Kolmogorov equations related to Path-dependent Options

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[Kolmogorov] (1934)

$$\partial_{x_1}^2 u + x_1 \partial_{x_2} u = \partial_t u \qquad (x,t) \in \mathbb{R}^2 \times \mathbb{R}$$

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Applications: Geometric average Asian Options

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \log(S) \frac{\partial V}{\partial A}$$

(change of variable $S = e^x$)

Arithmetic average Asian Options

$$\frac{\partial V}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + S \frac{\partial V}{\partial A}$$

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$$(x,t) \in \mathbb{R}^N \times \mathbb{R}$$

$$Lu := \sum_{i,j=1}^{N} a_{ij}(x,t)\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u$$

$$A(x,t) = \big(a_{ij}(x,t)\big)_{i,j=1,\dots,N} \qquad A(x,t) = A^T(x,t) \geq 0$$

$$B = \big(b_{ij}\big)_{i,j=1,\dots,N}$$ constant matrix

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$$B = \big(b_{ij}\big)_{i,j=1,\dots,N}$$ constant matrix

Uniform ellipticity in $\mathbb{R}^m$:

$$\mu^{-1}\sum_{j=1}^{m} \xi_j^2 \leq \sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \leq \mu \sum_{j=1}^{m} \xi_j^2$$

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$$L_0 := \sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j} + \langle Bx, \nabla \rangle - \partial_t$$

The following statements are equivalent:

- Kalman condition: rank $\left(A^{1/2}, BA^{1/2}, \dots, B^{N-1}A^{1/2}\right) = N$
- $$\mathcal{C}(t) = \int_0^t e^{-sB} A e^{-sB^T} ds > 0$$

  in this case

  $$\Gamma_0(x,t) = \frac{(4\pi)^{-N/2}}{\sqrt{\det \mathcal{C}(t)}} e^{-\frac{1}{4}\langle \mathcal{C}^{-1}(t)x, x\rangle}$$

  is the fundamental solution of L_0

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$$L_0 := \sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j} + \langle Bx, \nabla \rangle - \partial_t, \quad A = \frac{1}{2}\sigma\,\sigma^*,$$

- The process X_t

$$dX_t = BX_t dt + \sigma dW_t,$$
$$X_0 = 0,$$

  is “non singular” and Γ_0 is its density
- L_0 satisfies the Hörmarder condition

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$$u(x,t) = -\iint \Gamma_0(x - e^{(t-\tau)B}\xi, t-\tau) f(\xi,\tau)\, d\xi\, d\tau + \int \Gamma_0(x - e^{tB}\xi, t)\varphi(\xi)\, d\xi$$

is a solution to the Cauchy problem

$$\begin{cases} L_0 u = f, & t > 0 \\ u = \varphi, & t = 0 \end{cases}$$

Lie group

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is a solution to the Cauchy problem

$$\begin{cases} L_0 u = f, & t > 0 \\ u = \varphi, & t = 0 \end{cases}$$

NON-EUCLIDEAN CHANGE OF VARIABLE:

$$(\xi,\tau) \circ (x,t) := \left(x + e^{-t\,B}\xi, t+\tau\right)$$

$$(x,t)^{-1} = \left(-e^{t\,B}x, -t\right),$$

$$(\xi,\tau)^{-1} \circ (x,t) = (x - e^{(t-\tau)B}\xi, t-\tau).$$

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Kolmogorov equation:

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e^{-tB} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

$$(\xi_1, \xi_2, \tau) \circ (x_1, x_2, t) = (\xi_1 + x_1, \xi_2 + x_2 - t\xi_1, t + \tau)$$

Translations:

Let $v(x_1, x_2, t) = u(\xi_1 + x_1, \xi_2 + x_2 - t\xi_1, t + \tau)$. Then

$$u_{x_1 x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad v_{x_1 x_1} + x_1 v_{x_2} = v_t$$

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$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad e^{-tB} = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

$$(\xi_1, \xi_2, \tau) \circ (x_1, x_2, t) = (\xi_1 + x_1, \xi_2 + x_2 - t\xi_1, t + \tau)$$

Translations:

Let $v(x_1, x_2, t) = u(\xi_1 + x_1, \xi_2 + x_2 - t\xi_1, t + \tau)$. Then

$$u_{x_1x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad v_{x_1x_1} + x_1 v_{x_2} = v_t$$

Dilations:

Let $w(x_1, x_2, t) = u(\lambda x_1, \lambda^3 x_2, \lambda^2 t)$. Then

$$u_{x_1x_1} + x_1 u_{x_2} = u_t \quad \Leftrightarrow \quad w_{x_1x_1} + x_1 w_{x_2} = w_t$$

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$$|(x_1, x_2, t)|_L = |x_1| + |x_2|^{1/3} + |t|^{1/2}$$

Hölder continuity:

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, \tau)| \leq C|(\xi_1, \xi_2, \tau)^{-1} \circ (x_1, x_2, t)|_L^\alpha$$

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Hölder continuity:

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, \tau)| \le C|(\xi_1, \xi_2, \tau)^{-1} \circ (x_1, x_2, t)|_L^\alpha$$

REMARK: Hölder continuity in x:

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, t)| \le C\left(|x_1 - \xi_1| + |x_2 - \xi_2|^{1/3}\right)^\alpha$$

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$$|(x_1, x_2, t)|_L = |x_1| + |x_2|^{1/3} + |t|^{1/2}$$

