Introduction

> "It was a blessing for him and science as he was aware of his own abilities and used them in a field where the results spread light in all directions. His genius thus included the instinct that is an essential quality for a mathematician who breaks entirely new ground."
>
> Sophus LIE about Evariste GALOIS, 1896

Up until the mid-century, it was generally accepted that a quite elementary knowledge of mathematics was necessary and sufficient to study and solve problems in engineering sciences (analytical and continuum mechanics, thermodynamics, electromagnetism, ...). The main tool was linear algebra but one could eventually enlarge *matrix calculus* with some differential geometry so as to successively obtain the already sophisticated *exterior calculus* and *tensor calculus* which are now known to be essential in order to fully understand electromagnetism in the framework of special relativity.

Meanwhile, analysis and topology have always competed with each other, their union finally culminating in functional analysis. It is also well known that subsidiary methods such as Legendre or Fourier transforms are also quite useful because of their important respective uses in the study of electrical networks, heat exchanges or wave propagation.

The remaining mathematics have even been called "*pure*" because of the lack of any "*a priori*" possibility to apply them.

During the last quarter of a century or so, things have changed drastically and it is now fashionable to say that almost all branches of mathematics could be used. As we shall see later in this introduction, such a new trend can be particularly well noticed in *control theory* with the introduction of commutative algebra and differential geometry in the seventies.

However, certain parts of mathematics seem to remain "*more pure than others*", in the sense that even now it seems very difficult to apply them in almost any domain of engineering sciences.

In order to convince the reader already familiar with control theory, we invite him to enter into any scientific library or bookshop and to consult the famous Bourbaki's "*Elements of mathematics*". This monumental axiomatic treatise is divided into "books", each of them being divided into "chapters" and we now refer to the book Algebra, chapter 10, called "*Homological algebra*", which, of course, cannot be read without a good knowledge of many of the previous chapters and also highly depends on many results of the book "*Commutative Algebra*".

After having a look at such a book, the astonished reader will discover tensor products everywhere, combined with diagrams which, in most cases are referring to the so-called *torsion* and *extension* modules that are the key concepts of such a book. Hence, even if the reader knows that, during the last ten years, *differential algebra*, that is algebra mixed with differential geometry, has been applied with success to control theory, he will be sure that at least this part of mathematics will never have any kind of link with control theory.

The main purpose of the present book is just to prove the exact contrary and this introduction will try to stress this fact with the least amount of mathematics. Hence, not only homological algebra will soon become a key piece of machinery for understanding the basic concepts of classical control theory, hereafter called *ordinary differential (OD) control theory* as it deals with ordinary differential equations (OD equations), but it will also become the essential tool that will make it possible to extend these concepts to partial differential equations (PD equations) thus creating for the first time *partial differential (PD) control theory*. More precisely, the problem will not be exactly to enlarge these concepts but rather to prove that a control theory can be created for PD equations which just produces classical control theory when restricted to OD equations. Of course, the price to pay for that is to become familiar with a new language, sparsely hidden through many places in the literature. For this reason, we have tried to make this book as self-contained as possible, caring about the notations in such a way as to adopt the standard notations to be found in a variety of textbooks and research papers. Also each main concept will be at once illustrated by a simple academic example, while more complicated exercises and problems will be stated and treated at the end of the corresponding chapters.

To be perfectly fair, we must say that, five years ago, when preparing our previous book in the same collection, which contains a special chapter on control theory, we had no idea at all about such a possibility of applying methods of homological algebra to control theory.

We now find it useful, in this introduction, to sketch a brief history of control theory, focusing mainly on the concept of "*controllability*" [15]. In particular, we shall try to explain why it has taken almost forty years in order to cut off this concept from its intuitive roots while explaining it fully by means of a new convenient algebraic framework. Our critical sketch is summarized in the diagram opposite which we now detail and illustrate.

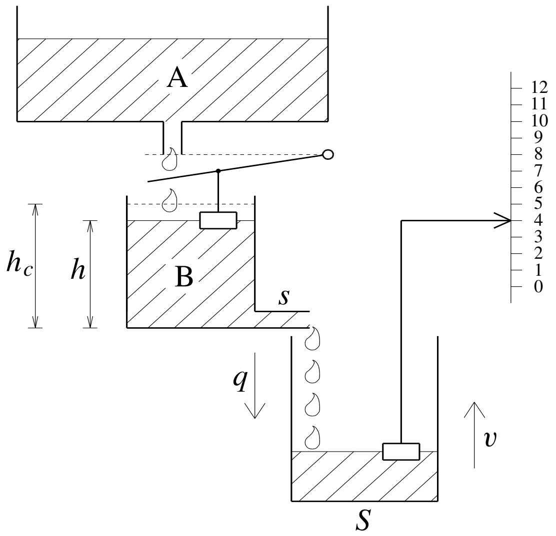
The concept of a system and the means of controlling it are very old. In ancient times, jugglers and minstrels were able to maintain in equilibrium on their nose an inverted pendulum made up of a rigid stick supporting a heavy mass.

A Brief History of Controllability

	OD equations	$\subset$	PD equations
1940	Transfer matrix		
1960	Functional definition		
1963	Kalman test		
1969	Hautus test		
1970	Localized parameters	$\longrightarrow$	Distributed parameters
1975	Bracket test		$\updownarrow$?
1980	Polynomial test		
1986		$\longrightarrow$	Formal definition
1994			Duality test
1996			Homological test

Similarly Chinese acrobats were able to support in their hands a long vertical pole on which a child was climbing up and down. In a simpler way, after some time, any child knows how to move himself on a swing in order to control the amplitude of the movement. It is also known that a falling cat can always move in order to fall on its legs and the study of this mechanical system is the subject of a recent extended literature (see other examples in [5, 56, 67, 73, 122]).

One of the first examples of a system with a "*feedback*" control is the "Clepsydre", an apparatus invented by the Greek mechanician Ktesibios (3[rd] century BC) and used in order to work as a clock by means of flowing water, along the following picture:

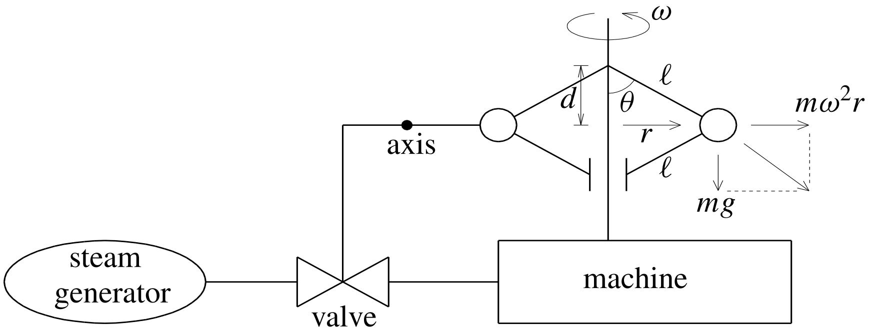


The big tank A delivers water into a smaller tank B from which the flow q per unit of time is used to raise up with speed v a floating sensor along a graduated time scale. According to Bernoulli theorem in hydrodynamics, v^2 is proportional to h, more precisely:

$$v^2 = 2gh(s/S)^2$$

where g is the local gravity, s/S is the ratio of respective sections and h is the height of water in the intermediate tank B used as a regulator. In order to have constant speed v, we must have a constant flow q and thus a constant reference height h_c in B. Therefore, an additional apparatus such as a floating system, eventually acting on a potentiometer or directly like the one drawn on the picture, closes or opens the bottom of A in a way proportional to $h - h_c$. Indeed, if $h < h_c$ then more water is coming from A as becomes clear from the dotted part of the picture.

Later on, during the industrial revolution of the last century brought on by the use of steam engines, it became necessary to control the corresponding systems. The simplest regulator has been discovered by J. Watt and works along the following picture:



Each mass m is in dynamical equilibrium when acted on by the tension of the upper bar, the centrifugal force $m\omega^2\ell\sin\theta$ and the weight mg. Accordingly, one has $m\omega^2\ell\sin\theta = mg\,\mathrm{tg}\,\theta$ and thus $\omega^2 d = \omega^2\ell\cos\theta = g$ as a bijective though non-linear relation between ω and θ. Again, we may fix a reference speed of rotation $\omega_c \leftrightarrow \theta_c \leftrightarrow d_c$ and act on a valve as in the picture, depending on the sign of $d - d_c$, in order to control the steam flowing from the generator to the engine.

In all these examples, the possibility to "control" the system by means of a mechanical apparatus was clear because of the effective possibility to construct and test the efficiency of such a convenient apparatus.

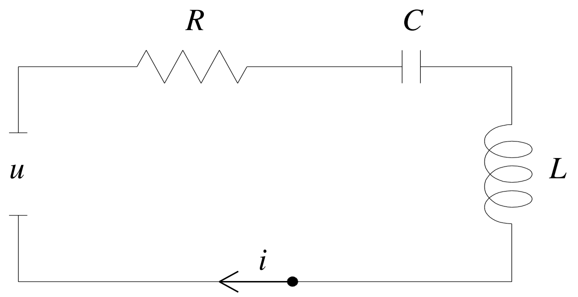
For more complicated systems, as we shall see, the situation will be more delicate and the possibility to "control" a system will depend on the answer to the following two questions:

1. What do we mean by "system"?
2. What do we mean by "control"?

Surprisingly, we shall discover little by little that no general answer has ever been given to these two questions, in the sense that, even now and in all textbooks, the answer to the second question highly depends on the answer to the first question. Of course, one of the main results of the present book is to provide a general answer by using quite unexpected tools from homological algebra in order to distinguish, among the class of all systems governed by PD equations, those which can be controlled from those which cannot be controlled. More precisely, a system that can be controlled will be called *controllable* while the other ones will be called *uncontrollable*.

For the moment, working along historical lines, we are still far from PD equations and go on studying systems involving quantities, called *control variables*, which are only functions of time and are governed by satisfying systems of OD equations.

One of the simplest examples is that of an RCL electrical circuit made up successively, as in the following picture:



by a resistance R in Ohm, a capacitance C in Farad and an inductance L in Henri. If $q = q(t)$ is the charge of the capacitance, the electric current $i = i(t)$ flowing through the circuit becomes such that $q(t) = \int_0^t i(s)ds$ or $i = \frac{dq}{dt} = \dot{q}$ and we obtain from the Kirchhoff law, by adding the successive differences of electric potential:

$$Ri + \frac{q}{C} + L\frac{di}{dt} = u, \qquad \frac{dq}{dt} = i$$

that is a system of two OD equations for the *outputs* q and i when the *input* u is given as a function of time. One must notice that u may not be totally arbitrary and can be a periodic signal with pulsation ω for which only the amplitude is arbitrary. Hence, we should add the OD equation $\ddot{u} + \omega^2 u = 0$. When u is given, one notices that the knowledge of i brings that of q by setting:

$$q = C\Big(u - Ri - L\frac{di}{dt}\Big)$$

while the knowledge of q brings that of i by derivation. More generally, one may measure any scalar function z of q, i and their derivatives which will therefore be called an *observable*. However, we notice that the choice of any observable $z = \alpha q + i$ does not allow in general to recover i and q separately. Indeed, we have:

$$\frac{dz}{dt} = \alpha \frac{dq}{dt} + \frac{di}{dt} = -\frac{1}{LC} q + \left(\alpha - \frac{R}{L}\right) i + \frac{1}{L} u.$$

Hence, the two linear equations in q and i are linearly independent unless:

$$\det \begin{pmatrix} \alpha & 1 \\ -\frac{1}{LC} & \alpha - \frac{R}{L} \end{pmatrix} = \alpha^2 - \frac{R}{L}\alpha + \frac{1}{LC} = 0.$$

For example, we may choose $\alpha = 1$ when $R = 2$, $L = C = 1$. We also notice that integrating the two previous first order OD equations is equivalent to integrating the following second order OD equation:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{du}{dt}$$

under the condition $1/C \neq 0$ because otherwise we could not recover q from i and should use instead the only first order OD equation for i:

$$L \frac{di}{dt} + Ri = u.$$

Such a difficulty does not occur if we consider the second order OD equation for q:

$$L \frac{d^2 q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = u.$$

Conversely, when one has a second order OD equation such as:

$$\ddot{y} - ay = \dot{u} + u$$

it is possible to replace it by the following system of two first order OD equations:

$$\dot{y}^1 = y^2 + u, \qquad \dot{y}^2 = ay^1 + u$$

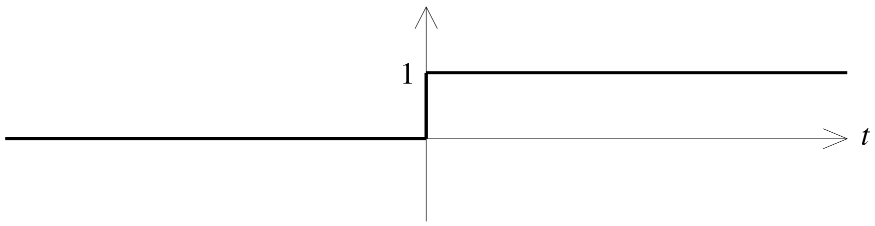
by setting $y = y^1$, a result showing that the system can be observed by means of y as we get at once $y^1 = y$, $y^2 = \dot{y} - u$.

