A Hybrid High-Order method for finite elastoplastic deformations within a logarithmic strain framework

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Context

- Finite plasticity within a logarithmic strain framework
  - non-linear measure of deformations (geometric nonlinearity)
  - non-linear stress-strain constitutive relation (material nonlinearity)
  - history of the deformations (irreversible phenomena)
- Presence of volumetric locking with primal H^1-conforming formulation due to plastic incompressibility
- An alternative : using mixed methods but more unknowns, more expensive to build, saddle-point problem to solve ...

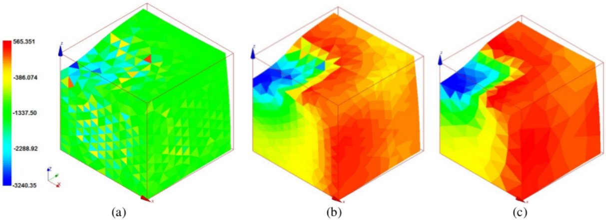


FIGURE 1 – Trace of the stress tensor for (a) P1 (b) P2 (c) P2/P1/P1

Main features of HHO methods

- Primal formulation

  $\Rightarrow$ More advantageous than mixed methods

- Abscence of volumetric locking

  $\Rightarrow$ More advantageous than primal FE methods

- Integration of the behavior law only at cell-based quadrature nodes

  $\Rightarrow$ More advantageous than discontinuous Galerkin (dG) methods

- Symmetric tangent matrix at each nonlinear solver iteration

  $\Rightarrow$ More advantageous than discontinuous Galerkin (dG) methods

- Implementation in the open-source libraries `disk++` and `code_aster`
  - https://github.com/wareHHOuse/diskpp
  - https://www.code-aster.org

Bibliography overview

Some references on primal formulations for finite plasticity without volumetric locking

- discontinuous Galerkin (dG)
  - [Liu, Wheeler, Dawson, Dean 13]
  - [Mc Bride, Reddy 09]
- Hybrid Methods
  - [Wulfinghoff, Bayat, Alipour, Reese 17]
  - [Krämer, Wieners, Wohlmuth, Wunderlich 16]
- Virtual Element Method (VEM)
  - [Chi, Beirão da Veiga, Paulino 17]
  - [Hudobivnik, Aldakheel, Wriggers 19]

Key ideas of **H**ybrid **H**igh-**O**rder (HHO) methods

- Primal formulation with cells and faces unknowns (poly. of order $k \geq 1$)
- Local reconstruction and stabilization
  - Gradient tensor field reconstructed in $\mathbb{P}_d^k(T; \mathbb{R}^{d \times d})$
  - Stabilization connecting cell and faces unknowns
- References
  - diffusion problem [Di Pietro, Ern, Lemaire, CMAM 14]
  - quasi-incompressible linear elasticity [Di Pietro, Ern, CMAME 15]
  - hyperelasticity with large deformations [Abbas, Ern, NP, CM 18]
  - plasticity with small deformations [Abbas, Ern, NP, CMAME 19]

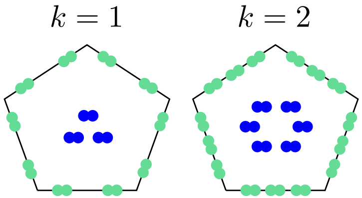


FIGURE 2 – face (green) and cell (blue) unknowns (2D)

Features of HHO methods

- Support of polyhedral meshes (with possibly nonconforming interfaces)
- Arbitrary approximation order $k \geq 1$
  - h^{k+1} convergence in energy-norm (linear elasticity)
  - h^{k+2} convergence in L^2-norm with elliptic regularity
- Attractive computational costs
  - cell unknowns are eliminated locally by static condensation
  - compact stencil for globally coupled face unknowns (only neighboring faces)
  - reduced size $N_{dofs}^{hho} \approx k^2 \#(\text{faces})$ vs. $N_{dofs}^{dG} \approx k^3 \#(\text{cells})$
- Local principle of virtual work (equilibrated tractions)
- Symmetric tangent problem at each nonlinear solver iteration
- Dimension-independent construction
- HHO methods are closely related to HDG and ncVEM
  - [Cockburn, Di Pietro, AE 16]

