Hybrid High-Order methods for nonlinear solid mechanics

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22.10.19

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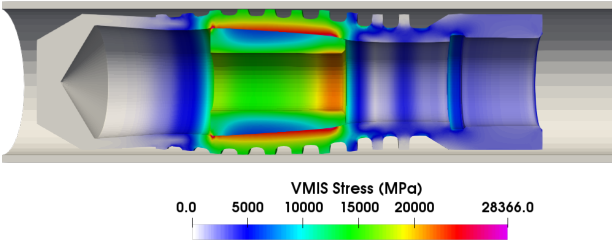
Outline

1. Industrial context
2. Introduction to Hybrid High-Order methods (HHO)
3. Contact and Tresca friction
4. Plasticity in small and finite deformations
5. Conclusions and perspectives

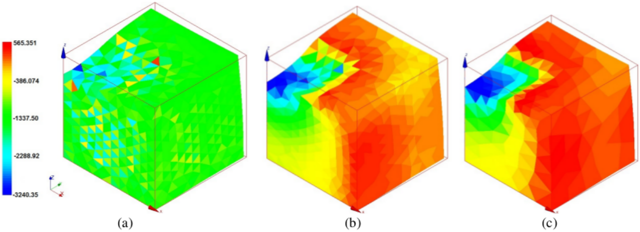
Industrial context

- Extend the lifetime of the nuclear power plants
- Accurate and robust numerical simulations with `code_aster`
- Strongly nonlinear mechanical problems to solve
  - nonlinear measure of deformations (geometric nonlinearity)
  - nonlinear stress-strain constitutive relation (material nonlinearity)
  - contact and friction (boundary nonlinearity)
- Industrial example : Notch plug



Numerical locking

- Presence of volumetric locking with primal H^1-conforming formulation due to plastic incompressibility
- An alternative : using mixed methods but more unknowns, more expensive to build, saddle-point problem to solve ...
- Example : pinching of a cube



Trace of the stress tensor for (a) P1 (b) P2 (c) P2/P1/P1

- or using a primal formulation without volumetric locking

Locking-free primal formulations

- discontinuous Galerkin (dG)
  - second order elliptic pb. [Arnold, Brezzi, Cockburn, Marini 01]
  - linear elasticity [Hansbo & Larson 03]
- Hybridizable Discontinuous Galerkin (HDG)
  - second order elliptic pb. [Cockburn, Gopalakrishnan, Lozarov 09]
  - linear elasticity [Soon, Cockburn, Stolarski 09]
- Hybrid High-Order (HHO) $\Leftarrow$ this thesis
  - diffusion problem [Di Pietro, Ern, Lemaire 14]
  - linear elasticity [Di Pietro & Ern 15]
- Virtual Element Method (VEM)
  - linear elasticity [Beirão da Veiga, Brezzi, Marini 13]
  - second order elliptic pb. [Beirão da Veiga, Brezzi, Marini, Russo 16]
- Strong connection between HDG and HHO [Cockburn, Di Pietro, Ern 16]

Main features of HHO for nonlinear solid mechanics

- More advantageous than mixed methods

  $\Rightarrow$ Primal formulation

- More advantageous than FE methods

  $\Rightarrow$ Absence of volumetric locking

- More advantageous than dG methods

  $\Rightarrow$ Integration of the behavior law only at cell-based quadrature nodes

  $\Rightarrow$ Symmetric tangent matrix at each nonlinear solver iteration

- Implementation in the open-source libraries `disk++`
  - `https://github.com/wareHHOuse/diskpp` (linear PDEs)

- Pave the way to HDG methods

Contributions

Publications

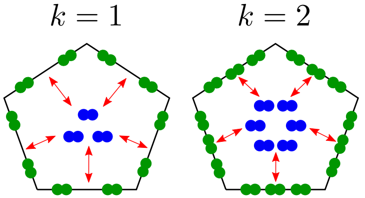
- 3 articles published
  - Hyperelasticity [Abbas, Ern, NP 18 (Comp. Mech.)]
  - Plasticity with small deformations [Abbas, Ern, NP 19 (CMAME)]
  - Plasticity in finite deformations [Abbas, Ern, NP 19 (IJNME)]
- 1 article submitted
  - Tresca friction with a Nitsche method [Chouly, Ern, NP (SISC)]

Softwares

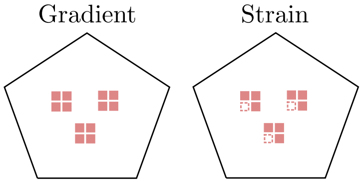
- Implementation from scratch in `code_aster` (integrated in version 15.0.8)
- Implementation of the nonlinear mechanical module in `disk++`

Key ideas of **H**ybrid **H**igh-**O**rder (HHO) methods

- Discontinuous (non-conforming) method
- Primal formulation with cell and face unknowns (poly. of order $k \geq 1$)
  - cell unknowns are eliminated locally by static condensation
- Local gradient/strain reconstruction (poly. of order $k \geq 1$)
  - h^{k+1} convergence in energy-norm (linear elasticity)
- Stabilization connecting cell and face unknowns

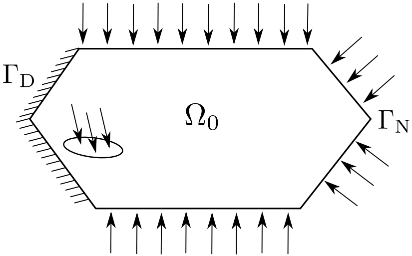


cell and face unknowns (2D)



gradient/strain unknowns (2D, k=1)

Linear elasticity problem

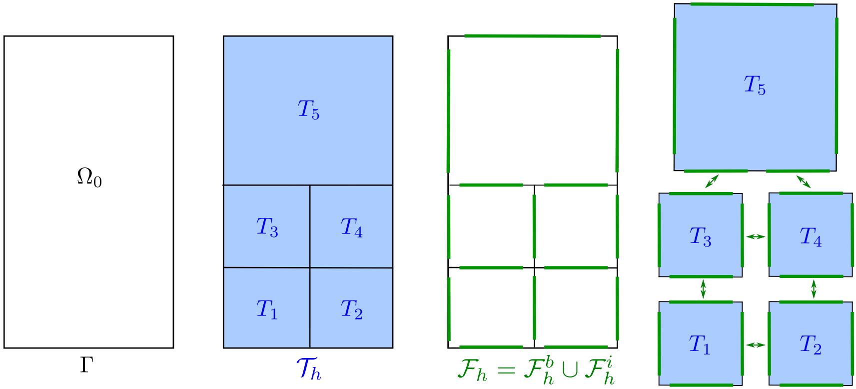


- $\Omega_0 \in \mathbb{R}^d$ (d=2,3) : a bounded connected polyhedron
- $\underline{f}$ and $\underline{g}_\mathrm{N}$: given volumetric and surface (on Γ_N) loads
- $\underline{u}_\mathrm{D}$: a given imposed displacement (on Γ_D)

$$\begin{cases} \text{Find } \underline{u} \in \underline{V}_\mathrm{D} := \left\{ \underline{v} \in H^1(\Omega_0;\mathbb{R}^d) \ : \ \underline{v} = \underline{u}_\mathrm{D} \text{ on } \Gamma_\mathrm{D} \right\} \text{ s.t. } \forall \underline{v} \in \underline{V}_0 \\ 2\mu(\varepsilon(\underline{u}),\varepsilon(\underline{v}))_{\mathbf{L}^2(\Omega_0)} + \lambda(\nabla\cdot\underline{u}, \nabla\cdot\underline{v})_{L^2(\Omega_0)} = (\underline{f},\underline{v})_{\underline{L}^2(\Omega_0)} + (\underline{g}_\mathrm{N},\underline{v})_{\underline{L}^2(\Gamma_\mathrm{N})}. \end{cases}$$

