Sensor Network Localization and Dimension Reduction

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Recall Conic LP

$$(CLP) \quad \text{minimize} \quad \mathbf{c} \bullet \mathbf{x}$$
$$\text{subject to} \quad \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ \mathbf{x} \in K,$$

where K is a closed and pointed convex cone.

Linear Programming (LP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $K = \mathcal{R}^n_+$

Second-Order Cone Programming (SOCP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $K = SOC = \{\mathbf{x} : \ x_1 \geq \|\mathbf{x}_{-1}\|_2\}$.

Semidefinite Programming (SDP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{S}^n$ and $K = \mathcal{S}^n_+$

p-Order Cone Programming (POCP): $\mathbf{c}, \mathbf{a}_i, \mathbf{x} \in \mathcal{R}^n$ and $K = POC = \{\mathbf{x} : \ x_1 \geq \|\mathbf{x}_{-1}\|_p\}$.

Here, $\mathbf{x}_{-1}$ is the vector $(x_2; ...; x_n) \in R^{n-1}$.

Dual of Conic LP

The dual problem to

$$(CLP) \quad \text{minimize} \quad \mathbf{c} \bullet \mathbf{x}$$
$$\text{subject to} \quad \mathbf{a}_i \bullet \mathbf{x} = b_i, i = 1, 2, ..., m, \ \mathbf{x} \in K.$$

is

$$(CLD) \quad \text{maximize} \quad \mathbf{b}^T \mathbf{y}$$
$$\text{subject to} \quad \sum_i^m y_i \mathbf{a}_i + \mathbf{s} = \mathbf{c}, \ \mathbf{s} \in K^*,$$

where $y \in \mathcal{R}^m$, $\mathbf{s}$ is called the dual slack vector/matrix, and K^* is the dual cone of K. The former is called the primal problem, and the latter is called dual problem.

Theorem 1 *The dual of the dual is the primal.*

CLP Duality Theorems

The weak duality theorem shows that a feasible solution to either problem yields a bound on the value of the other problem. We call $\mathbf{c} \bullet \mathbf{x} - \mathbf{b}^T \mathbf{y}$ the duality gap.

Corollary 1 *Let* $\mathbf{x}^* \in \mathcal{F}_p$ *and* $(\mathbf{y}^*, \mathbf{s}^*) \in \mathcal{F}_d$. *Then,* $\mathbf{c} \bullet x^* = \mathbf{b}^T y^*$ *implies that* $\mathbf{x}^*$ *is optimal for (CLP) and* $(\mathbf{y}^*, \mathbf{s}^*)$ *is optimal for (CLD).*

Is the reverse also true? That is, given $\mathbf{x}^*$ optimal for (CLP), then there is $(\mathbf{y}^*, \mathbf{s}^*)$ feasible for (CLD) and $\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*$?

This is called the Strong Duality Theorem.

“True” when $K = \mathcal{R}^n_+$, that is, the polyhedral cone case, but it may fail in general.

SDP Example with a Duality Gap

$$\mathbf{c} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

and

$$\mathbf{b} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

When Strong Duality Theorems Holds for CLP

Theorem 2 *(Strong duality theorem) Let $\mathcal{F}_p$ and $\mathcal{F}_d$ be non-empty and at least one of them has an interior. Then, x^* is optimal for (CLP) and (y^*, s^*) is optimal for (CLD) if any only if*

$$\mathbf{c} \bullet \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^*.$$

Theorem 3 *(CLP duality theorem) If one of (CLP) or (CLD) is unbounded then the other has no feasible solution.*

If (CLP) and (CLD) are both feasible, then both have bounded optimal objective values and the optimal objective values may have a duality gap.

If one of (CLP) or (CLD) has a strictly or interior feasible solution and it has an optimal solution, then the other is feasible and has an optimal solution with the same optimal value.

Optimality Conditions for SDP

$$\begin{array}{rcl} \mathbf{c} \bullet X - \mathbf{b}^T \mathbf{y} & = & 0 \\ \mathcal{A}X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array}, \quad (1)$$

or

$$\begin{array}{rcl} XS & = & \mathbf{0} \\ \mathcal{A}X & = & \mathbf{b} \\ -\mathcal{A}^T \mathbf{y} - S & = & -\mathbf{c} \\ X, S & \succeq & \mathbf{0} \end{array} \quad (2)$$

Here

$$\mathcal{A}\mathbf{x} = (\mathbf{a}_1 \bullet \mathbf{x}; ...; \mathbf{a}_m \bullet \mathbf{x}) \in \mathcal{R}^m \text{ and } \mathcal{A}^T \mathbf{y} = \sum_i^m y_i \mathbf{a}_i.$$

Sensor Network Localization

Given $\mathbf{a}_k \in \mathbf{R}^d$, $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in \mathbf{R}^d$ such that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = d_{ij}^2, \ \forall\, (i,j) \in N_x, \ i < j,$$
$$\|\mathbf{a}_k - \mathbf{x}_j\|^2 = \hat{d}_{kj}^2, \ \forall\, (k,j) \in N_a,$$

(ij) ((kj)) connects points $\mathbf{x}_i$ and $\mathbf{x}_j$ (or $\mathbf{a}_k$ and $\mathbf{x}_j$) with an edge whose Euclidean length is d_{ij} (or $\hat{d}_{kj}$).

Does the system have a localization or realization of all $\mathbf{x}_j$'s? Is the localization unique? Is there a certification for the solution to make it reliable or trustworthy? Is the system partially localizable with certification?

The SNL problem is closely related to Data Dimension Reduction, Molecular Confirmation, Graph Realization/Embedding, etc.. and it is one of the major topics in Dada Sciences.