Plasticity problem with small deformations

- Let $\Omega_0 \in \mathbb{R}^d$ (d=2,3), be a bounded connected polyhedron
- Let $\underline{\boldsymbol{f}}$ and $\underline{\boldsymbol{t}}$ be given volumetric and surface (on Γ_n) loads
- Let $\underline{\boldsymbol{u}}_d$ be a given imposed displacement (on Γ_d)
- History of the deformations $\rightarrow$ we introduce the internal state variables $\underline{\chi}$
- For all $1 \leq n \leq N$, find $\underline{\boldsymbol{u}}^n \in V_d := \{\underline{\boldsymbol{v}} \in H^1(\Omega_0; \mathbb{R}^d) \,|\, \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}_d \text{ on } \Gamma_d\}$ s.t.

$$\int_{\Omega_0} \boldsymbol{\sigma}(\underline{\boldsymbol{u}}^n) : \boldsymbol{\varepsilon}(\underline{\boldsymbol{v}})\, d\Omega_0 = \int_{\Omega_0} \underline{\boldsymbol{f}}^n \cdot \underline{\boldsymbol{v}}\, d\Omega_0 + \int_{\Gamma_n} \underline{\boldsymbol{t}}^n \cdot \underline{\boldsymbol{v}}\, d\Gamma \text{ for all } \underline{\boldsymbol{v}} \in V_0$$

and

$$\boldsymbol{\sigma}(\underline{\boldsymbol{u}}^n) = \mathrm{SMALL\_PLASTICITY}(\underline{\chi}^{n-1}, \boldsymbol{\varepsilon}(\underline{\boldsymbol{u}}^{n-1}), \boldsymbol{\varepsilon}(\underline{\boldsymbol{u}}^n))$$

where $\mathrm{SMALL\_PLASTICITY}$ is a generic behavior integrator

Local DOFs space

- Let $\mathcal{M}^h := (\mathcal{T}^h, \mathcal{F}^h)$ be a mesh of Ω_0 with $\mathcal{T}^h$ the set of cells and $\mathcal{F}^h$ the set of (planar) faces
- Let a polynomial degree $k \geq 1$; for all $T \in \mathcal{T}^h$, set

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underbrace{\mathbb{P}^k_d(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local face dofs}}.$$

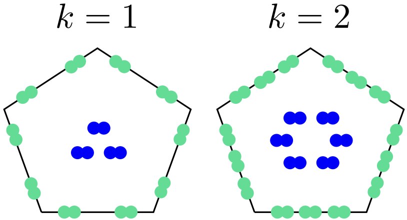


FIGURE 3 – Local DOFs for $k = 1, 2$. Cell unknowns are eliminated by static condensation

Symmetric strain reconstruction

$$\boldsymbol{E}_T^k : \mathbb{P}_d^k(T;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d) \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})}_{\text{local strain space}}$$

- The reconstructed strain $\boldsymbol{E}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T})$ solves

  $$(\boldsymbol{E}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}), \boldsymbol{\tau})_{\boldsymbol{L}^2(T)} := (\boldsymbol{\nabla}^{\mathrm{sym}} \underline{\boldsymbol{v}}_T, \boldsymbol{\tau})_{\boldsymbol{L}^2(T)} + (\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T}, \boldsymbol{\tau}\, \underline{\boldsymbol{n}}_T)_{\underline{\boldsymbol{L}}^2(\partial T)}$$

  for all $\boldsymbol{\tau} \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

  - local scalar mass-matrix of size $\binom{k+d}{k}$ (ex : $k = 2$, $d = 3 \Longrightarrow$ size $= 10$)
  - $\boldsymbol{E}_T^k$ depends only on the geometry of T (for k fixed)

Stabilization operator

- However, $\boldsymbol{E}^k_T(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) = \mathbf{0} \not\Rightarrow \underline{\boldsymbol{v}}_T = \underline{\boldsymbol{v}}_{\partial T} = \mathrm{cst}$
  $\Rightarrow$ We have to "connect" the traces of the cell unknowns to the face unknowns

- We penalize the quantity $\underline{\boldsymbol{S}}^k_{\partial T}(\underbrace{\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T}}_{:=\delta_{\partial T}}) \in \mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)$ s.t.

