon\delta)\}$$

i.e. (3.3) is proved.

Case B: justification of (H1)-(H2).

Here we consider a general Lipschitz bounded open set Ω satisfying assumptions (A1), (A2) of theorem 1.2. We can mollify this open set Ω such that it gives a bigger and smooth open set Ω^η where η is the mollification parameter such that $\Omega^\eta = \Omega$ for $\eta = 0$. This smooth open set Ω^η still satisfies assumptions (A1), (A2) uniformly in η small enough. We can in particular consider the minimizer u_λ^η of the energy

$$E^\eta(u) = \int_{\Omega^\eta} F\left(|\nabla u|^2) + u^2\right)$$

on the convex set

$$K_\lambda^\eta = \left\{u \in H^1(\Omega^\eta), \quad u \ge \lambda \quad \text{on} \quad \Omega^\eta, \quad u = \lambda_0 \quad \text{on} \quad \partial\Omega^\eta\right\}$$

This minimizer u_λ^η satisfies (H1)-(H2), and then (3.1),(3.2),(3.3).

Taking the limit $\eta \to 0$, we can extract (by Ascoli-Arzela theorem) a convergent subsequence $u_\lambda^\eta \to u$ such that u still satisfies (3.1),(3.2),(3.3).

We have the

Lemma 3.6 *The limit u is the minimizer u_λ of the energy E on K_λ.*

This ends the proof of proposition 3.1.

Proof of lemma 3.6

Let us recall that by (3.2), u_λ^η is bounded in $W^{1,\infty}$ uniformly in η small enough. Let

$$\tilde{u}_\lambda = \begin{cases} \lambda_0 & \text{on} \quad \Omega^\eta \backslash \Omega \\ \\ u_\lambda & \text{on} \quad \Omega \end{cases}$$

By construction, we have

$$E^\eta(\tilde{u}_\lambda) \geq E^\eta(u_\lambda^\eta)$$

At the limit $\eta = 0$, we get

$$E(u_\lambda) \geq E(u)$$

The uniqueness of the minimizer u_λ proves that $u = u_\lambda$. This ends the proof of the lemma 3.6.

4 Regularity of the free boundary near $\partial\Omega$: proof of theorem 1.2

We will prove theorem 1.2, thanks to Caffarelli result (lemma 2.2) applied to a particular blow-up sequence.

Case $F(q) = q$

If theorem 1.2 is false, then there exist a sequence of reals $\varepsilon_n = \sqrt{2\left(\frac{\lambda_0 - \lambda^n}{\lambda^n}\right)} \to 0$ and a sequence of singular points $X_{\lambda^n} \in \partial\left\{u_{\lambda^n} = \lambda^n\right\}$. Because of proposition 3.1, we have $\text{dist}(X_{\lambda^n}, \partial\Omega) > c\varepsilon_n$. Then we define

$$w^{\varepsilon_n}(X) = \frac{u_{\lambda^n}(X_{\lambda^n} + \varepsilon_n X) - \lambda^n}{\lambda^n \varepsilon_n^2}$$

We have

$$\begin{cases} \Delta w^{\varepsilon_n} = 1 + \varepsilon_n^2 w^{\varepsilon_n} \quad \text{on} \quad \{w^{\varepsilon_n} > 0\} \\ \\ 0 \leq w^{\varepsilon_n} \leq \frac{1}{2} \end{cases}$$

Now from proposition 3.1 we have the following L^∞ bound on the second derivatives:

$$\left|D^2 w^{\varepsilon_n}(X)\right| \leq C \quad \text{for} \quad \text{dist}(X_{\lambda^n} + \varepsilon_n X, \partial\Omega) \geq c\varepsilon_n$$