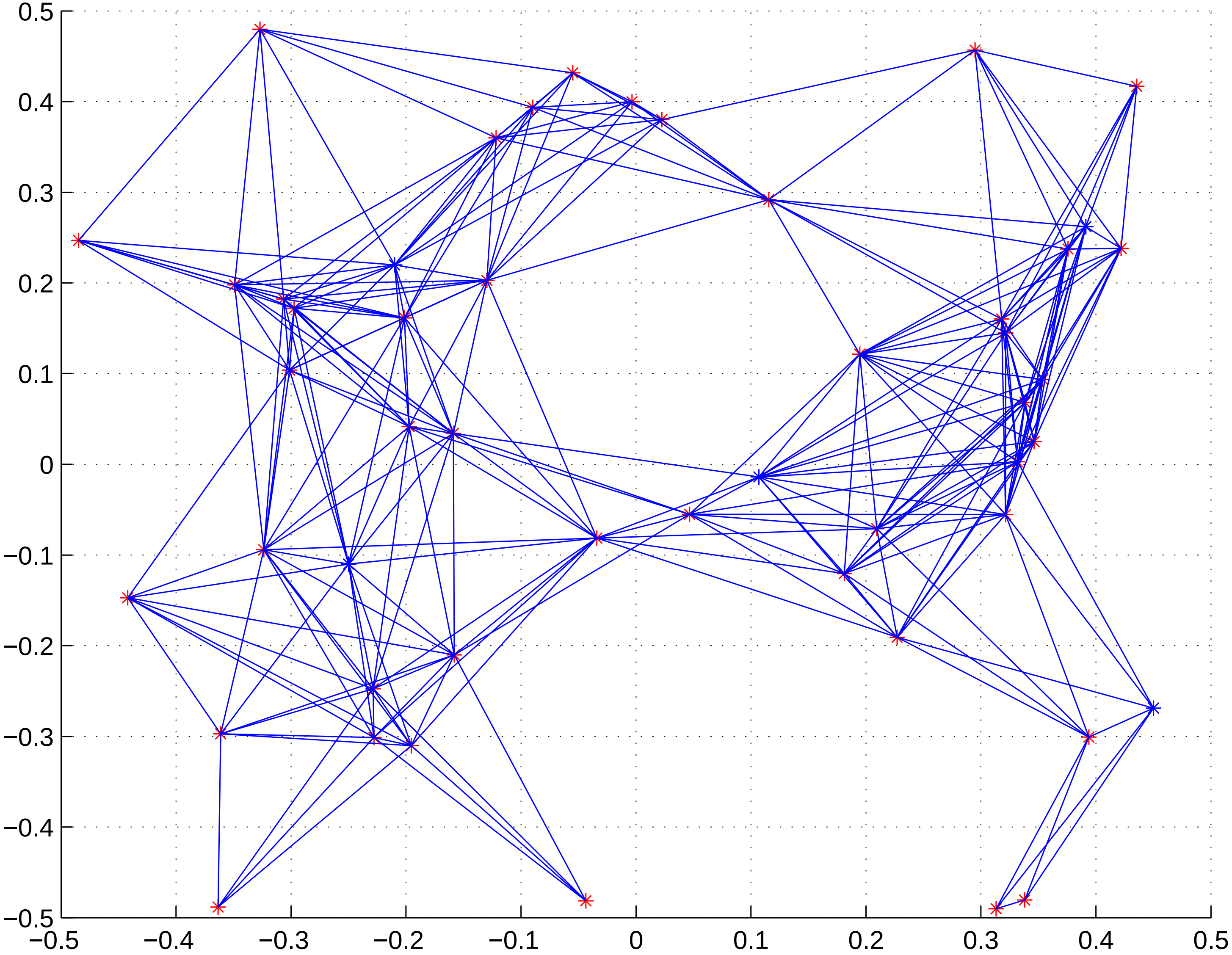


Figure 1: 50-node 2-D Sensor Localization

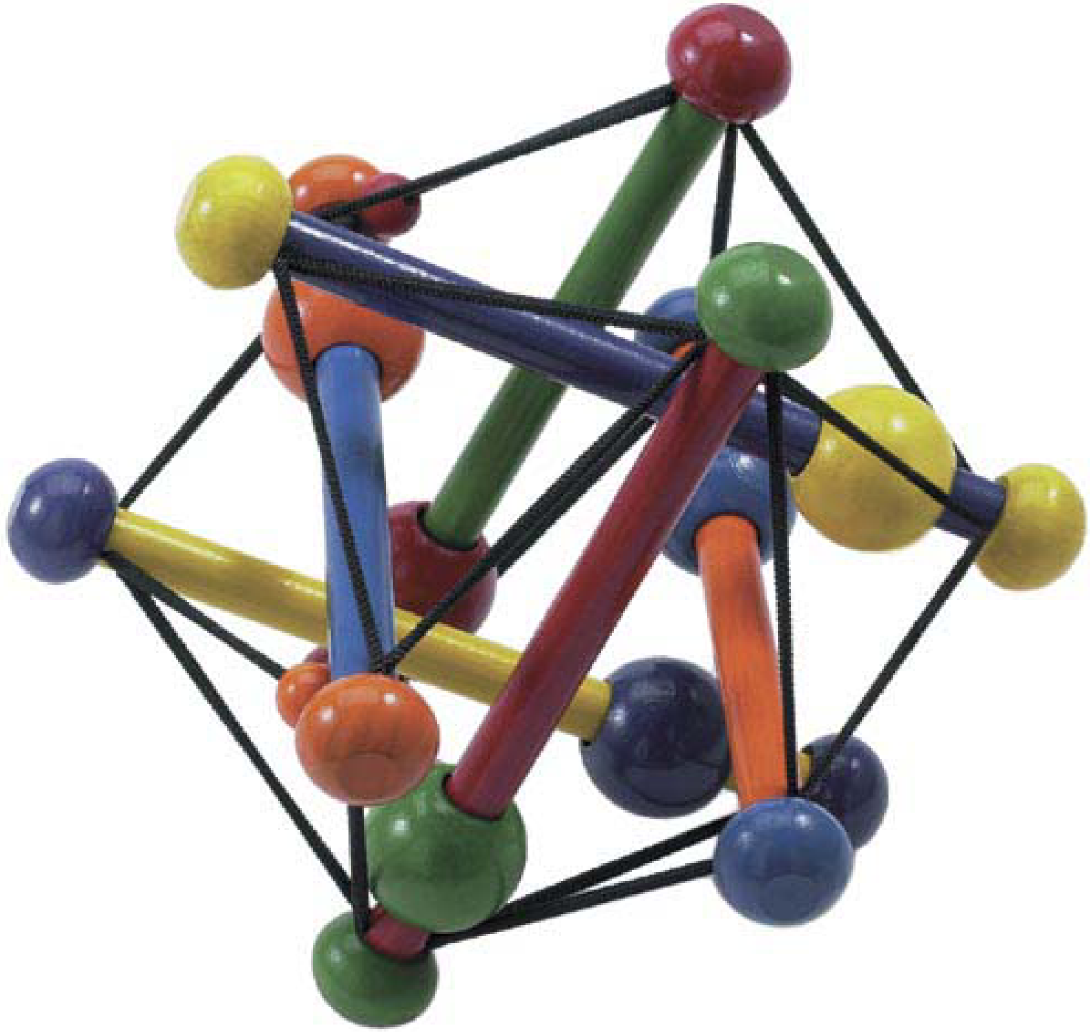


Figure 2: A 3-D Tensegrity Graph Realization; provided by Anstreicher

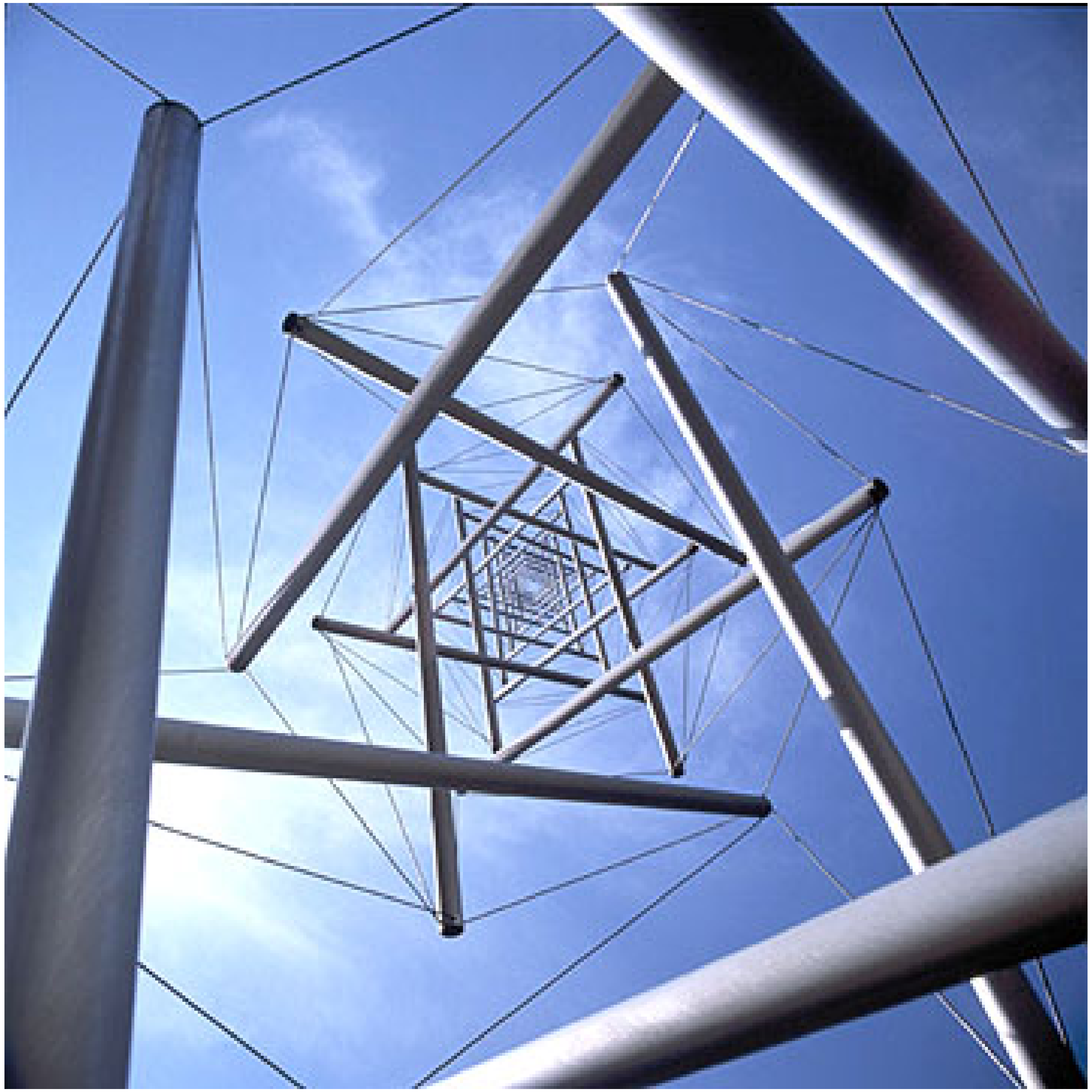


Figure 3: Tensegrity Graph: A Needle Tower; provided by Anstreicher

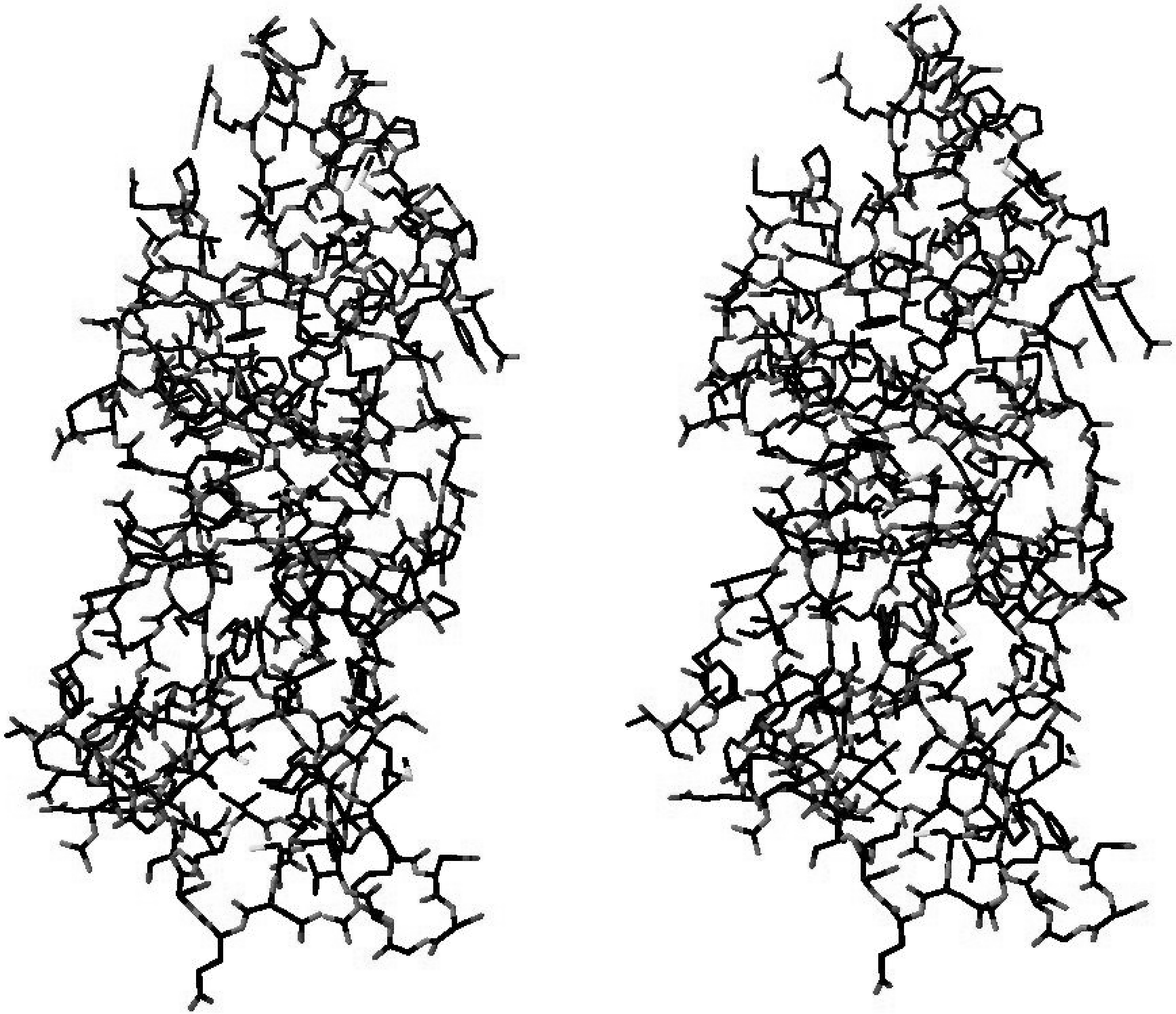


Figure 4: Molecular Conformation: 1F39(1534 atoms) with 85% of distances below $6\mathbf{r}$A and 10% noise on upper and lower bounds

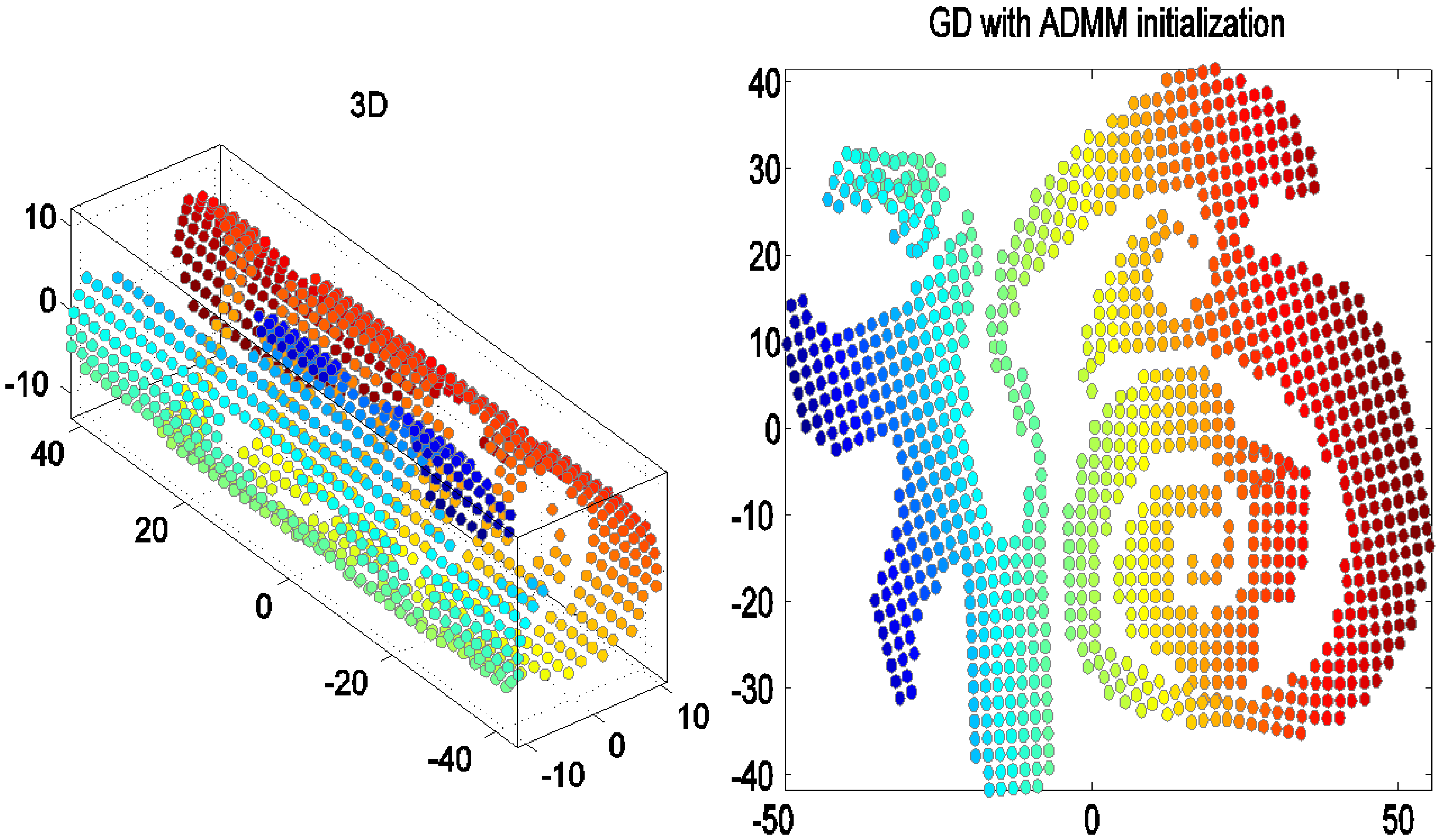


Figure 5: Dimension Reduction: Unfolding Scroll of Happiness

Variable Matrix Representation

Let $X = [\mathbf{x}_1\ \mathbf{x}_2\ \ldots\ \mathbf{x}_n]$ be the $d \times n$ matrix that needs to be determined and $\mathbf{e}_j$ be the vector of all zero except 1 at the jth position. Then

