A Hybrid High-Order method for incremental associative plasticity with small deformations

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Context

- Associative plasticity with small deformations
  - non-linear stress-strain constitutive relation (material nonlinearity)
  - history of the deformations (irreversible phenomena)
- Presence of volumetric-locking with primal H^1-conforming formulation due to plastic incompressibility
- An alternative : using mixed methods but more unknowns, more expensive to build, saddle-point problem to solve ...

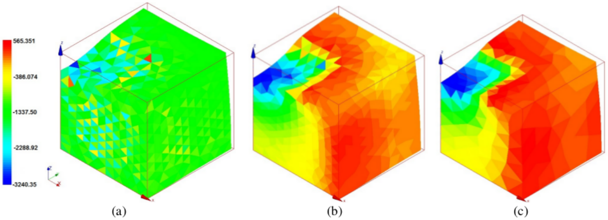


Figure 1 – Trace of the stress tensor for (a) P1 (b) P2 (c) P2/P1/P1

Main features of the proposed method

- Primal formulation

  ⇒ More advantageous than mixed methods

- Absence of volumetric-locking

  ⇒ More advantageous than primal methods

- Integration of the behavior law only at cell-based quadrature nodes

  ⇒ More advantageous than discontinuous Galerkin (dG) methods

- Implementation in the open-source libraries *disk++* and `code_aster`
  - code : https ://github.com/wareHHOuse/diskpp
  - code : https ://www.code-aster.org

Some references for infinitesimal plasticity

Some references on primal formulations without volumetric-locking

- discontinuous Galerkin (dG)
  - [Noels, Radovitzsky 06]
  - [ten Eyck, Lew 06]
- Hybrid Methods
  - [Wulfinghoff, Bayat, Alipour, Reese 17]
  - [Krämer, Wieners, Wohlmuth, Wunderlich 16]
- Virtual Element Method (VEM)
  - [Chi, Beirão da Veiga, Paulino 17]
  - [Wriggers, Reddy, Rust, Hudobivnik 17]

Key ideas of **H**ybrid **H**igh-**O**rder (HHO) methods

- Primal formulation with cells and faces unknowns
- Local reconstruction and stabilization
  - Symmetric gradient tensor field reconstructed in $\mathbb{P}_d^k(T; \mathbb{R}_{\mathrm{sym}}^{d\times d})$
  - Stabilization connecting cell and faces unknowns
- References
  - diffusion problem [Di Pietro, Ern, Lemaire, CMAM 14]
  - quasi-incompressible linear elasticity [Di Pietro, Ern, CMAME 15]
  - nonlinear elasticity with small def. [Botti, Di Pietro, Sochala, SINUM 17]
  - hyperelasticity with finite deformations [Abbas, Ern, Pignet, CM 18]

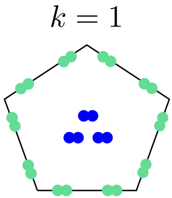


Figure 2 – Face (green) and Cell (blue) unknowns

Features of HHO methods

- Support of polytopal meshes (with possibly nonconforming interfaces)
- Arbitrary approximation order $k \geq 1$
  - h^{k+1} convergence in energy-norm
  - h^{k+2} convergence in L^2-norm with elliptic regularity
- Dimension-independent construction
- Attractive computational costs
  - Compact stencil (only neighbourhood faces)
  - Cell unknowns are eliminated locally by static condensation
  - Reduced size $N_{dofs}^{hho} \approx k^2 \mathrm{card}(\mathcal{F}^h)$ vs. $N_{dofs}^{dG} \approx k^3 \mathrm{card}(\mathcal{T}^h)$
- Local principle of virtual work (equilibrated tractions)
- HHO methods are bridged to HDG and ncVEM
  - [Cockburn, Di Pietro, Ern 16]

Plasticity problem

- Let $\Omega_0 \in \mathbb{R}^d$ (d=2,3), be a bounded connected polytopal domain
- Let $\underline{\boldsymbol{f}}$ and $\underline{\boldsymbol{t}}$ be given volumetric and surface (on Γ_n) loads
- Let $\underline{\boldsymbol{u}}_D$ be a given imposed displacement on Γ_d
- History of the deformations : $\rightarrow$ we introduce the internal state variables $\underline{\chi}$
- For all $1 \leq n \leq N$, find $\underline{\boldsymbol{u}}^n \in V_{\mathrm{D}} := \{\underline{\boldsymbol{v}} \in H^1(\Omega_0; \mathbb{R}^d) \,|\, \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}}_D \text{ on } \Gamma_d\}$ s.t.