Consequently from lemma 2.2, there exists a subsequence which converges to a convex function w^0 defined on Ω_0, where Ω_0 is the limit of the sets $\frac{1}{\varepsilon_n}(\Omega - X_{\lambda^n})$ (for an extracted subsequence). Moreover w^0 satisfies

$$\begin{cases} \Delta w^0 = 1 \quad \text{on} \quad \left\{w^0 > 0\right\} \\ \\ 0 \le w^0 \le \frac{1}{2} \quad \text{and} \quad \left|D^2 w^0(X)\right| \le C \quad \text{for} \quad \text{dist}(X, \partial\Omega_0) \ge c \end{cases}$$

Because Ω satisfies an interior cone condition (A2), Ω_0 inherits the same property. Moreover because we have made a blow-up close to the fixed boundary $\partial\Omega$, we deduce that Ω_0 contains an infinite cone $\mathcal{C}_0$ with a non-empty interior. Now we have two cases (see lemma 2.2):

i) the interior of the coincidence set of the blow-up limit is empty, and then the closure $\overline{\{w^0 > 0\}}$ contains the cone $\mathcal{C}_0$. It is then sufficient to take a ball $B_r \subset \mathcal{C}_0$ with r large enough such that (by the nondegeneracy lemma 2.3)

$$\sup_{B_r} w^0 \ge \frac{r^2}{2n}$$

which is in contradiction with $0 \le w^0 \le \frac{1}{2}$.

ii) the interior of the coincidence set of the blow-up limit is not empty, and then 0 is a regular point for w^0, and also a regular point for $w^{\varepsilon'_n}$ for ε'_n small enough. This means that X_{λ^n} are regular points for u_{λ^n}. Contradiction.

Case F general

In this case we introduce the operator (for $\varepsilon = \sqrt{2\left(\frac{\lambda_0 - \lambda}{\lambda}\right)}$)

$$A_\varepsilon(w) = a\left(\left(\frac{\lambda_0}{1 + \frac{\varepsilon^2}{2}}\right)\varepsilon\nabla w\right) D^2 w - \varepsilon^2 w \tag{4.1}$$

where $a(p) = F'(p^2)Id + 2F''(p^2)p \otimes p$. Then we have

$$\begin{cases} A_{\varepsilon_n}(w^{\varepsilon_n}) = 1 \quad \text{on} \quad \{w^{\varepsilon_n} > 0\} \\ \\ 0 \le w^{\varepsilon_n} \le \frac{1}{2} \end{cases}$$

A generalization of previous Caffarelli results to more general linear elliptic operators

$$L = \alpha_{ij}\partial_{ij} + \beta_i \partial_i + \gamma$$

is available in [8]. This allows to get similar results in the same way for our general case. This ends the proof of theorem 1.2.

5 Stability: proof of theorem 1.4

In this section we will prove theorem 1.4 on stability. A similar result is already known in the linear case (see for instance the book of Rodrigues [22] for general results of stability). In our case we use the approach of Caffarelli [8].

Proof of theorem 1.4

Let us assume that the theorem is false. Then for a compact set $\mathcal{K} \subset\subset \mathcal{K}^*$ we can find a sequence $(\lambda^n)_n$ such that $\lambda^n \to \lambda^*$ and a sequence of singular points $(X_{\lambda^n})_n$ of the free boundaries $\partial \{u_{\lambda^n} = \lambda^n\} \cap \mathcal{K}$. Up to extract a subsequence we can assume

$$X_{\lambda^{n'}} \longrightarrow X_{\lambda^*} \in \{u_{\lambda^*} = \lambda^*\} \cap \mathcal{K}$$

where we have used the continuity of the map

$$\lambda \longmapsto u_\lambda$$

The continuity of this map is a consequence of the L^∞ bound on the gradient of u_λ uniformly in λ (see (3.2)). This continuity easily follows by a classical argument from Ascoli-Arzela theorem, and the uniqueness of the solutions u_λ for each λ.
Let us recall that for $\varepsilon = \sqrt{2\left(\frac{\lambda_0-\lambda}{\lambda}\right)}$ we have (the operator A_ε is defined in (4.1))

$$A_\varepsilon(w_\lambda) = 1 \quad \text{on} \quad \{w_\lambda > 0\}$$

where for some point $X_\lambda \in \Omega$:

$$w_\lambda(X) = \frac{u_\lambda(X_\lambda + \varepsilon X) - \lambda}{\lambda \varepsilon^2}$$