$$
\begin{array}{rcl}
\mathbf{x}_i - \mathbf{x}_j = X(\mathbf{e}_i - \mathbf{e}_j) & \text{and} & \mathbf{a}_k - \mathbf{x}_j = [I\ \ X](\mathbf{a}_k; -\mathbf{e}_j); \\
d_{ij}^2 = \|\mathbf{x}_i - \mathbf{x}_j\|^2 & = & (\mathbf{e}_i - \mathbf{e}_j)^T X^T X (\mathbf{e}_i - \mathbf{e}_j), \\
\hat{d}_{kj}^2 = \|\mathbf{a}_k - \mathbf{x}_j\|^2 & = & (\mathbf{a}_k; -\mathbf{e}_j)^T [I\ \ X]^T [I\ \ X] (\mathbf{a}_k; -\mathbf{e}_j) \\
 & = & (\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j).
\end{array}
$$

Or, equivalently,

$$
\begin{array}{rcl}
(\mathbf{e}_i - \mathbf{e}_j)^T Y (\mathbf{e}_i - \mathbf{e}_j) & = & d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j, \\
(\mathbf{a}_k; -\mathbf{e}_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (\mathbf{a}_k; -\mathbf{e}_j) & = & \hat{d}_{kj}^2,\ \forall\, k, j \in N_a, \\
Y & = & X^T X.
\end{array}
$$

SDP Relaxation and SDP Standard Form

Relax $Y = X^T X$ to $Y \succeq X^T X$. The matrix inequality is equivalent to

$$Z := \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq \mathbf{0}.$$

Matrix Z has rank at least d; if it's d, then $Y = X^T X$, and the converse is also true.

The SDP relaxation becomes: Find a symmetric matrix $Z \in \mathbf{R}^{(d+n)\times(d+n)}$ such that

$$\begin{aligned}
&Z_{1:d,1:d} = I \\
&(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i, j \in N_x, \ i < j, \\
&(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\, k, j \in N_a, \\
&Z \succeq \mathbf{0}.
\end{aligned}$$

If every sensor point is connected, directly or indirectly, to an anchor point, then the solution set must be bounded.

Sensor Localization SDP Relaxation in 2D

$$
\begin{aligned}
&(1;0;\mathbf{0})(1;0;\mathbf{0})^T \bullet Z = 1, \ (w_1) \\
&(0;1;\mathbf{0})(0;1;\mathbf{0})^T \bullet Z = 1, \ (w_2) \\
&(1;1;\mathbf{0})(1;1;\mathbf{0})^T \bullet Z = 2, \ (w_3) \\
&(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i,j \in N_x, \ i < j, \ (w_{ij}) \\
&(\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2, \ \forall\, k,j \in N_a, \ (\hat{w}_{kj}) \\
&Z \succeq \mathbf{0}.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{X} \\ \bar{X}^T & \bar{X}^T\bar{X} \end{pmatrix} = (I, \ \bar{X})^T(I, \ \bar{X}) \in S^{n+2}
$$

is a feasible rank-2 solution for the relaxation, where $\bar{X} = [\bar{\mathbf{x}}_1 \ \bar{\mathbf{x}}_2 \ \ldots \ \bar{\mathbf{x}}_n]$ and $\bar{\mathbf{x}}_j$ is the true location of sensor j (if the distance measurements are accurate).

The Dual of the SDP Relaxation in 2D

$$\begin{array}{ll} \text{min} & w_1 + w_2 + 2w_3 + \sum_{i<j \in N_x} w_{ij} d_{ij}^2 + \sum_{k,j \in N_a} \hat{w}_{kj} \hat{d}_{kj}^2 \\ \text{s.t.} & w_1 (1;0;\mathbf{0})(1;0;\mathbf{0})^T + w_2 (0;1;\mathbf{0})(0;1;\mathbf{0})^T + w_3 (1;1;\mathbf{0})(1;1;\mathbf{0})^T + \\ & \sum_{i<j \in N_x} w_{ij} (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T + \sum_{k,j \in N_a} \hat{w}_{kj} (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \succeq \mathbf{0} \end{array}$$

Variable $\hat{w}_{kj}$ represent internal/tensional force on edge ij; and dual objective can be interpreted as the potential energy of the network.

The left-hand matrix, also in S^{n+2}, is called the stress matrix.

Since the primal is feasible, the minimal value of the dual is not less than 0. Note that all $\mathbf{0}$ is an minimal solution for the dual. Thus, there is no duality gap.

Is a non-trivial optimal dual solution attainable?

Duality Theorem for SNL

Theorem 4 *Let $\bar{Z}$ be a feasible solution for SDP and $\bar{U}$ be an optimal stress matrix of the dual. Then,*

1. *complementarity condition holds: $\bar{Z} \bullet \bar{U} = 0$ or $\bar{Z}\bar{U} = \mathbf{0}$;*
2. *Rank$(\bar{Z})$ + Rank$(\bar{U}) \leq 2 + n$;*
3. *Rank$(\bar{Z}) \geq 2$ and Rank$(\bar{U}) \leq n$.*

An immediate result from the theorem is the following:

Corollary 2 *If an optimal dual stress matrix has rank n, then every solution of the SDP has rank 2, that is, the SDP relaxation solves the original problem exactly. Such a sensor network with distance information is called Strongly Localizable (SL).*

Physical interpretation: All stresses or internal forces are balanced at every sensor point.

Theoretical Analyses on Sensor Network Localization

A sensor network is 2-Universally-Localizable (UL), weaker than SL, if there is a unique localization in $\mathbf{R}^2$ and there is no $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|x_i - x_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(a_k; \mathbf{0}) - x_j\|^2 = \hat{d}_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are simply augmented to $(a_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