Mesh notation

- $\mathcal{T}_h$: set of cells ; $\mathcal{F}_h$: set of (planar) faces
- Mesh $\mathcal{M}_h := (\mathcal{T}_h, \mathcal{F}_h)$

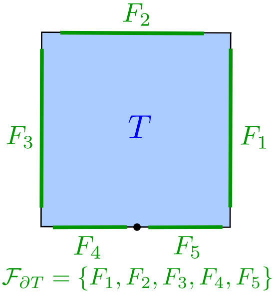


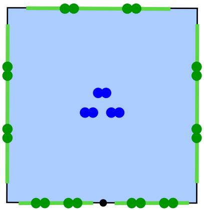
Mesh $\mathcal{M}_h$ composed of 5 cells and 15 faces

Local DOFs space

- $\mathcal{F}_{\partial T}$: set of mesh faces of cell T
- Let a polynomial degree $k \geq 1$; for all $T \in \mathcal{T}_h$, set

$$\underline{\hat{v}}_T := (\underline{v}_T, \underline{v}_{\partial T}) \in \underline{\hat{U}}_T^k := \underbrace{\mathbb{P}_d^k(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local face dofs}}$$





$\underline{\hat{U}}_T^k$

Symmetric strain reconstruction

$$\boldsymbol{E}_T^k : \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^d) \times \mathbb{P}_{d-1}^k(\mathcal{F}_{\partial T};\mathbb{R}^d)}_{=:\underline{\hat{U}}_T^k} \to \underbrace{\mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})}_{\text{local strain space}}$$

- The reconstructed strain $\boldsymbol{E}_T^k(\underline{\hat{v}}_T) \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ solves

$$(\boldsymbol{E}_T^k(\underline{\hat{v}}_T), \boldsymbol{\tau})_{\boldsymbol{L}^2(T)} := -(\underline{v}_T, \underline{\nabla} \cdot \boldsymbol{\tau})_{\boldsymbol{L}^2(T)} + (\underline{v}_{\partial T}, \boldsymbol{\tau}\, \underline{n}_T)_{\underline{L}^2(\partial T)}$$

  for all $\boldsymbol{\tau} \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$

  - mimic an integration by parts
  - local scalar mass-matrix of size $\binom{k+d}{k}$ (ex : $k = 2$, $d = 3 \Longrightarrow$ size $= 10$)

- Local interpolation operator : $\underline{\hat{I}}_T^k(\underline{v}) = (\underline{\Pi}_T^k(\underline{v}), \underline{\Pi}_{\partial T}^k(\underline{v}_{|\partial T})) \in \underline{\hat{U}}_T^k$

- Commuting property :

$$\boldsymbol{E}_T^k(\underline{\hat{I}}_T^k(\underline{v})) = \boldsymbol{\Pi}_T^k(\varepsilon(\underline{v})), \quad \forall \underline{v} \in H^1(T;\mathbb{R}^d)$$

Stabilization operator

- "Connect" the face unknowns to the trace of the cell unknowns
- We penalize the quantity $\underline{v}_{\partial T} - \underline{v}_{T|\partial T}$ in a least-squares sense
- HHO-stabilization operator $\underline{S}^k_{\partial T}(\hat{\underline{v}}_T) \in \mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)$ s.t.

$$\underline{S}^k_{\partial T}(\hat{\underline{v}}_T) := \underline{\Pi}^k_{\partial T}\big(\underbrace{\underline{v}_{\partial T} - \underline{v}_{T|\partial T}}_{\text{HDG term}} - \underbrace{(\boldsymbol{I}_d - \underline{\Pi}^k_T)\underline{R}^{k+1}_T(\underline{0}, \underline{v}_{\partial T} - \underline{v}_{T|\partial T})}_{\text{HHO correction}}\big)$$

$\underline{\Pi}^k_{\partial T}$: L^2-projector on $\mathbb{P}^k_{d-1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)$; $\underline{\Pi}^k_T$: L^2-projector on $\mathbb{P}^k_d(T; \mathbb{R}^d)$
$\underline{R}^{k+1}_T$: higher-order reconstructed displacement field in $\mathbb{P}^{k+1}_d(T; \mathbb{R}^d)$

- The HHO correction ensures high-order error estimates $\mathcal{O}(h^{k+1})$ on polyhedral meshes (instead of $\mathcal{O}(h^k)$)
- Stability :

$$\|\boldsymbol{\varepsilon}(\underline{v}_T)\|^2_{\boldsymbol{L}^2(T)} + h_T^{-1}\|\underline{v}_{\partial T} - \underline{v}_{T|\partial T}\|^2_{\underline{L}^2(\partial T)} \lesssim \|\boldsymbol{E}^k_T(\hat{\underline{v}}_T)\|^2_{\boldsymbol{L}^2(T)} + h_T^{-1}\|\underline{S}^k_{\partial T}(\hat{\underline{v}}_T)\|^2_{\underline{L}^2(\partial T)}$$

Local Galerkin contribution

- Local stress reconstruction : For all $\hat{\underline{v}}_T \in \hat{\underline{U}}_T^k$,

$$\boldsymbol{\sigma}(\hat{\underline{v}}_T) := 2\mu \boldsymbol{E}_T^k(\hat{\underline{v}}_T) + \lambda D_T^k(\hat{\underline{v}}_T)\boldsymbol{I}_d \in \mathbb{P}_d^k(T;\mathbb{R}^{d\times d}_{\mathrm{sym}})$$

- Local Galerkin contribution

$$\hat{a}_T^G(\hat{\underline{v}}_T, \hat{\underline{w}}_T) := \underbrace{2\mu(\boldsymbol{E}_T^k(\hat{\underline{v}}_T), \boldsymbol{E}_T^k(\hat{\underline{w}}_T))_{\boldsymbol{L}^2(T)} + \lambda(D_T^k(\hat{\underline{v}}_T), D_T^k(\hat{\underline{w}}_T))_{L^2(T)}}_{\text{FEM-like stiffness term}}$$

$$+ \underbrace{{\color{red}2\mu h_T^{-1}}(\underline{S}_{\partial T}^k(\hat{\underline{v}}_T), \underline{S}_{\partial T}^k(\hat{\underline{w}}_T))_{\underline{L}^2(\partial T)}}_{\text{stabilization term}}$$

