Decomposing Dynamic Programming equations

From global to nodal value functions

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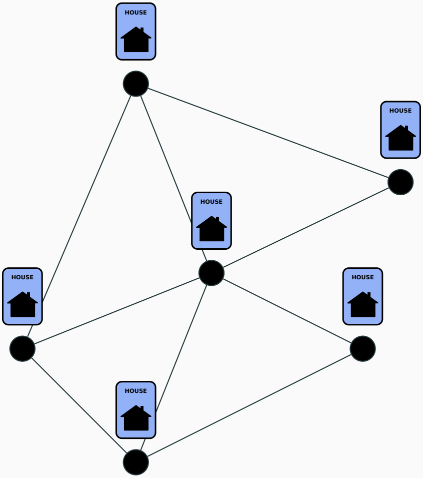
Motivation

We consider a *peer-to-peer* community,
where different buildings exchange energy



Optimization upper and lower bounds by decomposition

Decompose optimization problem with coupling constraints

Let, for $i \in [\![1, N]\!]$

- $\mathcal{C}^i$ be a Hilbert space
- $u^i \in \mathbb{U}^i$ be a decision variable
- $J^i : \mathbb{U}^i \to \mathbb{R}$ be a local objective
- $\Theta^i : \mathbb{U}^i \to \mathcal{C}^i$ be a mapping
- $S \subset \mathcal{C}^1 \times \cdots \times \mathcal{C}^N$ be a set

We consider the following problem

$$V^\sharp = \inf_{u^1, \cdots, u^N} \sum_{i=1}^N J^i(u^i)$$
$$\text{s.t. } \underbrace{\big(\Theta^1(u^1), \cdots, \Theta^N(u^N)\big) \in S}_{\text{coupling constraint}}$$

Price and resource value functions provide bounds

We define for $i \in [\![1, N]\!]$

- The *local price value function*

$$\underline{V}^i[\lambda^i] = \min_{u^i} J^i(u^i) + \left\langle \lambda^i , \Theta^i(u^i) \right\rangle , \ \forall \lambda^i \in (\mathcal{C}^i)^\star$$

- The *local resource value function*

$$\overline{V}^i[r^i] = \min_{\substack{u^i \\ \Theta^i(u^i) = r^i}} J^i(u^i) , \ \forall r^i \in \mathcal{C}^i$$

Theorem

For any

- *admissible price* $\lambda = (\lambda^1, \cdots, \lambda^N) \in S^o = \left\{ \lambda \in \mathcal{C}^\star \mid \left\langle \lambda , r \right\rangle \leq 0 , \ \forall r \in \mathcal{C} \right\}$
- *admissible resource* $r = (r^1, \cdots, r^N) \in S$

$$\sum_{i=1}^N \underline{V}^i[\lambda^i] \leq V^\sharp \leq \sum_{i=1}^N \overline{V}^i[r^i]$$

Application to stochastic optimal control

We now consider the stochastic optimal control problem

$$V_0^\sharp(x_0) = \min_{\mathbf{X},\mathbf{U}} \mathbb{E}\Big[\sum_{i=1}^{N} \sum_{t=0}^{T-1} \underbrace{L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1})}_{\text{instantaneous cost}} + K^i(\mathbf{X}_T^i)\Big]$$

$$\begin{aligned} \text{s.t. } & \mathbf{X}_{t+1}^i = g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) \;, \quad \mathbf{X}_0^i = x_0^i \\ & \sigma(\mathbf{U}_t^i) \subset \sigma(\mathbf{W}_0, \cdots, \mathbf{W}_t) \\ & \big(\Theta_t^1(\mathbf{X}_t^1, \mathbf{U}_t^1, \mathbf{W}_{t+1}), \cdots, \Theta_t^N(\mathbf{X}_t^N, \mathbf{U}_t^N, \mathbf{W}_{t+1})\big) \in S_t \end{aligned}$$

- $t = 0, \cdots, T$ are stages
- $\mathbf{W} = (\mathbf{W}_0, \cdots, \mathbf{W}_T)$ a global white noise process
- $\mathbf{X}^i = (\mathbf{X}_0^i, \cdots, \mathbf{X}_T^i)$ a local state process
- $\mathbf{U} = (\mathbf{U}_0^i, \cdots, \mathbf{U}_{T-1}^i)$ a local control process
- $g_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \times \mathbb{W}_{t+1} \to \mathbb{X}_{t+1}^i$ a local dynamics
- $L_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \times \mathbb{W}_{t+1} \to \mathbb{R}$ a local instantaneous cost

