Hybrid High-Order discretizations combined with Nitsche's method for contact with Tresca friction in small strain elasticity

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Context

- Linear elasticity with small deformations
- Presence of volumetric locking with primal H^1-conforming formulation in the incompressible limit
- An alternative : using mixed methods but more unknowns, more expensive to build, inf-sup condition, saddle-point problem to solve ...
- Non-linear contact and friction boundary conditions
- primal Nitsche formulation to impose weakly the contact/friction conditions (no Lagrange multiplier) [Chouly & Hild 13]

Main features of the proposed method

- Primal formulation

  $\Rightarrow$ Less globally coupled unknowns than in mixed methods and in discontinuous Galerkin (dG) methods

- Abscence of volumetric locking

  $\Rightarrow$ More advantageous than FE methods

- Optimal convergence rates in h^{k+1} in H^1-norm

  $\Rightarrow$ More advantageous than FE and mixed methods with globally coupled unknowns that are face-based polynomials of degree k

- Implementation in the open-source library `disk++` and industrial software `code_aster`
  - https://github.com/wareHHOuse/diskpp
  - https://www.code-aster.org

Key ideas of **H**ybrid **H**igh-**O**rder (HHO) methods

- Primal formulation with cells and faces unknowns (poly. of order $k \geq 1$)
- Local reconstruction and stabilization
  - Symmetric gradient tensor field reconstructed in $\mathbb{P}_d^k(T; \mathbb{R}_{\text{sym}}^{d \times d})$
  - Stabilization connecting cell and faces unknowns
- References
  - diffusion problem [Di Pietro, Ern, Lemaire, CMAM 14]
  - quasi-incompressible linear elasticity [Di Pietro, Ern, CMAME 15]
  - nonlinear elasticity with small def. [Botti, Di Pietro, Sochala, SINUM 17]
  - hyperelasticity with finite deformations [Abbas, Ern, NP, CM 18]
  - plasticity with small deformations [Abbas, Ern, NP, CMAME 19]

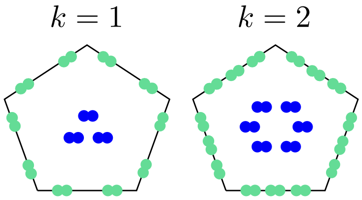


FIGURE 1 – face (green) and cell (blue) unknowns (2D)

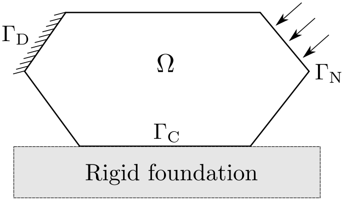
Features of HHO methods

- Support of polyhedral meshes (with possibly nonconforming interfaces)
- Arbitrary approximation order $k \geq 1$
  - h^{k+1} convergence in energy-norm
- Attractive computational costs
  - cell unknowns are eliminated locally by static condensation
  - compact stencil for globally coupled face unknowns (only neighboring faces)
  - reduced size $N_{dofs}^{hho} \approx k^2\#(\text{faces})$ vs. $N_{dofs}^{dG} \approx k^3\#(\text{cells})$
- Local principle of virtual work (equilibrated tractions)
- HHO methods are closely related to HDG and ncVEM
  - [Cockburn, Di Pietro, Ern 16]

Unilateral contact with Tresca friction

- Model problem ($2 \le d \le 3$)

$$\begin{aligned} -\underline{\nabla} \cdot \boldsymbol{\sigma}(\underline{u}) &= \underline{f} && \text{in } \Omega, \\ \boldsymbol{\varepsilon}(\underline{u}) &= \frac{1}{2}(\boldsymbol{\nabla}\underline{u} + (\boldsymbol{\nabla}\underline{u})^T) && \text{in } \Omega, \\ \boldsymbol{\sigma}(\underline{u}) &= 2\mu\,\boldsymbol{\varepsilon}(\underline{u}) + \lambda(\nabla\cdot\underline{u})\boldsymbol{I}_d && \text{in } \Omega, \\ \underline{u} &= \underline{0} && \text{on } \Gamma_{\mathrm{D}}, \\ \boldsymbol{\sigma}(\underline{u})\underline{n} &= \underline{t}_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}}, \end{aligned}$$