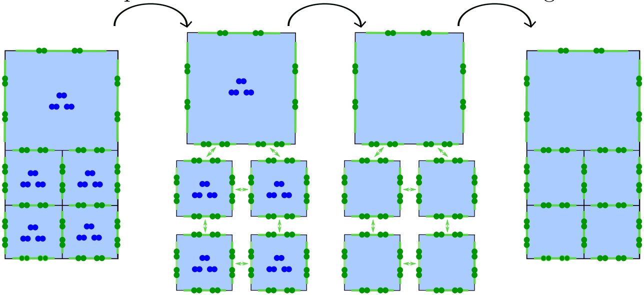
with the discrete divergence $D_T^k(\hat{\underline{v}}_T) := \mathrm{trace}(\boldsymbol{E}_T^k(\hat{\underline{v}}_T)) \in \mathbb{P}_d^k(T;\mathbb{R})$

- Local RHS

$$\hat{\ell}_T(\hat{\underline{v}}_T) := (\underline{f}, {\color{blue}\underline{v}_T})_{\underline{L}^2(T)} + (\underline{g}_{\mathrm{N}}, {\color{green}\underline{v}_{\partial T}})_{\underline{L}^2(\partial T \cap \Gamma_{\mathrm{N}})}$$

Global DOFs space

- Global DOFs : $\underline{\hat{u}}_h := (\underline{u}_{\mathcal{T}_h}, \underline{u}_{\mathcal{F}_h}) \in \underline{\hat{U}}_h^k := \underbrace{\mathbb{P}_d^k(\mathcal{T}_h;\mathbb{R}^d)}_{\text{global cells dofs}} \times \underbrace{\mathbb{P}_{d-1}^k(\mathcal{F}_h;\mathbb{R}^d)}_{\text{global faces dofs}}$



- Cellwise assembly (fully parallelizable)
- Face unknowns are uniquely defined
- Dirichlet boundary conditions are imposed strongly

$$\underline{\hat{U}}_{h,\mathrm{D}}^k := \left\{ \underline{\hat{u}}_h \in \underline{\hat{U}}_h^k \ : \underline{u}_F = \underline{\Pi}_F^k(\underline{u}_\mathrm{D}) \text{ on } \Gamma_\mathrm{D} \right\}$$

Global discrete problem (linear elasticity)

$$\begin{cases} \text{Find } \hat{\underline{u}}_h \in \hat{\underline{U}}_{h,\mathrm{D}}^k \text{ such that} \\ \hat{a}_h^G(\hat{\underline{u}}_h, \hat{\underline{v}}_h) = \hat{\ell}_h(\hat{\underline{v}}_h) \quad \forall\, \hat{\underline{v}}_h \in \hat{\underline{U}}_{h,0}^k \end{cases}$$

with

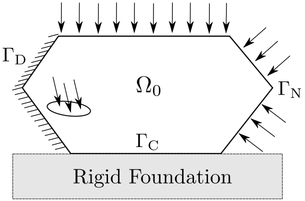
$$\hat{a}_h^G(\hat{\underline{u}}_h, \hat{\underline{v}}_h) := \sum_{T \in \mathcal{T}_h} \hat{a}_T^G(\hat{\underline{u}}_T, \hat{\underline{v}}_T) \text{ and } \hat{\ell}_h(\hat{\underline{v}}_h) := \sum_{T \in \mathcal{T}_h} \hat{\ell}_T(\hat{\underline{v}}_T)$$

- Well-posed problem
- Optimal convergence
  - h^{k+1}-convergence in energy-norm
  - h^{k+2}-convergence in L^2-norm with elliptic regularity
- Robustness in the incompressible limit ($\lambda \to +\infty$)

Unilateral contact with Tresca friction

- Small strain elasticity

$$
\begin{aligned}
-\underline{\nabla}\cdot\boldsymbol{\sigma}(\underline{u}) &= \underline{f} \quad \text{in } \Omega_0 \\
\boldsymbol{\sigma}(\underline{u}) = 2\mu\,\boldsymbol{\varepsilon}(\underline{u}) + \lambda(\nabla\cdot\underline{u})\boldsymbol{I}_d & \quad \text{in } \Omega_0 \\
+\text{BCs} &
\end{aligned}
$$

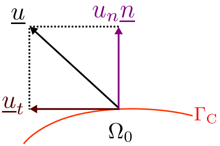


- Unilateral contact on Γ_C

$$
\begin{aligned}
u_n &\le 0 \\
\sigma_n(\underline{u}) &\le 0 \\
\sigma_n(\underline{u})\, u_n &= 0
\end{aligned}
$$

- Tresca friction on Γ_C $(s > 0)$

$$
\begin{cases}
|\underline{\sigma}_t(\underline{u})| \le s & \text{if } \underline{u}_t = \underline{0} \\
\underline{\sigma}_t(\underline{u}) = -s\,\dfrac{\underline{u}_t}{|\underline{u}_t|} & \text{otherwise}
\end{cases}
$$



Reformulation of Curnier–Alart

Proposition

Let two penalty parameters $\gamma_n > 0$ and $\gamma_t > 0$. The contact with Tresca friction conditions can be reformulated as follows :

$$\sigma_n(\underline{u}) = \left[\sigma_n(\underline{u}) - \gamma_n\, u_n\right]_{\mathbb{R}-}$$
$$\underline{\sigma}_t(\underline{u}) = \left[\underline{\sigma}_t(\underline{u}) - \gamma_t\, \underline{u}_t\right]_s$$

where $[\cdot]_{\mathbb{R}-}$ and $[\cdot]_s$ are projectors onto closed convex sets.

$$[x]_{\mathbb{R}-} := P_{(\mathbb{R}-,0)}(x)$$
$$[\underline{x}]_s := P_{\mathcal{B}(\underline{0},s)}(\underline{x})$$

Nitsche-FEM discretization

- Nitsche-FEM method can be seen as a consistent penalty method
  - Contact and friction conditions imposed weakly (no Lagrange multiplier)

- Conforming Nitsche-FEM discretization [Chouly & Hild 13]

$$\begin{cases} \text{Find } \underline{u}_h \in \underline{V}_h \text{ such that} \\ a_h(\underline{u}_h; \underline{v}_h) = \ell_h(\underline{v}_h) \quad \forall \underline{v}_h \in \underline{V}_h \end{cases}$$

with $a_h(\underline{v}_h; \underline{w}_h) := a_h^G(\underline{v}_h, \underline{w}_h) + a_h^N(\underline{v}_h; \underline{w}_h)$