Hölder continuity:

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, \tau)| \leq C|(\xi_1, \xi_2, \tau)^{-1} \circ (x_1, x_2, t)|_L^{\alpha}$$

REMARK: Hölder continuity in x:

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, t)| \leq C\left(|x_1 - \xi_1| + |x_2 - \xi_2|^{1/3}\right)^{\alpha}$$

Hölder continuity in (x, t):

$$|u(x_1, x_2, t) - u(\xi_1, \xi_2, \tau)| \leq C\Big(|x_1 - \xi_1| + |x_2 - \xi_2 - (\tau - t)\xi_1|^{1/3} + |t - \tau|^{1/2}\Big)^{\alpha}$$

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Theorem [P.] (1994), [Di Francesco, Pascucci] (2005)

$$Lu := \sum_{i,j=1}^{m} a_{ij}(x,t)\partial_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_i \partial_{x_j} u - \partial_t u$$

a_{ij} Hölder continuous, uniform ellipticity in $\mathbb{R}^m$, L_0 hypoelliptic. Then there exists a fundamental solution of L, and

$$\Gamma(x,t) \leq \frac{C^+}{\sqrt{\det \mathcal{C}(t)}} e^{-c^+ \langle \mathcal{C}^{-1}(t)x,x\rangle}.$$

- Previous results:
  [Weber] (1951), [Il'In] (1964), [Sonin] (1967),

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Proposition [Manfredini] (1997), [Di Francesco, P.] (2006)
Let u be a solution of $Lu = f$, in $\Omega \subset \mathbb{R}^N \times \mathbb{R}$.
If $a_{ij}, f \in C^\alpha_L$, then $u, u_{x_i}, u_{x_i x_j} \in C^\alpha_L$ (for $i, j = 1, \dots, m$) and

$$\left| u_{x_i x_j}(x,t) - u_{x_i x_j}(\xi,\tau) \right| \le C_K \left| (\xi,\tau)^{-1} \circ (x,t) \right|_L^\alpha$$

for every $(x,t)(\xi,\tau)$ in a compact set $K \subset \Omega$.

Also the directional derivative $\langle Bx, \nabla u \rangle - \partial_t u$ belongs to C^α_L.

- Related results for the Cauchy problem:
  [Lunardi] (1997), [Lorenzi] (2005), [Priola] (2007).

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- [Picone], [Fichera] (1956),
- [Oleĭnik, Radkevič] (1973),
- [Manfredini] (1997),

Elliptic regularization:

$\varepsilon^2 \Delta u_\varepsilon + L u_\varepsilon = f$ in Ω, $\varepsilon \to 0$.

Barriers at the boundary.

Boundary condition

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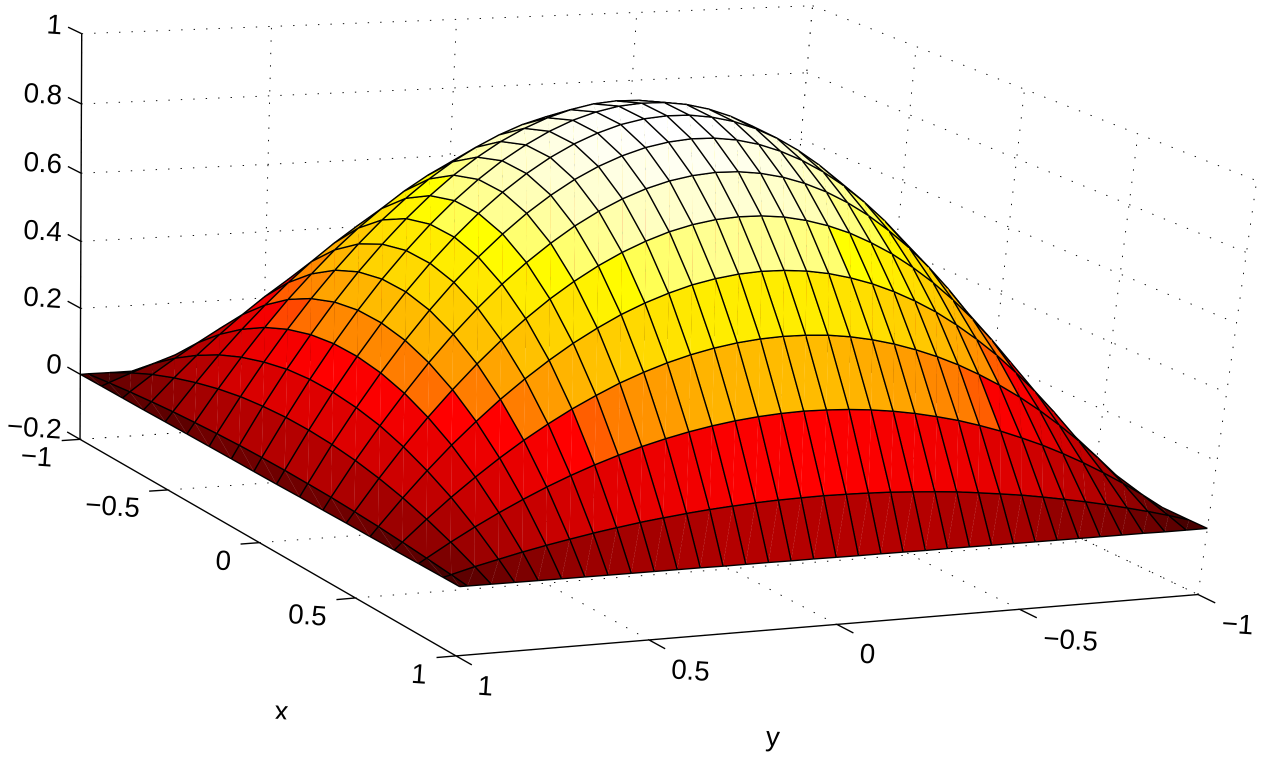