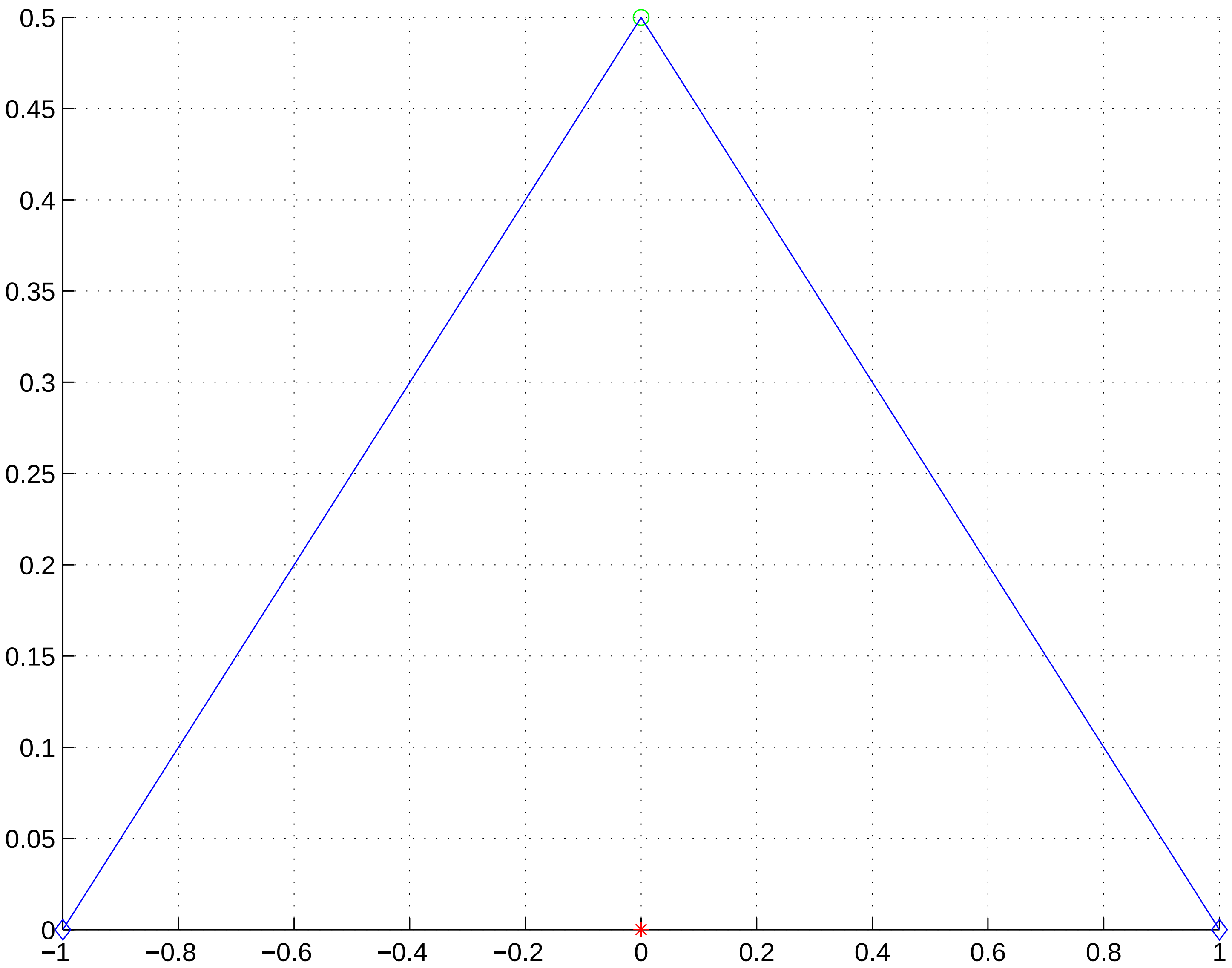


Figure 6: One sensor-Two anchors: Not localizable

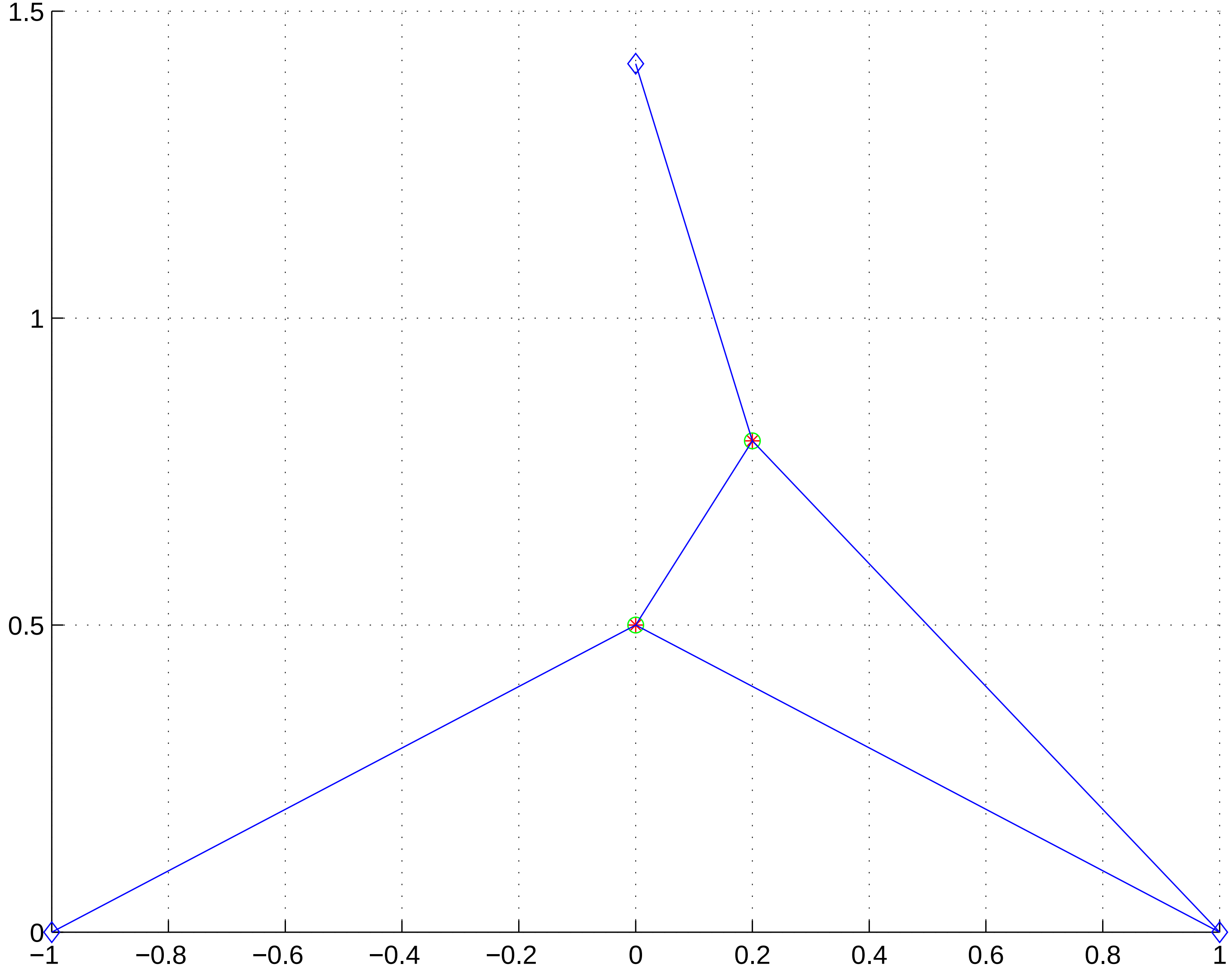


Figure 7: Two sensor-Three anchors: Strongly Localizable

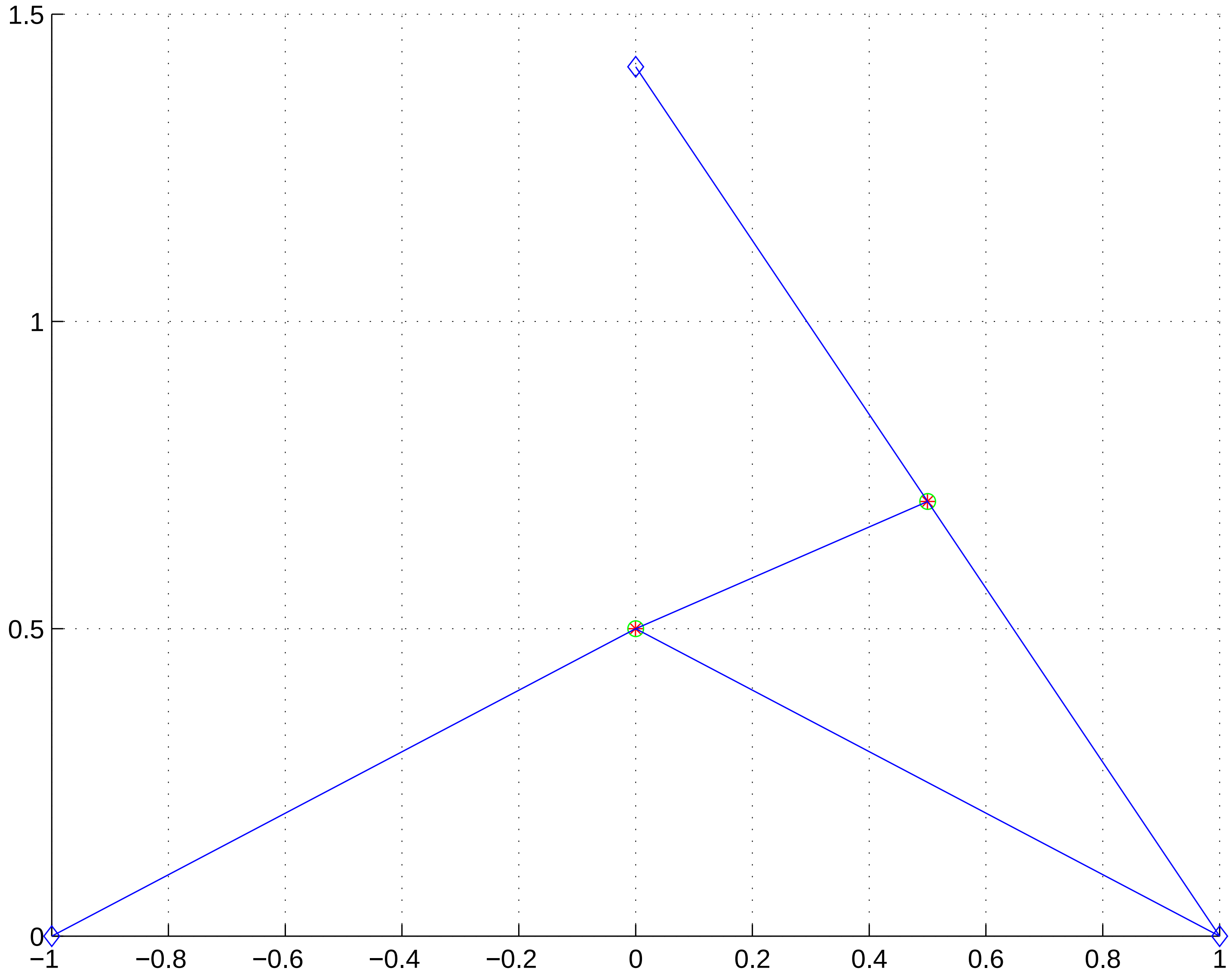


Figure 8: Two sensor-Three anchors: Localizable but not Strongly

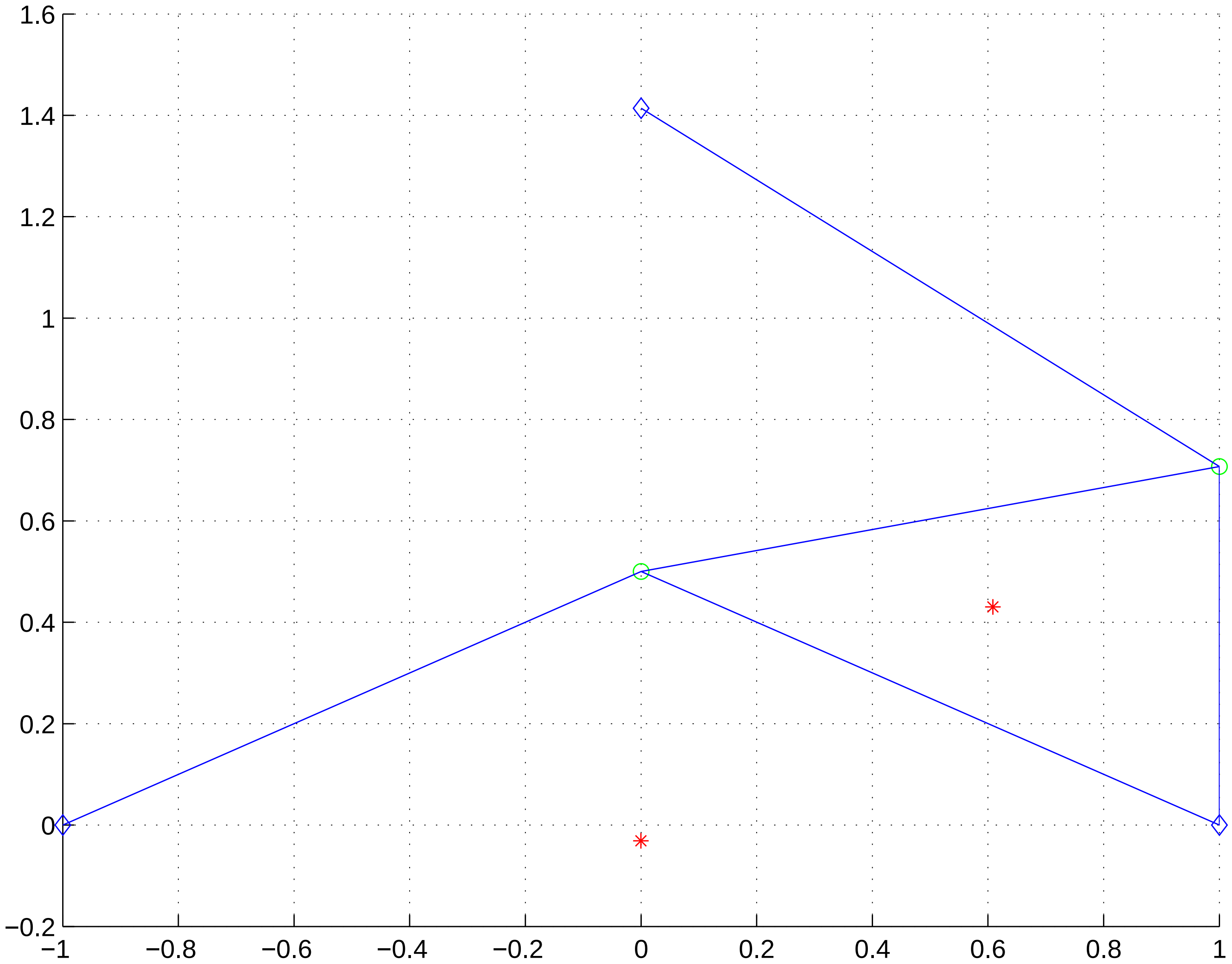


Figure 9: Two sensor-Three anchors: Not Localizable

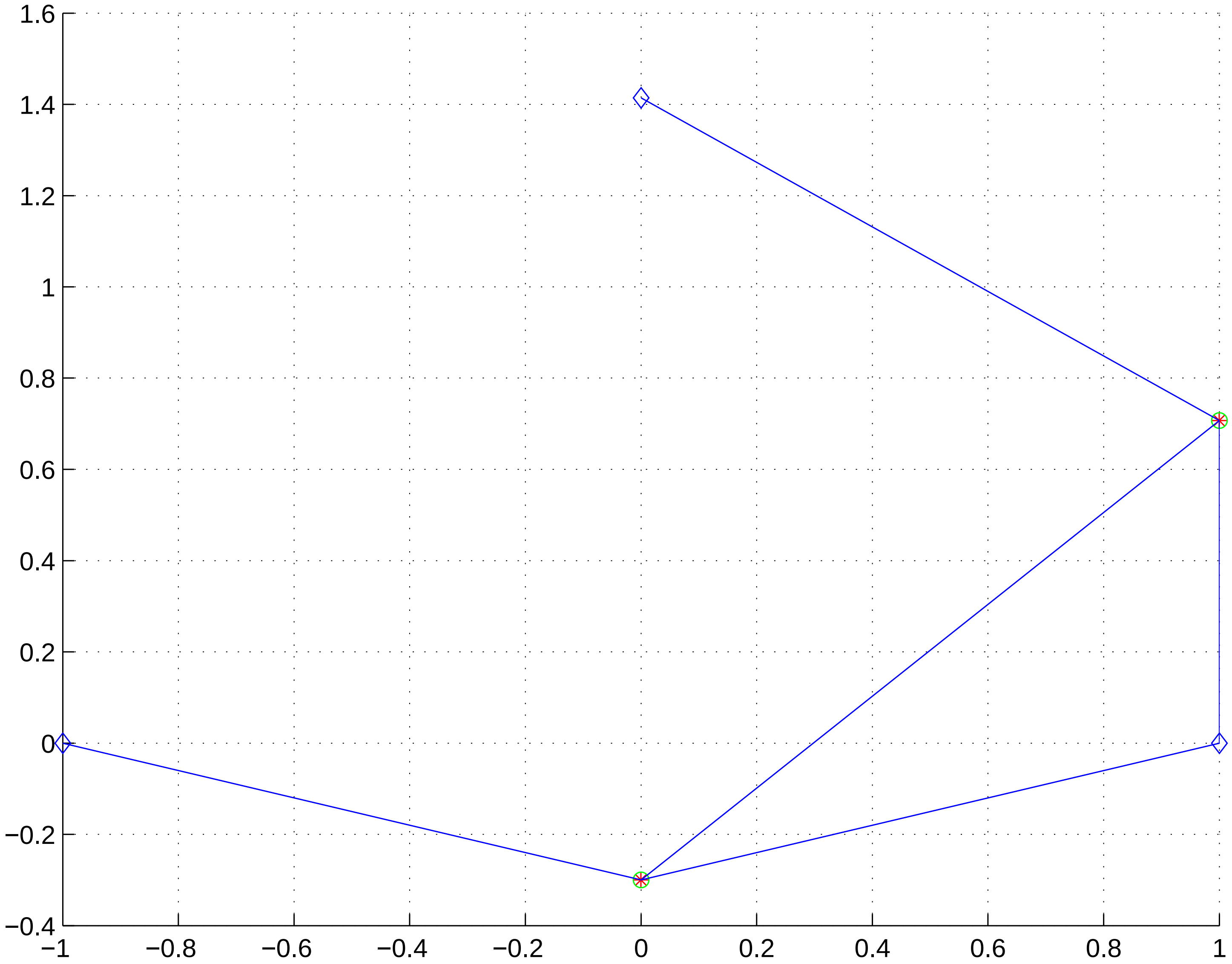


Figure 10: Two sensor-Three anchors: Strongly Localizable

UL Problems can be Localized by the SDP Relaxation

Theorem 5 *The following statements are equivalent:*

1. *The sensor network is 2-universally-localizable;*
2. *The max-rank solution of the SDP relaxation has rank 2;*
3. *The solution matrix has $Y = X^T X$ or $Tr(Y - X^T X) = 0$.*

For the following SNL problems:

- If every edge length is specified, then the sensor network is 2-universally-localizable (Schoenberg 1942);
- there is a sensor network (trilateral graph), with only $O(n)$ edge lengths specified, that is 2-universally-localizable (So 2007);
- if one sensor with its edge lengths to three anchors (in general positions) are specified, then it is 2-strongly-localizable (one of problems in HW2).

Universally-Localizable Problems (ULP)

Theorem 6 *The following SNL problems are Universally-Localizable:*

- *If every edge length is specified, then the sensor network is 2-universally-localizable (Schoenberg 1942).*
- *There is a sensor network (trilateral graph), with $O(n)$ edge lengths specified, that is 2-universally-localizable (So 2007).*
- *If one sensor with its edge lengths to at least three anchors (in general positions) specified, then it is 2-universally-localizable (So and Y 2005).*

ULPs can be localized in polynomial time

Theorem 7 *(So and Y 2005) The following statements are equivalent:*

1. *The sensor network is 2-universally-localizable;*
2. *The max-rank solution of the SDP relaxation has rank 2;*
3. *The solution matrix has $Y = X^T X$ or $Tr(Y - X^T X) = 0$.*

When an optimal dual (stress) slack matrix has rank n, then the problem is 2-strongly-localizable-problem (SLP). This is a sub-class of ULP.

Example: if one sensor with its edge lengths to three anchors (in general positions) are specified, then it is 2-strongly-localizable.

One sensor and three anchors

Find $x_1 \in \mathbf{R}^2$ such that

$$\|\mathbf{a}_k - \mathbf{x}_1\|^2 = \hat{d}_{kj}^2, \text{ for } k = 1, 2, 3,$$

Let $\bar{\mathbf{x}}_1$ be the true position of the sensor.

SDP Standard Form

$$
\begin{aligned}
&(1;0;0)(1;0;0)^T \bullet Z = 1,\\
&(0;1;0)(0;1;0)^T \bullet Z = 1,\\
&(1;1;0)(1;1;0)^T \bullet Z = 2,\\
&(\mathbf{a}_k;-1)(\mathbf{a}_k;-1)^T \bullet Z = \hat{d}_{k1}^2, \text{ for } k = 1,2,3,\\
&Z \succeq \mathbf{0}.
\end{aligned}
$$

$$
\bar{Z} = \begin{pmatrix} I & \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_1^T & \bar{x}_1^T \bar{x}_1 \end{pmatrix} = (I,\ \bar{\mathbf{x}}_1)^T (I,\ \bar{\mathbf{x}}_1)
$$

is a feasible rank-2 solution for the relaxation.

The dual slack matrix

$$\begin{pmatrix} \begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} + \sum_{k=1}^3 \hat{w}_{k1} \mathbf{a}_k \mathbf{a}_k^T & -\sum_{k=1}^3 \hat{w}_{k1} a_k \\ -(\sum_{k=1}^3 \hat{w}_{k1} a_k)^T & \hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} \end{pmatrix} \succeq \mathbf{0}.$$

Does an optimal slack matrix U have rank 1 with

$$w_1 + w_2 + 2w_3 + \sum_{k=1}^3 \hat{w}_{k1} \hat{d}_{k1}^2 = 0?$$

An optimal dual slack matrix

If we choose $w_\bullet$'s such that

$$\bar{U} = (-\bar{x}_1; 1)(-\bar{\mathbf{x}}_1; 1)^T,$$

then, $\bar{U} \succeq \mathbf{0}$ and $\bar{U} \bullet \bar{X} = 0$ so that $\bar{U}$ is an optimal slack matrix for the dual and its rank is 1.

How to select w's

We only need to consider choosing $\hat{w}$'s:

$$\sum_{k=1}^{3} \hat{w}_{k1}\mathbf{a}_k = \bar{\mathbf{x}}_1 \qquad \text{or} \qquad \sum_{k=1}^{3} \hat{w}_{k1}(\mathbf{a}_k - \bar{\mathbf{x}}_1) = \mathbf{0}$$

$$\hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1. \qquad \qquad \hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1.$$

This system always has a solution if $\mathbf{a}_k$ is not co-linear.

Then, select

$$\begin{pmatrix} w_1 + w_3 & w_3 \\ w_3 & w_2 + w_3 \end{pmatrix} = \bar{\mathbf{x}}_1\bar{\mathbf{x}}_1^T - \sum_{k=1}^{3} \hat{w}_{k1}\mathbf{a}_k\mathbf{a}_k^T$$

Other Conditions?

Even if $\mathbf{a}_k$ is co-linear, the system

$$\sum_{k=1}^{3} \hat{w}_{k1}(\mathbf{a}_k - \bar{\mathbf{x}}_1) = \mathbf{0}$$

$$\hat{w}_{11} + \hat{w}_{21} + \hat{w}_{31} = 1$$

may still have a solution $w_\bullet$?

Physical interpretation: $\hat{w}_{kj}$ is a stress/force on the edge and all stresses are balanced or at an equilibrium state. The objective represents the potential of the system.

Proof of Theorem 3

If the primal feasible matrix generated from the interior-point algorithm has rank 2, that is, $\bar{Y} = \bar{X}^T \bar{X}$ or the trace of $\bar{Y} = \bar{X}^T \bar{X}$ equal 0, then the feasible solution for the original problem is unique.

We now prove it's unique. First, every feasible matrix has rank at least 2 since $Y \succeq X^T X$.

Second, since the matrix solution computed from the interior-point algorithm has the maximal rank and it is 2, we conclude that every feasible matrix has rank exact 2.

Suppose that the system has two rank-2 feasible matrices:

$$Z_1 = \begin{pmatrix} I & X_1 \\ X_1^T & X_1^T X_1 \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} I & X_2 \\ X_2^T & X_2^T X_2 \end{pmatrix}$$

Consider $Z = \alpha Z_1 + \beta Z_2$, where $\alpha + \beta = 1$ and $\alpha, \beta > 0$. Then Z is a feasible solution and its rank must be 2.

$$Z = \begin{pmatrix} I & \alpha X_1 + \beta X_2 \\ \alpha X_1^T + \beta X_2^T & \alpha X_1^T X_1 + \beta X_2^T X_2 \end{pmatrix} =$$

$$\begin{pmatrix} I & \alpha X_1 + \beta X_2 \\ \alpha X_1^T + \beta X_2^T & (\alpha X_1 + \beta X_2)^T (\alpha X_1 + \beta X_2) \end{pmatrix}$$

Thus,

$$0 = \alpha X_1^T X_1 + \beta X_2^T X_2 - (\alpha X_1 + \beta X_2)^T (\alpha X_1 + \beta X_2) =$$

$$\alpha\beta (X_1 - X_2)^T (X_1 - X_2)$$

or

$$\|X_1 - X_2\| = 0.$$

Localize All Localizable Points

Theorem 8 *(So and Y 2005) If a problem (graph) contains a subproblem (subgraph) that is universally-localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank 2. That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.*

The proof is similar to the proof of Theorem 3 by removing the notes that is not localizable.

Implication: Diagonals of “co-variance” matrix

$$\bar{Y} - \bar{X}^T \bar{X},$$

$\bar{Y}_{jj} - \|\bar{x}_j\|^2$, can be used as a measure to see whether jth sensor’s estimated position is reliable or not.

Uncertainty Analysis and Confidence Measure

Alternatively, each x_j's can be viewed as uncertain points from the incomplete/uncertain distance measures. Then the solution to the SDP problem provides the first and second moment estimation (Bertsimas and Y 1998).

Generally, $\bar{x}_j$ is a point estimate of x_j and $\bar{Y}_{ij}$ is a point estimate $x_i^T x_j$.

Consequently,

$$\bar{Y}_{jj} - \|\bar{x}_j\|^2,$$

which is the individual variance estimation of sensor j, gives an interval estimation for its true position (Biswas and Y 2004).

Generically Universally-Localizability (GUL)

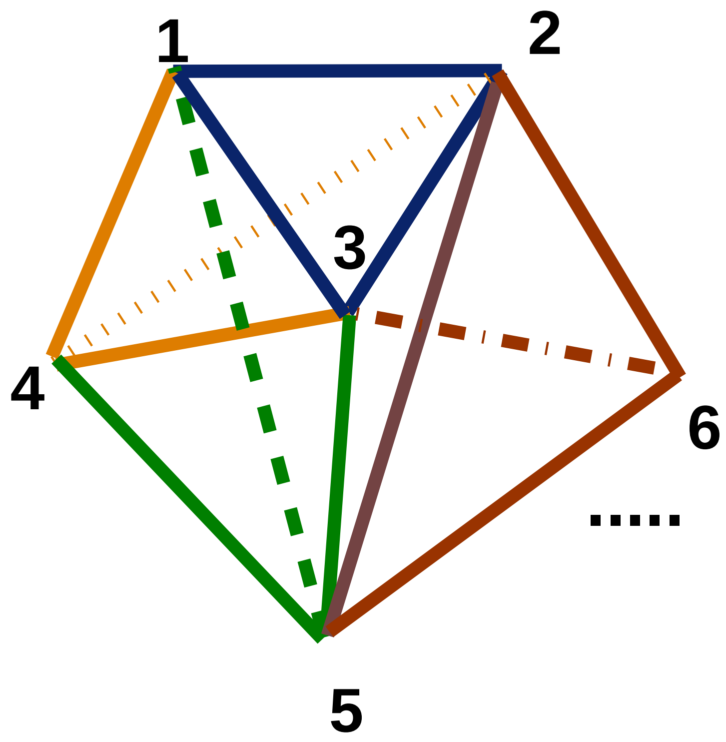
- The 2-universally-localizability depends on graph combinatorics as well as distance measurements d_{ij}.
- Is there a sparse graph that is generically 2-universally-localizable (GUL), that is, it depends on the graph combinatorics but independent of distance measurements?