- Galerkin contribution :

$$a_h^G(\underline{v}_h, \underline{w}_h) := 2\mu(\boldsymbol{\varepsilon}(\underline{v}_h), \boldsymbol{\varepsilon}(\underline{w}_h))_{\boldsymbol{L}^2(\Omega_0)} + \lambda(\nabla \cdot \underline{v}_h, \nabla \cdot \underline{w}_h)_{L^2(\Omega_0)}$$

- Contact/friction contribution with single penalty parameter $\gamma := \gamma_n = \gamma_t > 0$:

$$\begin{aligned} a_h^N(\underline{v}_h; \underline{w}_h) := &- (\tfrac{\theta}{\gamma}\underline{\sigma}_n(\underline{v}_h), \underline{\sigma}_n(\underline{w}_h))_{L^2(\Gamma_C)} \\ &+ (\tfrac{1}{\gamma}[\tau_n(\underline{v}_h)]_{\mathbb{R}^-}, (\tau_n + (\theta - 1)\sigma_n)(\underline{w}_h))_{L^2(\Gamma_C)} \\ &+ (\tfrac{1}{\gamma}[\underline{\tau}_t(\underline{v}_h)]_s, (\underline{\tau}_t + (\theta - 1)\underline{\sigma}_t)(\underline{w}_h))_{\underline{L}^2(\Gamma_C)} \end{aligned}$$

with $\theta \in \{-1, 0, 1\}$, $\tau_n(\underline{v}) := \sigma_n(\underline{v}) - \gamma h^{-1} v_n$ and $\underline{\tau}_t(\underline{v}) := \underline{\sigma}_t(\underline{v}) - \gamma h^{-1} \underline{v}_t$

Nitsche-HHO discretizations

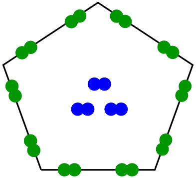
- Scalar contact problem with HHO discretization [Cascavita, Chouly, Ern 19]
  - Two variants :
    - Face-based $\underline{v}_{h|F} \to \underline{v}_F \in \mathbb{P}^k_{d-1}(F;\mathbb{R}^d)$
    - Cell-based $\underline{v}_{h|F} \to \underline{v}_{T|F} \in \mathbb{P}^{k+1}_d(T;\mathbb{R}^d)$
  - Sub-optimal convergence rates in H^1-norm for the face-based variant
  - Analysis only for $\theta = 1$
- Here the face-based variant is considered
  - Local enrichment on face dofs on Γ_{C}
    - face dofs of degree $(k+1)$ on the contact faces
    - Increase slightly the total number of face dofs
    - Optimal convergence rates in H^1-norm
  - Analysis for $\theta \in \{-1, 0, 1\}$
  - Tracking of the dependencies w.r.t μ, λ

Local DOFs space (contact modifications)

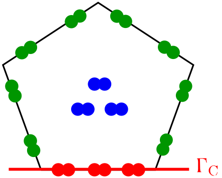
- $$\underbrace{\mathcal{F}_{\partial T}}_{\text{faces of T}} = \underbrace{\mathcal{F}_{\partial T}^{\setminus}}_{\text{other faces of T}} \cup \underbrace{\mathcal{F}_{\partial T}^{\mathrm{C}}}_{\text{contact faces of T}}$$
- Let a polynomial degree $k \geq 1$; for all $T \in \mathcal{T}_h$, set

$$\hat{\underline{v}}_T := (\underline{v}_T, \underline{v}_{\partial T}) \in \underbrace{\mathbb{P}_d^k(T; \mathbb{R}^d)}_{\text{local cell dofs}} \times \underbrace{\mathbb{P}_{d-1}^{k/k+1}(\mathcal{F}_{\partial T}; \mathbb{R}^d)}_{\text{local face dofs}}.$$

with $\mathbb{P}_{d-1}^{k/k+1}(\mathcal{F}_{\partial T}; \mathbb{R}^d) := \mathbb{P}_{d-1}^{k}(\mathcal{F}_{\partial T}^{\setminus}; \mathbb{R}^d) \times \mathbb{P}_{d-1}^{k+1}(\mathcal{F}_{\partial T}^{\mathrm{C}}; \mathbb{R}^d)$



(a) Cell without contact face



(b) Cell with contact face

Local DOFs for $k = 1$. Cell unknowns are eliminated by static condensation

Local contact/friction contributions

- Use face-based Nitsche-HHO method : $\underline{v}_F \in \mathbb{P}^{k+1}_{d-1}(F;\mathbb{R}^d)$ on Γ_{C}
- No change for the local Galerkin contribution (as in linear elasticity)
- Two penalty parameters $\gamma_n > 0$ and $\gamma_t > 0$
- Local contact/friction contribution on a contact cell

$$
\begin{aligned}
\hat{a}^N_T(\underline{\hat{v}}_T;\underline{\hat{w}}_T) :=& -\theta\frac{h_T}{\gamma_n}\left(\sigma_n(\underline{\hat{v}}_T),\sigma_n(\underline{\hat{w}}_T)\right)_{L^2(\partial T\cap\Gamma_{\mathrm{C}})} \\
&+\frac{h_T}{\gamma_n}\left(\left[\tau_n(\underline{\hat{v}}_T)\right]_{\mathbb{R}^-},(\tau_n+(\theta-1)\sigma_n)(\underline{\hat{w}}_T)\right)_{L^2(\partial T\cap\Gamma_{\mathrm{C}})} \\
&-\theta\frac{h_T}{\gamma_t}\left(\underline{\sigma}_t(\underline{\hat{v}}_T),\underline{\sigma}_t(\underline{\hat{w}}_T)\right)_{\underline{L}^2(\partial T\cap\Gamma_{\mathrm{C}})} \\
&+\frac{h_T}{\gamma_t}\left(\left[\underline{\tau}_t(\underline{\hat{v}}_T)\right]_s,(\underline{\tau}_t+(\theta-1)\underline{\sigma}_t)(\underline{\hat{w}}_T)\right)_{\underline{L}^2(\partial T\cap\Gamma_{\mathrm{C}})}
\end{aligned}
$$

with $\tau_n(\underline{\hat{v}}_T) := \sigma_n(\underline{\hat{v}}_T) - \gamma_n h_T^{-1} v_{\partial T,n}$ and $\underline{\tau}_t(\underline{\hat{v}}_T) := \underline{\sigma}_t(\underline{\hat{v}}_T) - \gamma_t h_T^{-1}\underline{v}_{\partial T,t}$

Global discrete problem and well-posedness

- Global discrete problem

$$\begin{cases} \text{Find } \underline{\hat{u}}_h \in \mathbb{P}_d^k(\mathcal{T}_h;\mathbb{R}^d) \times \mathbb{P}_{d-1}^{k/k+1}(\mathcal{F}_h;\mathbb{R}^d) \text{ s.t.} \\ \hat{a}_h(\underline{\hat{u}}_h;\underline{\hat{v}}_h) = \hat{\ell}_h(\underline{\hat{v}}_h) \quad \forall \underline{\hat{v}}_h \in \underline{\hat{U}}_{h,0}^k \end{cases}$$

with $\hat{a}_h(\underline{\hat{u}}_h;\underline{\hat{v}}_h) := \sum_{T\in\mathcal{T}_h} \left[\hat{a}_T^G(\underline{\hat{u}}_T,\underline{\hat{v}}_T) + \hat{a}_T^N(\underline{\hat{u}}_T;\underline{\hat{v}}_T)\right]$

Theorem (Well-posedness)

Let $k \geq 1$. Assume that the penalty parameters are such that

$$\min(\kappa^{-1}\gamma_n, 2\gamma_t) \geq 3(\theta+1)^2 C_{\mathrm{dt}}^2 \mu,$$

where $\kappa := \max(1, \frac{\lambda}{2\mu})$ and C_{dt} from a discrete trace inequality.