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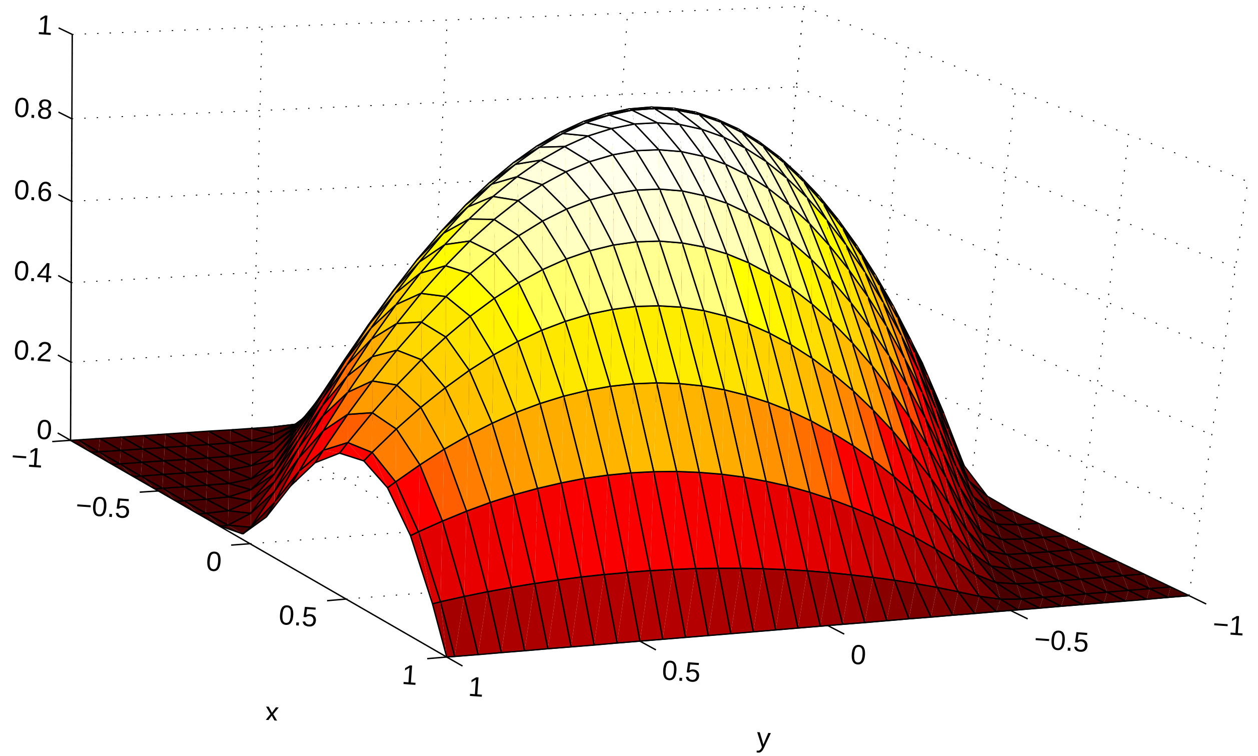

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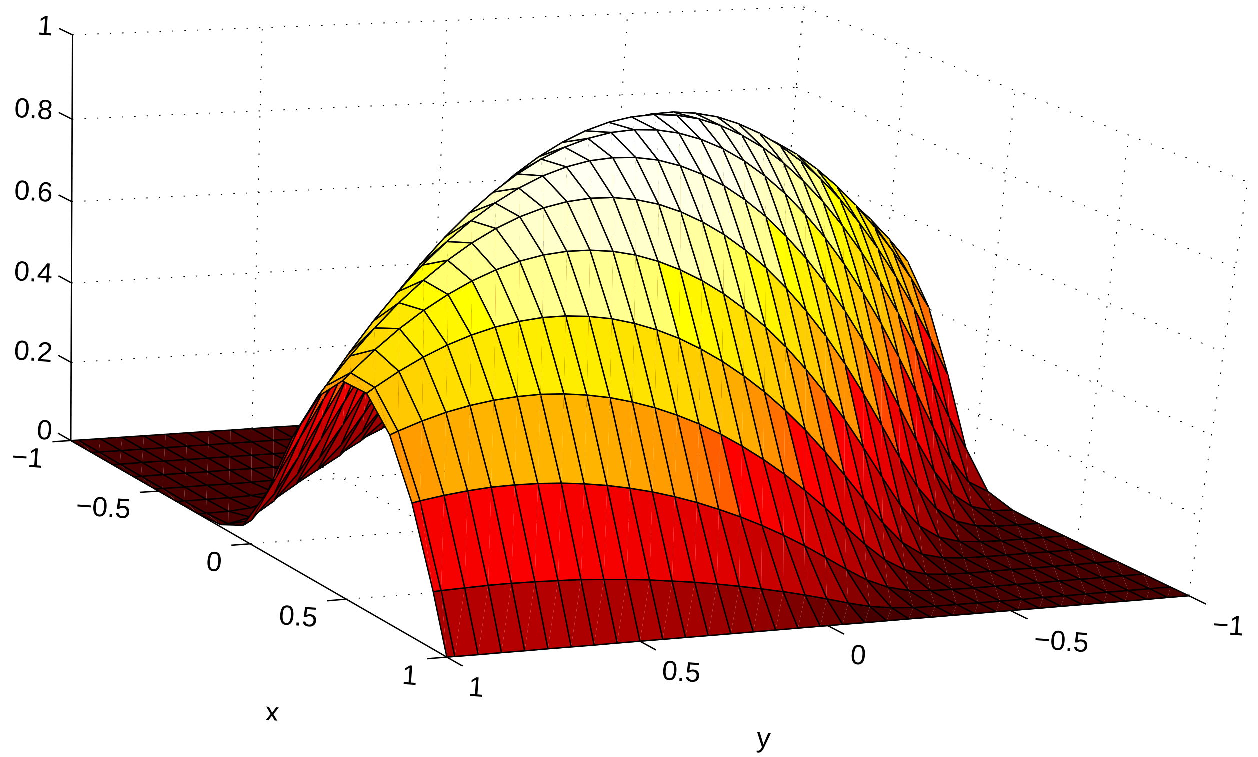

