Summer Course Project II: Sensor Network Localization

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Sensor Network Localization (SNL), also closely related to Data Dimension Reduction, Molecular Confirmation, Graph Realization, is one of major topics in Data Science. The SNL problem is: Given possible anchors $\mathbf{a}_k \in R^d$, distance information $d_{ij} \in N_x$, and $\hat{d}_{kj} \in N_a$, find $\mathbf{x}_i \in R^d$ for all i such that

$$\begin{aligned} \|\mathbf{x}_i - \mathbf{x}_j\|^2 &= d_{ij}^2,\ \forall\ (i,j) \in N_x,\ i < j, \\ \|\mathbf{a}_k - \mathbf{x}_j\|^2 &= \hat{d}_{kj}^2,\ \forall\ (k,j) \in N_a, \end{aligned} \tag{1}$$

where $(ij) \in N_x$ ($(kj) \in N_a$) connects points $\mathbf{x}_i$ and $\mathbf{x}_j$ ($\mathbf{a}_k$ and $\mathbf{x}_j$) with an edge whose Euclidean length is d_{ij} ($\hat{d}_{kj}$).

We establishes in class an SOCP relaxation for solving solve (1):: Find a symmetric matrix $\mathbf{x}_i$s such that

$$\begin{array}{ll} \min & \sum_i \mathbf{0}^T \mathbf{x}_i \\ \text{s.t.} & \|\mathbf{x}_i - \mathbf{x}_j\|^2 \le d_{ij}^2,\ \forall\ (i,j) \in N_x,\ i < j, \\ & \|\mathbf{a}_k - \mathbf{x}_j\|^2 \le \hat{d}_{kj}^2,\ \forall\ (k,j) \in N_a. \end{array} \tag{2}$$

We also establishes in class an SDP relaxation for solving solve (1):: Find a symmetric matrix $Z \in S^{d+n}$ such that

$$\begin{array}{ll} \min & \mathbf{0} \bullet Z \\ \text{s.t.} & Z_{1:d,1:d} = I, \\ & (\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)(\mathbf{0}; \mathbf{e}_i - \mathbf{e}_j)^T \bullet Z = d_{ij}^2,\ \forall\ i,j \in N_x,\ i < j, \\ & (\mathbf{a}_k; -\mathbf{e}_j)(\mathbf{a}_k; -\mathbf{e}_j)^T \bullet Z = \hat{d}_{kj}^2,\ \forall\ k,j \in N_a, \\ & Z \succeq \mathbf{0}. \end{array} \tag{3}$$

Note that $Z_{1:d,1:d} = I \in S^d$ can be realized through $d(d+1)/2$ linear equations.

There is a simple nonlinear least squares approach to solve (1):

$$\min \quad \sum_{(ij)\in N_x} \left(\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2\right)^2 + \sum_{(kj)\in N_a} \left(\|\mathbf{a}_k - \mathbf{x}_j\|^2 - d_{kj}^2\right)^2 \tag{4}$$

which is an unconstrained nonlinear minimization problem.

Question 1: Consider the problem on plane R^2 with one sensor and three anchors $\mathbf{a}_1 = (1;0)$, $\mathbf{a}_2 = (-1;0)$ and $\mathbf{a}_3 = (0;2)$. Suppose the Euclidean distances from the sensor to the three anchors are d_1, d_2 and d_3 respectively and known to us.Then, from the anchor and distance information, we can locate the sensor by finding $\mathbf{x} \in R^2$ such that

$$\|\mathbf{x} - \mathbf{a}_i\|^2 = d_i^2, \ i = 1, 2, 3.$$

Do the following numerical experimentations:

- Generate any sensor point in the convex hull of the three anchors, compute its distances to three anchors d_i, $i = 1, 2, 3$, respectively. Then solve the SOCP relaxation problem, that is,

$$\|\mathbf{x} - \mathbf{a}_i\|^2 \leq d_i^2, \ i = 1, 2, 3.$$

  Did you find the correct location? What happen if the sensor point is outside of the convex hull?