$$\underline{\boldsymbol{S}}^k_{\partial T}(\delta_{\partial T}) := \underline{\boldsymbol{\Pi}}^k_{\partial T}(\underbrace{\delta_{\partial T}}_{\text{HDG term}} - \underbrace{(\boldsymbol{I}_d - \underline{\boldsymbol{\Pi}}^k_T)\underline{\boldsymbol{D}}^{k+1}_T(\underline{\mathbf{0}}, \delta_{\partial T})}_{\text{high-order correction}})$$

$\underline{\boldsymbol{\Pi}}^k_{\partial T}$: L^2-projector on $\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)$; $\underline{\boldsymbol{\Pi}}^k_T$: L^2-projector on $\mathbb{P}^k_d(T; \mathbb{R}^d)$
$\underline{\boldsymbol{D}}^{k+1}_T$: higher-order reconstructed displacement field

- Different from the HDG-stabilization operator

- The high-order correction is a distinctive feature of HHO methods ensuring high-order error estimates on polyhedral meshes and linear model problems

Global discrete problem (small deformations)

For all $1 \leq n \leq N$, find

$$(\underline{\boldsymbol{u}}^n_{\mathcal{T}^h}, \underline{\boldsymbol{u}}^n_{\mathcal{F}^h}) \in \left\{ \prod_{T \in \mathcal{T}^h} \mathbb{P}^k_d(T;\mathbb{R}^d) \right\} \times \left\{ \prod_{F \in \mathcal{F}^h} \mathbb{P}^k_{d-1}(F;\mathbb{R}^d) \right\} \text{ s.t.}$$

$$\begin{aligned}
&\sum_{T \in \mathcal{T}^h} (\boldsymbol{\sigma}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}), \boldsymbol{E}^k_T(\delta\underline{\boldsymbol{v}}_T, \delta\underline{\boldsymbol{v}}_{\partial T}))_{\boldsymbol{L}^2(T)} \\
&+ \sum_{T \in \mathcal{T}^h} \beta h_T^{-1} (\underline{\boldsymbol{S}}^k_{\partial T}(\underline{\boldsymbol{u}}^n_{\partial T} - \underline{\boldsymbol{u}}^n_{T|\partial T}), \underline{\boldsymbol{S}}^k_{\partial T}(\delta\underline{\boldsymbol{v}}_{\partial T} - \delta\underline{\boldsymbol{v}}_{T|\partial T}))_{\underline{\boldsymbol{L}}^2(\partial T)} \\
&= \sum_{T \in \mathcal{T}^h} (\underline{\boldsymbol{f}}, \delta\underline{\boldsymbol{v}}_T)_{\underline{\boldsymbol{L}}^2(T)} + \sum_{F \in \mathcal{F}^h_{b,n}} (\underline{\boldsymbol{t}}, \delta\underline{\boldsymbol{v}}_F)_{\underline{\boldsymbol{L}}^2(F)}, \quad \forall\, (\delta\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta\underline{\boldsymbol{v}}_{\mathcal{F}^h})
\end{aligned}$$

and for all the quadrature points

$$\boldsymbol{\sigma}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}) = \mathrm{SMALL\_PLASTICITY}(\underline{\boldsymbol{\chi}}^{n-1}_T, \boldsymbol{E}^k_T(\underline{\boldsymbol{u}}^{n-1}_T, \underline{\boldsymbol{u}}^{n-1}_{\partial T}), \boldsymbol{E}^k_T(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}))$$

with $\beta \simeq 2\mu$ the stabilization parameter

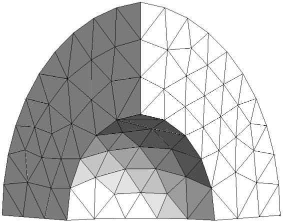
Numerical examples

- Nonlinear problem to solve (material nonlinearity)
- Iterative resolution with Newton's method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Implementation in the open-source libraries `disk++` and `code_aster`
- Verification on analytical solution :
  - Absence of volumetric locking due to plastic incompressibility
- Comparison to P^2 and $P^2/P^1/P^1$ (UPG) solutions [Al Akhrass et al. 2014]

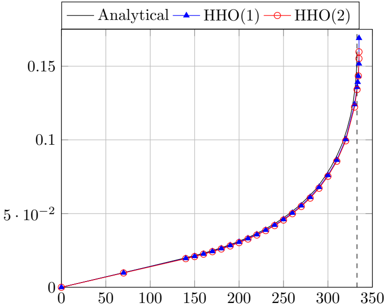
code_aster

Sphere under internal pressure I (small def.)