Saying intuitively for the moment that two control systems are *equivalent* whenever integrating one of them amounts to integrating the other, we obtain a kind of "*equivalence relation*" (reflexive, symmetric, transitive) and we understand at once from the previous examples that a correct definition of controllability must also be coherent with such an equivalence relation. In particular, the two preceding second order OD equations seem quite similar at first sight and a major problem during the period between the two world wars has been to find a procedure that could distinguish them. This is the so-called *transfer function* approach that we now sketch on the single input/single output (SISO) situation just encountered.

The *Laplace transform*:

$$f(t) \longrightarrow F(s) = \int_0^\infty e^{-st} f(t)dt$$

is the starting point of *Heaviside calculus*, very useful for the study of electrical circuits. Hence, the transform of the Heaviside unity step function $\Upsilon(t)$ represented by the following picture:



becomes simply:

$$\Upsilon(t) \longrightarrow \frac{1}{s}$$

and is of constant use whenever one suddenly connects an electrical circuit to a battery. The main property of the Laplace transform is:

$$\frac{df(t)}{dt} \longrightarrow sF(s)$$

as can easily be seen after an integration by part if we suppose that $f(0) = 0$. Conversely, when $F(s)$ is known, one can look for $f(t)$ by consulting lists of transformations.

Let us apply the Laplace transform to the RCL-circuit already considered. We obtain:

$$\left(Ls^2 + Rs + \frac{1}{C}\right)I = sU$$

that is to say:

$$I = \frac{s}{Ls^2 + Rs + \frac{1}{C}}U$$

whenever $I(s)$ and $U(s)$ are the respective transformed functions of $i(t)$ and $u(t)$. It seems that there could be a "*simplification*" whenever the numerator and the denominator in the above rational function of s may have a common factor. The only possibility is when the polynomial $Ls^2 + Rs + \frac{1}{C}$ is divisible by s, a fact which only occurs when $1/C = 0$. Such a situation could happen by itself, but cannot happen in the present situation if we take into account the above remark. Indeed, the relation:

$$I = \frac{1}{Ls + R} U$$

does not come from a "simplification" but is effectively true because of the structure of the electrical circuit. The previous difficulty does not occur if we consider the relation:

$$Q = \frac{1}{Ls^2 + Rs + \frac{1}{C}} U$$

as no factorization can be done.

Contrary to such a situation, the case of the other second order OD equation already considered is quite different. Indeed, we have:

$$(s^2 - a)Y = (s + 1)U \quad \Rightarrow \quad Y = \frac{s + 1}{s^2 - a} U.$$

As $s^2 - a = (s - 1)(s + 1) + (1 - a)$, we notice that a "simplification" may occur only when $a = 1$ and provides:

$$Y = \frac{1}{s - 1} U$$

that is to say the same relation as if we were starting from the first order OD equation:

$$\dot{y} - y = u$$

which nevertheless is more restrictive for u and y than the previous second order OD equation, in the sense that it has "less" solutions in y for a given u.

Collecting the previous results, we notice that input/output relations which can be defined by SISO systems may be classified into two categories:

1. Those for which a simplification is possible.

2. Those for which a simplification is not possible.

It remains to understand why we could call controllable the second class of control systems and how we could extend such a classification property to multi inputs/multi outputs (MIMO) systems defined by systems of PD equations, before we could look for any purely mathematical interpretation of it.

To clarify this point at the present stage, let us present two systems of PD equations well known by any engineer and ask whether they are controllable or not.

The first example will be that of Maxwell equations in the form:

$$\vec{\nabla} \wedge \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}, \qquad \vec{\nabla}.\vec{D} = \rho$$

where $\vec{H}$ is the magnetic induction, $\vec{D}$ the electric induction, $\vec{j}$ the density of electric current and ρ the density of electric charge. It is clear, in this example, that the input or *source* term is $(\vec{j}, \rho)$ while the output or electromagnetic induction is $(\vec{H}, \vec{D})$.

The second example will be made by the Euler equations for an incompressible fluid in hydrodynamics:

$$\vec{\nabla}.\vec{v} = 0, \qquad \frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v} + \vec{\nabla} p = 0$$

where $\vec{v}$ is the speed and p the pressure. In this example, by contrast, it is not clear at all how to define input and output.

The answer to the controllability question will constitute a good example of how pure mathematics can unify problems that seem at first sight unrelated.

It took about thirty years to answer the last question concerning controllability but people went first towards another way of looking at controllability, namely through *accessibility* that we now comment on.

Still considering linear SISO systems, if we transform any high order system in the output into a first order system by considering certain derivatives as new unknowns, we obtain in fact a dynamical system depending on the input and *eventually* its derivatives. We shall see later on that this point, rarely quoted in the literature, is quite important. In any case, in view of the preceding examples, derivatives of the input may effectively appear. Nevertheless, as we shall see in the second part of this book, when the order of derivatives of the input is smaller or equal to the order of derivatives of the output (*causality*) one can find techniques allowing to change the output in such a way that the derivatives of the input no longer appear in the corresponding dynamical system (see [122] for more details).

Using the "*dot*" notation for derivatives, let us consider the three following simple SISO systems:

$$\dot{y} - u = 0, \qquad \dot{y} - \dot{u} = 0, \qquad \ddot{y} - \dot{u} = 0$$

and study their respective behaviours with respect to the definition of accessibility in the following sense:

Definition. A dynamical system in the output will be said to have the accessibility property if, whenever two points are given in the output space, there exists an input providing a trajectory starting from the first point at time 0 and passing through the second point at a finite time T.

For the first system, if $v(t)$ is a primitive of $u(t)$, we have $y = v(t) + k$ and, $v(t)$ being arbitrary, we can always fulfill the two conditions:

$$y(0) = v(0) + k, \qquad y(T) = v(T) + k.$$

The first one fixes the integration constant k when $v(0)$ is given and the second one fixes $v(T)$ when T and $y(T)$ are given.

For the second system, we do not even need to find any primitive and must have at once:

$$y(0) = u(0) + k, \qquad y(T) = u(T) + k$$

while the same reasoning as above also works. Hence, from the accessibility point of view both systems seem to behave the same way. However, if we consider the new output/observable $z = y - u$ for the second system, we have $\dot{z} = 0$ and this system clearly does not possess the accessibility property as we must have $z(0) = z(T)$, $\forall T \geqslant 0$.

As for the third system, it exhibits the ambivalence of the two others. Indeed, one can transform it as the dynamical system:

$$\dot{y}^1 = y^2, \qquad \dot{y}^2 - \dot{u} = 0$$

or as the dynamical system:

$$\dot{y}^1 = y^2 + u, \qquad \dot{y}^2 = 0$$

by choosing in both cases $y = y^1$.

As there is "*a priori*" no reason to consider only first order dynamical systems in the outputs containing no derivative of the inputs, there is therefore no specific *intrinsic* link between the controllability property in the previous "simplification" sense and the accessibility property in the present sense, where by "intrinsic" we mean independent of the presentation of the system.

Despite this comment, a fundamental step ahead has been achieved by R.E. Kalman in the beautiful reference [107] of 1963, where he gives for the first time a controllability criterion for linear MIMO control systems of the form:

$$\dot{y} = Ay + Bu$$

where A is a square $m \times m$ constant matrix and B is a rectangular $m \times p$ constant matrix. As we shall see with more details in the second part of this book, the *Kalman test* is based on the possibility to have an explicit general solution of the above system in the form:

$$y(t) = \exp(tA)\, y(0) + \int_0^t \big(\exp(t-\tau)A\big) B\, u(\tau)\, d\tau$$

in order to test effectively the accessibility property. The Kalman test introduces the so-called “*controllability matrix*”:

$$\big(B, AB, A^2B, \dots, A^{m-1}B\big)$$

and says that the system is controllable if and only if the controllability matrix has full rank m. The test is coherent with the Cayley–Hamilton theorem stating the existence of numbers $\alpha_0, \dots, \alpha_m$ such that $\alpha_m A^m + \cdots + \alpha_1 A + \alpha_0 I = 0$, a polynomial identity related to the characteristic polynomial of A by the relation:

$$(-1)^m \alpha_m \det(A - \lambda I) \;\equiv\; \alpha_m \lambda^m + \cdots + \alpha_1 \lambda + \alpha_0.$$

Applying this result to the RCL-electrical circuit with $m = 2$, $p = 1$, we have:

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{1}{CL} & -\frac{R}{L} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix}$$
$$\det(B, AB) = \det \begin{pmatrix} 0 & \frac{1}{L} \\ \frac{1}{L} & \frac{-R}{L^2} \end{pmatrix} = -\frac{1}{L^2} \neq 0$$

and the system is controllable.

Applying this result to the second system gives successively with $m = 2$, $p = 1$:

$$A = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\det(B, AB) = \det \begin{pmatrix} 1 & 1 \\ 1 & a \end{pmatrix} = a - 1.$$

Hence the system is controllable if and only if $a \neq 1$. Indeed, when $a = 1$, setting $z = y^2 - y^1$ and subtracting the first OD equation from the second, we obtain $\dot{z} + z = 0$. Accordingly, if $y^1(0) = y^2(0)$, that is $z(0) = 0$, then $z(T) = 0 \Rightarrow y^1(T) = y^2(T), \forall T \geqslant 0$ and the trajectory follows the diagonal in the output space. As a byproduct, if the initial point is on the diagonal, the final point cannot be reached unless it is also on the diagonal and the system is not controllable in the Kalman sense.

Despite the crucial importance of this test, it raises at once two deep questions.

The first and main problem deals with the illogical fact that a concept defined in a functional way could be tested in a formal way. To understand such a difference, the reader must always have present in mind the difference existing between the operations that can be effected with a computer in the way of *numerical analysis* (finite element method, Galerkin method) only leading to *approximate results* or in the way of *symbolic calculus* (MACSYMA, MAPLE, MATHEMATICA packages) only leading to *exact results*.

The second problem, as we shall understand better in the next pages, is that the Kalman form $\dot{y} = Ay + Bu$ took its importance from the fact that, *in that case and in that case only*, controllability has something to do with accessibility. The new unknowns needed to transform a higher order input/output control system into a first order one with no derivative of input if possible, have been called "*state*". Also, as we shall see, the lack of any other test not referring to the intput/state/output formulation, led people to endow such a presentation of a control system with an importance that it will lose when controllability will be understood *by itself*, that is as a "*built-in*" *structural property* of the control system, independent of the separation of the control variables between inputs and outputs.

The *Hautus test* of 1969, though along the same line as the Kalman test, is for the first time appealing to operator/polynomial matrices [82]. It tells that a system $\dot{y} = Ay + Bu$ is controllable if and only if the matrix $(A - \lambda I, B)$ has rank m for every λ. In fact, as $A - \lambda I$ is a square matrix, its generic rank is m and can decrease only if λ is an eigenvalue of A. Hence the Hautus test amounts to say that the system is controllable if and only if the matrix $(A - \lambda I, B)$ has rank m for every eigenvalue λ of A or, equivalently, that the linear system $\eta A = \lambda\eta$, $\eta B = 0$, where η is a row vector and λ is an eigenvalue of A, has no solution other than $\eta = 0$. Let us just notice here that the Hautus test is coherent with the Kalman test, in the sense that, from $\eta A = \lambda\eta$ we deduce $\eta A^k = \lambda^k\eta$ and thus $\eta A^k B = \lambda^k \eta B = 0$ for $k = 0, 1, \dots, m - 1$, that is $\eta = 0$ whenever the controllability matrix has full rank m.