$$\int_{\Omega_0} \underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n) : \underline{\underline{\varepsilon}}(\underline{\boldsymbol{v}})\, d\Omega_0 = \int_{\Omega_0} \underline{\boldsymbol{f}}^n \cdot \underline{\boldsymbol{v}}\, d\Omega_0 + \int_{\Gamma_n} \underline{\boldsymbol{t}}^n \cdot \underline{\boldsymbol{v}}\, d\Gamma \text{ for all } \underline{\boldsymbol{v}} \in V_0,$$

and

$$\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n) = \mathrm{PLASTICITY}(\underline{\chi}, \underline{\underline{\varepsilon}}(\underline{\boldsymbol{u}}^{n-1}), \underline{\underline{\varepsilon}}(\underline{\boldsymbol{u}}^n)).$$

where $\mathrm{PLASTICITY}$ is a generic behavior integrator

Local DOFs space

- Let $M^h := (\mathcal{T}^h, \mathcal{F}^h)$ be a mesh of Ω_0 with $\mathcal{T}^h$ the set of cells and $\mathcal{F}^h$ the set of (planar) faces
- Let a polynomial degree $k \geq 1$, for all $T \in \mathcal{T}^h$

$$(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) \in \underbrace{\mathbb{P}^k_d(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local faces dofs}}.$$

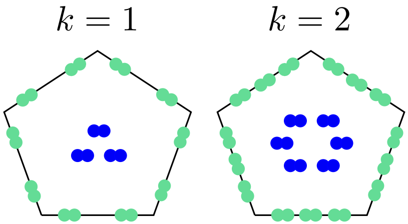


Figure 3 – Local DOFs for $k = 1, 2$. Cell unknowns are eliminated by static condensation

Symmetric strain reconstruction

$$\underline{\underline{E}}_T^k : \mathbb{P}_d^k(T;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d) \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\text{sym}})}_{\text{local strain space}}$$

- The reconstructed strain $\underline{\underline{E}}_T^k(\underline{v}_T, \underline{v}_{\partial T})$ solves, $\forall \underline{\underline{\tau}} \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\text{sym}})$

$$(\underline{\underline{E}}_T^k(\underline{v}_T, \underline{v}_{\partial T}), \underline{\underline{\tau}})_{\underline{\underline{L}}^2(T)} = (\underline{\underline{\nabla}}^{\text{sym}} \underline{v}_T, \underline{\underline{\tau}})_{\underline{\underline{L}}^2(T)} + (\underline{v}_{\partial T} - \underline{v}_T, \underline{\underline{\tau}}\, \underline{n}_T)_{\underline{L}^2(\partial T)}.$$

  - well-posed local Neumann problem
  - local scalar mass-matrix of size $\binom{k+d}{k}$ to invert (ex : $k = 2$, $d = 3$, size $= 10$)
  - $\underline{\underline{E}}_T^k$ depends only on the geometry of T (for k fixed)

Stabilization operator

- Problem : $\underline{\underline{\boldsymbol{E}}}_T^k(\underline{\boldsymbol{v}}_T, \underline{\boldsymbol{v}}_{\partial T}) = \underline{\underline{\boldsymbol{0}}} \nRightarrow \underline{\boldsymbol{v}}_T = \underline{\boldsymbol{v}}_{\partial T} = \text{cste}$

  $\Rightarrow$ We have to "connect" the cell and faces unknowns

- Hence, we penalize the difference between the faces unknowns and the trace of the cell unknowns : $\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T}) \in \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T}; \mathbb{R}^d)$,

$$\underline{\boldsymbol{S}}_{\partial T}^k(\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T}) = \underline{\boldsymbol{\Pi}}_{\partial T}^k(\underbrace{\underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T}}_{\text{HDG term}} - \underbrace{(\underline{\underline{\boldsymbol{I}}}_d - \underline{\boldsymbol{\Pi}}_T^k)\underline{\boldsymbol{D}}_T^{k+1}(\underline{\boldsymbol{0}}, \underline{\boldsymbol{v}}_{\partial T} - \underline{\boldsymbol{v}}_{T|\partial T})}_{\text{high-order correction}})$$

  where $\underline{\boldsymbol{\Pi}}_{\partial T}^k$ is the L^2-projector on ∂T, $\underline{\boldsymbol{\Pi}}_T^k$ the L^2-projector on T, and $\underline{\boldsymbol{D}}_T^{k+1}$ is a reconstructed displacement field