Using the adaptation of the nondegeneracy lemma 2.3 (see Caffarelli [8]) for general linear elliptic operators, we get the existence of a constant $c_0 > 0$ such that

$$\sup_{B_r(X_{\lambda^n})} (u_{\lambda^n}(X) - \lambda^n) \geq c_0 r^2$$

Then at the limit we get

$$\sup_{B_r(X_{\lambda^*})} (u_{\lambda^*}(X) - \lambda^*) \geq c_0 r^2$$

which proves that $X_{\lambda^*} \in \partial \{u_{\lambda^*} = \lambda^*\}$. In particular because X_{λ^*} is a regular point for u_{λ^*}, i.e. 0 is a regular point for w_{λ^*}, we get that the blow-up sequence

$$w^{\delta}_{\lambda^*}(X) = \frac{w_{\lambda^*}(\delta X)}{\delta^2}$$

converges (up to extraction of a subsequence) to a blow-up limit of regular type (see theorem 2.1; for an extension to general linear elliptic operators, see Caffarelli [8]):

$$w^0_{\lambda^*}(X) = \frac{1}{2} \left(\max\left(< X, \nu_{X_{\lambda^*}} >, 0\right)\right)^2$$

We realize that the origin 0 is obviously a regular point of $w^0_{\lambda^*}$. Finally we can consider the other blow-up sequence:

$$w^{\delta^n}_{\lambda^n}(X) = \frac{w_{\lambda^n}(\delta^n X)}{(\delta^n)^2}$$

Because for $\delta^n = \delta$ fixed and $\lambda^n \to \lambda^*$, this sequence of functions converges to $w^{\delta}_{\lambda^*}$, we see that we can choose a sequence $(\delta^n)_n$ slowly decreasing to zero such that

$$w^{\delta^n}_{\lambda^n} \longrightarrow w^0_{\lambda^*}$$

Then applying an adaptation of lemma 2.2 (see Caffarelli [8]) still true for general linear elliptic operators, we deduce from the fact that 0 is a regular point for the blow-up limit of $w^{\delta^n}_{\lambda^n}$, that 0 is also a regular point for $w^{\delta^n}_{\lambda^n}$ for n large enough. This means that X_{λ^n} is a regular point for u_{λ^n}. Contradiction. This ends the proof of theorem 1.4.

6 Hausdorff measure of the free boundary: proof of theorem 1.5

In this section we give the proof of theorem 1.5, which is an adaptation of a method of Caffarelli presented in the linear case in [9, 19]. We perform the proof in two steps.

Step 1

For the function $u = u_\lambda$, let

$$O^\eta = \{X \in \Omega, \quad |\nabla u(X)| < \eta \quad \text{and} \quad u(X) > \lambda\}$$

For a function $u \geq \lambda$, we note

$$1_{\{u>\lambda\}}(X) = \begin{cases} 1 & \text{if} \quad u(X) > \lambda \\ 0 & \text{if} \quad u(X) = \lambda \end{cases}$$

Lemma 6.1 (Estimate in a neighbourhood of the free boundary)

If

$$\begin{cases} \nabla \cdot (F'(|\nabla u|^2)\nabla u) = u \ 1_{\{u>\lambda\}} & \textit{on} \quad \Omega \\ \\ u \geq \lambda > 0 & \textit{on} \quad \partial\Omega \\ \\ |D^2 u(X)| \leq M & \textit{on} \quad \{X \in \Omega, \quad dist(X, \partial\Omega) \geq c\varepsilon\} \end{cases}$$

then for all compact $\mathcal{K} \subset \{X \in \Omega, \quad dist(X, \partial\Omega) \geq c\varepsilon\}$ such that $\partial\mathcal{K}$ is C^1, there is a constant $C = C(M)$, such that

$$|O^\eta \cap \mathcal{K}| \leq \eta C \lambda^{-2}(|\mathcal{K}| + \mathcal{H}^{n-1}(\partial\mathcal{K}))$$

where $|\mathcal{K}|$ is the volume of $\mathcal{K}$ and $\mathcal{H}^{n-1}(\partial\mathcal{K})$ is the $(n-1)$ dimensional Hausdorff measure of its perimeter.