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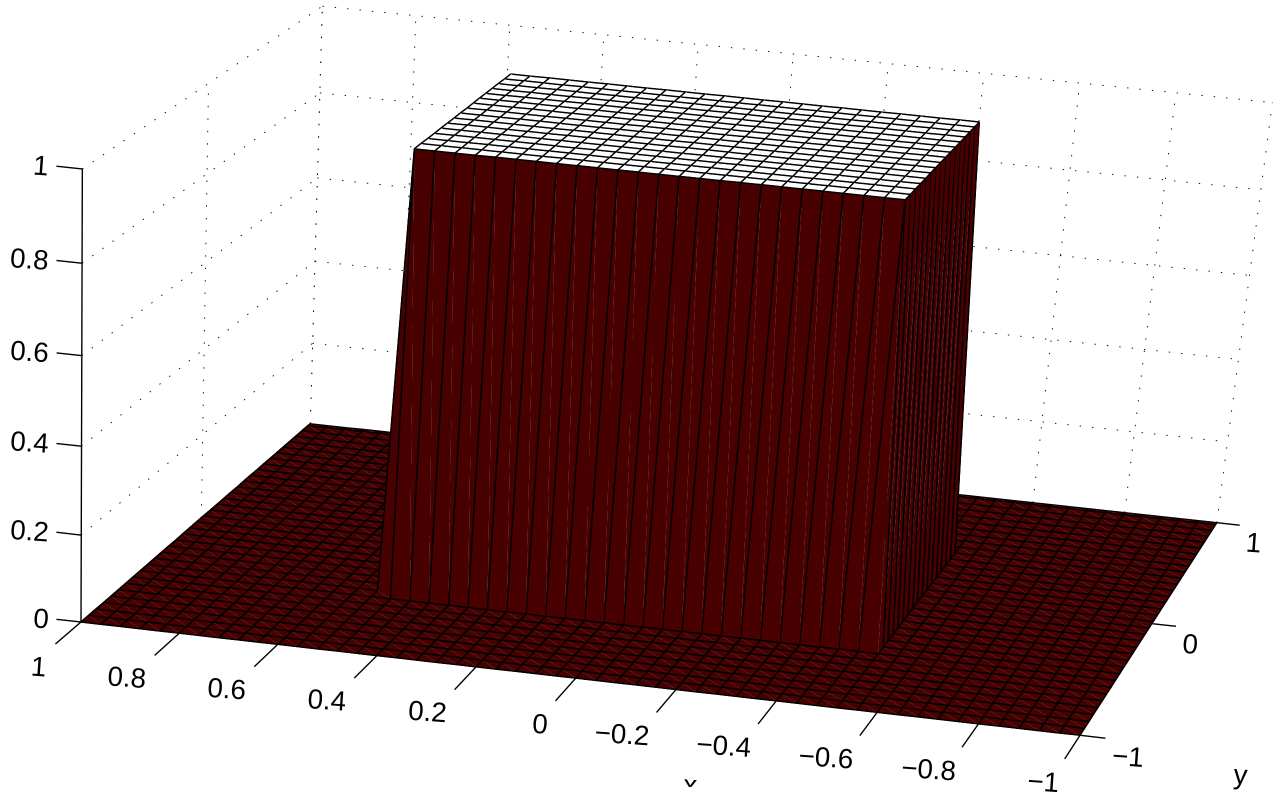