Then, the global discrete problem is well-posed

Theorem (H^1-error estimate)

Let $k \geq 1$. Assume that the penalty parameters are such that

$$\min(\kappa^{-1}\gamma_n, 2\gamma_t) \geq 3\Big((\theta+1)^2 + \epsilon(4 + (\theta-1)^2)\Big) C_{\mathrm{dt}}^2 \mu \quad \textit{with } \epsilon \in (0,1]$$

Assume $\underline{u} \in H^{1+r}(\Omega_0; \mathbb{R}^d)$ *and* $\nabla\cdot\underline{u} \in H^r(\Omega_0; \mathbb{R})$, $r \in (\frac{1}{2}, k+1]$. *Then,*

$$\sum_{T\in\mathcal{T}_h} \Big(2\mu \| \boldsymbol{\varepsilon}(\underline{u}) - \boldsymbol{E}_T^k(\underline{\hat{u}}_T) \|^2_{\boldsymbol{L}^2(T)} + \lambda \| \nabla\cdot\underline{u} - D_T^k(\underline{\hat{u}}_T)\|^2_{L^2(T)} \Big)$$

$$+ \frac{\epsilon}{2(1+\epsilon)} \sum_{T\in\mathcal{T}_h^C} \Big(\frac{h_T}{\gamma_n} \| [\tau_n(\underline{u})]_{\mathbb{R}^-} - [\tau_n(\underline{\hat{u}}_T)]_{\mathbb{R}^-} \|^2_{L^2(\partial T^C)} + \frac{h_T}{\gamma_t} \| [\underline{\tau}_t(\underline{u})]_s - [\underline{\tau}_t(\underline{\hat{u}}_T)]_s \|^2_{\underline{L}^2(\partial T^C)} \Big)$$

$$\lesssim \sum_{T\in\mathcal{T}_h} \Big(\Big[2\mu + \frac{1}{\epsilon}\Big(\frac{\mu^2\kappa^2}{\gamma_n} + \frac{\mu^2}{\gamma_t} + \gamma_n \Big) \Big] h_T^{2r} |\underline{u}|^2_{\underline{H}^{1+r}(T)} + \frac{1}{2\mu} \lambda^2 h_T^{2r} |\nabla\cdot\underline{u}|^2_{H^r(T)} \Big).$$

- Robustness in the inco. limit for unilateral contact (only $\theta = -1$)
  - $\forall\theta$ and $\epsilon \approx 1$: $\gamma_t \approx \mu$ and $\gamma_n \approx \mu\kappa$
  - $\theta = -1$ and $\epsilon \approx \kappa^{-1}$: $\gamma_t \approx \mu$ and $\gamma_n \approx \mu$ (independent of κ i.e. λ)
- Robustness in the inco. limit for bilateral contact $u_n = 0$ on Γ_{C} ($\forall\theta$)
- Smoothness assumption : $\underline{u} \in H^{\frac{5}{2}-\eta}, \eta > 0$ ($r = \frac{3}{2} - \eta$)
  - Maximal convergence rate is $\mathcal{O}(h^{\frac{3}{2}-\eta})$ and is reached for $k = 1$

Numerical examples

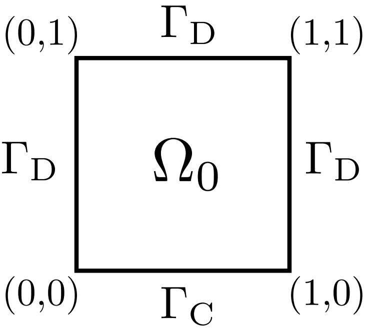
- Nonlinear problem to solve (contact and friction nonlinearities)
- Iterative resolution with a semi-smooth Newton's method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Verification on analytical solution :
  - Optimal convergence rates in H^1-norm
  - Absence of volumetric locking in the incompressible limit
- Comparison to mixed methods [Bostan & Han 06]
- Industrial application

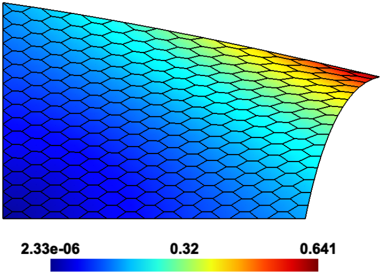
Manufactured solution

- Manufactured solution

$$u_x = \left(1 + \frac{1}{1+\lambda}\right) x e^{x+y}, \quad u_y = \left(-1 + \frac{1}{1+\lambda}\right) y e^{x+y}.$$

- Friction coefficient $s = \frac{\mu}{6}\frac{\lambda+2}{\lambda+1}x^2$





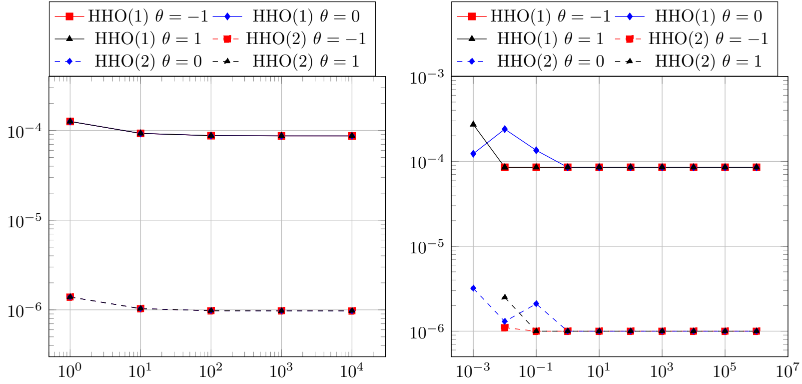
Manufactured solution : convergence rates

Mesh size h	$k=1$		$k=2$	
	H^1-error	order	H^1-error	order
3.33e-1	5.423e-3	-	4.406e-4	-
1.75e-1	1.380e-3	2.13	5.871e-5	3.13
9.06e-2	3.472e-4	2.08	7.620e-6	3.07
4.60e-2	8.694e-5	2.05	9.719e-7	3.04