- Unilateral contact conditions on Γ_{C}

$$\begin{aligned} u_n &\le 0, \\ \sigma_n(\underline{u}) &\le 0 \\ \sigma_n(\underline{u})\, u_n &= 0. \end{aligned}$$

- Tresca conditions on Γ_{C} (s : given threshold)

$$\begin{cases} |\underline{\sigma}_t(\underline{u})| \le s, & \text{if } \underline{u}_t = \underline{0}, \\ \underline{\sigma}_t(\underline{u}) = -\,s\,\dfrac{\underline{u}_t}{|\underline{u}_t|} & \text{otherwise,} \end{cases}$$

with $\underline{u} = u_n\underline{n} + \underline{u}_t$ and $\boldsymbol{\sigma}(\underline{u})\underline{n} = \sigma_n(\underline{u})\underline{n} + \underline{\sigma}_t(\underline{u})$

Reformulation of Curnier–Alart

Proposition (Reformulation of contact and friction conditions)

Let be $\gamma_0 > 0$. The contact with Tresca friction conditions can be reformulated as follows :

$$\sigma_n(\underline{u}) = \left[\sigma_n(\underline{u}) - \gamma_0\, u_n\right]_{\mathbb{R}-},$$
$$\underline{\sigma}_t(\underline{u}) = \left[\underline{\sigma}_t(\underline{u}) - \gamma_0\, \underline{u}_t\right]_s.$$

where $[\cdot]_{\mathbb{R}-}$ and $[\cdot]_s$ are projectors onto closed convex sets.

$$[x]_{\mathbb{R}-} := P_{(\mathbb{R}-,0)}(x),$$
$$[\underline{x}]_s := P_{\mathcal{B}(\underline{0},s)}(\underline{x}).$$

Nitsche-FEM discretization I

- First work for contact problem [Chouly & Hild 13]
- Conforming Nitsche-FEM discretization of the model problem

$$\begin{cases} \text{Find } \underline{u}_h \in \underline{V}_{h,0} \text{ such that} \\ a_{\gamma,h}(\underline{u}_h; \underline{v}_h) = \ell_{\gamma,h}(\underline{v}_h) \qquad \forall \underline{v}_h \in \underline{V}_{h,0}, \end{cases}$$

where $\underline{V}_{h,0}$ is a FEM-space and

$$a_{\gamma,h}(\underline{v}_h, \underline{w}_h) := \int_\Omega \boldsymbol{\sigma}(\underline{v}_h) : \varepsilon(\underline{w}_h)\, d\Omega - \int_{\Gamma_\mathrm{C}} \frac{1}{\gamma_0} \boldsymbol{\sigma}(\underline{v}_h)\underline{n} \cdot \boldsymbol{\sigma}(\underline{w}_h)\underline{n}\, d\Gamma$$

$$+ \int_{\Gamma_\mathrm{C}} \frac{1}{\gamma_0} \left[\phi_\gamma^n(\underline{v}_h)\right]_{\mathbb{R}^-} \phi_\gamma^n(\underline{w}_h)\, d\Gamma + \int_{\Gamma_\mathrm{C}} \frac{1}{\gamma_0} \left[\underline{\phi}_\gamma^t(\underline{v}_h)\right]_s \cdot \underline{\phi}_\gamma^t(\underline{w}_h)\, d\Gamma,$$

$$\ell_{\gamma,h}(\underline{w}_h) := \int_\Omega \underline{f} \cdot \underline{w}_h\, d\Omega + \int_{\Gamma_\mathrm{N}} \underline{t}_\mathrm{N} \cdot \underline{w}_h\, d\Gamma$$

with $\phi_\gamma^n(\underline{v}) := \sigma_n(\underline{v}) - \gamma_0 v_n$ and $\underline{\phi}_\gamma^t(\underline{v}) := \underline{\sigma}_t(\underline{v}) - \gamma_0 \underline{v}_t$.