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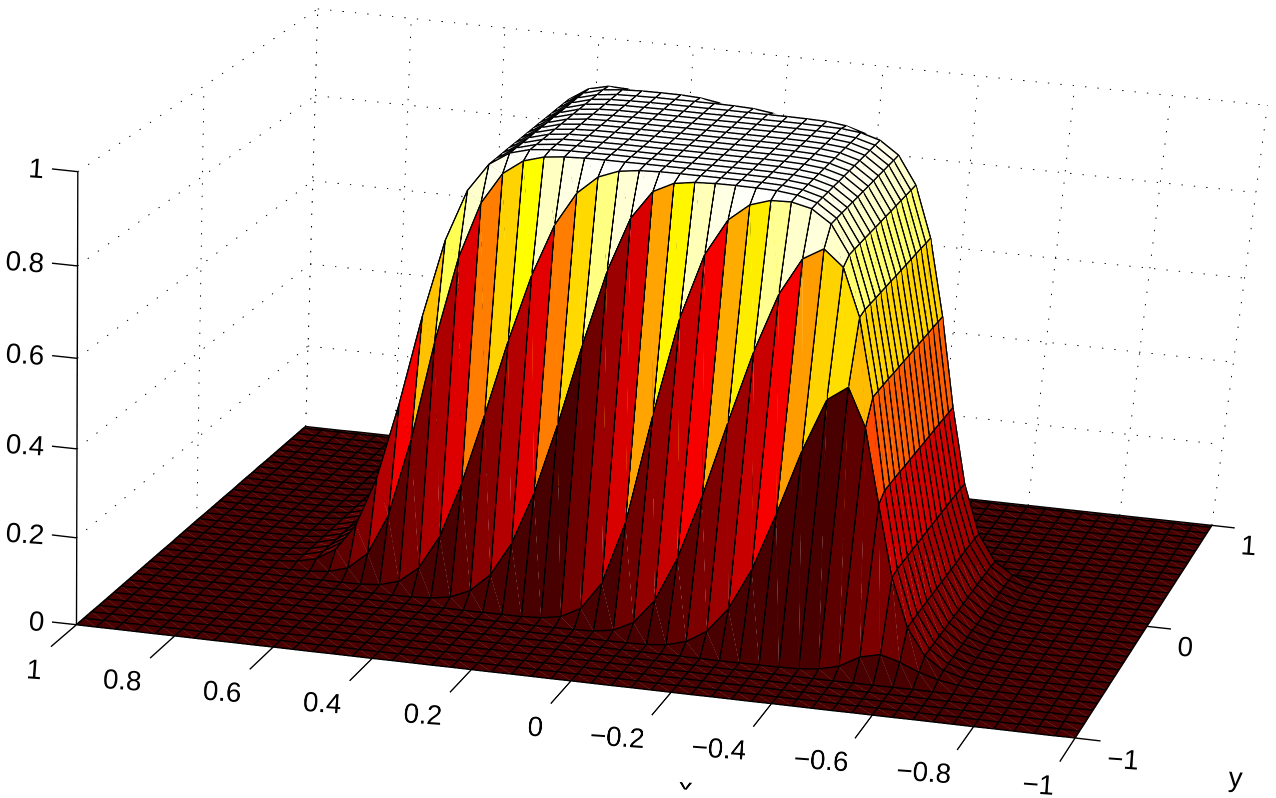

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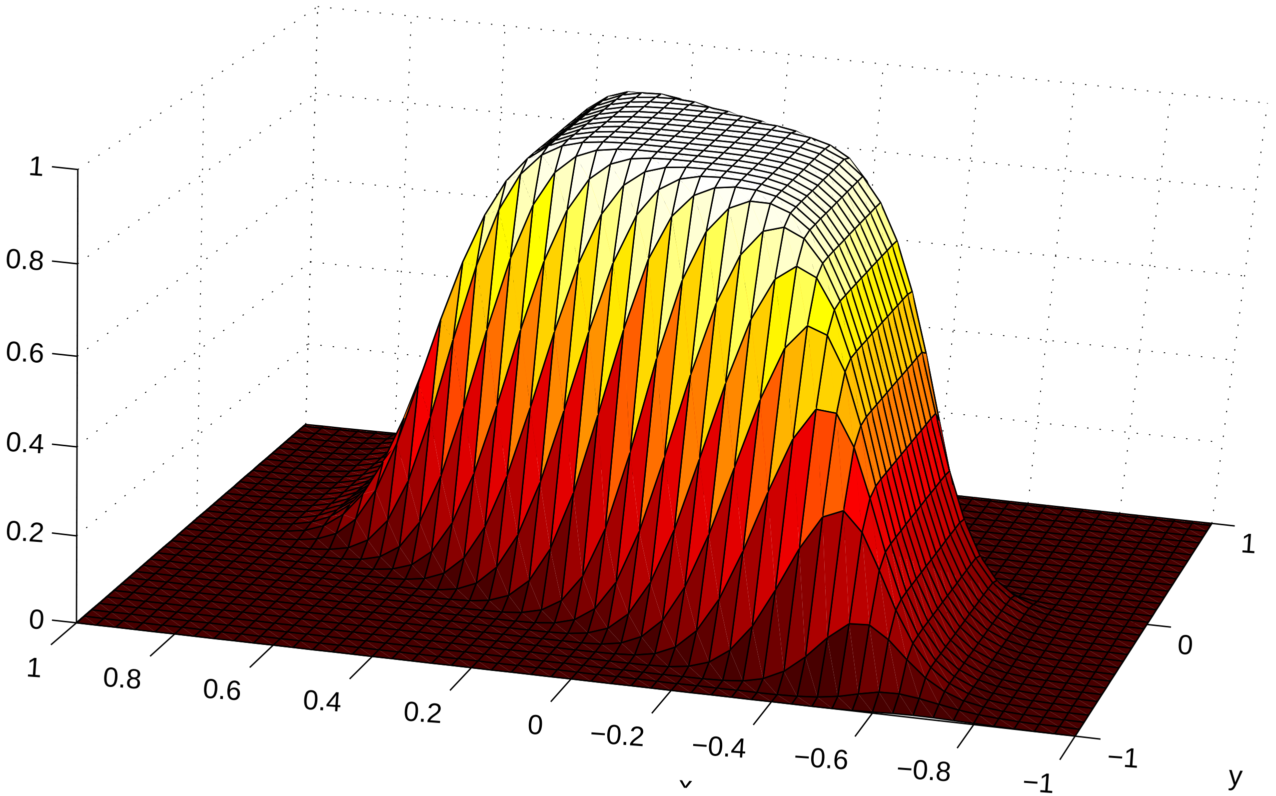