Remark 6.2 (The Hausdorff measure)

Let us recall the definition of the Hausdorff measure. If U is a set, let

$$\text{diam }(U) = \sup_{X,X' \in U} |X' - X|$$

Then for $s \geq 0$ and a set A let

$$\mathcal{H}^s_\delta(A) = c_s \inf_{\{\{U^i\}_i, \ A \subset \cup_i U^i, \text{diam } U^i \leq \delta\}} \sum_i (\text{diam } U^i)^s$$

which is a nondecreasing function of δ. Then the s-dimensional Hausdorff measure is

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A)$$

The constant c_s must be chosen such that the Hausdorff measure coincides with the Lebesgue measure of $\mathbf{R}^s$ if $s \in \mathbf{N}$.

Proof of lemma 6.1

Because $F' \in C^{1,1}$, we have $h_i \in C^{0,1}$ where

$$h_i = \begin{cases} -\eta & \text{if } F'\nabla_i u \leq -\eta \\ \\ F' \cdot \nabla_i u & \text{if } |F'\nabla_i u| \leq \eta \\ \\ \eta & \text{if } F'\nabla_i u \geq \eta \end{cases} \tag{6.1}$$

We note X_i the vector field defined by $X_i = \nabla_i(F'\nabla u) \in L^\infty$. Then the Stokes formula gives :

$$\int_{\mathcal{K}} \nabla h_i \cdot X_i = \int_{\partial\mathcal{K}} h_i(X_i \cdot n) - \int_{\mathcal{K}} h_i(\nabla \cdot X_i) \tag{6.2}$$

But $\nabla \cdot X_i = \nabla_i(\nabla \cdot (F'\nabla u)) = \nabla_i u$ on $\{u > \lambda\}$, and $h_i = 0$ on $\{u = \lambda\}$. Then

$$\int_{O^\eta \cap \mathcal{K}} \nabla(F'\nabla_i u) \cdot \nabla_i(F'\nabla u) \quad \leq \quad \eta C(M)(|\mathcal{K}| + \mathcal{H}^{n-1}(\partial\mathcal{K})) \tag{6.3}$$

But

$$\nabla(F'\nabla_i u) \cdot \nabla_i(F'\nabla u) = [\nabla_i(F'\nabla_i u)]^2 + \sum_{k \neq i} [F' D^2_{ik} u]^2 + O(|\nabla u|^2)$$

and

$$\left| \int_{O^\eta \cap \mathcal{K}} O(|\nabla u|^2) \right| \quad \leq \quad \eta C(M)|\mathcal{K}|$$

Making the sum $\sum_i$, we get

$$\int_{O^\eta \cap \mathcal{K}} \sum_i \left(\nabla_i(F'\nabla_i u)\right)^2 \quad \leq \quad \eta C(M)(|\mathcal{K}| + \mathcal{H}^{n-1}(\partial\mathcal{K})) \tag{6.4}$$

But

$$\sum_i \left(\nabla_i(F'\nabla_i u)\right)^2 \quad \geq \quad \left(\frac{\nabla \cdot (F'\nabla u)}{2}\right)^2 \quad \geq \quad \frac{u^2}{4} \quad \geq \quad \frac{\lambda^2}{4}$$

and then we get the expected result.