Nitsche-FEM method II

- Nitsche-FEM method can be seen as a consistent penalty method

$$a_{\gamma,h}(\underline{v}_h, \underline{w}_h) := \int_\Omega \boldsymbol{\sigma}(\underline{v}_h) : \boldsymbol{\varepsilon}(\underline{w}_h)\, d\Omega - \int_{\Gamma_C} \underbrace{\frac{1}{\gamma_0}\boldsymbol{\sigma}(\underline{v}_h)\underline{n}\cdot\boldsymbol{\sigma}(\underline{w}_h)\underline{n}\, d\Gamma}_{symmetry}$$

$$\text{(Contact)} \quad + \int_{\Gamma_C} \frac{1}{\gamma_0}\left[\sigma_n(\underline{v}_h) - \gamma v_{h,n}\right]_{\mathbb{R}^-} \Big(\underbrace{\sigma_n(\underline{w}_h)}_{penalty} - \underbrace{\gamma_0 w_{h,n}}_{consistency}\Big)\, d\Gamma$$

$$\text{(Tresca)} \quad + \int_{\Gamma_C} \frac{1}{\gamma_0}\left[\underline{\sigma}_t(\underline{v}_h) - \gamma_0 \underline{v}_{h,t}\right]_s \cdot \Big(\underbrace{\underline{\sigma}_t(\underline{w}_h)}_{penalty} - \underbrace{\gamma_0 \underline{w}_{h,t}}_{consistency}\Big)\, d\Gamma$$

- Non-symmetric variant available ($\boldsymbol{\sigma}(\underline{w}_h)\underline{n} \to \theta\boldsymbol{\sigma}(\underline{w}_h)\underline{n}$, $\theta \in \{-1, 0, 1\}$)
- Scalar contact problem with Nitsche-HHO discretization [Cascavita, Chouly, Ern 19]

Local DOFs space (equal-order case)

- Let $\mathcal{M}^h := (\mathcal{T}_h, \mathcal{F}_h)$ be a mesh of Ω with $\mathcal{T}_h$ the set of cells and $\mathcal{F}_h$ the set of (planar) faces
- $\mathcal{F}_{\partial T}$ the set of faces of the cell T
- Let a polynomial degree $k \geq 1$; for all $T \in \mathcal{T}_h$, set

$$\underline{\hat{v}}_T := (\underline{v}_T, \underline{v}_{\partial T}) \in \underbrace{\mathbb{P}^k_d(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local face dofs}}.$$

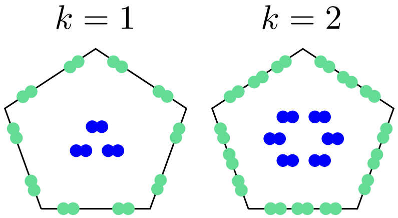


FIGURE 2 – Local DOFs for $k = 1, 2$. Cell unknowns are eliminated by static condensation

Symmetric strain reconstruction $\boldsymbol{E}_T^k$

$$\boldsymbol{E}_T^k : \mathbb{P}_d^k(T;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d) \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})}_{\text{local strain space}}$$

- The reconstructed strain $\boldsymbol{E}_T^k(\underline{v}_T, \underline{v}_{\partial T})$ solves

$$(\boldsymbol{E}_T^k(\underline{\hat{v}}_T), \boldsymbol{\tau})_{\boldsymbol{L}^2(T)} := -(\underline{v}_T, \nabla\cdot\boldsymbol{\tau})_{\underline{L}^2(T)} + (\underline{v}_{\partial T}, \boldsymbol{\tau}\,\underline{n}_T)_{\underline{L}^2(\partial T)}$$

for all $\boldsymbol{\tau} \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

  - "Mimic" integration by parts
  - local scalar mass-matrix of size $\binom{k+d}{k}$ (ex : $k = 2$, $d = 3 \Longrightarrow$ size $= 10$)
  - $\boldsymbol{E}_T^k$ depends only on the geometry of T (for k fixed)

Stabilization operator $\underline{S}^k_{\partial T}$

- However, $\boldsymbol{E}^k_T(\hat{\underline{v}}_T) = \mathbf{0} \not\Rightarrow \underline{v}_T = \underline{v}_{\partial T} = \underline{\mathrm{cst}}$
  $\Rightarrow$ We have to "connect" the trace of the cell unknowns to the face unknowns