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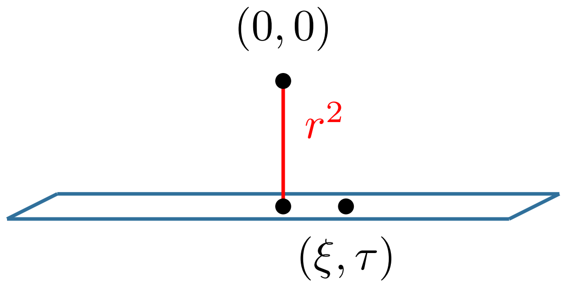
THEOREM [P.] (1995) [Di Francesco, P.] (2006) Let u be a non-negative solution to $Lu = 0$ (with a_{ij} Hölder continuous). Then



$$u(\xi,\tau) \leq C\, u(x_0,t_0)\ \forall(\xi,\tau) = (x_0,t_0) \circ \big(\delta_r(x,-1)\big) : |x| \leq 1.$$

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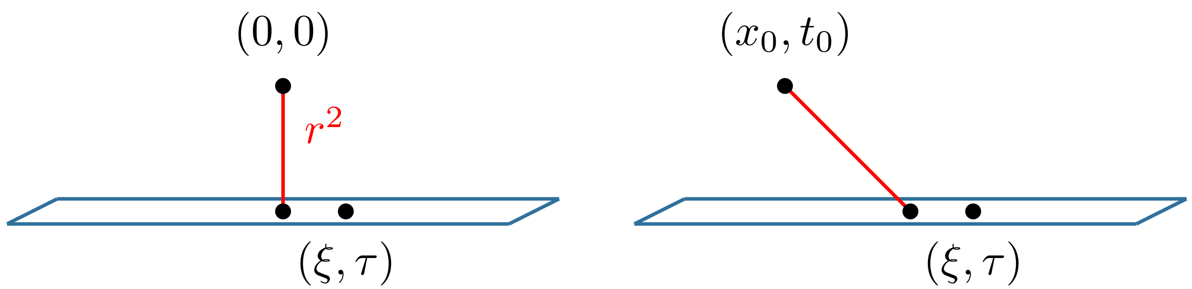

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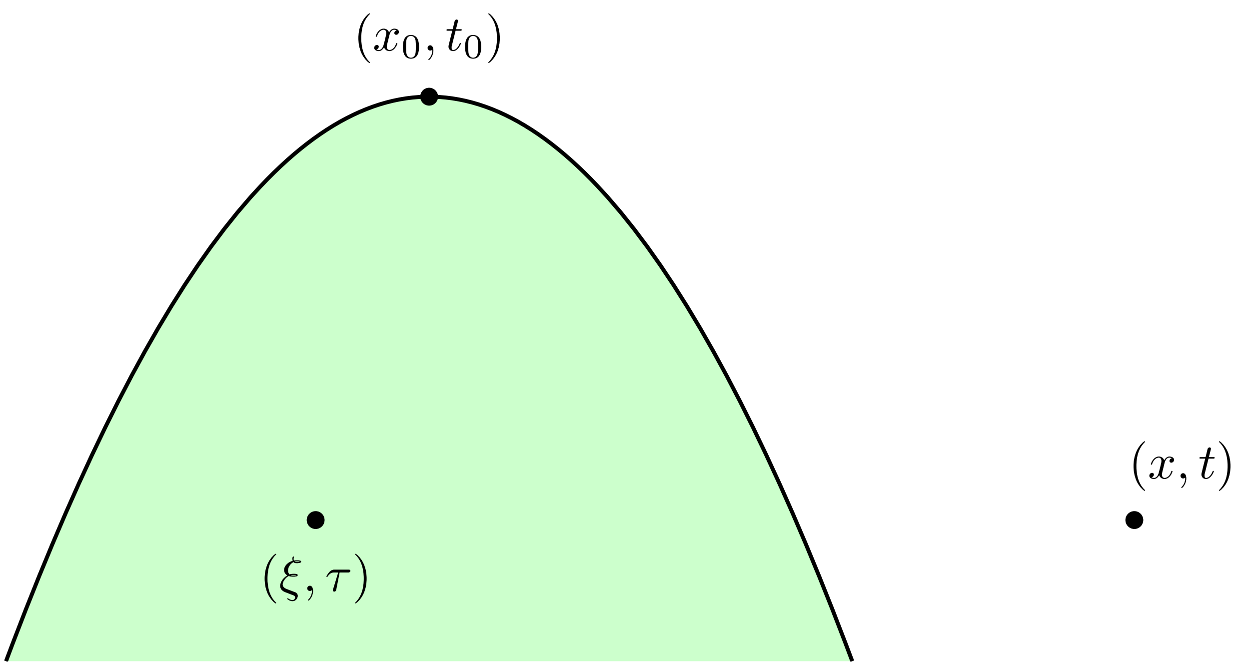
L uniformly parabolic operator. Then