- Different to the HDG-stabilization operator

- The high-order correction is a distinctive feature of HHO methods

Global discrete problem

For all $1 \leq n \leq N$, find $(\underline{\boldsymbol{u}}^n_{\mathcal{T}^h}, \underline{\boldsymbol{u}}^n_{\mathcal{F}^h}) \in \left\{ \bigtimes_{T \in \mathcal{T}^h} \mathbb{P}^k_d(T; \mathbb{R}^d) \right\} \times \left\{ \bigtimes_{F \in \mathcal{F}^h} \mathbb{P}^k_{d-1}(F; \mathbb{R}^d) \right\}$

$$
\begin{aligned}
&\sum_{T \in \mathcal{T}^h} (\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}), \underline{\underline{\boldsymbol{E}}}^k_T(\delta \underline{\boldsymbol{v}}_T, \delta \underline{\boldsymbol{v}}_{\partial T}))_{\underline{\underline{\boldsymbol{L}}}^2(T)} \\
&+ \sum_{T \in \mathcal{T}^h} \beta h_T^{-1} (\underline{\boldsymbol{S}}^k_{\partial T}(\underline{\boldsymbol{u}}^n_{\partial T} - \underline{\boldsymbol{u}}^n_{T|\partial T}), \underline{\boldsymbol{S}}^k_{\partial T}(\delta \underline{\boldsymbol{v}}_{\partial T} - \delta \underline{\boldsymbol{v}}_{T|\partial T}))_{\underline{\boldsymbol{L}}^2(\partial T)} \\
&= \sum_{T \in \mathcal{T}^h} (\underline{\boldsymbol{f}}, \delta \underline{\boldsymbol{v}}_T)_{\underline{\boldsymbol{L}}^2(T)} + \sum_{F \in \mathcal{F}^h_{b,n}} (\underline{\boldsymbol{t}}, \delta \underline{\boldsymbol{v}}_F)_{\underline{\boldsymbol{L}}^2(F)}, \quad \forall (\delta \underline{\boldsymbol{v}}_{\mathcal{T}^h}, \delta \underline{\boldsymbol{v}}_{\mathcal{F}^h}).
\end{aligned}
$$

and for all quadrature points

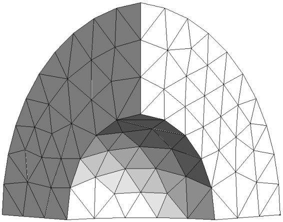
$$
\underline{\underline{\boldsymbol{\sigma}}}(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}) = \mathrm{PLASTICITY}(\underline{\boldsymbol{\chi}}_T, \underline{\underline{\boldsymbol{E}}}^k_T(\underline{\boldsymbol{u}}^{n-1}_T, \underline{\boldsymbol{u}}^{n-1}_{\partial T}), \underline{\underline{\boldsymbol{E}}}^k_T(\underline{\boldsymbol{u}}^n_T, \underline{\boldsymbol{u}}^n_{\partial T}))
$$

with $\beta \simeq 2\mu$ an user-dependent stabilization parameter

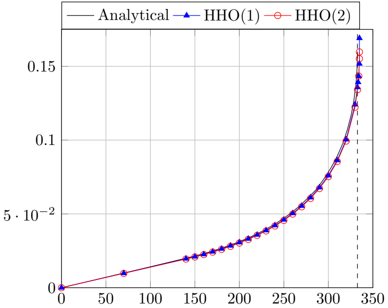
Numerical examples

- Nonlinear problem to solve (material nonlinearity)
- Iterative resolution with a Newton method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Implementation in the open-source libraries *disk++* and `code_aster`
- Verification on analytical solution :
  - Absence of volumetric-locking due to plastic incompressibility
- Comparison to P^2 and $P^2/P^1/P^1$ (UPG) methods

- Perfect J_2-plasticity
- Increase the internal pressure until the limit load



(a) Mesh.