Step 2

The Hausdorff measure is bounded from above by:

$$\mathcal{H}^{n-1}(\Gamma) \quad \leq \quad \lim_{\eta \to 0} \inf_{\{B_\eta(Y_i)\}} \frac{1}{\eta} \sum_i |B_\eta(Y_i)| \tag{6.5}$$

where $\Gamma = \partial\{u=\lambda\}$ is the free boundary, and where $\{B_\eta(Y_i)\}_i$ is a recovering of Γ by balls of center Y_i on Γ and of radius η.

From proposition 3.1, we know that

$$u(X)-\lambda \geq c\varepsilon^2 \quad \text{while} \quad \text{dist}(X,\partial\Omega) < c\varepsilon \quad \text{where} \quad \varepsilon = \sqrt{2\left(\frac{\lambda_0-\lambda}{\lambda}\right)}$$

which in particular implies

$$\text{dist}\left(\{u=\lambda\},\partial\Omega\right) \quad \geq \quad c\varepsilon$$

Now starting from a point Y_i on $\partial\{u=\lambda\}$ we have from (3.3)

$$u(X)-\lambda \quad \leq \quad \frac{1}{2}C\,|X-Y_i|^2 \quad \text{while} \quad \text{dist}(X,\partial\Omega) \geq c\varepsilon$$

Therefore we get

$$\text{dist}(B_\eta(Y_i),\partial\Omega) \geq c\varepsilon \quad \text{while} \quad \frac{1}{2}C\eta^2 < c\varepsilon^2$$

i.e. for η small enough.

Then for such η we have

$$B_\eta(Y_i)\cap\{u>\lambda\} \quad \subset \quad B_\eta(Y_i)\cap\{u>\lambda, \quad |\nabla u|\leq C\eta\} \quad \subset \quad B_\eta(Y_i)\cap O^{C\eta}$$

From the nondegeneracy lemma 2.3, we deduce the existence of a real $\gamma\in(0,1)$ such that

$$|B_\eta(Y_i)\cap\{u>\lambda\}| \geq \gamma\,|B_\eta(Y_i)|$$

As a consequence we get

$$|B_\eta(Y_i)| \leq \gamma^{-1}\left|B_\eta(Y_i)\cap O^{C\eta}\right|$$

Thus

$$\begin{aligned}
\eta^{-1}\textstyle\sum_i |B_\eta(Y_i)| &\leq \eta^{-1}\gamma^{-1}\textstyle\sum_i |B_\eta(Y_i)\cap O^{C\eta}| \\
&\leq \eta^{-1}\gamma^{-1}\textstyle\int_\Omega \sum_i 1_{B_\eta(Y_i)}1_{O^{C\eta}} \\
&\leq \eta^{-1}\gamma^{-1}\sup(\textstyle\sum_i 1_{B_\eta(Y_i)})\int_\Omega 1_{O^{C\eta}} \\
&\leq \eta^{-1}\gamma^{-1}\sup(\textstyle\sum_i 1_{B_\eta(Y_i)})\left|O^{C\eta}\right| \\
&\leq \gamma^{-1}C_nC'\lambda^{-2}\left(|\mathcal{K}_\varepsilon| + \mathcal{H}^{n-1}(\partial\mathcal{K}_\varepsilon)\right)
\end{aligned}$$

where we have used the fact that we can always use locally finite recovering $\{B_\eta(Y_i)\}_i$ such that $\sum_i 1_{B_\eta(Y_i)} \le C_n$ where the constant only depends on the dimension n. On the other hand we have applied lemma 6.1 introducing a smooth compact set $\mathcal{K}_\varepsilon$ such that

$$\mathcal{K}_\varepsilon \quad \subset \quad \{X \in \Omega, \quad 2c\varepsilon \ge \mathrm{dist}(X, \partial\Omega) \ge c\varepsilon\}$$

In fact $\mathcal{K}_\varepsilon$ can be seen as a smooth approximation of $\partial\Omega$. Consequently we get

$$\mathcal{H}^{n-1}(\Gamma) \le C$$

where the constant C only depends on Ω, λ_0 and F, and is uniform with respect to $\lambda \in [0, \lambda_0]$. This proves theorem 1.5.

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