A Hybrid High-Order method for incremental associative plasticity with small deformations

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Context

- Associative plasticity with small deformations
  - non-linear stress-strain constitutive relation (material nonlinearity)
  - history of the deformations (irreversible phenomena)
- Presence of volumetric-locking with primal H^1-conforming formulation due to plastic incompressibility
- An alternative : using mixed methods but more unknowns, more expensive to build, saddle-point problem to solve ...

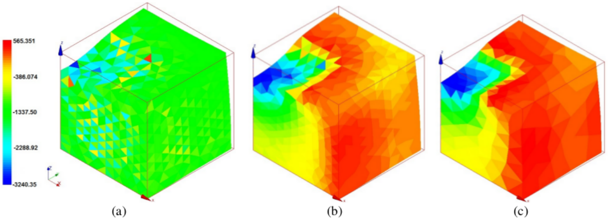


Figure 1 – Trace of the stress tensor for (a) P1 (b) P2 (c) P2/P1/P1

Main features of the proposed method

- Primal formulation

  $\Rightarrow$ More advantageous than mixed methods

- Absence of volumetric-locking

  $\Rightarrow$ More advantageous than primal methods

- Integration of the behavior law only at cell-based quadrature nodes

  $\Rightarrow$ More advantageous than discontinuous Galerkin (dG) methods

- Implementation in the open-source library *disk++*
  - code : https ://github.com/datafl4sh/diskpp

Key ideas of **H**ybrid **H**igh-**O**rder (HHO) methods

- Primal formulation with cells and faces unknowns
- Local reconstruction and stabilization
  - Symmetric gradient tensor field reconstructed in $\mathbb{P}_d^k(T; \mathbb{R}^{d\times d}_{\mathrm{sym}})$
  - Stabilization connecting cell and faces unknowns
- References
  - diffusion problem [Di Pietro, Ern, Lemaire, CMAM 14]
  - quasi-incompressible linear elasticity [Di Pietro, Ern, CMAME 15]
  - nonlinear elasticity with small def. [Botti, Di Pietro, Sochala, SINUM 17]
  - hyperelasticity with finite deformations [Abbas, Ern, Pignet, CM 18]

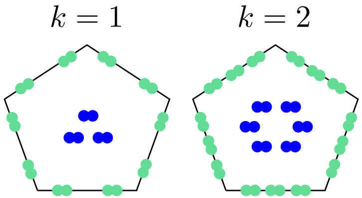


Figure 2 – Face (green) and Cell (blue) unknowns

Features of HHO methods

- Support of polytopal meshes (with possibly nonconforming interfaces)
- Arbitrary approximation order $k \geq 1$
  - h^{k+1} convergence in energy-norm
  - h^{k+2} convergence in L^2-norm with elliptic regularity
- Dimension-independent construction
- Attractive computational costs
  - Compact stencil (only neighbourhood faces)
  - Cell unknowns are eliminated locally by static condensation
  - Reduced size $N_{dofs}^{hho} \approx k^2 \mathrm{card}(\mathcal{F}^h)$ vs. $N_{dofs}^{dG} \approx k^3 \mathrm{card}(\mathcal{T}^h)$
- Local principle of virtual work (equilibrated tractions)
- HHO methods are bridged to HDG and ncVEM
  - [Cockburn, Di Pietro, Ern 16]

Plasticity problem

- Let $\Omega_0 \in \mathbb{R}^d$ (d=2,3), be a bounded connected polytopal domain
- Let $\underline{\boldsymbol{f}}$ and $\underline{\boldsymbol{t}}$ be given volumetric and surface (on Γ_n) loads
- Let $\underline{\boldsymbol{u}}_D$ be a given imposed displacement on Γ_d
- History of the deformations : $\rightarrow$ we introduce the internal state variables $\underline{\chi}$
- For all $1 \leq n \leq N$, find $\underline{\boldsymbol{u}}^n \in V_\mathrm{D} := \{\underline{\boldsymbol{v}} \in H^1(\Omega_0; \mathbb{R}^d) \,|\, \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}_D \text{ on } \Gamma_d\}$ s.t.

$$\int_{\Omega_0} \underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n) : \underline{\underline{\varepsilon}}(\underline{\boldsymbol{v}})\, d\Omega_0 = \int_{\Omega_0} \underline{\boldsymbol{f}}^n \cdot \underline{\boldsymbol{v}}\, d\Omega_0 + \int_{\Gamma_n} \underline{\boldsymbol{t}}^n \cdot \underline{\boldsymbol{v}}\, d\Gamma \text{ for all } \underline{\boldsymbol{v}} \in V_0,$$

and

$$\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n) = \mathrm{PLASTICITY}(\underline{\chi}, \underline{\underline{\varepsilon}}(\underline{\boldsymbol{u}}^{n-1}), \underline{\underline{\varepsilon}}(\underline{\boldsymbol{u}}^n)).$$

where PLASTICITY is a generic behavior integrator

Local DOFs space

- Let $M^h := (\mathcal{T}^h, \mathcal{F}^h)$ be a mesh of Ω_0 with $\mathcal{T}^h$ the set of cells and $\mathcal{F}^h$ the set of faces
- Let a polynomial degree $k \geq 1$, for all $T \in \mathcal{T}^h$

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underbrace{\mathbb{P}_d^k(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local faces dofs}}.$$

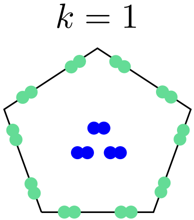


Figure 3 – Local DOFs for $k = 1, 2$. Cell unknowns are eliminated by static condensation

Symmetric strain reconstruction + stabilization

$$\underline{\underline{\boldsymbol{E}}}_T^k : \mathbb{P}_d^k(T;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d) \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}_{\mathrm{sym}}^{d\times d})}_{\text{local strain space}}$$