- We penalize the quantity $\underline{S}^k_{\partial T}(\underbrace{\underline{v}_{\partial T} - \underline{v}_{T|\partial T}}_{:=\delta_{\partial T}\hat{\underline{v}}_T}) \in \mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T};\mathbb{R}^d)$ s.t.

$$\underline{S}^k_{\partial T}(\delta_{\partial T}\hat{\underline{v}}_T) := \underline{\Pi}^k_{\partial T}(\underbrace{\delta_{\partial T}\hat{\underline{v}}_T}_{\text{HDG term}} - \underbrace{(\boldsymbol{I}_d - \underline{\Pi}^k_T)\underline{D}^{k+1}_T(\underline{0}, \delta_{\partial T}\hat{\underline{v}}_T)}_{\text{high-order correction}})$$

$\underline{\Pi}^k_{\partial T}$: L^2-projector on $\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T};\mathbb{R}^d)$; $\underline{\Pi}^k_T$: L^2-projector on $\mathbb{P}^k_d(T;\mathbb{R}^d)$
$\underline{D}^{k+1}_T$: higher-order reconstructed displacement field

- Different from the HDG-stabilization operator

- The high-order correction is a distinctive feature of HHO methods ensuring high-order error estimates on polyhedral meshes and linear model problems

Local Galerkin contribution

- Local Galerkin contribution

$$\hat{a}_T^G(\underline{\hat{v}}_T, \underline{\hat{w}}_T) := 2\mu(\boldsymbol{E}_T^k(\underline{\hat{v}}_T), \boldsymbol{E}_T^k(\underline{\hat{w}}_T))_{\boldsymbol{L}^2(T)} + \lambda(D_T^k(\underline{\hat{v}}_T), D_T^k(\underline{\hat{w}}_T))_{L^2(T)}$$
$$+ 2\mu h_T^{-1}(\underline{S}_{\partial T}^k(\delta_{\partial T}\underline{\hat{v}}_T), \underline{S}_{\partial T}^k(\delta_{\partial T}\underline{\hat{w}}_T))_{\underline{L}^2(\partial T)},$$

with the discrete divergence $D_T^k(\underline{\hat{v}}_T) := \mathrm{trace}(\boldsymbol{E}_T^k(\underline{\hat{v}}_T)) \in \mathbb{P}_d^k(T;\mathbb{R})$.

- Local stress reconstruction

$$\boldsymbol{\sigma}_T^k(\underline{\hat{v}}_T) := 2\mu \boldsymbol{E}_T^k(\underline{\hat{v}}_T) + \lambda D_T^k(\underline{\hat{v}}_T)\boldsymbol{I}_d \in \mathbb{P}_d^k(T;\mathbb{R}_{\mathrm{sym}}^{d\times d})$$

Local contact/friction contributions

- $T(F)$ is the unique cell which contain the contact face F
- Use face-based Nitsche-HHO : $\underline{v}_h \to \underline{v}_F$
- Local contact contribution on a contact face ($\gamma = \gamma_0 h_F^{-1}$)

$$\begin{aligned}\hat{a}_F^N(\underline{\hat{v}}_{T(F)}, \underline{\hat{w}}_{T(F)}) := &- \gamma_0^{-1} h_F \left(\boldsymbol{\sigma}_{T(F)}^k(\underline{\hat{v}}_{T(F)})\underline{n}_{TF}, \boldsymbol{\sigma}_{T(F)}^k(\underline{\hat{w}}_{T(F)})\underline{n}_{TF}\right)_{\underline{L}^2(F)} \\ &+ \gamma_0^{-1} h_F \left(\left[\hat{\phi}_\gamma^n(\underline{\hat{v}}_{T(F)})\right]_{\mathbb{R}^-}, \hat{\phi}_\gamma^n(\underline{\hat{w}}_{T(F)})\right)_{L^2(F)} \\ &+ \gamma_0^{-1} h_F \left(\left[\underline{\hat{\phi}}_\gamma^t(\underline{\hat{v}}_{T(F)})\right]_s, \underline{\hat{\phi}}_\gamma^t(\underline{\hat{w}}_{T(F)})\right)_{\underline{L}^2(F)}\end{aligned}$$