H^1-error and convergence order vs. h for $\theta = 1$

- Optimal h^{k+1}-convergence rates in H^1-norm

Manufactured solution : robustness

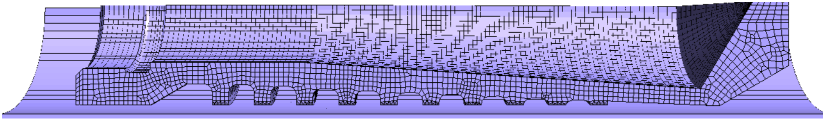


(a) H^1-error vs. λ ($\gamma_n = \gamma_t = 2\mu$)

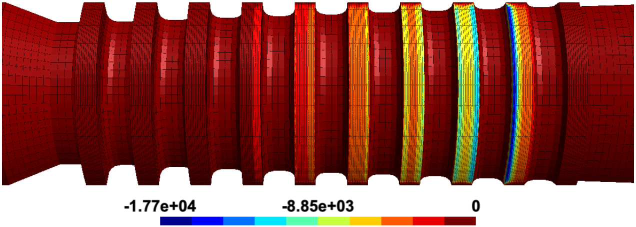
(b) H^1-error vs. $\frac{\gamma_n}{2\mu}$ ($\lambda = 1000$ and $\gamma_t = 2\mu$)

- Absence of volumetric locking in the incompressible limit
- H^1-error independent of γ_n if $\gamma_n \geq 2\mu$ (and $\gamma_t = 2\mu$)

Industrial application : Notch plug



Mesh composed of 21,200 hexahedra and 510 prisms in the reference configuration



Normal stress σ_n in the contact zone (in MPa) for $k = 1$ and $\theta = 1$

- Results in agreement with `code_aster` and Coulomb friction

Main difficulties for plasticity problems

- Irreversibility of the plastic deformations
- Plastic incompressibility
  - Volumetric locking for H^1-conforming FEM
- Strongly nonlinear problems
  - Constitutive law
  - Finite deformations
- Loss of coercivity (softening materials)

Bibliography overview

Some references on primal formulations for plasticity without volumetric locking

- discontinuous Galerkin (dG)
  - [Mc Bride, Reddy 09]
  - [Liu, Wheeler, Dawson, Dean 13]
- Hybrid discontinuous Galerkin with conforming traces (Hybrid dG)
  - [Wulfinghoff, Bayat, Alipour, Reese 17]
- Hybrid weakly conforming method (Hybrid WCM)
  - [Krämer, Wieners, Wohlmuth, Wunderlich 16]
- Virtual Element Method (VEM)
  - [Chi, Beirão da Veiga, Paulino 17]
  - [Hudobivnik, Aldakheel, Wriggers 19]
- No HDG methods

Plasticity model (small def.)

- Framework of generalized standard materials [Halphen & Nguyen 75]
- Linearized strain tensor

$$\varepsilon(\underline{v}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$$

- Plastic strain tensor and incompressibility

$$\varepsilon^p \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \ \text{ and } \ \mathrm{trace}(\varepsilon^p) = 0$$

- Additive decomposition

$$\varepsilon^e := \varepsilon - \varepsilon^p$$

- The internal state is described by $\varepsilon, \varepsilon^p$ and a set of internal variables

$$\underline{\alpha} := (\alpha_1, \cdots, \alpha_m) \in \mathbb{R}^m$$

- Generalized internal variables

$$\underline{\chi} := \{\varepsilon^p, \underline{\alpha}\}$$

Plasticity problem (small def.)

- $\Omega_0 \in \mathbb{R}^d$ (d=2,3) : bounded connected polyhedron
- Pseudo-time stepping : $n = 1, \dots, N$ (history of the deformations)
- Find $\underline{u}^n \in V_d := \{\underline{v} \in H^1(\Omega_0; \mathbb{R}^d) \,|\, \underline{v} = \underline{u}^n_{\mathrm{D}} \text{ on } \Gamma_{\mathrm{D}}\}$ s.t.

$$\int_{\Omega_0} \boldsymbol{\sigma}^n : \boldsymbol{\varepsilon}(\underline{v}) \, d\Omega_0 = \ell(\underline{v}) \text{ for all } \underline{v} \in V_0$$

and

$$\boldsymbol{\sigma}^n = \mathrm{SMALL\_PLASTICITY}(\underline{\chi}^{n-1}, \boldsymbol{\varepsilon}(\underline{u}^{n-1}), \boldsymbol{\varepsilon}(\underline{u}^n))$$

where $\mathrm{SMALL\_PLASTICITY}$ is the given behavior integrator

- Constitutive algorithm : radial return mapping
- Many examples in `code_aster`

Global discrete problem (small def.)

- No modification for HHO operators $\boldsymbol{E}_T^k$ and $\underline{S}_{\partial T}^k$ (as in linear elasticity)
- For all $1 \leq n \leq N$, find $\underline{\hat{u}}_h^n \in \underline{\hat{U}}_{h,\mathrm{D}}^k$ such that

$$\sum_{T \in \mathcal{T}_h} (\boldsymbol{\sigma}^n, \boldsymbol{E}_T^k(\underline{\hat{v}}_T))_{\boldsymbol{L}^2(T)} + \sum_{T \in \mathcal{T}_h} \beta h_T^{-1} (\underline{S}_{\partial T}^k(\underline{\hat{u}}_T^n), \underline{S}_{\partial T}^k(\underline{\hat{v}}_T))_{\underline{L}^2(\partial T)}$$
$$= \hat{\ell}_h(\underline{\hat{v}}_h), \quad \forall\, \underline{\hat{v}}_h \in \underline{\hat{U}}_{h,0}^k$$

and for all the cell-quadrature points

$$\boldsymbol{\sigma}^n = \mathrm{SMALL\_PLASTICITY}(\underline{\chi}_T^{n-1}, \boldsymbol{E}_T^k(\underline{\hat{u}}_T^{n-1}), \boldsymbol{E}_T^k(\underline{\hat{u}}_T^n))$$

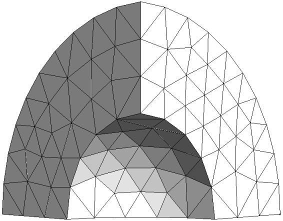
with $\beta \simeq 2\mu$ the user-dependent stabilization parameter

Numerical examples (small def.)

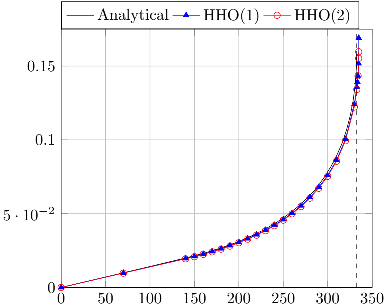
- Nonlinear problem to solve (material nonlinearity)
- Iterative resolution with Newton's method
- Static condensation performed at each Newton's iteration
- Offline computations (gradient and stabilization operators precomputed)
- Symmetric tangent matrix at each nonlinear solver iteration
- Verification on analytical solution :
  - Absence of volumetric locking due to plastic incompressibility
- Comparison to P^2 and $P^2/P^1/P^1$ (Displacement/Pressure/Dilatation) solutions

Sphere under internal pressure I (small def.)