Obtaining bounds for the global problem

Theorem

For any

- *admissible price process* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^1, \cdots, \boldsymbol{\lambda}^N) \in S^o$
- *admissible resource process* $\mathbf{R} = (\mathbf{R}^1, \cdots, \mathbf{R}^N) \in S$

$$\sum_{i=1}^{N} \underline{V}_0^i[\boldsymbol{\lambda}^i](x_0^i) \leq V_0(x_0) \leq \sum_{i=1}^{N} \overline{V}_0^i[\mathbf{R}^i](x_0^i)$$

Price local value function

$$\underline{V}_0^i[\boldsymbol{\lambda}^i](x_0^i) = \min_{\mathbf{X}^i, \mathbf{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) + \big\langle \boldsymbol{\lambda}_t^i \,, \Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) \big\rangle + K^i(\mathbf{X}_T^i)\Big]$$

$$\text{s.t. } \mathbf{X}_{t+1}^i = g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) \;, \quad \mathbf{X}_0^i = x_0^i$$

$$\sigma(\mathbf{U}_t^i) \subset \sigma(\mathbf{W}_0, \cdots, \mathbf{W}_t)$$

Resource local value function

$$\overline{V}_0^i[\mathbf{R}^i](x_0^i) = \min_{\mathbf{X}^i, \mathbf{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T-1} L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) + K^i(\mathbf{X}_T^i)\Big]$$

$$\text{s.t. } \mathbf{X}_{t+1}^i = g_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) \;, \quad \mathbf{X}_0^i = x_0^i$$

$$\sigma(\mathbf{U}_t^i) \subset \sigma(\mathbf{W}_0, \cdots, \mathbf{W}_t)$$

$$\Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}) = \mathbf{R}_t^i$$

Mixing price/resource and temporal decompositions

$$\sum_{i=1}^{N} \underline{V}_0^i[\boldsymbol{\lambda}^i](x_0^i) \leq V_0(x_0) \leq \sum_{i=1}^{N} \overline{V}_0^i[\mathbf{R}^i](x_0^i)$$

Price decomposition

- Fix a deterministic price $\lambda = (\lambda^1, \cdots, \lambda^N)$
- Obtain $\underline{V}_0^i[\lambda^i](x_0^i)$ by Dynamic Programming

$$\underline{V}_t^i(x_t^i) = \min_{u_t^i} \mathbb{E}\big[L_t(x_t^i, u_t^i, \mathbf{W}_{t+1}) + \langle \lambda_t^i, \Theta_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}) \rangle + \underline{V}_{t+1}^i(g_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1})\big]$$

- Return the value functions $\{\underline{V}_t^i\}$

Resource decomposition

- Fix a deterministic resource $r = (r^1, \cdots, r^N)$
- Obtain $\overline{V}_0^i[r^i](x_0^i)$ by Dynamic Programming

$$\overline{V}_t^i(x_t^i) = \min_{u_t^i} \mathbb{E}\big[L_t(x_t^i, u_t^i, \mathbf{W}_{t+1}) + \overline{V}_{t+1}^i(g_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1})\big] \quad \text{s.t. } \Theta_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}) = r_t^i$$

- Return the value functions $\{\overline{V}_t^i\}$

Deducing two control policies

Once value functions $\underline{V}_t^i$ and $\overline{V}_t^i$ computed, we define

- the global price policy

$$\underline{\pi}_t(x_t^1, \cdots, x_t^N) \in \mathop{\arg\min}_{u_t^1, \cdots, u_t^N} \mathbb{E}\Big[\sum_{i=1}^N L_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}) + \underline{V}_{t+1}^i(\mathbf{X}_{t+1}^i)\Big]$$
$$\text{s.t. } \mathbf{X}_{t+1}^i = g_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}) \ , \quad \forall i \in [\![1, N]\!]$$
$$\big(\Theta_t(x_t^1, u_t^1, \mathbf{W}_{t+1}), \cdots, \Theta_t(x_t^N, u_t^N, \mathbf{W}_{t+1})\big) \in S_t$$

- the global resource policy

$$\overline{\pi}_t(x_t^1, \cdots, x_t^N) \in \mathop{\arg\min}_{u_t^1, \cdots, u_t^N} \mathbb{E}\Big[\sum_{i=1}^N L_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}) + \overline{V}_{t+1}^i(\mathbf{X}_{t+1}^i)\Big]$$
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$$\big(\Theta_t(x_t^1, u_t^1, \mathbf{W}_{t+1}), \cdots, \Theta_t(x_t^N, u_t^N, \mathbf{W}_{t+1})\big) \in S_t$$

Where are we where are we heading to?