(b) Radial displacement vs. internal pressure.

Numerical example : sphere under internal pressure II

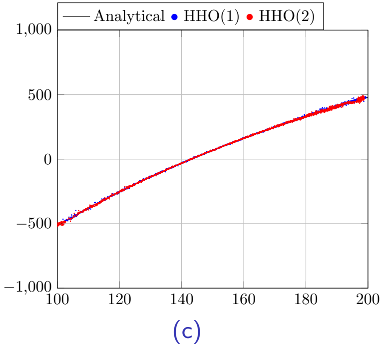
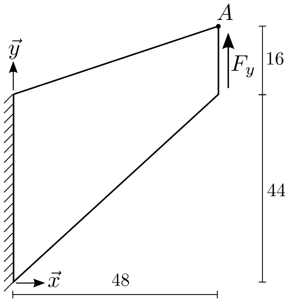


Figure 4 – Trace of the stress tensor at the quadrature points at the limit load

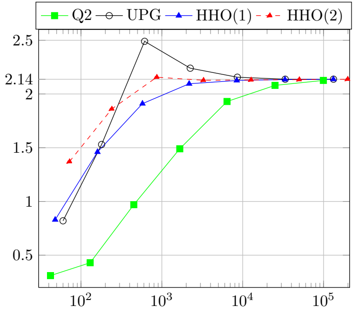
- Absence of volumetric-locking for HHO and mixed (UPG) methods

Numerical example : quasi-incompressible Cook's membrane

- Linear isotropic hardening with J_2-plasticity ($\nu = 0.4999$)



(a) Geometry and BC.

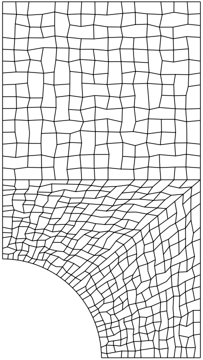


(b) Displacement of A vs. #dofs.

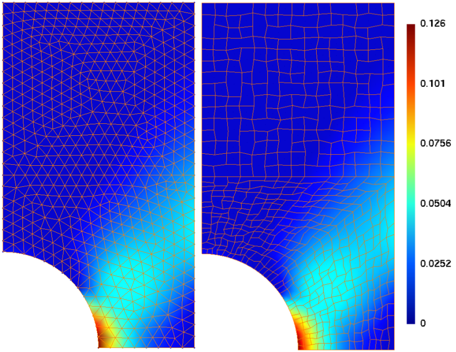
- HHO(2) out-performs compare to mixed (UPG) methods (same displacement order)

Perforated strip under uniaxial extension

- Combined linear kinematic and isotropic hardening with J_2-plasticity



(c) Polygonal mesh



(d) Equivalent plastic strain with HHO(2)

- Support of polyhedral meshes

Unstabilized HHO method on simplicial meshes

- How to choose the stabilization parameter β :
  - $\beta = 2\mu$ is the value for the linear elasticity
  - No general theory on the choice of β
  - If β is too small $\Rightarrow$ Difficulties to converge
  - If β is too large $\Rightarrow$ The system is ill-conditioned
- Original idea for dG : [John, Neilan, Smears 16]
  - Based on the properties of the Raviart–Thomas space
- Gradient reconstruction in $\mathbb{P}_d^{k+1}(T; \mathbb{R}^{d\times d}_{sym})$ (larger space)
  - ex : $k = 2$, $d = 3$, size $= 20$ for $\mathbb{P}_d^{k+1}(T; \mathbb{R}^{d\times d})$ vs 10 for $\mathbb{P}_d^{k}(T; v^{d\times d})$
- No additional stabilization is needed ($\beta \equiv 0$)
- Lower convergence rates (h^k in energy-norm and h^{k+1} in L^2-norm)
- Comparable numerical cost vs. stabilized HHO methods

Conclusions and perspectives

- Conclusion :
  - Adaptation of HHO methods to associative plasticity with small deformations
  - Primal formulation
  - Absence of volumetric-locking

- Perspectives of this work :
  - Extension to finite plasticity
  - Introduction of contact and friction using Nitsche's method (with F. Chouly)

Thank you for your attention

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Reference : M. Abbas, A. Ern and NP, "A Hybrid High-Order method for incremental associative plasticity with small deformations", Comput. Methods. Appl. Mech. Engrg. (2018)