with

$$\begin{aligned}\hat{\phi}_\gamma^n(\underline{\hat{v}}_{T(F)}) &:= \sigma_{n,T(F)}^k(\underline{\hat{v}}_{T(F)}) - \gamma_0 h_F^{-1} v_{F,n} \in \mathbb{P}_{d-1}^k(F; \mathbb{R}) \\ \underline{\hat{\phi}}_\gamma^t(\underline{\hat{v}}_{T(F)}) &:= \underline{\sigma}_{t,T(F)}^k(\underline{\hat{v}}_{T(F)}) - \gamma_0 h_F^{-1} \underline{v}_{F,t} \in \mathbb{P}_{d-1}^k(F; \mathbb{R}^{d-1})\end{aligned}$$

Global discrete problem and well-posedness

- Global discrete problem

$$\begin{cases} \text{Find } \underline{\hat{u}}_h \in \mathbb{P}_d^k(\mathcal{T}_h;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h;\mathbb{R}^d)), \text{ s.t.} \\ \hat{a}_{\gamma,h}(\underline{\hat{v}}_h, \underline{\hat{w}}_h) = \hat{\ell}_{\gamma,h}(\underline{\hat{w}}_h) \qquad \forall \underline{v}_h \in \underline{\hat{U}}_{h,0}^k, \end{cases}$$

with

$$\hat{a}_{\gamma,h}(\underline{\hat{v}}_h, \underline{\hat{w}}_h) := \sum_{T\in\mathcal{T}_h} \underbrace{\hat{a}_T^G(\underline{\hat{v}}_T, \underline{\hat{w}}_T)}_{\text{Galerkin}} + \sum_{F\in\mathcal{F}_h^{\mathrm{b,C}}} \underbrace{\hat{a}_F^N(\underline{\hat{v}}_{T(F)}, \underline{\hat{w}}_{T(F)})}_{\text{Nitsche}}$$

$$\hat{\ell}_{\gamma,h}(\underline{\hat{w}}_h) := \sum_{T\in\mathcal{T}_h} (\underline{f}, \underline{w}_T)_{\underline{L}^2(T)} + \sum_{F\in\mathcal{F}_h^{\mathrm{b,N}}} (\underline{t}_{\mathrm{N}}, \underline{w}_F)_{\underline{L}^2(F)}.$$

- The discrete problem is well-posed if γ_0 is large enough
- Sub-Optimal convergence rates in H^1-norm ($O(h^k)$) [Cascavita, Chouly, Ern 19]

A new idea to recover optimal convergence

- Where is the problem in the analysis ?

$$\sum_{F \in \mathcal{F}_h^{\mathrm{b,C}}} h_F^{-1} \| \underline{u} - \underline{\Pi}_F^k(\underline{u}) \|_{\underline{L}^2(F)}^2 \lesssim h^{2k} |\underline{u}|_{\underline{H}^{k+1}(\mathcal{T}_h)}$$

- In [Cascavita, Chouly, Ern 19], this is cured by increasing the degree of the cell unknowns (see also [Burman, Ern 18], but this trick here breaks the robustness w.r.t. incompressibility

- The new idea is to use face unknowns of degree $k+1$ on the contact faces $\mathcal{F}_h^{\mathrm{b,C}}$

- Increase slightly the total number of face dofs

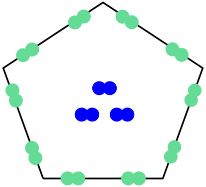
- Minor modifications of the HHO operators $\boldsymbol{E}_T^k$ and $\underline{S}_{\partial T}^k$

Local DOFs space (contact modifications)