Theorem 9 *(So 2007, Zhu, So and Y 2009)*

- *A graph is generically 2-universally-localizable if it contains a spanning subgraph that is generically 2-universally-localizable.*
- *The union of two generically 2-universally-localizable graphs connected with at least 3 nodes is generically 2-universally-localizable.*
- *The spanning d-trilateration graph in dimension d is generically d-universally-localizable.*
- *The information-theoretical complexity of the spanning d-trilateration graph is near-optimal with only $(d+1) \cdot n$ distance measurements, where the freedom dimension of the problem is $d \cdot n$.*

2-Trilateration Graph

Let $n \geq 1$ be integers with $n \geq 3$. An n–node graph $G = (V, E)$ is called a 2-trilateration graph if there exists an ordering $\{1, 2, \ldots, n\}$ of the notes in V (called trilateration ordering) such that (i) the first 3 notes form a complete graph, and (ii) every vertex $j \geq 4$ is connected to at least 3 of the nodes from $1, 2, \ldots, j-1$.



More Localizability of Trilateration Graphs

Theorem 10 *The spanning d-trilateration graph in dimension d is d-strongly-localizable if and only if the nodes are in general positions.*

Note the difference between “generic” and “general”.

Objective Regularization for Low-rank SNL

One typically maximizes or minimizes the the trace of Z:

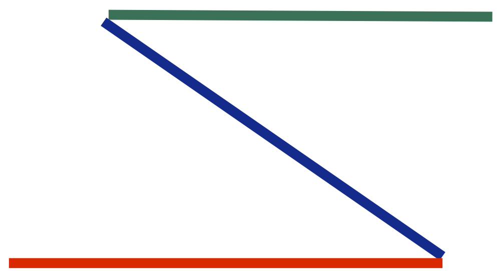
$$
\begin{array}{ll}
\text{Maximize} & I \bullet Z \\
\text{Subject to} & (1;0;\mathbf{0})(1;0;\mathbf{0})^T \bullet Z = 1, \\
& (0;1;\mathbf{0})(0;1;\mathbf{0})^T \bullet Z = 1, \\
& (1;1;\mathbf{0})(1;1;\mathbf{0})^T \bullet Z = 2, \\
& (\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)(\mathbf{0};\mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2, \ \forall\, i,j \in N_x,\ i<j, \\
& (\mathbf{a}_k;-\mathbf{e}_j)(\mathbf{a}_k;-\mathbf{e}_j)^T \bullet Z = d_{kj}^2, \ \forall\, k,j \in N_a, \\
& Z \succeq \mathbf{0}.
\end{array}
$$

Tensegrity (Tensional-Integrity) Objective for SNL: a 1D problem

Anchor-free SNL:

$$\begin{aligned} (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T \bullet Y &= d_{ij}^2, \ \forall\, (i,j) \in E,\ i < j, \\ Y &\succeq \mathbf{0}. \end{aligned}$$

For certain graphs, to select a subset edges to maximize and/or a subset of edges to minimize is guaranteed to finding the lowest rank SDP solution – Tensegrity Method.



The Chain Graph Example

Consider:

$$\begin{array}{ll} \text{max} & \mathbf{e}_3\mathbf{e}_3 \bullet Y \\ \text{s.t.} & \mathbf{e}_1\mathbf{e}_1^T \bullet Y = 1, \\ & (\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)^T \bullet Y = 1, \\ & (\mathbf{e}_2 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_3)^T \bullet Y = 1, \\ & Y \succeq \mathbf{0} \in \mathcal{M}^3, \end{array}$$

The dual is

$$\begin{array}{ll} \text{min} & y_1 + y_2 + y_3 \\ \text{s.t.} & y_1\mathbf{e}_1\mathbf{e}_1^T + y_2(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)^T + y_3(\mathbf{e}_2 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_3)^T - S = \mathbf{e}_3\mathbf{e}_3, \\ & S \succeq \mathbf{0} \in \mathcal{M}^3, \end{array}$$

Interpretation of the Dual

What's the interpretation of the dual? What's the interpretation of complementarity?

Dual variables are stresses (internal-forces) on edges, and the objective is the total potential of the graph. At the optimal solution, stresses reach an equilibrium or balanced state with an external force added at node 3.

$$(1;\ 2;\ 3)(1;\ 2;\ 3)^T \bullet S = 0 \quad \text{implies} \quad S(1;\ 2;\ 3) = \mathbf{0};$$

or

$$\left(3\mathbf{e}_1\mathbf{e}_1^T + 3(\mathbf{e}_1 - \mathbf{e}_2)(\mathbf{e}_1 - \mathbf{e}_2)^T + 3(\mathbf{e}_2 - \mathbf{e}_3)(\mathbf{e}_2 - \mathbf{e}_3)^T - \mathbf{e}_3\mathbf{e}_3\right)(1;\ 2;\ 3) = \mathbf{0},$$

that is,

$$3\mathbf{e}_1 - 3(\mathbf{e}_1 - \mathbf{e}_2) - 3(\mathbf{e}_2 - \mathbf{e}_3) = 3\mathbf{e}_3.$$

Rank of the Dual Slack Matrix: Strongly Localizability

$$X = (1,\ 2,\ 3)(1,\ 2,\ 3)^T, \quad \text{and} \quad \mathbf{e}_3\mathbf{e}_3 \bullet X = 9.$$

The dual slack matrix

$$S = \begin{pmatrix} y_1 + y_2 & -y_2 & 0 \\ -y_2 & y_2 + y_3 & -y_3 \\ 0 & -y_3 & y_3 - 1 \end{pmatrix},$$

where $(y_1 = 3,\ y_2 = 3,\ y_3 = 3)$ make the dual slack matrix

$$S = \begin{pmatrix} 6 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 2 \end{pmatrix}.$$

The rank of the matrix is 2, indicating the max-rank primal solution matrix is 1.