- Now try the SDP relaxation, that is, let

$$Z := \begin{pmatrix} I & \mathbf{x} \\ \mathbf{x}^T & Y \end{pmatrix} \in S^3,$$

  and solve

$$\begin{aligned} (1;0;0)(1;0;0)^T \bullet Z &= 1, \\ (0;1;0)(0;1;0)^T \bullet Z &= 1, \\ (1;1;0)(1;1;0)^T \bullet Z &= 2, \\ (\mathbf{a}_i;-1)(\mathbf{a}_i;-1)^T \bullet Z &= d_i^2, \ i = 1, 2, 3, \\ Z &\succeq \mathbf{0} \in S^3. \end{aligned}$$

  Did you find the correct location everywhere on the plane?

You can download CVX or MOSEK directly to solve these numerical problems.

Question 2: Furthermore, run some randomly generated problems (in 1D and 2D) with (2 and 3) anchors with a few sensors, where we assume that the Euclidean distance is exactly known when any two points is closer than a given range.

- Compare the three approaches for localization quality?

- Use the SOCP or SDP solution $\bar{X} = [\bar{\mathbf{x}}_1, \ \bar{\mathbf{x}}_2, \ ...., \ \bar{\mathbf{x}}_n]$ as the initial solution for approach (4) and apply the Steepest Descent Method for a number of steps.

In practical problems, there are often noises in the distance information. To deal with possible noises, the SDP relaxation approach (3) can be modified to minimize the L_1 norm of the errors:

$$\begin{array}{ll} \min & \sum_{ij\in N_x}(\delta'_{ij}+\delta''_{ij})+\sum_{kj\in N_a}(\hat{\delta}'_{kj}+\hat{\delta}''_{kj}) \\ \text{s.t.} & Z_{1:d,1:d}=I, \\ & (\mathbf{0};\mathbf{e}_i-\mathbf{e}_j)(\mathbf{0};\mathbf{e}_i-\mathbf{e}_j)^T\bullet Z+\delta'_{ij}-\delta''_{ij}=d_{ij}^2,\ \forall\, i,j\in N_x,\ i<j, \\ & (\mathbf{a}_k;-\mathbf{e}_j)(\mathbf{a}_k;-\mathbf{e}_j)^T\bullet Z+\hat{\delta}'_{kj}-\hat{\delta}''_{kj}=\hat{d}_{kj}^2,\ \forall\, k,j\in N_a, \\ & Z\succeq \mathbf{0}. \end{array} \tag{5}$$

The SDP solution from the relaxation

$$\bar{Z}=\begin{pmatrix} I & \bar{X} \\ \bar{X}^T & \bar{Y} \end{pmatrix}$$

often may not be rank d so that $\bar{X}\in R^{d\times n}$ cannot be the best possible localization of the n sensors.

Question 3: Generate noise distance data and use the SDP solution $\bar{X}=[\bar{\mathbf{x}}_1,\ \bar{\mathbf{x}}_2,\ ....,\ \bar{\mathbf{x}}_n]$ as the initial solution for approach (4) and apply the Steepest Descent Method for a number of steps.

- How is the final solution come out after steepest descent?
- What about if you start the Steepest Descent with a random initial solution?

References

[1] P. Biswas and Y. Ye, "Semidefinite programming for ad hoc wireless sensor network localization," in *Proceedings of the third international symposium on Information processing in sensor networks*, ACM Press, 2004, pp. 46–54.

[2] Pratik Biswas, T-C Liang, K-C Toh, Y. Ye, T-C Wang, "Semidefinite programming approaches for sensor network localization with noisy distance measurements," IEEE transactions on automation science and engineering, 3(4), 2006, pp. 360-371.

[3] Pratik Biswas, Hamid Aghajan, Y. Ye, "Integration of angle of arrival information for multimodal sensor network localization using semidefinite programming," 39th Asilomar Conference on Signals, Systems and Computers, 2005.

[4] L. Doherty, L. E. Ghaoui, and K. S. J. Pister, "Convex position estimation in wireless sensor networks." in *Proceedings of IEEE Infocom*, Anchorage, Alaska, April 2001, pp. 1655 –1663.

[5] Mingjie Gao, Ka-Fai Cedric Yiu, Sven Nordholm, Y. Ye, "On a New SDP-SOCP Method for Acoustic Source Localization Problem," ACM Transactions on Sensor Networks (TOSN), 12(4), 2016.

[6] A. So. A Semidefinite Programming Approach to the graph realization problem: Theory, Applications and Extensions *Phd Thesis, Stanford University*, 2007.