$\Gamma(x,t) \geq c\, t^{-\frac{N}{2}} e^{-C\,\frac{|x|^2}{t}}$ [Moser] (1964) [Aronson & Serrin] (1967)



$u(\xi,\tau) \leq C\, u(x_0,t_0)$

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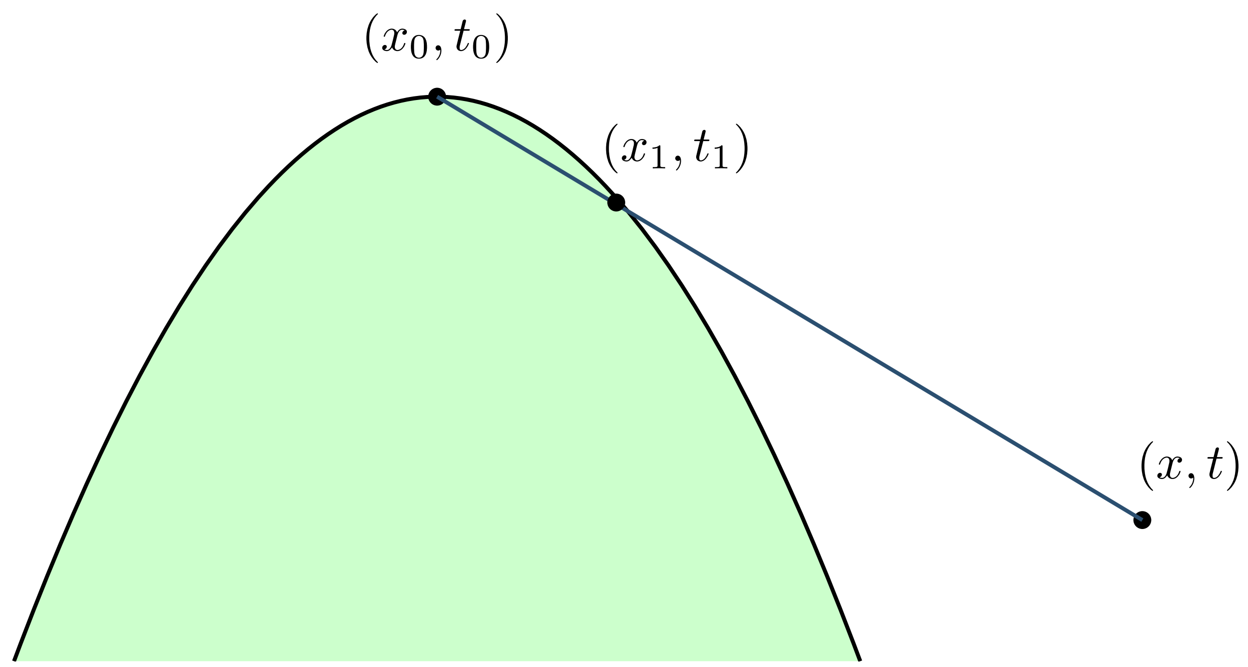

$$u(x_1,t_1) \leq C\, u(x_0,t_0)$$

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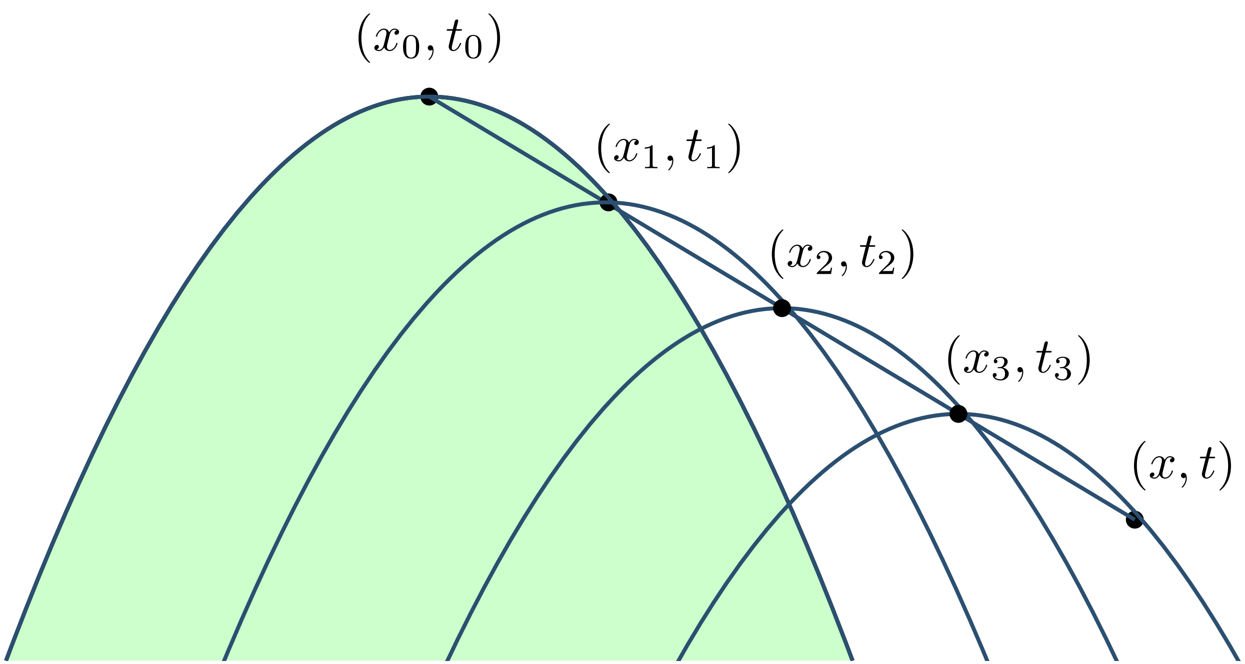

$u(x,t) \leq C^k\, u(x_0,t_0) \qquad |\lambda|^2(t_0 - t) \leq k < |\lambda|^2(t_0-t) + 1$

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- L–admissible path:

$$\begin{cases} \gamma' = B\gamma + A\lambda \\ \gamma(0) = x, \qquad \gamma(t) = 0 \end{cases}$$

(an L–admissible γ exists, by the Kalman condition).