A More Complicated Example

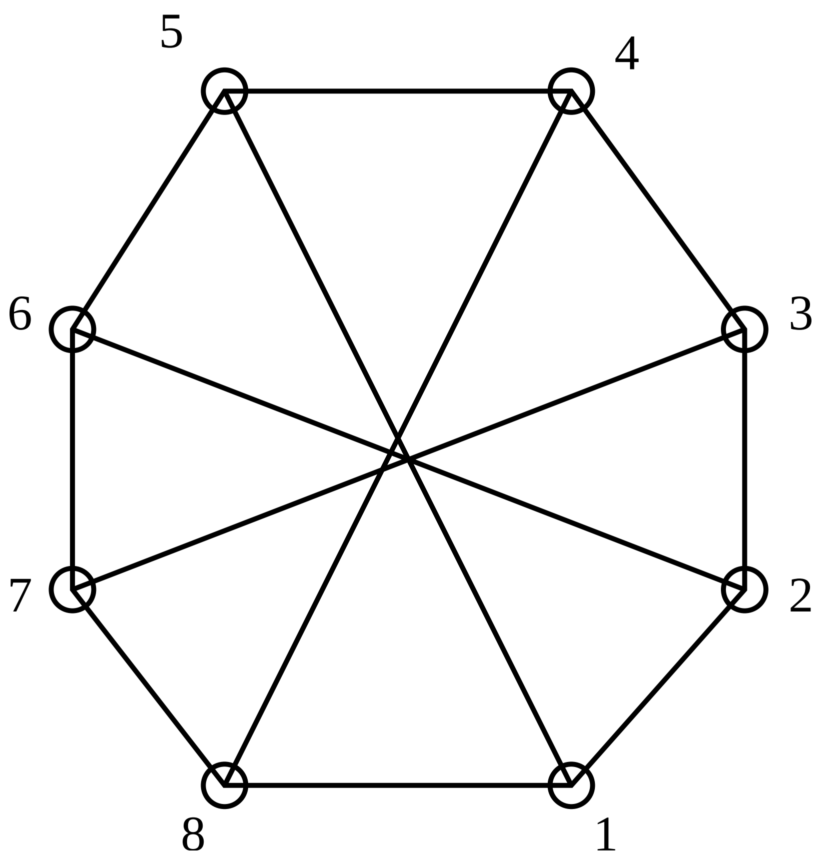


Figure 11: Maximization of non-edge $(1, 4)$ will work

Extension to d-Triangulation Graph, Davood et al. 2011

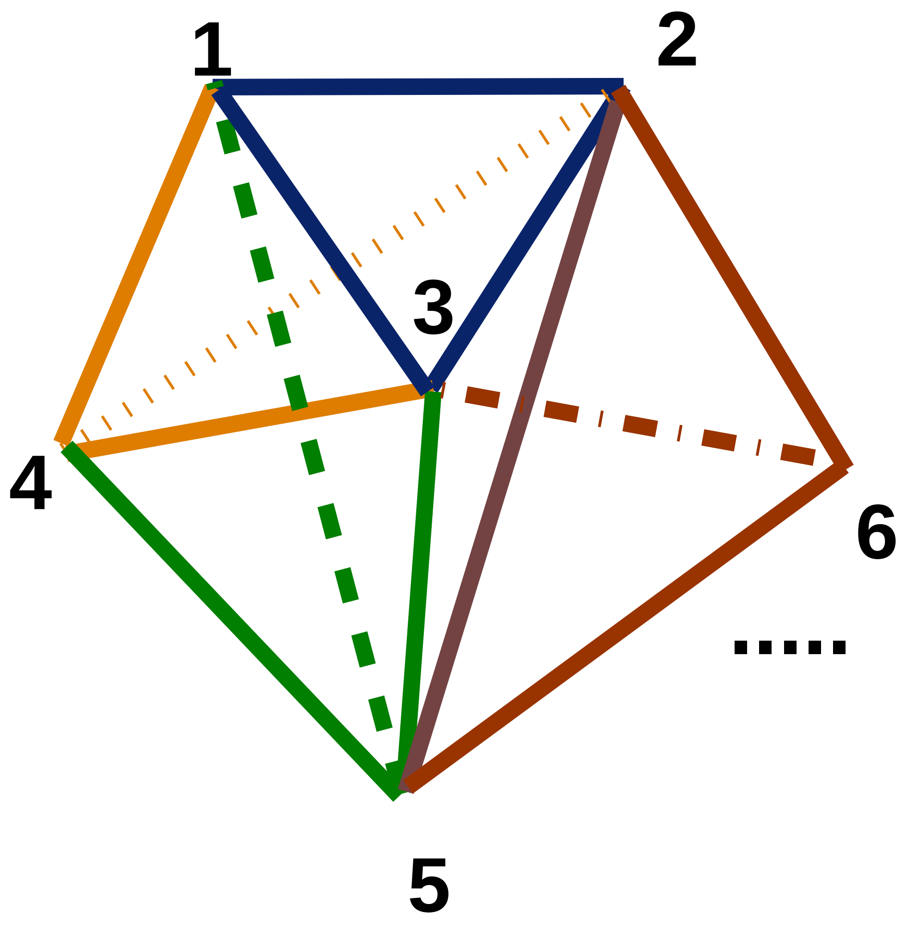


Figure 12: Maximization of the sum of diagonal non-edges

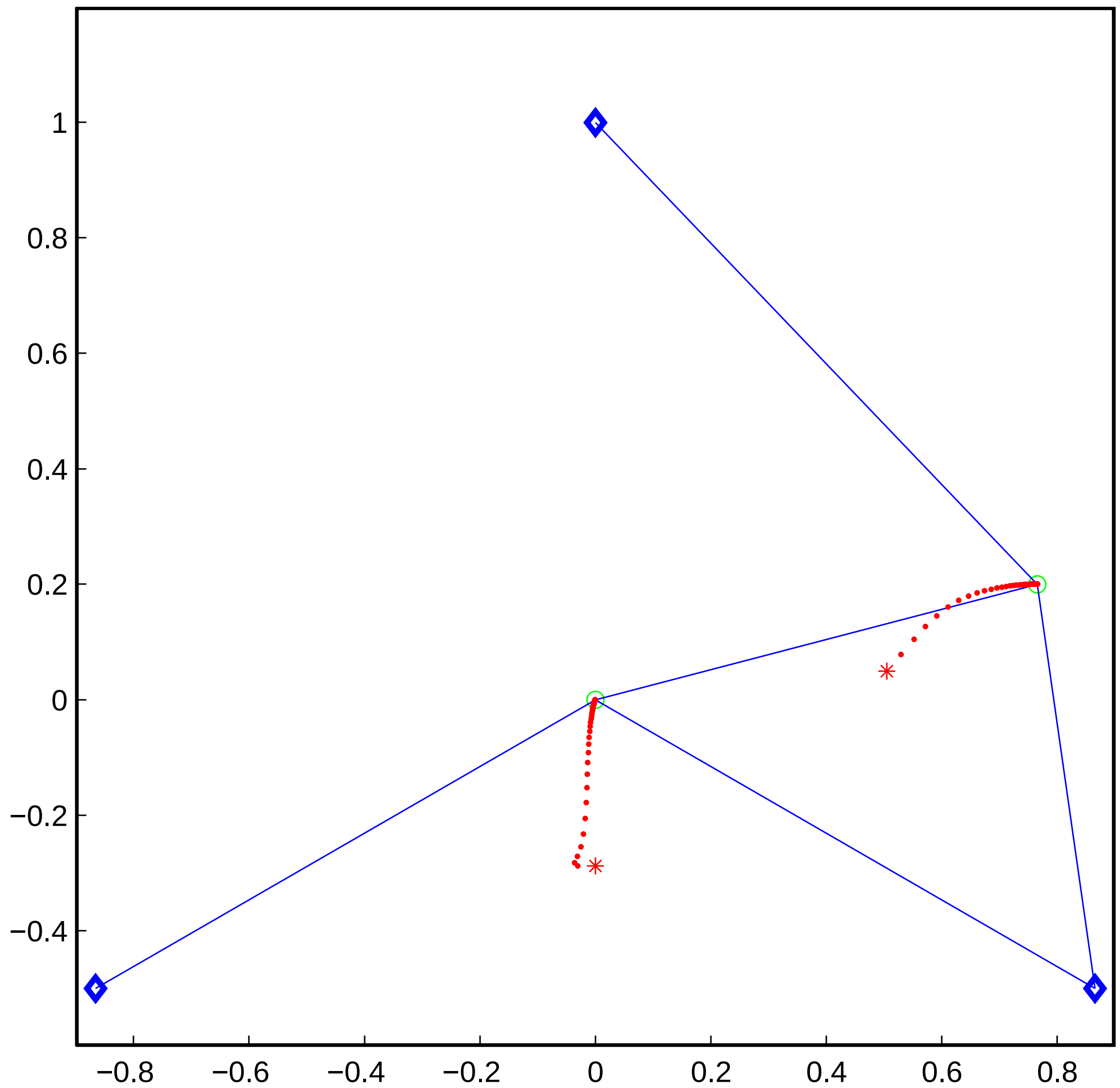
SDP Solution Post Rank Reduction: A Practical Method

- When measurement noises exist, the SDP solution almost always has a high rank. How to round the high-rank solution into a low rank?

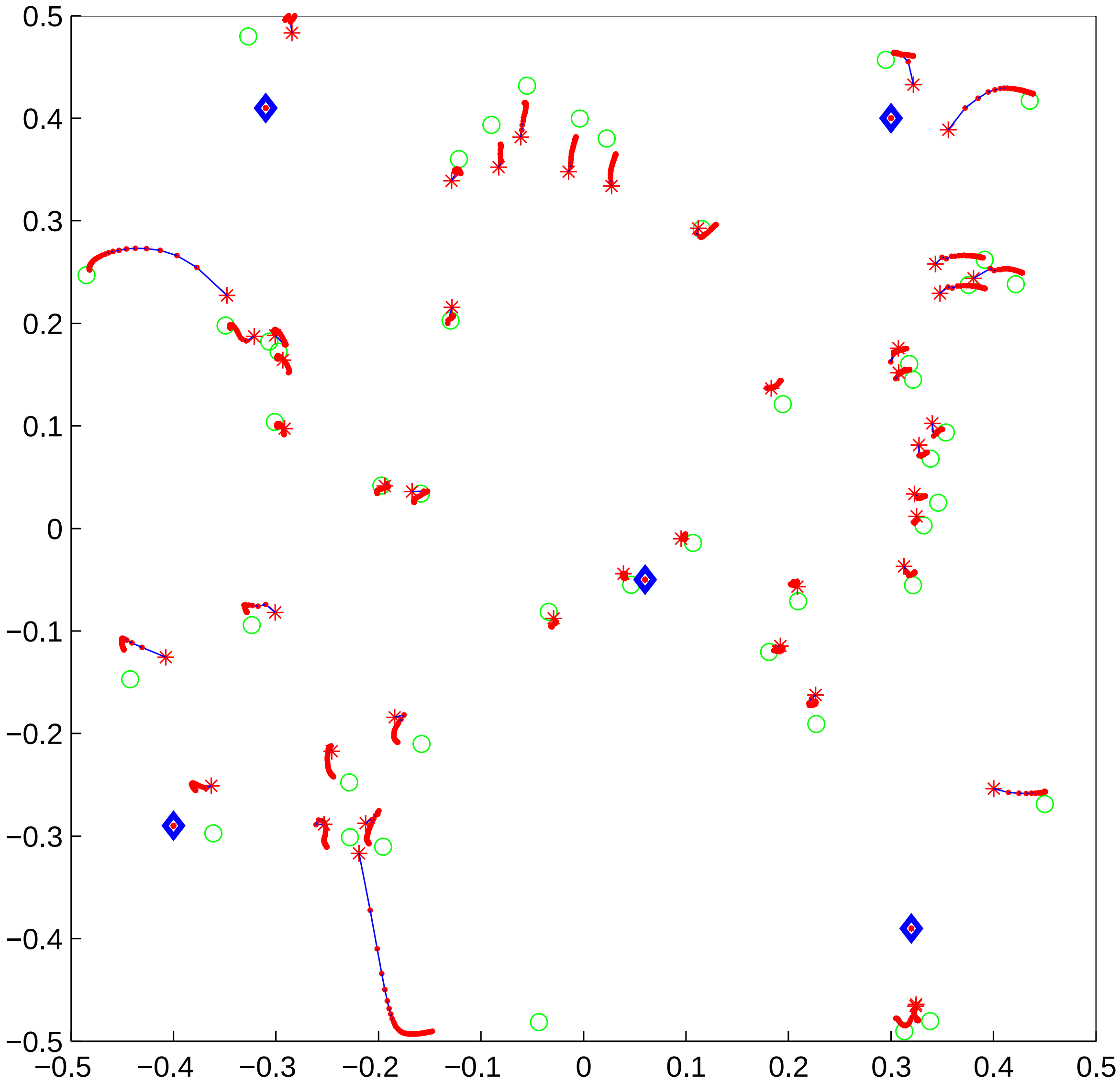
- Gradient-based local search: using the SDP solution projection as the initial point, apply the steepest descent method to further minimize

$$\sum_{(i,j)\in N_x} (\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2)^2 + \sum_{(k,j)\in N_a} (\|\mathbf{a}_k - \mathbf{x}_j\|^2 - d_{kj}^2)^2$$

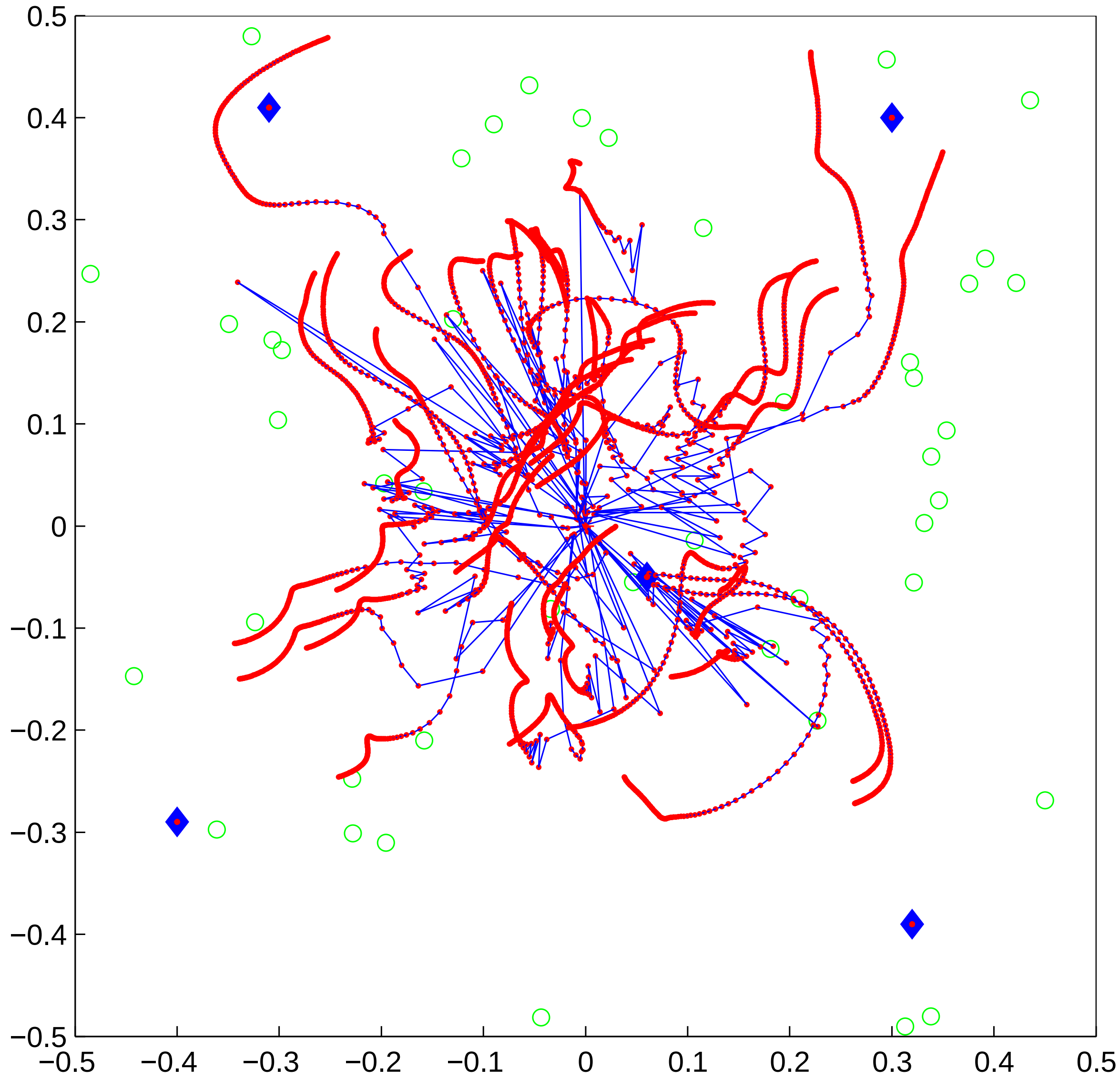
Gradient Search Trajectories: an Example