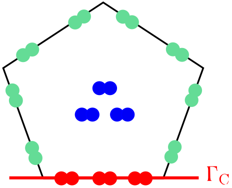
- $\underbrace{\mathcal{F}_{\partial T}}_{\text{faces of T}} = \underbrace{\mathcal{F}_{\partial T}^{\setminus}}_{\text{other faces of T}} \cup \underbrace{\mathcal{F}_{\partial T}^{\mathrm{b,C}}}_{\text{contact faces of T}}$
- Let a polynomial degree $k \geq 1$; for all $T \in \mathcal{T}_h$, set

$$\hat{\underline{v}}_T := (\underline{v}_T, \underline{v}_{\partial T}) \in \underbrace{\mathbb{P}_d^k(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}_{d-1}^{k/k+1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local face dofs}}.$$

with $\mathbb{P}_{d-1}^{k/k+1}(\mathcal{F}_{\partial T}; \mathbb{R}^d) := \mathbb{P}_{d-1}^{k}(\mathcal{F}_{\partial T}^{\setminus}; \mathbb{R}^d) \times \mathbb{P}_{d-1}^{k+1}(\mathcal{F}_{\partial T}^{\mathrm{b,C}}; \mathbb{R}^d)$



(a) Cell without contact face



(b) Cell with contact face

FIGURE 3 – Local DOFs for $k = 1$. Cell unknowns are eliminated by static condensation

Global discrete problem and well-posedness

- Global discrete problem

$$\begin{cases} \text{Find } \hat{\underline{u}}_h \in \mathbb{P}_d^k(\mathcal{T}_h;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\mathrm{b,D}} \cup \mathcal{F}_h^{\mathrm{b,N}};\mathbb{R}^d) \times \mathbb{P}_{d-1}^{k+1}(\mathcal{F}_h^{\mathrm{b,C}};\mathbb{R}^d), \text{ s.t.} \\ \hat{a}_{\gamma,h}(\hat{\underline{v}}_h, \hat{\underline{w}}_h) = \hat{\ell}_{\gamma,h}(\hat{\underline{w}}_h) \qquad \forall \underline{v}_h \in \hat{\underline{U}}_{h,0}^k, \end{cases}$$

with

$$\hat{\phi}_\gamma^n(\hat{\underline{w}}_{T(F)}) := \sigma_{n,T(F)}^k(\hat{\underline{w}}_{T(F)}) - \gamma_0 h_F^{-1} w_{F,n} \in \mathbb{P}_{d-1}^{k+1}(F;\mathbb{R})$$
$$\hat{\underline{\phi}}_\gamma^t(\hat{\underline{w}}_{T(F)}) := \underline{\sigma}_{t,T(F)}^k(\hat{\underline{w}}_{T(F)}) - \gamma_0 h_F^{-1} \underline{w}_{F,t} \in \mathbb{P}_{d-1}^{k+1}(F;\mathbb{R}^{d-1})$$

- The discrete problem is well-posed if γ_0 is large enough

- We have now this term to bound on the contact faces

$$\sum_{F \in \mathcal{F}_h^{\mathrm{b,C}}} h_F^{-1} \|\underline{u} - \underline{\Pi}_F^{k+1}(\underline{u})\|_{\underline{L}^2(F)}^2 \lesssim h^{2k+2} |\underline{u}|_{\underline{H}^{k+2}(\mathcal{T}_h)}^2$$

a priori H^1-error estimate

Theorem (H^1-error estimate)

Let $k \geq 1$. Assume that $\underline{u} \in H^{1+\nu}(\Omega;\mathbb{R}^d)$ with $\nu > \frac{1}{2}$ and γ_0 large enough. Then, there exists $C > 0$ independent of h, μ, λ and γ_0 such that

$$\sum_{T\in\mathcal{T}_h} \|\varepsilon(\underline{u} - \underline{D}_T^{k+1}(\underline{\hat{u}}_T))\|^2_{\boldsymbol{L}^2(T)} \leq C \sum_{T\in\mathcal{T}_h} h_T^{2t} \left\{ (2\mu)^2 |\underline{u}|^2_{\underline{H}^{1+t}(T)} + \lambda^2 |\nabla\cdot\underline{u}|^2_{H^t(T)} \right\}.$$

with $t := \min(k+1, \nu)$

- The maximum regularity is $H^{\frac{5}{2}}(\mathcal{T}_h;\mathbb{R}^d)$ if transition between contact and no contact