- Perfect J_2-plasticity
- Increase the internal pressure until the limit load
- Analytical solution available

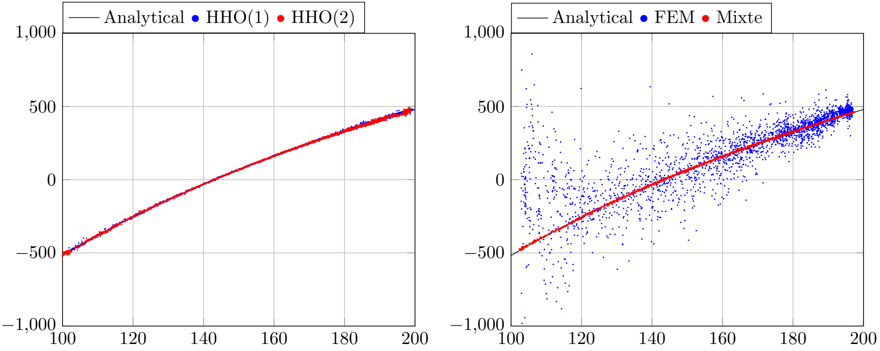


(a) Mesh



(b) Radial displ. vs. internal pressure

Sphere under internal pressure II (small def.)

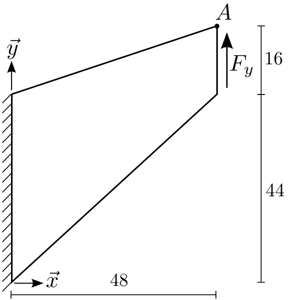


Trace of the stress tensor at all the quadrature points at the limit load

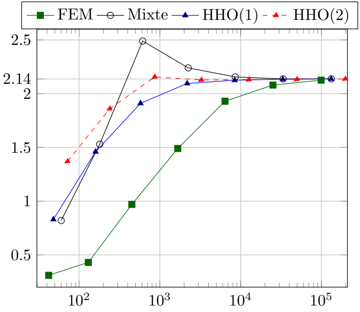
- Absence of volumetric locking for HHO and mixed methods

Quasi-incompressible Cook's membrane (small def.)

- Linear isotropic hardening with J_2-plasticity ($\nu = 0.4999$)



(a) Geometry and BC.

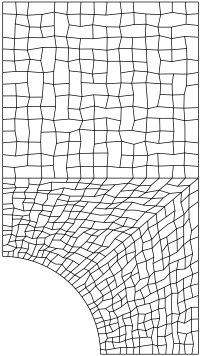


(b) Displacement of A vs. #dofs.

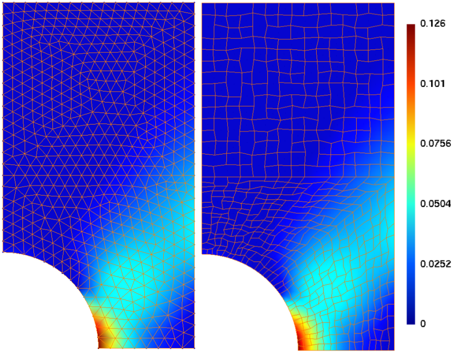
- HHO(2) outperforms mixed methods (same displacement order)

Perforated strip under uniaxial extension (small def.)

- Combined linear kinematic and isotropic hardening with J_2-plasticity



(a) Polygonal mesh



(b) Equivalent plastic strain with HHO(2)

- Excellent matching between triangle and polygonal meshes

Extension to finite deformations

- Extension to finite deformations using the logarithmic strain framework
- Logarithmic strain tensor

$$\boldsymbol{E} := \frac{1}{2} \ln \boldsymbol{F}^T \boldsymbol{F} \in \mathbb{R}^{d \times d}_{\mathrm{sym}}$$

- Additive decomposition (elastic $\boldsymbol{E}^e$ and plastic $\boldsymbol{E}^p$ parts)

$$\boldsymbol{E}^e := \boldsymbol{E} - \boldsymbol{E}^p \in \mathbb{R}^{d \times d}_{\mathrm{sym}}$$

Algorithm 1 Given $\underline{\chi}^{n-1}, \boldsymbol{F}^{n-1}, \boldsymbol{F}^n$, Return Piola-Kirchhoff 1 tensor $\boldsymbol{P}^n$

```
1: procedure FINITE_PLASTICITY(χ^{n-1}, F^{n-1}, F^n)
2:     Solve eigenvalue pb. E^m := 1/2 ln(F^{m,T} F^m), m ∈ {n-1, n}
3:     Compute T^n = SMALL_PLASTICITY(χ^{n-1}, E^{n-1}, E^n).
4:     return P^n = T^n : (∂_F E)^n
5: end procedure
```

1: **procedure** $\mathrm{FINITE\_PLASTICITY}(\underline{\chi}^{n-1}, \boldsymbol{F}^{n-1}, \boldsymbol{F}^n)$
2: Solve eigenvalue pb. $\boldsymbol{E}^m := \frac{1}{2} \ln(\boldsymbol{F}^{m,T} \boldsymbol{F}^m)$, $m \in \{n-1, n\}$
3: Compute $\boldsymbol{T}^{\mathrm{n}} = \mathrm{SMALL\_PLASTICITY}(\underline{\chi}^{n-1}, \boldsymbol{E}^{n-1}, \boldsymbol{E}^n)$.
4: **return** $\boldsymbol{P}^n = \boldsymbol{T}^n : (\partial_{\boldsymbol{F}} \boldsymbol{E})^n$
5: **end procedure**

- For HHO methods, the only modification is the gradient reconstruction $\boldsymbol{G}^k_T \in \mathbb{P}^k_d(T; \mathbb{R}^{d \times d})$ (to replace $\boldsymbol{E}^k_T \in \mathbb{P}^k_d(T; \mathbb{R}^{d \times d}_{\mathrm{sym}})$)

Global discrete problem (finite def.)