SDP/Gradient Search: Start from the SDP solution



No-SDP/Gradient Search: Start from a Random Point



Large Scale Problems: Distributed Computation

1. Partition the anchors into a number of clusters according to their geographical positions or connectivity.

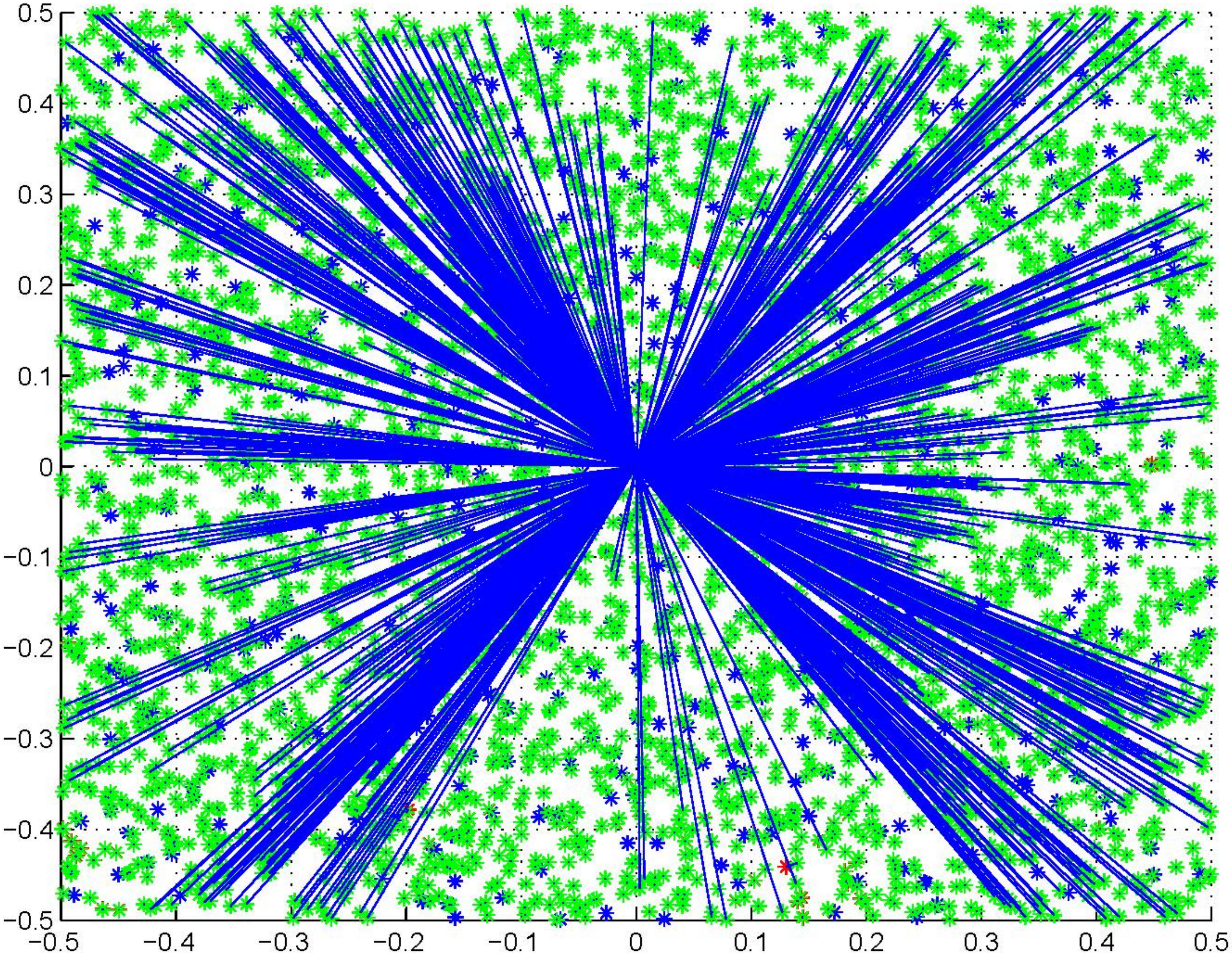
2. Assign un-positioned sensors to clusters. Note that a sensor may be assigned into multiple clusters and some sensors are not assigned into any cluster.

3. For each cluster of anchors and unknown sensors, formulate the error minimization problem for that cluster, and solve the resulting SDP model if the number of anchors is more than 2.

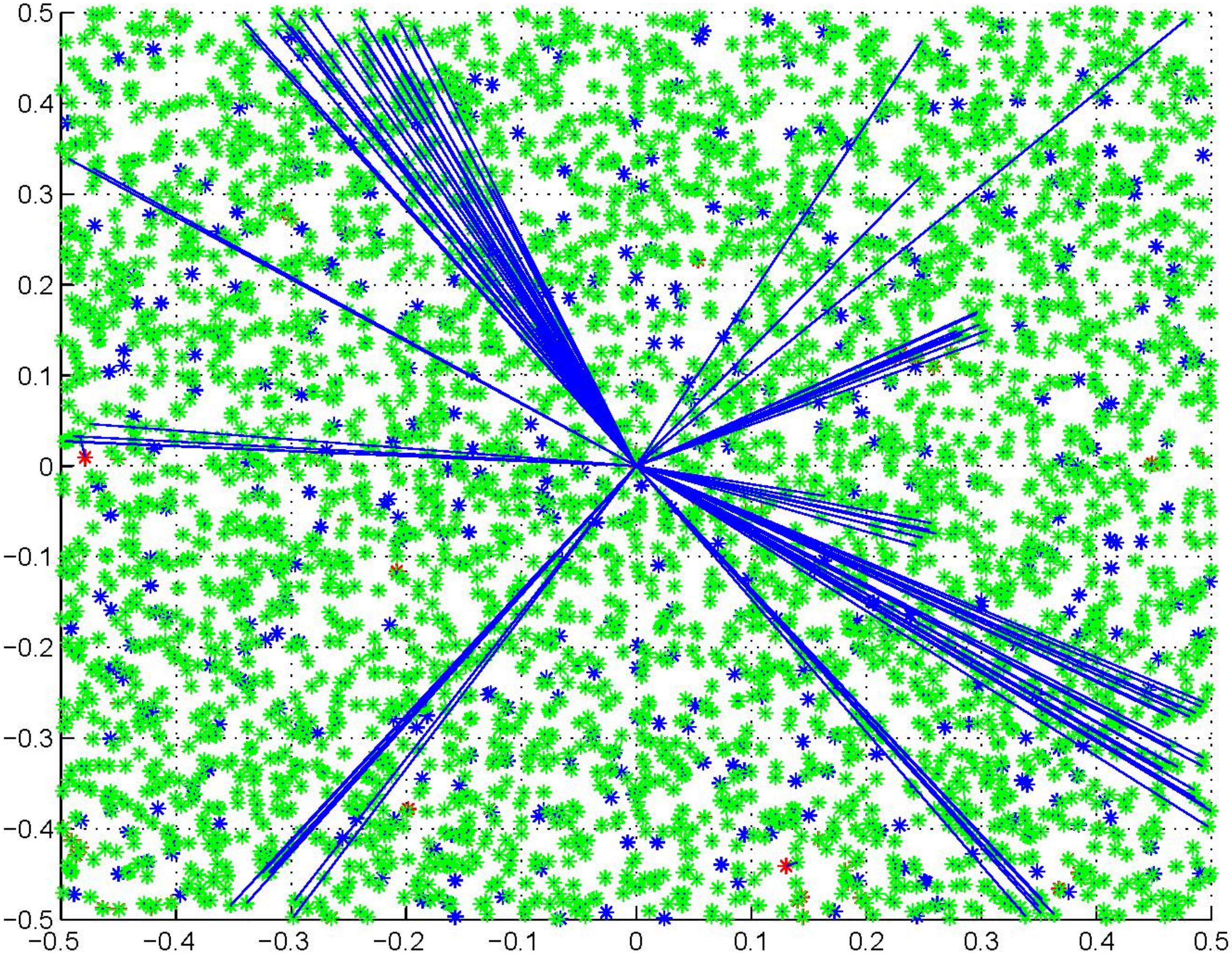
4. After solving each SDP model, check the individual trace for each unknown sensor in the model. If it is below a predetermined small tolerance, label the sensor as positioned and its estimation $\bar{x}_j$ becomes an " anchor".

5. Consider positioned sensors as anchors and return to Step 1 to start the next round of estimation.

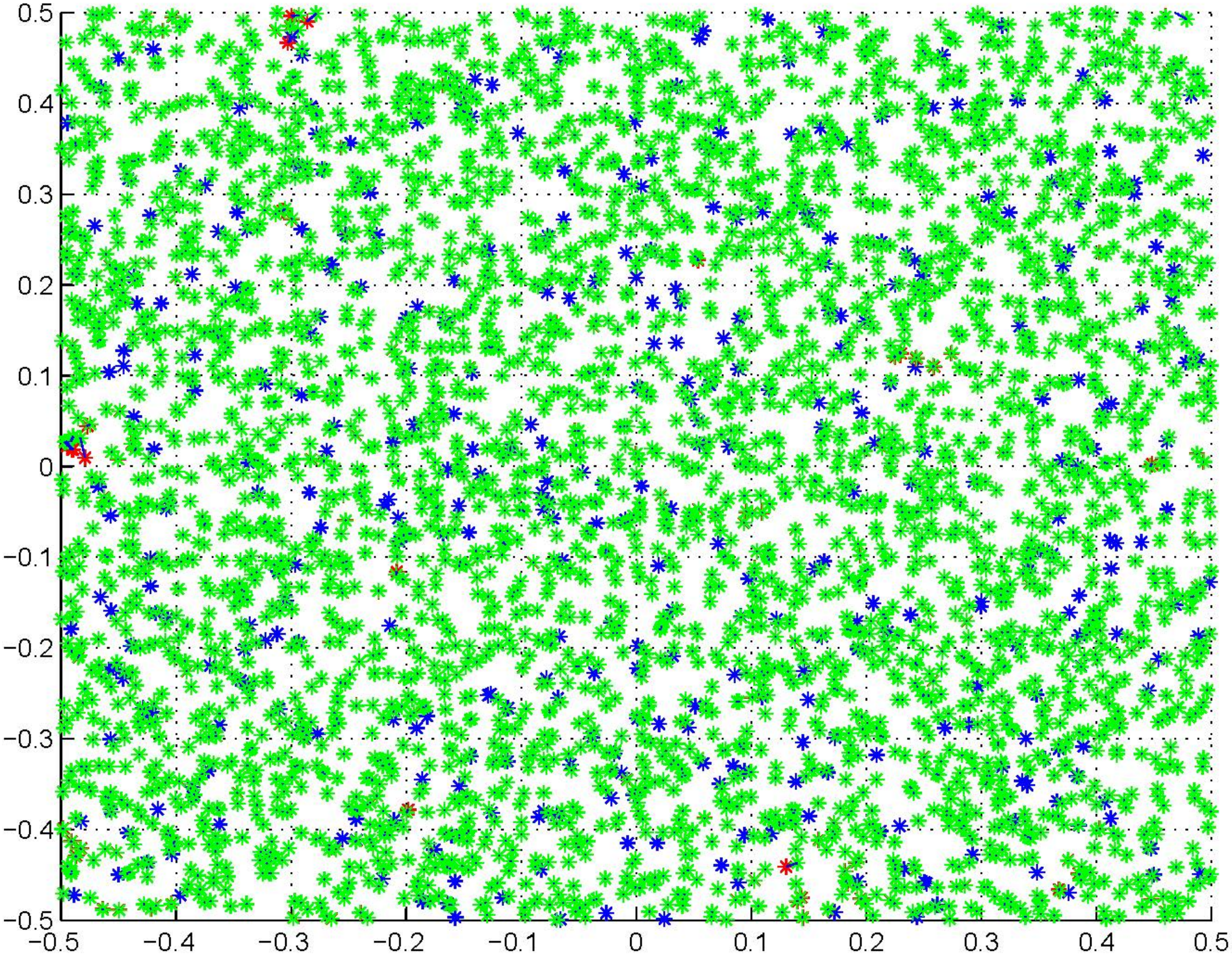
A distributed example: Step 1



A distributed example: Step 2



A distributed example: Step 4



Some Ongoing Work on SNL

- Other measurements could be distance intervals, absolute angles, relative angles, path-distances, etc (Biswas 2007).
- Other Problems: Localization based on Time-Series Data Measurement, Structural Knowledges (nodes on a sphere), etc.
- More systematic objective regularizations?
- More theorems on UL and SL?
- More low-rank SDP applications.