In the previous example, the rank of the matrix:

$$\begin{pmatrix} -\lambda & 1 & 1 \\ a & -\lambda & 1 \end{pmatrix}$$

is 2 whenever $\lambda^2 - a = 0$, unless $\lambda = -1$ and thus $a = \lambda^2 = 1$.

It is interesting to pass to the operator point of view, though such an alternative approach is lacking from the paper of Hautus. Indeed, writing out the system in the form:

$$-\dot{y} + Ay + Bu = 0$$

we may substitute s in place of the operator $\frac{d}{dt}$ and consider the correspondence:

$$\left(-I\frac{d}{dt} + A, B\right) \leftrightarrow \left(-sI + A, B\right)$$

between *operator matrices* and *polynomial matrices*, without any reference to the Laplace transform any more. The generic rank in s of the resulting matrix is of course m because $-sI + A$ is a square $m \times m$ matrix. Hence, *control systems can be separated into two classes*:

1. The ones for which the rank is m for any s.

2. The ones for which the rank becomes less than m for certain s.

As before, in this second case, we must have $\mathrm{rank}(-sI + A) < m$, thus we must have $\det(-sI + A) = 0$ and s must be an eigenvalue of A as in the Hautus test but in a different framework, a reason allowing to call controllable the systems of the first class as a way to generalize for MIMO systems the separation principle already met for SISO systems.

Coming back to the two preceding examples, we obtain for the RCL-circuit the following polynomial matrix (up to sign):

$$\begin{pmatrix} s & -1 & 0 \\ Ls + R & \frac{1}{C} & -1 \end{pmatrix}$$

which has rank 2 for any s because the determinant of the submatrix made by the second and third columns is equal to 1. Similarly, we obtain for the second example the polynomial matrix:

$$\begin{pmatrix} -s & 1 & 1 \\ a & -s & 1 \end{pmatrix}$$

which has generic rank 2 and rank 1 only when the determinants of the 3 submatrices are nul, that is to say:

$$s^2 - a = 0, \quad s + a = 0, \quad s + 1 = 0 \qquad \Rightarrow \qquad a = 1.$$

The above comment is important for two reasons. First, it proves that controllability is a structural property of a control system that does not depend on the separation of the control variables between input and output. Second, it proves that controllability can be extended to higher order situations and solves at least our problem concerning the transfer matrix approach. Indeed, if we consider now a SISO system:

$$P\left(\frac{d}{dt}\right)y + Q\left(\frac{d}{dt}\right)u = 0$$

the corresponding polynomial 1 row matrix is:

$$\big(P(s),\, Q(s)\big)$$

and finding s such that $P(s) = 0$, $Q(s) = 0$ amounts to say that $P(s)$ and $Q(s)$ have a common root. In the contrary case, that is if $P(s)$ and $Q(s)$ have no common root and are relatively prime, Hilbert theorem or, in this case, a succession of euclidean divisions, provides polynomials $A(s)$ and $B(s)$ such that:

$$P(s)A(s) + Q(s)B(s) \equiv 1$$

as a Bezout identity equivalent to the lack of simplification in the transfer function.

It is only during the period 1970–1980 that the preceding ideas have been extended to MIMO systems and we refer the reader to the pioneering reference [20] by H. Blomberg and R. Ylinen.

Let us consider the MIMO system:

$$P\Big(\frac{d}{dt}\Big)y + Q\Big(\frac{d}{dt}\Big)u = 0$$

where now P is a square $m \times m$ operator matrix and Q is a rectangular $m \times p$ operator matrix. *By analogy with the case of the SISO systems*, we shall say that this system is controllable if and only if the corresponding polynomial matrices $P(s)$ and $Q(s)$ are *left coprime*, that is one can find an $m \times m$ polynomial matrix $A(s)$ and a $p \times m$ polynomial matrix $B(s)$ such that one has the generalized Bezout identity:

$$P(s)A(s) + Q(s)B(s) \equiv I_m$$

where I_m stands for the $m \times m$ identity matrix. As a generalization of the Hautus test, we notice that for all values of s we have:

$$\eta P(s) = 0, \quad \eta Q(s) = 0 \qquad \Rightarrow \qquad \eta = 0$$

and thus the rank of the total polynomial matrix $(P(s), Q(s))$ is equal to its generic rank m for *every* s. Moreover, we can write the preceding matrix relation in the form:

$$\big(P(s),\, Q(s)\big)\begin{pmatrix} A(s) \\ B(s) \end{pmatrix} \equiv I_m$$

showing that the above definition of controllability does not depend on the separation of the control variables between input and output. We notice that, introducing the adjoint matrices and changing s to $-s$, we have the equivalent identity:

$$\big(\tilde{A}(-s),\, \tilde{B}(-s)\big)\begin{pmatrix} \tilde{P}(-s) \\ \tilde{Q}(-s) \end{pmatrix} \equiv I_m$$

that will get a meaning from the next pages.

In the language of operator matrices, we shall say that the linear operator $\big(P(\frac{d}{dt}), Q(\frac{d}{dt})\big)$ sending a vector with $m+p$ function components to a vector with m function components, *which is surjective because we have assumed that the generic rank of* $(P(s), Q(s))$ *was equal to* m, admits a *lift*.

Until now, we hope that the reader still does not see what homological algebra has to do with controllability and, for a moment, we pass to the study of nonlinear control systems along with the following simple illustrative SISO example:

$$u\dot{y} - \dot{u} = a = \text{cst}.$$

The basic problem is to define and understand controllability for such a system and to look for the corresponding critical values of the constant parameter a involved in the system.

A first idea is to look for accessibility. However, even if $a = 0$, we may integrate and find as before that the initial and final conditions:

$$y(0) = \log u(0) + k, \qquad y(T) = \log u(T) + k$$

can always be satisfied. Hence we have the same trouble as for linear systems but without the possibility now to use the transfer function approach.

A second idea is to linearize the system and to consider the linearized system:

$$u\dot{Y} + \dot{y}U - \dot{U} = 0.$$

This can be done by giving a "small" variation $U = \delta u$, $Y = \delta y$ to u and y or, more precisely, to substitute $u + \varepsilon U$ and $y + \varepsilon Y$ in place of u and y in such a way as to compute the derivative with respect to ε for $\varepsilon = 0$. It seems that this system no longer depends on a but we must not forget that the coefficients now depend on the solution $(y(t), u(t))$ of the given control system which depends on a. In that situation we have a linear system indeed but with variable coefficients and the previous method does not work because the Laplace transform cannot be used as before. Hence it should remain to extend the operator/polynomial matrix approach to the variable coefficients case. For this, we have to transform the purely "*matrix*" approach to a purely "*operator*" approach. Surprisingly, we have not found any trace of this alternative in the literature. Mixing together the control variables under the name ξ, we may represent the control system by the system of OD equations $\mathcal{D}\xi = 0$ where $\mathcal{D}$ is an operator matrix, with the property that $\mathcal{D}$ is a *surjective operator* in the sense that the system with second member $\mathcal{D}\xi = \eta$ does not bring any compatibility condition on η by eliminating ξ. For simplicity, in the sequel we shall omit the word "matrix" and speak only about "operator". The question is now to discover the condition on $\mathcal{D}$ for the existence of an operator or *differential lift* $\mathcal{P}$ such that $\mathcal{D} \circ \mathcal{P} = I$, that is, if we write $\mathcal{D}\xi = \eta$ for any ξ, we must have the *identity* $\mathcal{D} \circ \mathcal{P}\eta \equiv \eta$ for any η. The crucial idea will be to introduce the *adjoint operator* $\widetilde{\mathcal{D}} = \mathrm{ad}(\mathcal{D})$ of $\mathcal{D}$ along the three following simple rules also valid for PD operators:

1. The adjoint of a zero order operator or matrix of coefficients is just the transposed matrix.
2. The adjoint of ∂_i is $-\partial_i$, hence the adjoint of $\frac{d}{dt}$ is $-\frac{d}{dt}$.
3. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are differential operators that can be composed, then $\widetilde{\mathcal{D}_2 \circ \mathcal{D}_1} = \widetilde{\mathcal{D}}_1 \circ \widetilde{\mathcal{D}}_2$.

Equivalently, one can multiply the equations on the left by the row vector λ and integrate by part the scalar function $\lambda\mathcal{D}\xi$ in order to "push" all the derivatives from ξ to λ. We shall write symbolically:

$$\langle \lambda, \mathcal{D}\xi \rangle = \langle \widetilde{\mathcal{D}}\lambda, \xi \rangle + \mathrm{div}(\ )$$

by using the Stokes formula.

Accordingly, taking the adjoint, we obtain the relation:

$$\widetilde{\mathcal{P}} \circ \widetilde{\mathcal{D}} = \tilde{I} = I$$

because the transpose of the identity matrix I is again the identity matrix I. But $\widetilde{\mathcal{P}}$ is now on the left of $\widetilde{\mathcal{D}}$ and its interpretation is quite different as it becomes the *left inverse* of $\mathcal{D}$. Equivalently, we discover that $\widetilde{\mathcal{D}}$ admits a left inverse if and only if it is injective and this result will bring about the necessity to introduce the delicate concept of *formal integrability*.

To let the reader understand it, we start with one of the best examples we know, namely the third order system of PD equations in one unknown y:

$$\begin{cases} \partial_{222}y + x^2 y = u \\ \partial_{11}y + \partial_2 y = v \end{cases}$$

with two independent variables (x^1, x^2) and second members (u, v).

Differentiating the first PD equation with respect to x^1 twice, we get:

$$\partial_{11222}y + x^2\partial_{11}y = \partial_{11}u.$$

Differentiating the second PD equation with respect to x^2 three times, we get:

$$\partial_{11222}y + \partial_{2222}y = \partial_{222}v.$$

Differentiating the first PD equation with respect to x^2, we get:

$$\partial_{2222}y + x^2\partial_2 y + y = \partial_2 u.$$

Hence, we obtain by linear combination:

$$x^2\partial_{11}y + x^2\partial_2 y + y = \partial_{11}u + \partial_2 u - \partial_{222}v$$

and thus:

$$y = \partial_{11}u + \partial_2 u - \partial_{222}v - x^2 v.$$

It follows that:

$$u = 0, \quad v = 0 \qquad \Rightarrow \qquad y = 0$$

and the corresponding operator is injective while the preceding result provides a differential inverse. This inverse is not necessarily unique as one may add expressions involving the two compatibility conditions in (u, v) which are of sixth order (exercise). For the sake of completeness and later use, we provide them as follows:

$$\begin{aligned} A &\equiv \partial_{11222}u + \partial_{2222}u - \partial_{222222}v - 2x^2\partial_{222}v \\ &\qquad\qquad + x^2\partial_{11}u + x^2\partial_2 u - u - 3\partial_{22}v - (x^2)^2 v = 0 \\ B &\equiv \partial_{111}u + 2\partial_{112}u + \partial_{22}u - \partial_{11222}v - \partial_{2222}v - x^2\partial_{11}v - x^2\partial_2 v - v = 0 \end{aligned}$$

and we finally have the unique compatibility condition (exercise):

$$\partial_{222}B - \partial_{11}A - \partial_2 A + x^2 B \equiv 0.$$

In our previous book we have also quoted and treated an example given by M. Janet in 1920 as follows (see [90, 93] for more examples):

$$\begin{cases} \partial_{33}y - x^2\partial_{11}y = 0 \\ \partial_{22}y = 0. \end{cases}$$

It is not evident at first sight that the space of solutions is a 12-dimensional vector space over the constants but one can discover this fact by a direct integration of the second PD equation, substituting the resulting expression $y = A(x^1, x^3)x^2 + B(x^1, x^3)$ in the first PD equation and separating the variables (exercise).