- Perfect J_2-plasticity
- Increase the internal pressure until the limit load
- Analytical solution available



(a) Mesh



(b) Radial displ. vs. internal pressure

Sphere under internal pressure II (small def.)

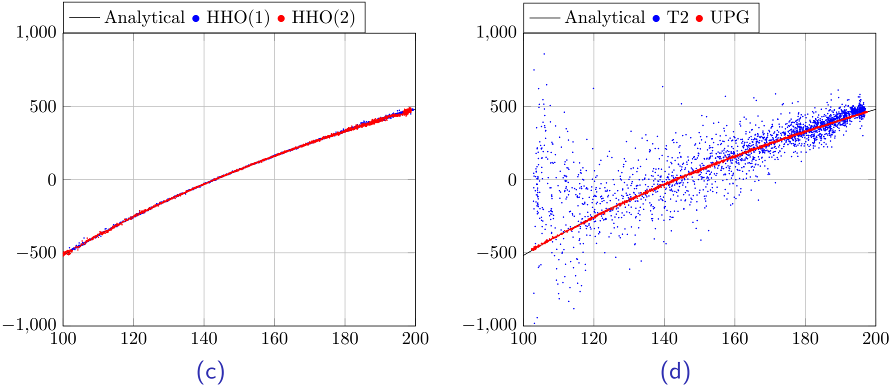


FIGURE 4 – Trace of the stress tensor at the quadrature points at the limit load

- Absence of volumetric locking for HHO and mixed (UPG) methods

Extension to finite deformations

- Extension to finite deformations using the logarithmic strain framework
- Logarithmic strain tensor $\boldsymbol{E}^{\log} = \frac{1}{2} \ln \boldsymbol{F}^T \boldsymbol{F} \in \mathbb{R}^{d \times d}_{\text{sym}}$
- Additive decomposition (elastic $\boldsymbol{E}^{\log,\text{e}}$ and plastic $\boldsymbol{E}^{\log,\text{p}}$ parts)

$$\boldsymbol{E}^{\log} = \boldsymbol{E}^{\log,\text{e}} + \boldsymbol{E}^{\log,\text{p}}$$

Algorithm 1 Computation of $\boldsymbol{P}^{\text{new}}$ (given $\underline{\chi}, \boldsymbol{F}, \boldsymbol{F}^{\text{new}}$)

1: **procedure** $\text{FINITE\_PLASTICITY}(\underline{\chi}, \boldsymbol{F}, \boldsymbol{F}^{new})$
2: Set $\boldsymbol{E}^{\log} = \frac{1}{2} \ln(\boldsymbol{F}^T \boldsymbol{F})$ and $\boldsymbol{E}^{\log,\text{new}} = \frac{1}{2} \ln(\boldsymbol{F}^{\text{new},T} \boldsymbol{F}^{\text{new}})$
3: Compute $\boldsymbol{T}^{\text{new}} = \text{SMALL\_PLASTICITY}(\underline{\chi}, \boldsymbol{E}^{\log}, \boldsymbol{E}^{\log,\text{new}})$.
4: **return** $\boldsymbol{P}^{\text{new}} = \boldsymbol{T}^{\text{new}} : (\partial_{\boldsymbol{F}} \boldsymbol{E}^{\log})^{\text{new}}$
5: **end procedure**

- For HHO methods, the only modification is the gradient reconstruction $\boldsymbol{G}^k_T \in \mathbb{P}^k_d(T; \mathbb{R}^{d \times d})$ (to replace $\boldsymbol{E}^k_T \in \mathbb{P}^k_d(T; \mathbb{R}^{d \times d}_{\text{sym}})$)

Global discrete problem (finite deformations)