For all $1 \leq n \leq N$, find $\underline{\hat{u}}_h^n \in \underline{\hat{U}}_{h,\mathrm{D}}^k$ such that

$$\sum_{T\in\mathcal{T}_h} (\boldsymbol{P}^n, \boldsymbol{G}_T^k(\underline{\hat{v}}_T))_{\boldsymbol{L}^2(T)} + \sum_{T\in\mathcal{T}_h} \beta h_T^{-1} (\underline{S}_{\partial T}^k(\underline{\hat{u}}_T^n), \underline{S}_{\partial T}^k(\underline{\hat{v}}_T))_{\underline{L}^2(\partial T)}$$
$$= \hat{\ell}_h(\underline{\hat{v}}_h), \quad \forall\, \underline{\hat{v}}_h \in \underline{\hat{U}}_{h,0}^k$$

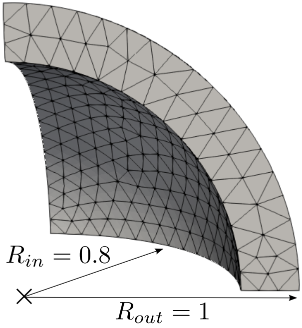
and for all the cell-quadrature points

$$\boldsymbol{P}^n = \mathrm{FINITE\_PLASTICITY}(\underline{\chi}_T^{n-1}, \boldsymbol{F}_T^k(\underline{\hat{u}}_T^{n-1}), \boldsymbol{F}_T^k(\underline{\hat{u}}_T^n))$$

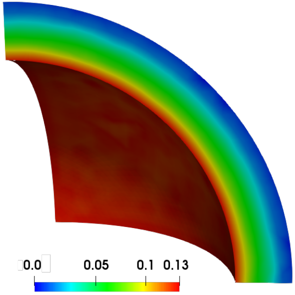
with $\boldsymbol{F}_T^k = \boldsymbol{G}_T^k + \boldsymbol{I}_d$ and $\beta \simeq 2\mu$ the stabilization parameter (no general theory)

Quasi-incomp. sphere under internal forces I (finite def.)

- Perfect J_2-plasticity
- $\nu = 0.499$
- Increase the internal radial force until the limit load
- Analytical solution available for $\nu = 0.5$

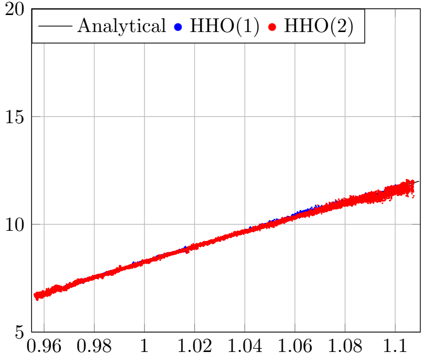


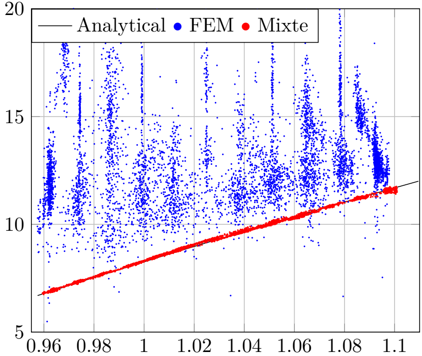
(a) 1580 tetrahedra



(b) Equivalent plastic strain p - HHO(1)

Quasi-incomp. sphere under internal forces II (finite def.)



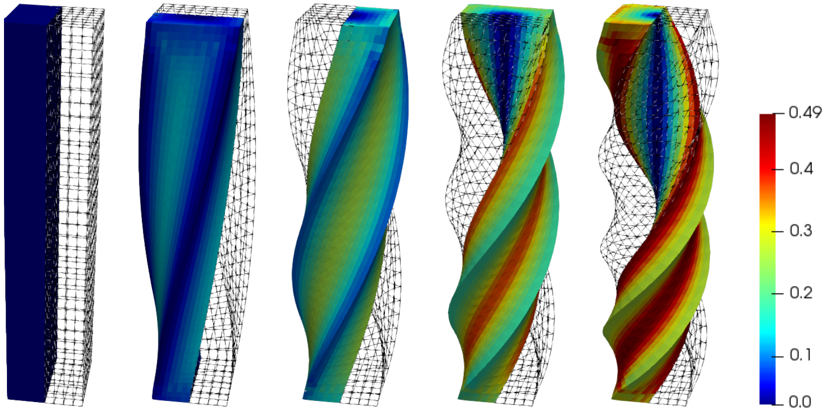


Trace of the stress tensor at all the quadrature points at the limit load

- Absence of volumetric locking for HHO and mixed methods

Torsion of a square-section bar (finite def.)

- Nonlinear isotropic hardening with J_2-plasticity

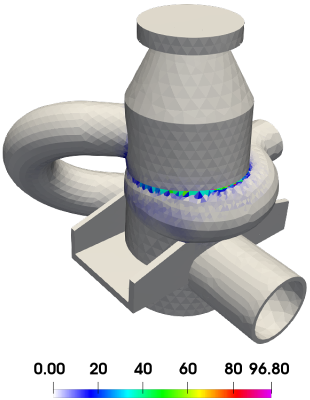


(a) $\Theta = 0^\circ$ (b) $\Theta = 90^\circ$ (c) $\Theta = 180^\circ$ (d) $\Theta = 270^\circ$ (e) $\Theta = 360^\circ$

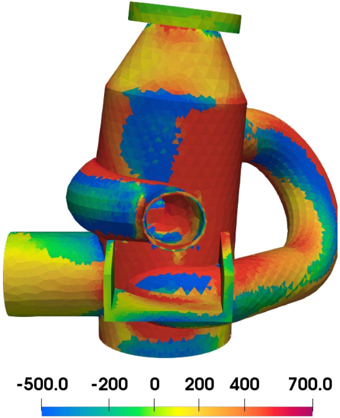
Equivalent plastic strain p for HHO(1) for different rotation angles Θ.

Industrial application : pump under internal forces (fin. def)

- Linear isotropic hardening with J_2-plasticity



(a) Equivalent plastic strain p (in %)

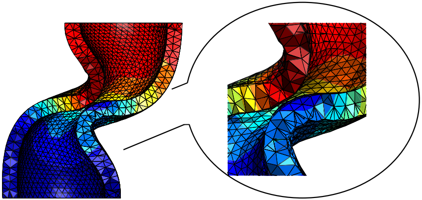


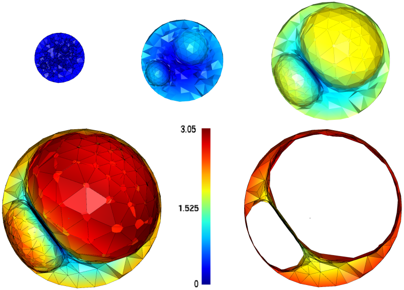
(b) von Mises stress (in MPa)

Mesh composed of 23,837 tetrahedra and results for HHO(1 ;2)

Conclusions

- HHO methods for finite plasticity, hyperelasticity and contact
- Primal formulation without of volumetric locking
- Numerous test-cases passed successfully
- Implementation in `code_aster` and `disk++`





Hyperelastic computations

Perspectives

- *a priori* error estimates
  - plasticity with small deformations [Djoko & al 07]
  - Nitsche-HHO with Coulomb friction [Hild & Renard 12] [Chouly & al 19]
- Extension to dynamic problems [Hauret & Le Tallec 06] [Stanglmeier & al 16]
- Non-local plasticity and damage models [McBride & Reddy 09] [Zhang & al 18]

Non-local damage model and mixed method [Chen 19 (PhD)]

- Industrial applications with `code_aster`

Perspectives

- *a priori* error estimates
  - plasticity with small deformations [Djoko & al 07]
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Thank you for your attention