- First, we have obtained upper and lower bounds
  for global optimization problems with coupling constraints
  thanks to two spatial decomposition schemes
  - Price decomposition
  - Resource decomposition

- Second, with proper coordinating price and resource processes
  we have computed the upper and lower bounds
  by Dynamic Programming (temporal decomposition)

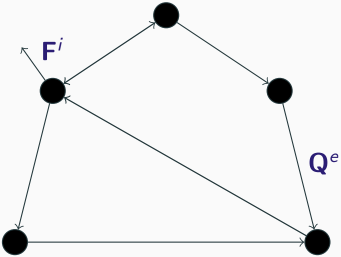
- With the upper and lower Bellman value functions,
  we have deduced two online policies

- Now, we will apply these decomposition schemes to a graph problem

Nodal decomposition of a network optimization problem

Modeling flows between nodes

Graph $G = (\mathcal{V}, \mathcal{E})$



- $\mathbf{Q}_t^e$ flow through edge e,
- $\mathbf{F}_t^i$ flow imported at node i

Let A be the *node-edge* incidence matrix

At each time $t \in [\![0, T-1]\!]$, Kirchhoff current law couples nodal and edge flows

$$A\mathbf{Q}_t + \mathbf{F}_t = 0$$

Writing down the nodal problem

We aim at minimizing the nodal costs over the nodes $i \in \mathcal{V}$

$$J^i_{\mathcal{V}}(\mathbf{F}^i) = \min_{\mathbf{X}^i, \mathbf{U}^i} \mathbb{E}\Big[\sum_{t=0}^{T-1} \underbrace{L^i_t(\mathbf{X}^i_t, \mathbf{U}^i_t, \mathbf{W}_{t+1})}_{\text{instantaneous cost}} + K^i(\mathbf{X}^i_T) \Big]$$

subject to, for all $t \in [\![0, T-1]\!]$

i) The nodal dynamics constraint (for battery and hot water tank)

$$\mathbf{X}^i_{t+1} = g^i_t(\mathbf{X}^i_t, \mathbf{U}^i_t, \mathbf{W}_{t+1})$$

ii) The non-anticipativity constraint (future remains unknown)

$$\sigma(\mathbf{U}^i_t) \subset \sigma(\mathbf{W}_0, \cdots, \mathbf{W}_t)$$

iii) The load balance equation (production + import = demand)

$$\Delta^i_t(\mathbf{X}^i_t, \mathbf{U}^i_t, \mathbf{F}^i_t, \mathbf{W}_{t+1}) = 0$$

Transportation costs are decoupled in time

At each time step $t \in [\![0, T-1]\!]$, we define the edges cost as the sum of the costs of flows $\mathbf{Q}_t^e$ through the edges e of the grid

$$J_{\mathcal{E}}^e(\mathbf{Q}) = \mathbb{E}\Big(\sum_{t=0}^{T-1} l_t^e(\mathbf{Q}_t^e) \Big)$$

Global optimization problem

The *nodal cost* $J_{\mathcal{V}}$ aggregates the costs at all nodes i

$$J_{\mathcal{V}}(\mathbf{F}) = \sum_{i \in \mathcal{V}} J_{\mathcal{V}}^{i}(\mathbf{F}^{i})$$

and the *edge cost* $J_{\mathcal{E}}$ aggregates the edges costs at all time t

$$J_{\mathcal{E}}(\mathbf{Q}) = \sum_{e \in \mathcal{E}} J_{\mathcal{E}}^{e}(\mathbf{Q}^{e})$$

The global optimization problem writes

$$V^{\sharp} = \min_{\mathbf{F},\mathbf{Q}} J_{\mathcal{V}}(\mathbf{F}) + J_{\mathcal{E}}(\mathbf{Q})$$
$$\text{s.t. } A\mathbf{Q} + \mathbf{F} = 0$$

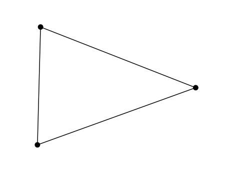
What do we plan to do?