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  (an L–admissible γ exists, by the Kalman condition).

- $\Gamma(x,t) \geq \frac{c}{t^{Q/2}} e^{-C \int_0^t |\lambda(s)|^2 ds}$

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(an L–admissible γ exists, by the Kalman condition).

- $\Gamma(x,t) \geq \frac{c}{t^{Q/2}} e^{-C \int_0^t |\lambda(s)|^2 ds}$
- the Pontryagin M. P. gives the optimal cost

$$\min_{\gamma} \int_0^t |\lambda(s)|^2 ds = \langle \mathcal{C}^{-1}(t)x, x \rangle$$

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Theorem [P.] (1997), [Di Francesco, P.] (2006)
There exist two positive constants C^-, c^- such that

$$\Gamma(x,t) \geq \frac{c^-}{\sqrt{\det \mathcal{C}(t)}} e^{-C^- \langle \mathcal{C}^{-1}(t)x, x\rangle}.$$

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Let u be a solution to the Cauchy problem

$$\begin{cases} Lu = 0 & \text{t>0} \\ u = 0 & \text{t=0} \end{cases}$$

Theorem [Di Francesco, Pascucci] (2005)

$$\iint_{\mathbb{R}^N \times]0,T[} |u(x,t)| e^{-C|x|^2} dx\, dt < +\infty \quad \Rightarrow \quad u \equiv 0$$

Theorem [Di Francesco, P.] (2006)

$$u(x,t) \geq 0 \quad \Rightarrow \quad u \equiv 0$$

Pointwise bounds

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Divergence form operators:

$$Lu := \sum_{i,j=1}^{m} \partial_{x_i} \left(a_{ij}(x,t) \partial_{x_j} u \right) + \langle Bx, \nabla u \rangle - \partial_t u$$

a_{ij} measurable, uniform ellipticity in $\mathbb{R}^m$, L_0 hypoelliptic.

We say that u is a weak solution if

$$\int -\langle A(x,t)\nabla u, \nabla \psi \rangle + \left(\langle Bx, \nabla u \rangle - \partial_t u \right) \psi = 0$$

- Theorem [Pascucci, P.] (2004) [Cinti, Pascucci, P.] (2008):

$$\| u \|_{L^\infty(H_r)} \le \frac{C}{(R-r)^Q} \int_{H_R} u(x,t)\, dx\, dt$$

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Caccioppoli inequality: use $\psi = u\varphi^2$ as a test function:

$$\int_{H_r} \langle A\nabla u, \nabla u\rangle \varphi^2 dx\, dt \leq C(R, r, \varphi) \int_{H_R} u^2 dx\, dt$$

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$$\int_{H_r} \langle A\nabla u, \nabla u\rangle \varphi^2 dx\, dt \le C(R, r, \varphi) \int_{H_R} u^2 dx\, dt$$

Sobolev inequality:

$$L_0 u = \text{div}\left((A_0 - A(x,t))\nabla u\right) + Lu$$

$$u(x,t) = \int \Gamma_0(x,t,\xi,\tau)\text{div}\left((A_0 - A(\xi,\tau))\nabla u(\xi,\tau)\right) d\xi\, d\tau =$$

$$- \int \nabla_\xi \Gamma_0(x,t,\xi,\tau)\left((A_0 - A(\xi,\tau))\nabla u(\xi,\tau)\right) d\xi\, d\tau$$

Moser iteration

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Caccioppoli inequality: use $\psi = u\varphi^2$ as a test function:

$$\int_{H_r} \langle A\nabla u, \nabla u\rangle \varphi^2 dx\, dt \le C(R,r,\varphi) \int_{H_R} u^2 dx\, dt$$

Sobolev inequality:

$$L_0 u = \text{div}\,((A_0 - A(x,t))\nabla u) + Lu$$

$$u(x,t) = \int \Gamma_0(x,t,\xi,\tau)\text{div}\,((A_0 - A(\xi,\tau))\nabla u(\xi,\tau))\, d\xi\, d\tau =$$

$$-\int \nabla_\xi \Gamma_0(x,t,\xi,\tau)\,((A_0 - A(\xi,\tau))\nabla u(\xi,\tau))\, d\xi\, d\tau$$

Then $\|u\|_{L^{2+\frac{4}{Q}}(H_r)} \le C_{R,r} \|\nabla u\|_{L^2(H_R)}$

Upper bound for Γ

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COROLLARY: [Pascucci, P.] (2003)

$$\Gamma(x,t) \le \frac{C^+}{\sqrt{\det \mathcal{C}(t)}} e^{-c^+ \langle \mathcal{C}^{-1}(t)x, x\rangle}.$$

REMARK: No Poincaré inequality.