To recapitulate, when a system of PD equations is given by equations of order q, we may need to differentiate many times in order to get all the possible PD equations of order $0, 1, 2, \ldots$ that could be used for an explicit (symbolic) computation of the formal power series solutions. For checking the difficulty, the reader can study the Janet example with $(x^2)^r$ in place of x^2 for $r = 0, 1, 2, 3$. Of course, the ultimate situation for a linear system of PD equations is when one can find so many new lower order PD equations that the only possible solution is the zero solution which always exists and this is just the case we have considered above. Examining all the differentiations/combinations needed, eventually gives the desired differential inverse and the corresponding differential lift is thus obtained by taking the adjoint operator.

This procedure will be applied to our variable coefficients linearized system to obtain:

$$\lambda(u\dot{Y} + \dot{y}U - \dot{U}) = -(u\dot{\lambda} + \dot{u}\lambda)Y + (\dot{\lambda} + \dot{y}\lambda)U + \frac{d}{dt}\big(\lambda uY - \lambda U\big)$$

and we have therefore:

$$\widetilde{\mathcal{D}}\lambda = 0 \quad \Leftrightarrow \quad \dot{\lambda} + \dot{y}\lambda = 0, \quad u\dot{\lambda} + \dot{u}\lambda = 0.$$

Multiplying the first OD equation by u and subtracting from it the second OD equation, we finally get the zero order OD equation $a\lambda = 0$. Hence $\widetilde{\mathcal{D}}$ is injective if and only if $a \neq 0$ and thus $\mathcal{D}$ admits a lift if and only if $a \neq 0$ too.

Finally, we notice that the surjectivity of $\mathcal{D}$, always implicitly assumed in classical control theory, is an essential assumption and it remains to know what to do if we do not assume the surjectivity of $\mathcal{D}$ any longer. Also, if we can hope that the injectivity of $\widetilde{\mathcal{D}}$ brings controllability in the OD case, it is not sure at all (and not true indeed as we shall see) that such a result remains valid in the PD case.

A third idea is to try to bring the control system to a form close to the Kalman form by finding out a nonlinear equivalent dynamical system where the derivative of u should not appear. In our example, this can be done as follows:

$$\begin{cases} \dot{y}^1 = ay^2 - v \\ \dot{y}^2 = y^2 v \end{cases}$$

by setting $y^2 = 1/u$ and we are led to examine nonlinear dynamical systems of the form:

$$\dot{y} = f(y) + g(y)u$$

waiting for results that *must* be coherent with the special linear case

$$f(y) = Ay, \qquad g(y) = B.$$

We now have one vector field f with m components and p vector fields g with m components too by setting:

$$f = f(y)\frac{\partial}{\partial y}, \qquad g = g(y)\frac{\partial}{\partial y}$$

and the problem is to know what to do with vectors in place of matrices.

It happens that the only possible intrinsic construction with two vector fields $\xi = \xi^i(x)\partial_i$ and $\eta = \eta^j(x)\partial_j$ on an n-dimensional manifold X with local coordinates (x^i) where $i = 1, \dots, n$ is to construct their *bracket* as a new vector field defined by the following formula:

$$\big([\xi,\eta]\big)^i(x) = \xi^r(x)\partial_r\eta^i(x) - \eta^r(x)\partial_r\xi^i(x)$$

which is easily seen not to depend on the coordinate system and has therefore an intrinsic meaning. We let the reader check by himself as an exercise the *Jacobi identity*:

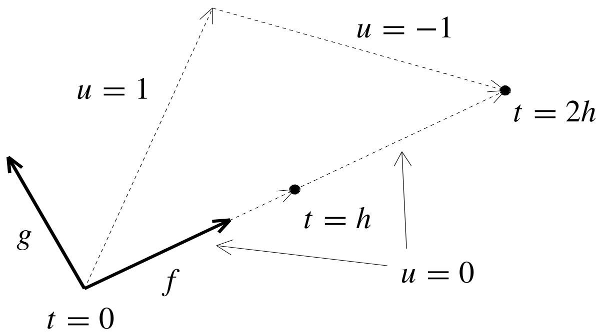
$$[\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]] = 0, \qquad \forall \xi,\eta,\zeta.$$

By differentiating the preceding control system, we get:

$$\begin{aligned}\ddot{y} &= \frac{\partial f}{\partial y}\dot{y} + \frac{\partial g}{\partial y}\dot{y}u + g\dot{u}\\ &= f\frac{\partial f}{\partial y} + \Big(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\Big)u + g\frac{\partial g}{\partial y}uu + g\dot{u}.\end{aligned}$$

The "$+$" sign in the coefficient of u seems surprising compared to the "$-$" sign of the bracket but comes from the fact that a second order derivative does not transform like a vector when one changes the coordinates y.

In order to make the bracket appearing, let us compare the trajectory without control obtained by setting $u = 0$ between $t = 0 \Rightarrow y = y(0) = y_0$ and $t = 2h$ with the trajectory followed by setting $u = 1$ from $t = 0$ to $t = h$ and $u = -1$ from $t = h$ to $t = 2h$, assuming that $h \ll 1$ as in the following picture:



The Taylor development gives:

$$
\begin{aligned}
y(h) &= y_0 + h\big(f(y_0) + g(y_0)\big) + \\
&\qquad \frac{h^2}{2}\Big(f\frac{\partial f}{\partial y} + \Big(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\Big) + g\frac{\partial g}{\partial y}\Big)(y_0) + \cdots \\
y(2h) &= y(h) + h\big(f(y(h)) - g(y(h))\big) + \\
&\qquad \frac{h^2}{2}\Big(f\frac{\partial f}{\partial y} - \Big(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\Big) + g\frac{\partial g}{\partial y}\Big)(y(h)) + \cdots
\end{aligned}
$$

By addition and substitution, we get:

$$y(2h) = y_0 + 2hf(y_0) + h^2\Big(2f(y_0)\frac{\partial f(y_0)}{\partial y} - [f, g](y_0)\Big) + \cdots$$

while the trajectory corresponding to $u = 0$ should be:

$$\bar{y}(2h) = y_0 + 2hf(y_0) + 2h^2 f(y_0)\frac{\partial f(y_0)}{\partial y} + \cdots$$

and we obtain by subtraction the striking relation:

$$\bar{y}(2h) - y(2h) = h^2[f, g](y_0) + \cdots$$

showing out the importance of the bracket. Now, considering successively ($u_1 = 1$, $u_2 = 0$) from $t = 0$ to $t = h$, ($u_1 = 0$, $u_2 = 1$) from $t = h$ to $t = 2h$, ($u_1 = -1$, $u_2 = 0$) from $t = 2h$ to $t = 3h$, ($u_1 = 0$, $u_2 = -1$) from $t = 3h$ to $t = 4h$ should similarly lead to the relation:

$$y(4h) = y_0 + h^2[g_1, g_2](y_0) + \cdots$$

Collecting the last two relations, we discover that, contrary to what is claimed in many books [88, 147], accessibility has to do with the vector field f while controllability, a property depending on what happens just by changing the input, has to do with vector fields such as:

$$g_1, g_2, [f, g_1], [f, g_2], [g_1, g_2], \ldots$$

at least. This is the true reason for which people usually make a distinction between the so-called "*weak controllability distribution*", namely the *distribution* or collection of vector spaces at each point generated by f, g_1, g_2 and all their possible brackets, on one side, and the so-called "*strong controllability distribution*", namely the collection of vector spaces at each point generated by $g_1, g_2, [g_1, g_2], \ldots$ and closed with respect to taking the bracket with f, that is also containing $[f, g_1], [f, g_2], \dots$ It should be emphasized that *only the strong controllability distribution is coherent with the Kalman test for linear systems* because, in that case, we have:

$$f(y) = Ay, \quad g(y) = B \qquad \Rightarrow \qquad [g_1, g_2] = 0, \quad [f, g] = -AB, \quad \dots$$

Applying this result to our preceding example, we have:

$$f = ay^2 \frac{\partial}{\partial y^1}, \quad g = -\frac{\partial}{\partial y^1} + y^2 \frac{\partial}{\partial y^2} \qquad \Rightarrow \qquad [f, g] = -f.$$

The distribution generated by g and $[f, g] = -f$ has generic rank 2 unless $a = 0$ but, again, though such a result is coherent with the one obtained by linearization, once more it does not give any idea about a possible formal definition of controllability.

Let us prove that the use of the adjoint operator brings at once the necessity to use the strong controllability distribution and *not* the weak one. For this, linearizing the system $\dot{y} = f(y) + g(y)u$, while using the same capital letters for variations as above, we obtain:

$$-\dot{Y} + \frac{\partial f}{\partial y} Y + u \frac{\partial g}{\partial y} Y + gU = 0.$$

Multiplying by a test row vector λ on the left and integrating by part, we discover at once that the kernel or solution space of the adjoint operator is made by the λ solutions of the system of OD equations:

$$\dot{\lambda} + \lambda \frac{\partial f}{\partial y} + \lambda u \frac{\partial g}{\partial y} = 0, \qquad \lambda g = 0.$$

This system is *not* formally integrable in general because, differentiating the zero order OD equation we obtain:

$$\dot{\lambda} g + \lambda \Big(f \frac{\partial g}{\partial y} + ug \frac{\partial g}{\partial y} \Big) = 0.$$

Multiplying the first order OD equation by g on the right and subtracting, we get the new zero order OD equation:

$$\lambda \big([f, g] + u[g, g] \big) = 0.$$

If the maximum number of linearly independent zero order OD equations for λ were stabilizing after $(r-1)$ differentiations, differentiating once more, we should get a new OD equation of the form $\lambda u^{(r)}[g, g] + \cdots = 0$ which should be a linear combination of the preceding "saturated" OD equations. As u is arbitrary, such an equality separately implies:

$$\lambda[f, g] = 0, \qquad \lambda[g, g] = 0$$

Starting afresh with these new OD equations for λ, we discover that the number of zero order OD equations for λ only depends on the rank or dimension of the distribution generated by the vector fields $g, [f, g], [g, g], \ldots$

Independently of the previous algebraic/geometric approach to control theory that we have sketched, J.L. Lions started around 1970 the theory of *distributed parameter systems* as a tentative to study the control theory of PD equations (see [56]).

The example of heating a room will be convenient for pointing out a few criticisms, despite the fundamental importance of the method in principle for engineering applications such as cooling or regulating a nuclear plant, stopping the vibrations of long antennas or structures folding out from satellites, to consider only two specific examples.

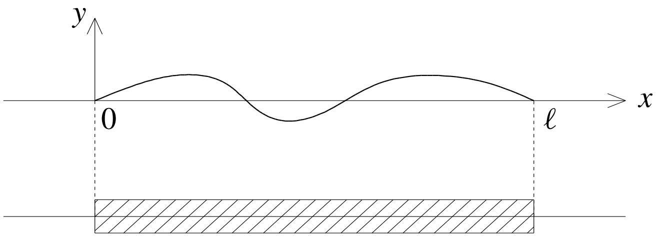
Considering for simplicity the 1-dimensional case of temperature propagating along a bar of length ℓ, the unknown will be the temperature $y = y(x, t)$ satisfying the linear PD control system:

$$\frac{\partial y}{\partial t} = c\frac{\partial^2 y}{\partial x^2} + u$$

where the input $u = u(x, t)$ is describing the heating source. Such a PD equation is an example of the more general linear distributed control system, also called *evolution equation*:

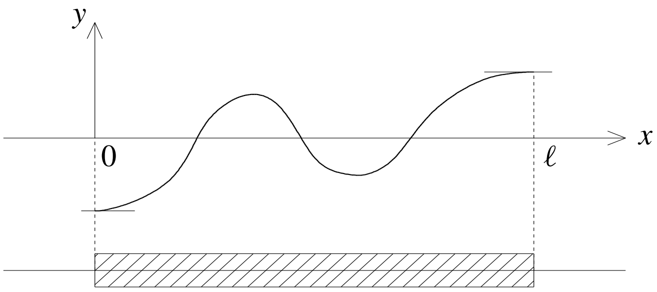
$$\frac{\partial y}{\partial t} = Ay + Bu$$

where now A and B are linear differential operators with respect to the space coordinates x on an open set $\Omega \subset \mathbb{R}^n$. Such a system of PD equations is always completed by *initial conditions* $y(x, 0) = y_0(x)$ on Ω and by *boundary conditions* which can be of the *Dirichlet type* with the function y given on $]0, T[\, \times \Gamma$ where Γ is the boundary of Ω, or of the *Neumann type* with the normal gradient $\partial y/\partial n$ given on $]0, T[\, \times \Gamma$. In our example, $y(0, t)$ and $y(\ell, t)$ are given in the Dirichlet case while $\partial y/\partial x$ is given for $x = 0, x = \ell$ in the Neumann case according to the two following pictures:



$\Omega =]0, \ell[$

Dirichlet condition: $y(0, t) = 0$, $y(\ell, t) = 0$



$\Omega =]0, \ell[$

Neumann condition: $\frac{\partial y}{\partial x} = 0$ for $x = 0, x = \ell$

The input can be explicit as in the present example or can be given only through the boundary conditions. Two typical examples could be respectively with $\Omega =]0, 1[$:

$$\frac{\partial y}{\partial t} = c\frac{\partial^2 y}{\partial x^2}, \qquad y(x, 0) = 0, \qquad y(0, t) = 0, \qquad y(1, t) = u(t),$$

$$\frac{\partial y}{\partial t} = c\frac{\partial^2 y}{\partial x^2}, \qquad y(x, 0) = 0, \qquad \frac{\partial y}{\partial x}(0, t) = 0, \qquad \frac{\partial y}{\partial x}(1, t) = u(t).$$

By copying what was done for classical control systems in 1970, people used to extend the definition of controllability from the finite dimensional situation to the infinite dimensional situation by using functional analysis in the case of an evolution equation [56]:

Definition. The distributed system:

$$\frac{\partial y}{\partial t} = Ay + Bu, \qquad y(x, 0) = y_0(x)$$

is said to be *approximately controllable* if the output $y_u(x, T)$ for $T > 0$ spans a dense subspace of $L^2(\Omega)$, the space of square integrable functions over Ω, whenever the input u takes all possible values in the input space.

Roughly, starting from $y_0(x)$ in $L^2(\Omega)$, for example $y_0(x) = 0$, one can always find an input such that $y_u(x, T)$ becomes arbitrarily close to a given final value $y_T(x)$ in $L^2(\Omega)$.

It is clear that such a definition, which depends on the control system, is as far as possible from a formal definition because not only it also depends on the boundary conditions but it also highly depends on the choice of the input signal space.

We shall illustrate this definition by using the last example considered with $\Omega =]0, 1[$ and $u \in L^2\big(]0, T[\big)$ in order to exhibit the role of the adjoint operator. Indeed, multiplying the evolution equation by a test function λ and integrating by part, we may consider the adjoint system:

$$\frac{\partial\lambda}{\partial t} + c\frac{\partial^2\lambda}{\partial x^2} = 0, \qquad \frac{\partial\lambda}{\partial x}(0, t) = 0, \qquad \frac{\partial\lambda}{\partial x}(1, t) = 0.$$

If the control system were not controllable in the previous functional sense, then the set of $y_u(x, T)$ should not be dense in $L^2\big(]0, 1[\big)$ and, according to the Hahn–Banach theorem, we could find a function $\eta(x)$ such that $\eta \neq 0$, $\eta \in L^2(\Omega)$ and:

$$\int_0^1 \eta(x) y_u(x, T)\, dx = 0, \qquad \forall u \in L^2\big(]0, T[\big).$$

Now we have:

$$\lambda\Big(\frac{\partial y}{\partial t} - c\frac{\partial^2 y}{\partial x^2}\Big) = -\Big(\frac{\partial\lambda}{\partial t} + c\frac{\partial^2\lambda}{\partial x^2}\Big)y + \frac{\partial}{\partial t}(\lambda y) + c\frac{\partial}{\partial x}\Big(y\frac{\partial\lambda}{\partial x} - \lambda\frac{\partial y}{\partial x}\Big).$$

An integration on space-time (x, t) gives

$$\int_0^T \int_0^1 \Big[\frac{\partial}{\partial t}(\lambda y) + c\frac{\partial}{\partial x}\Big(\frac{\partial\lambda}{\partial x}y - \lambda\frac{\partial y}{\partial x}\Big)\Big]\, dx\, dt = 0$$

and thus

$$\int_0^1 \Big[\lambda y\Big]_0^T dx + c\int_0^T \Big[\frac{\partial\lambda}{\partial x}y - \lambda\frac{\partial y}{\partial x}\Big]_0^1 dt = 0.$$

Taking into account the initial and boundary conditions, we obtain:

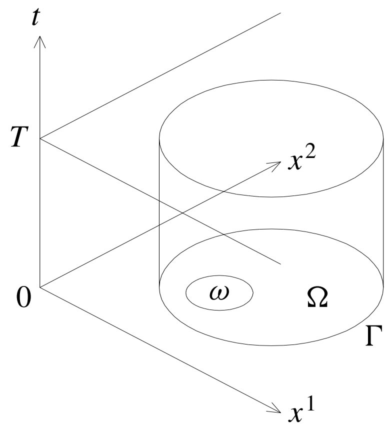
$$\int_0^1 \lambda(x, T) y_u(x, T)\, dx = c\int_0^T \lambda(1, t)u(t)\, dt.$$

Choosing now $\lambda(x, t)$ such that $\lambda(x, T) = \eta(x)$, we must have:

$$\int_0^T \lambda(1, t)u(t)\, dt = 0, \qquad \forall u \in L^2\big(]0, T[\big)$$

and such a condition is known to imply $\lambda(1, t) = 0$ on $]0, T[$. We shall admit that such a condition on λ also implies $\lambda(x, t) = 0$ everywhere and in particular $\eta(x) = \lambda(x, T) = 0$ leading to a contradiction.

Let us now consider the case $\Omega \subset \mathbb{R}^2$ as in the following picture where ω is an arbitrarily small open subset of Ω and $\Delta = \partial_{11} + \partial_{22}$ is the Laplacian operator.



Let us consider the heat equation:

$$\frac{\partial y}{\partial t} - \Delta y = u\chi_\omega$$

where χ_ω is the characteristic function of ω, equal to 1 on ω and 0 outside. We assume the initial condition $y(x, 0) = 0$ and the boundary condition $y = 0$ on $\Gamma \times]0, T[$. To fix ideas, we also assume that $u \in L^2\big(\omega \times]0, T[\big)$. Then it is known that $y_u(x, T)$ is uniquely defined in $L^2(\Omega)$. We prove that $y_u(x, T)$ spans a dense subset of $L^2(\Omega)$ when u spans $L^2\big(\omega \times]0, T[\big)$. As before, let $\eta \neq 0$, $\eta \in L^2(\Omega)$ be such that:

$$\int_\Omega y_u(x, T)\eta(x)\, dx = 0, \qquad \forall u \in \text{input space}$$

with $dx = dx^1 dx^2$ and introduce the solution $\lambda(x, t)$ of the adjoint system:

$$\begin{aligned} &\frac{\partial \lambda}{\partial t} + \Delta\lambda = 0 \quad \text{in } \Omega \times]0, T[\\ &\lambda(x, T) = \eta(x), \quad \lambda = 0 \quad \text{on } \Gamma \times]0, T[. \end{aligned}$$

Integrating the relation:

$$\lambda\Big(\frac{\partial y}{\partial t} - \Delta y\Big) = -\Big(\frac{\partial \lambda}{\partial t} + \Delta\lambda\Big)y + \frac{\partial}{\partial t}(\lambda y) + \vec{\nabla}.\big(y\vec{\nabla}\lambda - \lambda\vec{\nabla}y\big)$$

over the cylinder $\Omega \times]0, T[$, we obtain:

$$\int_\Omega \big[y\lambda\big]_0^T\, dx = \int_{\Omega\times]0,T[} \lambda u \chi_\omega \, dx\, dt = 0$$

for any $u \in L^2\big(\omega,]0, T[\big)$, that is to say: $\lambda = 0$ on $\omega \times]0, T[$. It follows that $\lambda = 0$ in $\Omega \times]0, T[$ and thus $\eta = 0$ as before, leading to a contradiction.

With the same notations, let us now consider the following nonlinear system given by A. Bamberger and J. Henry in 1977:

$$\frac{\partial y}{\partial t} - \Delta y + y^3 = u\chi_\omega$$

with the same initial and boundary conditions as before. We prove that $y_u(x, T)$ spans a set which is far from being dense in $L^2(\Omega)$ whenever u spans $L^2\big(\omega \times]0, T[\big)$. Let us introduce a function $\psi = \psi(x)$ such that $\psi = 0$ on ω, $\psi > 0$ outside $\overline{\omega}$, $\int_\Omega \psi^{-3}\big|\vec{\nabla}\psi\big|^4 dx = c < \infty$ where $\overline{\omega}$ is the closure of ω and ψ is smooth enough so that the integrations by part which follow are valid. Multiplying the previous PD equation by ψy and integrating on Ω, the right side cancels and we obtain:

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \psi y^2 dx + \int_\Omega \psi\big|\vec{\nabla} y\big|^2 dx + \int_\Omega y\vec{\nabla}\psi.\vec{\nabla} y\, dx + \int_\Omega \psi y^4 dx = 0.$$

We have the estimation:

$$\begin{aligned}\Big|\int_\Omega y\vec{\nabla}\psi.\vec{\nabla} y dx\Big| &= \Big|\int_\Omega \Big(\psi^{\frac14} y\Big)\Big(\psi^{\frac12}\vec{\nabla} y\Big).\Big(\psi^{-\frac34}\vec{\nabla}\psi\Big)dx\Big| \\ &\leqslant \int_\Omega \psi\big|\vec{\nabla} y\big|^2 dx + \int_\Omega \Big(\psi^{\frac12} y^2\Big)\Big(\psi^{-\frac32}|\vec{\nabla}\psi|^2\Big)dx \\ &\leqslant \int_\Omega \psi y^4 dx + \int_\Omega \psi\big|\vec{\nabla} y\big|^2 dx + \int_\Omega \psi^{-3}\big|\vec{\nabla}\psi\big|^4 dx\end{aligned}$$

and, looking at the sign of the various integrals, we obtain finally the estimate:

$$\int_\Omega \psi\big|y_u(x, T)\big|^2 dx \leqslant 2cT = \text{cst}$$

which proves our desired result. From this example, J.L. Lions has conjectured that "*the addition of a damping or viscosity term could diminish controllability*".

In any case, the comparison between the results obtained for OD control systems (*localized sense*) and those obtained for PD control systems (*distributed sense*) clearly shows that, in the first framework, controllability is a built-in property of the control system which can be eventually defined in terms of accessibility as a property of trajectories for a very particular class of control systems though it does not depend on initial conditions through formal tests, while, in the second framework, it is a typical functional property which surely depends on the control system but does not seem to be a built-in property because it also heavily depends on the boundary conditions and cannot certainly be tested in a formal way.

The reader must not forget that the theory of distributed control systems, being born around 1970, at a period when people did not even know or feel that classical controllability was not depending on the separation of the control variables between input and output, could not evolve in another way, especially when it was motivated by specific engineering problems.

To summarize the results so far obtained, we arrive at the two following possibilities:

- Controllability is a *functional property* that must admit a *functional definition* and can be tested by the methods of *functional analysis*.

- Controllability is a *formal property* that must admit a *formal definition* and can be tested by a *formal test* (rank of matrices, distributions, . . .).

If the first point of view is good, the existing definition is only valid for dynamical systems (OD equations) and evolution equations (PD equations) but nothing is known for other types of systems. Also it is strange that it can be tested in a formal way for OD equations only.

If the second point of view if good, it remains to find out the formal definition involved in a manner coherent with all the cases already met by considering OD equations as just particular cases of PD equations. Of course, in this approach, the natural extension to PD equations becomes *very different* from the distributed approach as opposed to the localized approach.

Our purpose is now to explain that *such a new point of view does exist indeed* and gives rise to a "*control theory for PD equations*" which does not seem at first sight to have any relation with the already existing one that we have just described, even though it is clear that links will be established in the future.

Our first task will be to explain the formal definition of controllability that we gave for the first time in 1986 (C.R. Acad. Sci. Paris, t. 302, 547–550) and which is valid for all types of OD as well as PD control systems. For this purpose, we forget for a few pages about control theory and present the basic concepts of differential algebra that are needed in order to understand the origin of this definition.