For all $1 \leq n \leq N$, find

$$(\underline{\boldsymbol{u}}^n_{\mathcal{T}^h}, \underline{\boldsymbol{u}}^n_{\mathcal{F}^h}) \in \left\{ \prod_{T\in\mathcal{T}^h} \mathbb{P}^k_d(T;\mathbb{R}^d) \right\} \times \left\{ \prod_{F\in\mathcal{F}^h} \mathbb{P}^k_{d-1}(F;\mathbb{R}^d) \right\} \text{ s.t.}$$

$$\begin{aligned}
&\sum_{T\in\mathcal{T}^h} (\boldsymbol{P}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}), \boldsymbol{G}^k_T(\delta\underline{\boldsymbol{v}}_T, \delta\underline{\boldsymbol{v}}_{\partial T}))_{\boldsymbol{L}^2(T)} \\
&+ \sum_{T\in\mathcal{T}^h} \beta h_T^{-1} (\underline{\boldsymbol{S}}^k_{\partial T}(\underline{\boldsymbol{u}}^n_{\partial T} - \underline{\boldsymbol{u}}^n_{T|\partial T}), \underline{\boldsymbol{S}}^k_{\partial T}(\delta\underline{\boldsymbol{v}}_{\partial T} - \delta\underline{\boldsymbol{v}}_{T|\partial T}))_{\underline{\boldsymbol{L}}^2(\partial T)} \\
&= \sum_{T\in\mathcal{T}^h} (\underline{\boldsymbol{f}}, \delta\underline{\boldsymbol{v}}_T)_{\underline{\boldsymbol{L}}^2(T)} + \sum_{F\in\mathcal{F}^h_{b,n}} (\underline{\boldsymbol{t}}, \delta\underline{\boldsymbol{v}}_F)_{\underline{\boldsymbol{L}}^2(F)}, \quad \forall\, (\delta\underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta\underline{\boldsymbol{v}}_{\mathcal{F}^h})
\end{aligned}$$

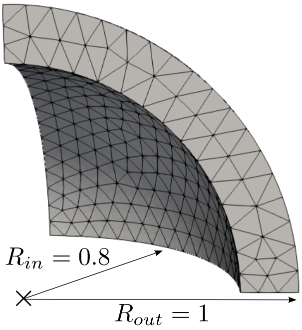
and for all the quadrature points

$$\boldsymbol{P}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}) = \mathrm{FINITE\_PLASTICITY}(\underline{\boldsymbol{\chi}}^{n-1}_T, \boldsymbol{F}^k_T(\underline{\boldsymbol{u}}^{n-1}_T, \underline{\boldsymbol{u}}^{n-1}_{\partial T}), \boldsymbol{F}^k_T(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}))$$

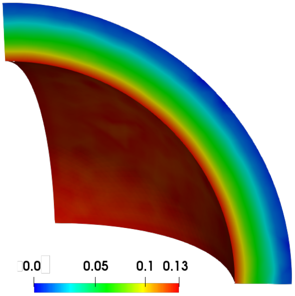
with $\beta \simeq 2\mu$ the stabilization parameter and $\boldsymbol{F}^k_T = \boldsymbol{G}^k_T + \boldsymbol{I}_d$

Quasi-incompressible sphere under internal pressure I

- Perfect J_2-plasticity ($\nu = 0.499$)
- Increase the internal pressure until the limit load
- Analytical solution available



(a) 1580 tetrahedra



(b) Equivalent plastic strain p - HHO(1)

Quasi-incompressible sphere under internal pressure II

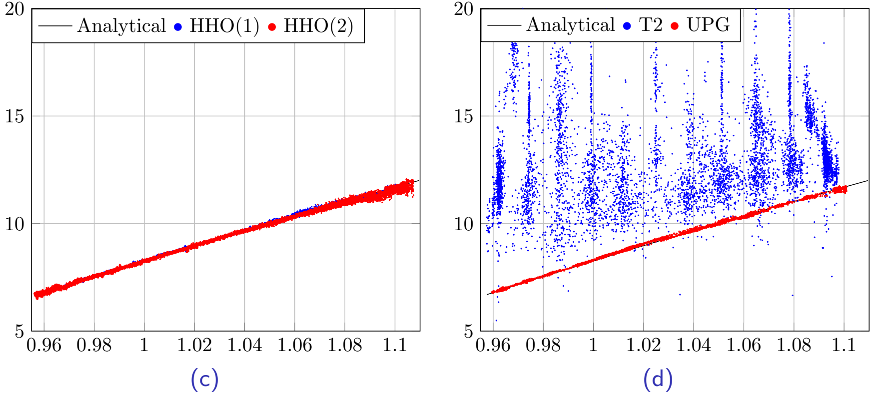
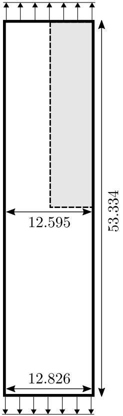