Numerical examples

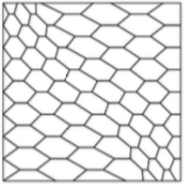
- Nonlinear problem to solve (contact nonlinearity)
- Iterative resolution with Newton's method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Implementation in the open-source libraries `disk++` and `code_aster`
- Verification on analytical solution :
  - Optimal convergence rates in H^1-norm
  - Absence of volumetric locking in the incompressible limit
- Comparison to mixed methods [Bostan & Han 06]

Manufactured solution

- $\Omega = (0,1)^2$ with $\Gamma_{\mathrm{C}} = (0,1) \times \{0\}$ and $\Gamma_{\mathrm{D}} = \{0,1\} \times (0,1) \cup (0,1) \times \{1\}$
- Manufactured solution

$$u_x = \left(1 + \frac{1}{1+\lambda}\right) x e^{x+y}, \quad u_y = \left(-1 + \frac{1}{1+\lambda}\right) y e^{x+y}.$$

- Friction coefficient $s = \frac{\mu}{6} x^2 \frac{\lambda+2}{\lambda+1}$
- Test on hexagonal meshes (convergence and robustness)



Manufactured solution : convergence rates

- Test for $k = 1$ and $k = 2$
- Lamé parameter $\mu = 2$ and $\lambda = 2000$
- Optimal convergence rates in H^1-norm

h	$k = 1$		$k = 2$	
	H^1-error	Order	H^1-error	Order
3.33e-1	5.423e-3	-	4.406e-4	-
1.75e-1	1.380e-3	2.13	5.871e-5	3.13
9.06e-2	3.472e-4	2.08	7.618e-6	3.07
4.60e-2	8.694e-5	2.05	9.720e-7	3.04

TABLE 1 – 2D manufactured solution : H^1-error vs. h.

Manufactured solution : robustness

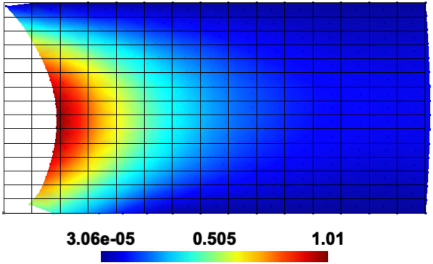
- Test for $k = 1$ and $k = 2$
- Fixed hexagonal mesh
- Absence of volumetric locking in the incompressible limit

λ	ν	$k = 1$	$k = 2$
1	0.17	1.261e-4	1.390e-6
10	0.42	9.276e-5	1.032e-6
100	0.49	8.753e-5	9.776e-7
1000	0.499	8.699e-5	9.727e-7
10000	0.4999	8.694e-5	9.742e-7

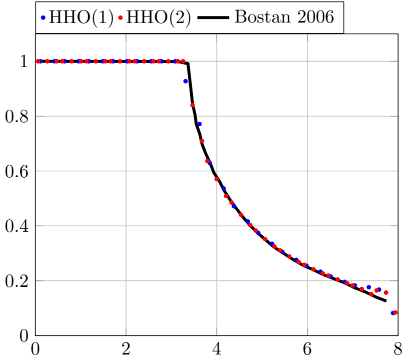
TABLE 2 – 2D manufactured solution : H^1-error vs. λ for a hexagonal mesh (h =4.60e-2).

Comparison with mixed method [Bostan & Han 06]

- Comparison with mixed-FEM method [Bostan & Han 06]
- Transition contact/no contact and stick/slip
- Results are in agreement with mixed-FEM method



(a) Euclidian norm of the discrete solution on the deformed configuration $k = 1$.



(b) $\|\underline{\sigma}^k_{t,T(F)}\|/s$ at the quadrature points

Conclusions and perspectives

- Conclusions
  - HHO methods for contact problem with Tresca friction
  - Primal formulation
  - Optimal convergence rates in H^1-norm
  - Absence of volumetric locking
- Perspectives
  - a priori analysis for Coulomb friction
  - Plastic behavior law

Thank you for your attention