Roughly speaking, by a *ring* A we mean a set of elements $a, b, c, \dots$ closed under the operations denoted by "$+$" for addition, "$-$" for subtraction and "$\times$" for multiplication, with the additional property that the multiplication is associative and distributive with respect to the addition. In the sequel, we shall always suppose that all the rings considered contain the ring $\mathbb{Z}$ of relative integers as a subring, that is as a subset closed under the above operations.

Similarly, by a *field* K we mean a set of elements $a, b, c, \ldots$ closed under the operations denoted by "$+$" for addition, "$-$" for subtraction, "$\times$" for multiplication and "$/$" or ":" for division by nonzero elements. We shall always suppose in this book that all the fields considered will contain the subfield $\mathbb{Q}$ of rational numbers as a subset closed under the above operations. By an *extension* L of K or simply by an extension L/K we mean a field L containing the field K as a subfield.

Historically, algebra started with the search for roots of polynomial equations. The only root of $ay+b=0$ with $a, b \in K$ is simply $y = -\frac{b}{a} \in K$ from the definition of K. As "*zero*" and "*negative numbers*" were not considered before the sixteenth century, it is only during the Renaissance period that the general solution of the quadratic $ay^2 + by + c = 0$ and cubic $y^3 + py + q = 0$ algebraic equations has been known by means of "extracting" respectively square and cubic radicals. Proper symbols have been introduced to simplify the explicit writing of numbers that could not be known exactly or could not be calculated. For example, the positive root of $y^2 - 2 = 0$ is denoted by $\sqrt{2}$ to be read "*square root of 2*" in order to avoid writing down its explicit value, namely 1,414 . . . . Similarly, the root of $y^2 + 1 = 0$ is denoted by "i" and called an imaginary quantity.

More generally, if y is an indeterminate, we shall denote by $K[y]$ the ring of polynomials in y with coefficients in K. A polynomial $P \in K[y]$ is called *irreducible* if it cannot be the product of two polynomials in $K[y]$. The reason for introducing such a definition is that if $P = P_1 P_2$, then looking for the roots of P amounts to looking for the roots of P_1 *and* P_2 *separately*, which both have degree less than that of P and are "*simpler*" in a certain sense. Many practical tests have been invented for checking the irreducibility of a polynomial, among which we quote the well known Eisenstein criterion. More generally, if $y = (y^1, \ldots, y^m)$ is a family of indeterminates, we may still denote simple by $K[y]$ the polynomial ring $K[y^1, \ldots, y^m]$ and pay attention to the *ideal* $(P_1, \ldots, P_r)$ generated by r polynomials $P_1, \ldots, P_r$ in $K[y]$, namely by the subset made by all the linear combinations of $P_1, \ldots, P_r$ with polynomial coefficients.

Then algebra passed from the study of a single polynomial equation $P(y) = 0$ in one indeterminate to the study of polynomial equations $P_1(y) = 0, \ldots, P_r(y) = 0$ in many indeterminates. The key concept became that of a *prime ideal*, namely an ideal $\mathfrak{p}$ in a ring A is said to be prime if, whenever a product $ab \in \mathfrak{p}$ for $a, b \in A$, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$.

From the previous lines it becomes clear that $\sqrt{2}$ is just a symbol for a number such that its square can be replaced by 2. Similarly i is just a symbol for a quantity such that its square can be replaced by -1 but the level of abstraction is higher. If now $\mathfrak{p}$ is an ideal in a ring A, we may introduce the *residue ring* $A/\mathfrak{p}$ with operations induced by those of A while taking into account that $\mathfrak{p}$ is an ideal. When $\mathfrak{p}$ is a prime ideal, then $A/\mathfrak{p}$ becomes an *integral domain*, that is does not possess any non zero zerodivisor. Fractions of the type a/b with $b \neq 0$ can be added according to the rule $a/b + c/d = (ad + bc)/bd$ and, when $a \neq 0$, the inverse of a/b is b/a, such a construction allowing to consider the "*field of fractions*" $Q(A/\mathfrak{p})$ of $A/\mathfrak{p}$ as a field containing A. If $A = K[y]$ as before, we have thus obtained an (abstract) extension L/K. Another description in this case is to introduce the image η^k of y^k for $k = 1, \ldots, m$ or simply the image η of y in the canonical projection $A[y] \to A[y]/\mathfrak{p}$ which is such, by definition, that $P(\eta) = 0, \forall P \in \mathfrak{p}$. We may set $K[y]/\mathfrak{p} = K[\eta]$ and $L = Q(K[y]/\mathfrak{p}) = K(\eta)$ becomes the field of rational functions in η with coefficients in K. The essential idea has been to associate a field extension L/K with any prime ideal $\mathfrak{p} \subset K[y]$. Of course, we discover that η is just a symbol that generalizes the more specific ones already met such as $\sqrt{2}$ and i. One can also view η as a *generic solution* of the equations $P(y) = 0, \forall P \in \mathfrak{p}$ or simply of $\mathfrak{p}$ in the sense that $P(\eta) = 0 \Leftrightarrow P \in \mathfrak{p}$.

In the case of a field extension L/K, the only possible *intermediate field* between K and L that can be constructed is the *algebraic closure* K_0 of K in L, namely the subfield of L made by all elements of L that are roots of a polynomial equation with coefficients in K.

Differential algebra deals with the possibility to add the word "*differential*" in front of all the concepts of algebra that we have just considered. We ask the reader to stop reading ahead for a while and to ask himself what controllability could have to do with "*differential algebraic closure*".

Differential algebra has been created by J.F. Ritt in 1930 [178] but the idea of a differential extension could be found in the thesis of J. Drach about a generalization to PD equations of the Galois theory, as early as 1898 [162]. Let us illustrate the main difficulties by following as closely as possible the preceding pages dealing with pure algebra.

To start with, a *differential ring* A is a ring with n commuting *derivations* ∂_i. More precisely, each ∂_i is an automorphism of A such that $\partial_i(a + b) = \partial_i a + \partial_i b$, $\partial_i(ab) = (\partial_i a)b + a\partial_i b$, $\forall a, b \in A$ and moreover $\partial_i\partial_j = \partial_j\partial_i$, $\forall i, j = 1, \ldots, n$. We shall speak about an *ordinary differential* (OD) ring if $n = 1$ and about a *partial differential* (PD) ring if $n \geqslant 2$. Similar definitions should hold for a *differential field* K and we add:

$$\partial_i\left(\frac{a}{b}\right) = \frac{(\partial_i a)b - a\partial_i b}{b^2} \qquad \forall a, b \neq 0 \in K.$$

By a *differential extension* L of K we mean a differential field L which is also an extension of K with derivatives d_i such that $d_i|_K = \partial_i, \forall i = 1, \ldots, n$.

Historically, differential algebra started with the search for solutions of certain

OD equations in one indeterminate with coefficients in the trivial differential field of *constants* $\mathbb{Q}$ or, more generally, in the differential field $\mathbb{Q}(x) = \mathbb{Q}(x^1, \dots, x^n)$ with $\partial_i = \partial/\partial x^i$. Denoting simply by y_x the usual differentiation of a *dependent variable* y as a function of the *independent variable* x, the simplest example is that of the linear OD equation $y_x - y = 0$ with analytic solution $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$ simply denoted by the symbol e^x. Also we may consider the linear inhomogeneous OD equation $(x+1)y_x - 1 = 0$ with analytic solution $y = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ simply denoted by the symbol $\log(x+1)$. Similarly, one may quote the well known linear second order Bessel OD equation:

$$y_{xx} + \frac{1}{x} y_x + \Big(1 - \frac{\nu^2}{x^2}\Big) y = 0$$

or Legendre OD equation:

$$(1 - x^2) y_{xx} - 2x y_x + \nu(1+\nu) y = 0$$

where ν is an arbitrary constant parameter. If ν is a positive integer, the solutions of the Legendre OD equation are the so-called Legendre polynomials:

$$1 \text{ for } \nu = 0, \quad x \text{ for } \nu = 1, \quad \frac{1}{2}(3x^2 - 1) \text{ for } \nu = 2, \quad \frac{1}{2}(5x^3 - 3x) \text{ for } \nu = 3, \quad \dots$$

which are orthogonal with respect to the integration of the product of any two of them on the segment $]-1, +1[$. Of course, as each Legendre polynomial belongs to $\mathbb{Q}[x]$, it is far from being a generic solution of the corresponding Legendre OD equation.

The question of irreducibility is much more tricky in the differential framework. In order to understand it fully because it is the key concept and difficulty for studying differential extensions, we briefly review a few classical examples among the best we know.

To start with, let us consider the algebraic OD equation $P(y, y_x) \equiv {y_x}^2 - 4y = 0$ where P is a polynomial in y and y_x therefore called a *differential polynomial*. Considering y as a function of x and y_x as its derivative, differentiating as usual we find at once $d_x P \equiv 2y_x(y_{xx} - 2) = 0$ and it seems that, if P is irreducible in the variables (y, y_x), then $d_x P$ becomes reducible in the variables (y, y_x, y_{xx}). Hence any "solution" of the OD equation $P = 0$ must satisfy either $y_x = 0$ and thus be the trivial solution $y = 0$, or $y_{xx} - 2 = 0$ and thus be a solution of the form $y = x^2 + ax - \frac{1}{4}a^2$ with $a = \text{cst}$ an arbitrary constant parameter. We also point out that the meaning of "$a = \text{cst}$" is not clear at all as a may belong to $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots$ or even be just a symbol such that $a_x = 0$. Also we notice that $y = 0$ does not belong to the second parametrized family of solutions and the set of all solutions therefore splits into two subsets, namely the trivial solution $y = 0$ on one side and the set of solutions of $(y_x)^2 - 4y = 0$, $y_{xx} - 2 = 0$ on the other side. The OD equation $y_{xxx} = 0$ is common to both families as it is an evident differential consequence of $y = 0$ or of $y_{xx} - 2 = 0$.

At first sight, it seems that the study of the OD equation $P \equiv {y_x}^2 - 4y^3 = 0$ is similar but we shall see that this if not true at all. Indeed, we have

$$d_x P \equiv 2y_x(y_{xx} - 6y^2)$$

but we cannot imagine to consider separately the solution $y = 0$ and the set of solutions of ${y_x}^2 - 4y^3 = 0$, $y_{xx} - 6y^2 = 0$ because now $y = 0$ is among this set. Also, after an easy but tedious computation, we obtain the tricky relation:

$$\begin{aligned}(y_{xx} - 6y^2)^4 &\equiv (y_{xx} - 6y^2)^2({y_{xx}}^2 - 12y^2 y_{xx} + 36y^4)\\ &\equiv \big(\frac{1}{2}y_{xx} - 6y^2\big)\big[(y_{xx} - 6y^2)d_{xx}P - (y_{xxx} - 12yy_x)d_x P\big]\\ &\qquad + \frac{9}{4}y(d_x P)^2 - 9y\big(y_{xx} - 6y^2\big)^2 P.\end{aligned}$$

Hence, among the differential consequences of $P = 0$ we have $(y_{xx} - 6y^2)^4 = 0$ and thus $y_{xx} - 6y^2 = 0$ for *any* solution, contrary to the situation in the previous example.

The case of algebraic PD equations is even worse. With one unknown y and two independent variables (x^1, x^2), let us consider the two differential polynomials:

$$P_1 \equiv y_{22} - \frac{1}{2}(y_{11})^2, \qquad P_2 \equiv y_{12} - y_{11}.$$

We obtain the relations:

$$\begin{aligned}(y_{11} - 1)y_{111} &\equiv d_2 P_2 - d_1 P_1 + d_1 P_2\\ (y_{111})^3 &\equiv y_{111}(d_{12}P_2 - d_{11}P_1 + d_{11}P_2) - y_{1111}(d_2 P_2 - d_1 P_1 + d_1 P_2).\end{aligned}$$

As before, we could believe at first that the set of solutions of $P_1 = 0$, $P_2 = 0$ splits into two subsets, namely the set of solutions of $y_{11} = 1$, $y_{12} = 1$, $y_{22} = \frac{1}{2}$ on one side and the set of solutions of $P_1 = 0$, $P_2 = 0$, $y_{111} = 0$ on the other side. The PD equation $y_{111} = 0$ is a differential consequence of the two systems at the same time and the first subset of solutions is contained into the second subset of solutions. However, according to the second relation above, any solution of $P_1 = 0$, $P_2 = 0$ also satisfies $(y_{111})^3 = 0$ and thus $y_{111} = 0$. This result showing that the previous splitting must not be taken into account, can only be discovered after two differentiations.