- The reconstructed strain $\underline{\underline{\boldsymbol{E}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})$ solves, $\forall \underline{\underline{\boldsymbol{\tau}}} \in \mathbb{P}_d^k(T;\mathbb{R}_{\mathrm{sym}}^{d\times d})$

$$(\underline{\underline{\boldsymbol{E}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}), \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} = (\underline{\underline{\boldsymbol{\nabla}}}^{\mathrm{sym}} \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}})_{\underline{\underline{\boldsymbol{L}}}^2(T)} + (\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_T, \underline{\underline{\boldsymbol{\tau}}}\, \underline{\boldsymbol{n}}_T)_{\underline{\boldsymbol{L}}^2(\partial T)}.$$

  - local scalar mass-matrix of size $\binom{k+d}{k}$ to invert (ex : $k = 2$, $d = 3$, size $= 10$)

- We penalize the difference between the faces unknowns and the trace of the cell unknowns : $\underline{\boldsymbol{\theta}} := \underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T} \in \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d)$,
  - Stabilization operator : $\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{\theta}}) \in \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d)$
  - Different to the HDG-stabilization operator

Global discrete problem

For all $1 \leq n \leq N$, find $(\underline{\boldsymbol{u}}^n_{\mathcal{T}^h}, \underline{\boldsymbol{u}}^n_{\mathcal{F}^h}) \in \left\{ \bigtimes_{T \in \mathcal{T}^h} \mathbb{P}^k_d(T; \mathbb{R}^d) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}^h} \mathbb{P}^k_{d-1}(F; \mathbb{R}^d) \right\}$

$$
\begin{aligned}
&\sum_{T \in \mathcal{T}^h} (\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}), \underline{\underline{\boldsymbol{E}}}^k_T(\delta \underline{\boldsymbol{v}}_T, \delta \underline{\boldsymbol{v}}_{\partial T}))_{\underline{\underline{\boldsymbol{L}}}^2(T)} \\
&+ \sum_{T \in \mathcal{T}^h} \beta h_T^{-1} (\underline{\boldsymbol{S}}^k_{\partial T}(\underline{\boldsymbol{u}}^n_{\partial T} - \underline{\boldsymbol{u}}^n_{T|\partial T}), \underline{\boldsymbol{S}}^k_{\partial T}(\delta \underline{\boldsymbol{v}}_{\partial T} - \delta \underline{\boldsymbol{v}}_{T|\partial T}))_{\underline{\boldsymbol{L}}^2(\partial T)} \\
&= RHS((\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})), \quad \forall\, (\delta \underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta \underline{\boldsymbol{v}}_{\mathcal{F}^h}).
\end{aligned}
$$

and

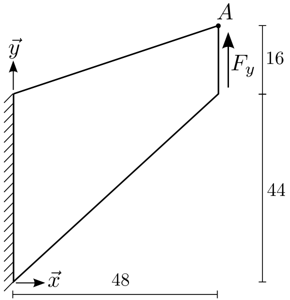
$$
\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}) = \mathrm{PLASTICITY}(\underline{\tilde{\boldsymbol{\chi}}}_T, \underline{\underline{\boldsymbol{E}}}^k_T(\underline{\boldsymbol{u}}^{n-1}_T, \underline{\boldsymbol{u}}^{n-1}_{\partial T}), \underline{\underline{\boldsymbol{E}}}^k_T(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}))
$$

with $\beta \simeq 2\mu$ an user-dependent stabilization parameter

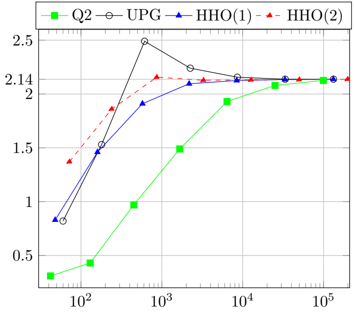
- Iterative resolution with a Newton method (SPD global system)

Numerical example : quasi-incompressible Cook's membrane

- Linear isotropic hardening with J_2-plasticity



(a) Geometry and BC.

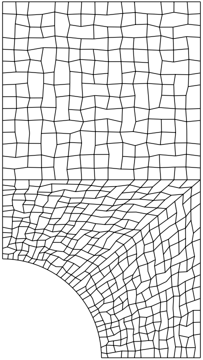


(b) Displacement of A vs. #dofs.

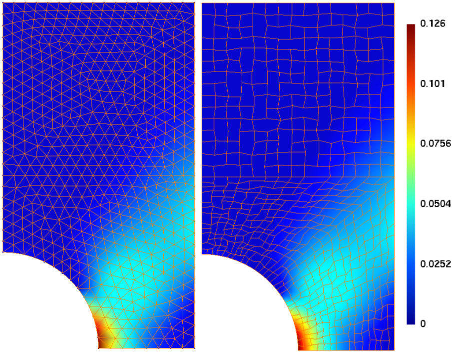
- Absence of volumetric-locking for HHO and mixed (UPG) methods

Perforated strip under uniaxial extension

- Combined linear kinematic and isotropic hardening with J_2-plasticity



(c) Polygonal mesh



(d) Equivalent plastic strain with HHO(2)

- Support of polyhedral meshes

Conclusions and perspectives

- Conclusion :
  - Adaptation of HHO methods to associative plasticity with small deformations
  - Absence of volumetric-locking

- Perspectives of this work :
  - Extension to finite plasticity
  - Introduction of contact and friction
  - Implementation in `code_aster` (in progress)

Thank you for your attention

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Reference : M. Abbas, A. Ern and NP, "A Hybrid High-Order method for incremental associative plasticity with small deformations", arXiv 18