- We have formulated a multistage stochastic optimization problem on a graph
- We will handle the coupling Kirchhoff constraints by the two methods presented earlier
  - Price decomposition
  - Resource decomposition
- We will show the scalability of decomposition algorithms (We solve problems up to 48 buildings)

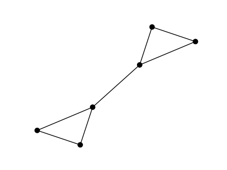
Numerical results on urban microgrids

We consider different urban configurations

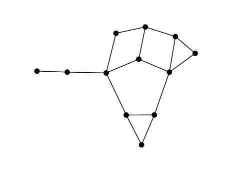
3-Nodes

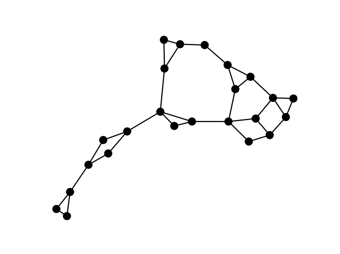


6-Nodes

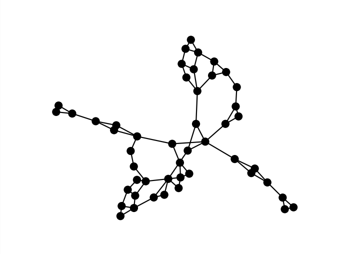


12-Nodes





24-Nodes



48-Nodes

Problem settings

- One day horizon at 15mn time step: $T = 96$
- Weather corresponds to a sunny day in Paris *(June 28th, 2015)*
- We mix three kind of buildings
  1. Battery + Electrical Hot Water Tank
  2. Solar Panel + Electrical Hot Water Tank
  3. Electrical Hot Water Tank

  and suppose that all consumers are commoners sharing their devices

Upper and lower bounds on the global problem

	Graph	**3-Nodes**	**6-Nodes**	**12-Nodes**	**24-Nodes**	**48-Nodes**
State dim.	$\lvert\mathbb{X}\rvert$	4	8	16	32	64
SDDP	time	1'	3'	10'	79'	453'
SDDP	LB	2.252	4.559	8.897	17.528	33.103
Price	time	6'	14'	29'	41'	128'
Price	LB	2.137	4.473	8.967	17.870	33.964
Resource	time	3'	7'	22'	49'	91'
Resource	UB	2.539	5.273	10.537	21.054	40.166

Upper and lower bounds on the global problem

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- For the **24-Nodes** problem

$$\begin{array}{ccccccc} \underline{V}_0[sddp] & \leq & \underline{V}_0[price] & \leq & V^\sharp & \leq & \overline{V}_0[resource] \\ 17.528 & \leq & 17.870 & \leq & V^\sharp & \leq & 21.054 \end{array}$$

Upper and lower bounds on the global problem

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- For the biggest instance, Price Decomposition is 3.5x as fast as SDDP

Policy evaluation by Monte Carlo simulation

Graph	**3-Nodes**	**6-Nodes**	**12-Nodes**	**24-Nodes**	**48-Nodes**
SDDP policy	2.26 ± 0.006	4.71 ± 0.008	9.36 ± 0.011	18.59 ± 0.016	35.50 ± 0.023
Price policy	2.28 ± 0.006	4.64 ± 0.008	9.23 ± 0.012	18.39 ± 0.016	34.90 ± 0.023
Gap	-0.9 %	+1.5%	+1.4%	+1.1%	+1.7%
Resource policy	2.29 ± 0.006	4.71 ± 0.008	9.31 ± 0.011	18.56 ± 0.016	35.03 ± 0.022
Gap	-1.3 %	0.0%	+0.5%	+0.2%	+1.2%

Price policy beats SDDP policy and resource policy

$$
\begin{array}{ccccccc}
V^\sharp & \leq & C[price] & \leq & C[resource] & \leq & C[sddp] \\
V^\sharp & \leq & 18.39 & \leq & 18.56 & \leq & 18.59
\end{array}
$$

Conclusion

Conclusion

- We have presented two algorithms that decompose, spatially then temporally, a global optimization problem under coupling constraints

- On this case study, decomposition beat SDDP for large instances ($\geq$ 24 nodes)
  - In time (3.5x faster)
  - In precision ($>$ 1% better)

- Can we obtain tighter bounds?
  If we select properly the resource and price processes $\mathbf{R}$ and $\boldsymbol{\lambda}$, among Markovian ones we can obtain nodal value functions (with an extended local state)