On the contrary, the differential polynomial $P \equiv {y_x}^2 - 1$ leads to $d_x P \equiv y_x y_{xx}$ and thus to

$$y_{xx} \equiv y_x d_x P - y_{xx} P$$

despite the fact that $P = P_1 P_2$ with $P_1 = y_x - 1$, $P_2 = y_x + 1$.

More generally, if $y = (y^1, \ldots, y^m)$ is a family of differential indeterminates, we may denote by y_q the set of derivatives up to order q of all the y^k for $k = 1, \ldots, m$ and introduce the ring $K\{y\}$ of differential polynomials as the limit for $q \to \infty$ of the polynomial ring $K[y_q]$. Each derivative can be considered as a new indeterminate (*jet coordinate*) and denoted by y^k_μ where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index such that we should have $\partial_\mu = (\partial_1)^{\mu_1} \ldots (\partial_n)^{\mu_n}$ for the usual differentiation. The ring $K\{y\}$ becomes a differential ring with n commuting derivations d_i if we set:

$$d_i y^k_\mu = y^k_{\mu+1_i} \qquad \text{with } \mu + 1_i = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n)$$

and $d_i|_K = \partial_i$. The field of quotients of $K\{y\}$, that is to say the field of rational functions in y_q for arbitrary q, will be denoted by $K\langle y\rangle$ and becomes a differential field under the rule:

$$d_i\Big(\frac{a}{b}\Big) = \frac{(d_i a)b - a d_i b}{b^2} \qquad \forall a, b \in K\{y\}.$$

A *differential ideal* in $K\{y\}$ will be an ideal stable under the d_i for $i = 1, \ldots, n$ and will be called a *prime differential ideal* if it is at the same time a prime ideal and a differential ideal. As before, when $\mathfrak{p} \subset K\{y\}$ is a prime differential ideal, the residue ring $K\{y\}/\mathfrak{p}$ is a differential integral domain and its field of fractions $L = Q(K\{y\}/\mathfrak{p})$ is a differential extension of K. Also the image η of y under the canonical projection $K\{y\} \to K\{y\}/\mathfrak{p}$ can be considered as a *generic solution* of $\mathfrak{p}$. We may set $K\{y\}/\mathfrak{p} = K\{\eta\}$ and $L = K\langle\eta\rangle$ becomes the differential field of differential rational functions in η, that is rational functions of η and all its derivatives. The essential idea has been to associate a differential field extension L/K with any prime differential ideal $\mathfrak{p} \subset K\{y\}$. Of course, we discover that η is just a symbol that generalizes the more specific ones already met such as e^x and *a major problem is thus to have a criterion in order to recognize whether a differential ideal is prime or not*. Surprisingly, though we have clearly formulated and illustrated this criterion in [162, 163] it is still not quoted or acknowledged today in the literature [22] because it deeply depends on the formal theory of systems of PD equations developed by D.C. Spencer and collaborators (D. Quillen, H. Goldschmidt) during the period 1965–1975 [76, 77, 78, 161, 173, 197].

According to the previous examples, among ideals we may call *perfect* or *radical* the ones that do not contain a power of an element without containing this element too. In polynomial rings, perfect ideals play a fundamental role as one can prove that they can be expressed as the intersection of a finite number of prime ideals. Such a result is coherent with Hilbert theorem saying that a polynomial Q becomes zero for all the solutions of given polynomials $P_1, \ldots, P_r$ if and only if a power of Q belongs to

$(P_1, \dots, P_r)$. It is rather a chance that perfect differential ideals can also be expressed as the intersection of a finite number of prime differential ideals despite the fact that we deal with an infinite number of indeterminates. Again, a criterion for recognizing perfect differential ideals does exist and set up most formal problems for deciding about the study of a differential ideal [163].

In the case of a differential extension L/K, the algebraic closure K_0 of K in L is a differential intermediate field. Indeed, if $\zeta \in L$ satisfies $P(\zeta) = 0$, we may suppose that the polynomial $P \equiv P(z)$ has no double root. The polynomials P and $P' = dP/dz$ are therefore relatively prime and one can find a couple (A, B) of polynomials such that $AP + BP' = 1$. We have now $d_i P \equiv P' d_i z + \partial_i P$ where $\partial_i P$ means that we only touch the coefficients of P in K. Combining these relations, we have:

$$d_i z \equiv B d_i P + A d_i z P - B \partial_i P$$

and substituting η in place of z, we get:

$$d_i \eta = -B(\eta) \partial_i P(\eta) \in K(\eta).$$

The only possible other intermediate differential field between K_0 and L will be the *differential algebraic closure* K' of K in L, namely the subset of L made by all the elements that are solutions of at least one differential polynomial with coefficients in K. The reader will notice through the proof given in the first part of the book that it is not evident at all that this subset is indeed a subfield and even a differential field.

It is only at this point that the link with controllability becomes clear if we use differential algebra as a motivating framework for defining new concepts. Indeed, let us consider a control system and call *observable* any (nontrivial true) function of the control variables and their derivatives. There are only two possibilities:

- This observable is *constrained* or *autonomous*, that is it satisfies at least one PD equation for itself.
- This observable is *free*, that is it does not satisfy any PD equation for itself.

In the first situation, we cannot "*act*" on the corresponding PD equation which only contains the observable as a single unknown and no other variable that could be used as an input variable. Hence it is clear in this case that the system is not controllable in any of the possible meanings we have already examined. However, it is not evident at all that the following formal definition is indeed absolutely equivalent to all of them and can be used as a formal definition of controllability.

Definition. A control system is controllable if and only if any observable is free.

The novelty is that such a definition proves that controllability is a built-in property of the control system that does not depend on the separation of the control variables between input and output, exactly as we claimed.

A first warning is to care about the definition of an observable by specifying the class of functions that can be considered. For example, the SISO algebraic control system $u\dot{y} - \dot{u} = 0$ is such that the observable $z = y - \log u$ satisfies $\dot{z} = 0$ but all algebraic observables are free.

A second warning will concern the previous notations. Indeed, the differential polynomial $P \equiv u\dot{y} - \dot{u}$ clearly generates a prime differential ideal in the differential ring $\mathbb{Q}\{u, y\}$ over the trivial differential ring $\mathbb{Q}$ and we should define the observable say ζ, in terms of the respective residual images ξ and η of u and y, as an element of the corresponding differential extension $\mathbb{Q}\langle \xi, \eta \rangle / \mathbb{Q}$. Hence, it could be misleading to define the observable as a function of the indeterminates directly. For this reason, with a slight abuse of language when no confusion may happen, observables will also be defined simply in terms of the differential indeterminates and the corresponding OD equation or PD equation, if it exists, will be found by means of differential elimination theory.

For the two above reasons, it becomes natural to believe that it will be particularly easy to study the case of linear control systems defined over a differential ring A, in terms of existing algebra, by introducing concepts from *module* theory.

We recall that a module M over a ring A or simply an A-module M is a commutative additive group with a map $A \times M \to M : (a, x) \to ax$ such that, $\forall a, b \in A$, $\forall x, y \in M$:

$$\begin{cases} a(x + y) = ax + ay \\ (ab)x = a(bx) \\ (a + b)x = ax + bx. \end{cases}$$

In all this book we shall only consider, unless specified, *finitely generated* modules, that is to say modules generated over A by a finite number of their elements. In that case, we call a non-zero element $x \in M$ a *torsion element* if one can find a non zerodivisor (CARE) $a \in A$ such that $ax = 0$. We say that M is *torsion-free* if it has no torsion element and M is *torsion* if any element of M is a torsion element. The reader must care about the fact that the ring $\mathbb{Z}/4\mathbb{Z}$ is torsion-free if we consider it as a module over itself though $2 \times 2 = 4 \equiv 0 \pmod 4$, as 2 is a zerodivisor and thus not a torsion element.

Now the basic idea is just to restrict all the preceding definitions and constructions from polynomials to linear combinations in the indeterminates in order to define by residue a *differential module* M over A or simply a differential A-module, whenever A is a differential ring. An additional idea is to introduce the non-commutative *ring* *of differential operators* $D = A[d_1, \ldots, d_n]$ or shortly $D = A[d]$ with multiplication satisfying $d_i d_j = d_j d_i$ for all $i, j = 1, \ldots, n$ and $d_i a = a d_i + \partial_i a$ in order to consider M simply as a D-module. We can now state the following restricted definition:

Definition. A linear control system is controllable if and only if the corresponding D-module is torsion-free.

We study a few explicit examples in order to show the coherence of the two preceding definitions with a few results already presented.

Let us start with a control system $\dot{y} = Ay + Bu$ in Kalman form and set $z = \lambda y + \mu_0 u + \mu_1 \dot{u} + \mu_2 \ddot{u} + \cdots$ where λ and the μ are constant (possibly complex) row vectors and the derivatives of the system equations have been used in order to compute $\dot{y}, \ddot{y}, \dddot{y}, \ldots$ in terms of $y, u, \dot{u}, \ddot{u}, \ldots$ only. We have:

$$\begin{aligned} \dot{z} &= \lambda A y + \lambda B u + \mu_0 \dot{u} + \mu_1 \ddot{u} + \cdots \\ \ddot{z} &= \lambda A^2 y + \lambda A B u + \lambda B \dot{u} + \mu_0 \ddot{u} + \mu_1 \dddot{u} + \cdots \end{aligned}$$

and therefore z can satisfy an OD equation by itself if and only if:

1. $0 = \mu_0 = \mu_1 = \mu_2 = \cdots$
2. $\lambda B = 0, \quad \lambda A B = 0, \quad \ldots \quad \lambda A^{m-1} B = 0.$

Not only we recover the controllability matrix from a completely different point of view but also understand why controllability and accessibility coincide on the Kalman type systems because any constrained observable must be of the form $z = \lambda y$ *in this case only*. Using the Cayley–Hamilton formula we have in any case:

$$\alpha_m z^{(m)} + \cdots + \alpha_1 \dot{z} + \alpha_0 z = 0$$

but it may happen that one can factorize the characteristic polynomial by using complex numbers. For example, if we look for a constrained observable satisfying $\left(\frac{d}{dt} - \nu\right) z \equiv \dot{z} - \nu z = 0$ we should have in that case:

$$\lambda A = \nu \lambda, \qquad \lambda B = 0$$

and we recover the Hautus test though we notice that λ is a complex vector whenever ν is a complex eigenvalue.

As an illustration, let us consider the system:

$$\begin{cases} \dot{y}^1 = y^1 + u \\ \dot{y}^2 = y^3 \\ \dot{y}^3 = -y^2 \end{cases} \quad \Rightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Setting $z = iy^2 + y^3$, we obtain $\dot{z} - iz = 0$ and the characteristic polynomial factorizes into $(\nu - 1)(\nu - i)(\nu + i)$. Nevertheless, the controllability matrix is:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and has rank 1. Hence, any *real* linear combination of y^2 and y^3, say $z = \lambda_2 y^2 + \lambda_3 y^3$ is a torsion element satisfying $\ddot{z} + z = 0$.