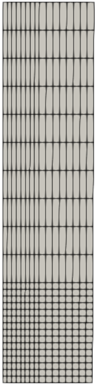
FIGURE 5 – Trace of the stress tensor at the quadrature points at the limit load

$\Rightarrow$ Absence of volumetric locking for HHO and mixed (UPG) methods

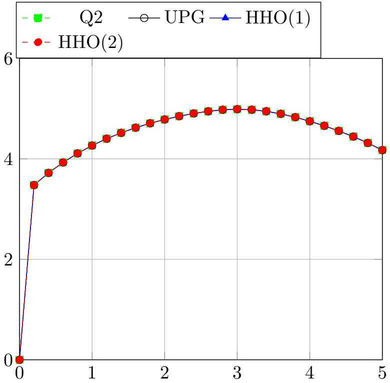
- Nonlinear isotropic hardening with J_2-plasticity



(a) Geometry

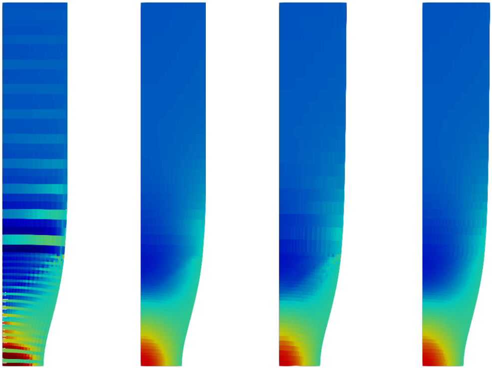


(b) Mesh



(c) Reaction vs. displacement

Necking of a 2D rectangular bar II



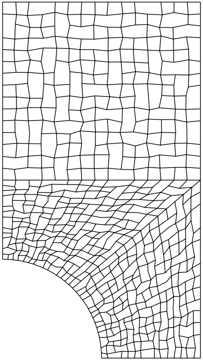
(d) Q2 (e) UPG (f) HHO(1) (g) HHO(2)

FIGURE 6 – Trace of the Cauchy stress tensor $\boldsymbol{\sigma}$ at the quadrature points on the final configuration.

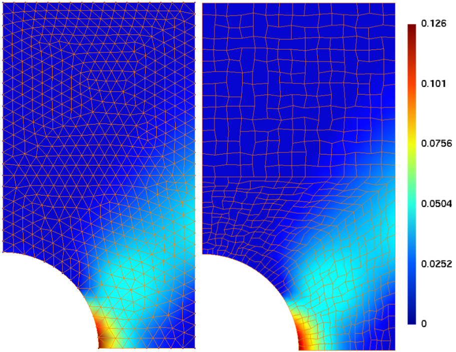
$\Rightarrow$ Absence of volumetric-locking for HHO and UPG methods

Perforated strip under uniaxial extension

- Combined linear kinematic and isotropic hardening with J_2-plasticity



(a) Polygonal mesh



(b) Equivalent plastic strain with HHO(2)

$\Rightarrow$ HHO supports polyhedral meshes

Conclusions and perspectives

- Conclusions :
  - HHO methods for finite plasticity (easy extension from small deformations)
  - Primal formulation
  - Absence of volumetric locking

- Perspectives :
  - Introduction of contact and friction using Nitsche's method (with F. Chouly)
  - Industrial applications with `code_aster`

- References :

  - M. Abbas, A. Ern, NP "A Hybrid High-Order method for incremental associative plasticity with small deformations", CMAME : 346 (2019) 891–912 ;
  - M. Abbas, A. Ern, NP, "A Hybrid High-Order method for finite elastoplastic deformations within a logarithmic strain framework", arXiv 1901.04480

Thank you for your attention