In the case of a nonlinear system of the form $\dot{y} = f(y) + g(y)u$, a constrained observable must be of the form $z = h(y)$ because of the same reasoning as before and we get successively [166]:

$$\begin{aligned}
&z = h(y) \\
&\dot{z} = f\frac{\partial h}{\partial y} + g\frac{\partial h}{\partial y}u && \Rightarrow \quad g\frac{\partial h}{\partial y} = 0 \\
&\ddot{z} = f\frac{\partial}{\partial y}\Big(f\frac{\partial h}{\partial y}\Big) + g\frac{\partial}{\partial y}\Big(f\frac{\partial h}{\partial y}\Big)u && \Rightarrow \quad g\frac{\partial}{\partial y}\Big(f\frac{\partial h}{\partial y}\Big) = 0
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \qquad & g\frac{\partial f}{\partial y}\frac{\partial h}{\partial y} + fg\frac{\partial}{\partial y}\Big(\frac{\partial h}{\partial y}\Big) \\
&= g\frac{\partial f}{\partial y}\frac{\partial h}{\partial y} + f\Big(\frac{\partial}{\partial y}\Big(g\frac{\partial h}{\partial y}\Big) - \frac{\partial g}{\partial y}\frac{\partial h}{\partial y}\Big) \\
&= [g, f]\frac{\partial h}{\partial y} = 0.
\end{aligned}$$

The above notation without indices could even be simplified by writing the operation of a vector on a function with a "dot" like $f \cdot h$ in order to use $g \cdot (f \cdot h) - f \cdot (g \cdot h) = [g, f] \cdot h$ and we let the reader fill in the details. Hence, $h(y)$ must be killed by $g, [f, g], \ldots$ and we find back the so-called strong controllability distribution. We are left with:

$$z = h(y) = h_0(y), \qquad \dot{z} = h_1(y), \qquad \ddot{z} = h_2(y), \ldots$$

and z satisfies a nonlinear OD equation of order at most equal to m obtained by eliminating y.

We consider now the case of the more general non-linear SISO system $u\dot{y} - \dot{u} = a$ over $\mathbb{Q}(a)$ where a is a constant arbitrary parameter, keeping the dot derivatives, we obtain successively:

$$z = f(y, u)$$

$$\begin{aligned}
\dot{z} &= \frac{\partial f}{\partial y}\dot{y} + \frac{\partial f}{\partial u}\dot{u} \\
&= \frac{\partial f}{\partial y}\frac{a}{u} + \Big(\frac{1}{u}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial u}\Big)\dot{u} \qquad\qquad \Rightarrow \frac{\partial f}{\partial y} + u\frac{\partial f}{\partial u} = 0 \\
\ddot{z} &= \frac{\partial^2 f}{\partial y \partial y}\frac{a}{u}\dot{y} + \frac{a}{u}\Big(\frac{\partial^2 f}{\partial u \partial y} - \frac{1}{u}\frac{\partial f}{\partial y}\Big)\dot{u} \\
&= \frac{\partial^2 f}{\partial y \partial y}\frac{a^2}{u^2} + \frac{a}{u}\Big(\frac{1}{u}\frac{\partial^2 f}{\partial y \partial y} + \frac{\partial^2 f}{\partial u \partial y} + \frac{\partial f}{\partial u}\Big)\dot{u} \\
&= \frac{\partial^2 f}{\partial y \partial y}\frac{a^2}{u^2} + \frac{a}{u}\frac{\partial f}{\partial u}\dot{u} \qquad\qquad \Rightarrow a\frac{\partial f}{\partial u} = 0.
\end{aligned}$$

Hence, there are two possibilities:

- $a \neq 0 \Rightarrow \frac{\partial f}{\partial u} = 0 \Rightarrow \frac{\partial f}{\partial y} = 0$ and z can only be a constant, that is there is no nontrivial constrained observable and the system is controllable.
- $a = 0 \Rightarrow \frac{\partial f}{\partial y} + u\frac{\partial f}{\partial u} = 0$. We may set $z = y - \log u$ and get $\dot{z} = 0$, that is the system is not controllable and any constrained observable is a function of $y - \log u$.

Coming back to the linear SISO system:

$$\ddot{y} - ay - u - \dot{u} = 0$$

and using the various derivatives of this OD equation in order to calculate $\ddot{y}, \dddot{y}, \dots$ in terms of $y, \dot{y}, u, \dot{u}, \ddot{u}, \dots$, we understand that any linear observable can be written as:

$$z = \dot{y} + \lambda y + \mu_0 u + \mu_1 \dot{u} + \cdots$$

up to a constant factor, leading to:

$$\begin{aligned}
\dot{z} &= \ddot{y} + \lambda\dot{y} + \mu_o\dot{u} + \mu_1\ddot{u} + \cdots \\
&= \lambda\dot{y} + ay + u + (\mu_0 + 1)\dot{u} + \mu_1\ddot{u} + \cdots
\end{aligned}$$

Hence, we must have *at least* $\mu_0 = -1, \mu_1 = 0, \dots$ in such a way that z could be constrained, that is becomes a torsion element for the corresponding $K[\frac{d}{dt}]$-module with $K = \mathbb{Q}(a)$. Going on differentiating in order to eliminate $y, \dot{y}, u$ among the successive derivatives $z, \dot{z}, \ddot{z}, \dddot{z}$ we get:

$$\begin{cases}
z = \dot{y} + \lambda y - u \\
\dot{z} = \lambda\dot{y} + ay + u \\
\ddot{z} = a\dot{y} + \lambda a y + \lambda u + (\lambda + 1)\dot{u} \quad \Rightarrow \quad \lambda = -1 \\
\dddot{z} = -a\dot{y} + a^2 y + au + (a - 1)\dot{u} \quad \Rightarrow \quad a = 1.
\end{cases}$$

Hence, it is only when $a = 1$ that we can have a torsion element $z = \dot{y} - y - u$ satisfying $\dot{z} + z = 0$. As we have indeed *separately*:

$$\Big(\frac{d}{dt}+1\Big)z \equiv \Big(\frac{d}{dt}+1\Big)\Big[\Big(\frac{dy}{dt}-y\Big)-u\Big]$$
$$\equiv \Big(\frac{d}{dt}+1\Big)\Big(\frac{d}{dt}-1\Big)y - \Big(\frac{d}{dt}+1\Big)u$$

the factorization by the operator $\frac{d}{dt}+1$ explains at once the so-called "simplification" of the transfer function described at the beginning of this introduction.

We now illustrate the new definition of controllability on PD control systems.

In the case of the Maxwell equations, we first notice that, contrary to the "tradition", the inputs are no longer independent but must satisfy the conservation PD equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}.\vec{j} = 0$$

as a differential consequence of the Maxwell system by itself. Now, any observable is a function of $(\vec{j}, \rho, \vec{H}, \vec{D})$ and their derivatives. However, using the Maxwell equations, we can express $(\vec{j}, \rho)$ through the derivatives of $(\vec{H}, \vec{D})$. Hence any observable can be expressed as a function of $(\vec{H}, \vec{D})$ and their derivatives *only*. But $(\vec{H}, \vec{D})$ can be given arbitrarily in the Maxwell equations as the first three equations will provide $\vec{j}$ while the fourth equation will provide ρ. Hence, any (scalar) observable is free and the Maxwell system is controllable.

We now establish the link that the new definition of controllability could provide with *turbulence*. A measure instrument is a technical device (manometer, thermometer, ...) allowing to measure a *scalar quantity* (pressure, temperature, ...). The result at a given point in a medium can be *stationary*, *unstable* or *turbulent*, that is constantly changing "a little" around an equilibrium value in an apparently random fashion. In a complex system involving many unknowns satisfying PD equations, if we do want to study turbulence it may be convenient to restrict our attention to the study of a single scalar observable by decoupling it from the other unknowns, if it can be constrained by at least one OD equation or PD equation. This can be done by means of differential elimination. Hence, at the opposite of a controllable system like the Maxwell system, we may find systems such that any observable (including each unknown) is constrained by at least one OD equation or PD equation. However, the differential elimination may be absolutely tedious and the resulting equation "very" nonlinear. The best example we know about such a situation is the Euler system of PD equations for incompressible fluids that we have already exhibited.

For a 1-dimensional flow, we have:

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0$$

and the two decoupling PD equations for v and p are:

$$\frac{\partial v}{\partial x} = 0, \qquad \frac{\partial^2 p}{\partial x^2} = 0.$$

For a 2-dimensional flow, we have:

$$\begin{aligned}
&\partial_1 v^1 + \partial_2 v^2 = 0\\
&\frac{\partial v^1}{\partial t} + v^1\partial_1 v^1 + v^2\partial_2 v^1 + \partial_1 p = 0\\
&\frac{\partial v^2}{\partial t} + v^1\partial_1 v^2 + v^2\partial_2 v^2 + \partial_2 p = 0.
\end{aligned}$$

Eliminating p by taking the curl and differentiating again with respect to x^2 while setting $x^0 = t$, we get:

$$\begin{aligned}
&v^1\partial_{112}v^2 + \partial_2 v^1\partial_{11}v^2 + v^2\partial_{122}v^2 + \partial_{012}v^2 + (2\partial_2 v^2 + \partial_1 v^1)\partial_{12}v^2\\
&+ (\partial_{22}v^2 + \partial_{12}v^1)\partial_1 v^2 - \partial_2 v^1\partial_{22}v^2 - 2\partial_{22}v^1\partial_2 v^2 - v^2\partial_{222}v^1\\
&- v^1\partial_{122}v^1 - \partial_{022}v^1 - 2\partial_2 v^1\partial_{12}v^1 - \partial_1 v^1\partial_{22}v^1 = 0.
\end{aligned}$$

Using the conservation of mass in order to keep the maximum number of derivatives of v^1 we get:

$$\begin{aligned}
&\partial_2 v^1\partial_{11}v^2 - (\partial_{112}v^1 + \partial_{222}v^1)v^2 - v^1\partial_{111}v^1 - \partial_{011}v^1\\
&- v^1\partial_{122}v^1 - \partial_{022}v^1 + \partial_1 v^1\partial_{11}v^1 - \partial_2 v^1\partial_{12}v^1 + \partial_1 v^1\partial_{22}v^1 = 0.
\end{aligned}$$

This is a PD equation of the form:

$$\partial_2 v^1\partial_{11}v^2 - A(v^1)v^2 - B(v^1) = 0$$

where A and B are nonlinear functions of v^1 and its derivatives up to order 3. Differentiating with respect to x^2 again, we get:

$$\partial_2 v^1\partial_{112}v^2 + \partial_{22}v^1\partial_{11}v^2 - A(v^1)\partial_2 v^2 - d_2A(v^1)v^2 - d_2B(v^1) = 0.$$

Using again the conservation of mass, we have:

$$-\partial_2 v^1\partial_{111}v^1 + \partial_{22}v^1\partial_{11}v^2 + A(v^1)\partial_1 v^1 - d_2A(v^1)v^2 - d_2B(v^1) = 0$$

that is a PD equation of the form:

$$\partial_{22}v^1\partial_{11}v^2 - C(v^1)v^2 - D(v^1) = 0$$

where C and D are nonlinear functions of v^1 and its derivatives up to order 4. Eliminating $\partial_{11}v^2$ among the two quasi-linear PD equations in v^2, we obtain v^2 as a rational function of v^1 and its derivatives up to order 4. Differentiating with respect to v^2 again, replacing $\partial_2 v^2$ by $-\partial_1 v^1$ and eliminating v^2 among the two resulting quasi-linear equations in v^2, we obtain the following decoupling PD equation of order 5 for v^1 only, where, for reducing the length of the formula, we have replaced everywhere $\partial_1 v^1$ by v^1_1 and so on:

$$
\begin{aligned}
&\big(v^1 v^1_2 v^1_{1122} + v^1 v^1_2 v^1_{2222} - v^1 v^1_{22} v^1_{112} - v^1 v^1_{22} v^1_{222}\big) v^1_{11122} + \\
&\big(-v^1 v^1_2 v^1_{1112} - v^1_2 v^1_{0112} - v^1 v^1_2 v^1_{1222} - v^1_2 v^1_{0222} + \big(v^1 v^1_{22} - 2(v^1_2)^2\big) v^1_{111} + \\
&\;2v^1_2 v^1_1 v^1_{112} + v^1_{22} v^1_{011} + \big(v^1 v^1_{22} - 2(v^1_2)^2\big) v^1_{122} + 2v^1_2 v^1_1 v^1_{222} + v^1_{22} v^1_{022} + \\
&\;\big(v^1_2 v^1_{12} - v^1_1 v^1_{22}\big) v^1_{11} + v^1_2 v^1_{22} v^1_{12} - v^1_1 (v^1_{22})^2\big) v^1_{11222} + \\
&\big(v^1_2 v^1_{1122} + v^1_2 v^1_{2222} - v^1_{22} v^1_{112} - v^1_{22} v^1_{222}\big) v^1_{01122} + \\
&\big(v^1 v^1_2 v^1_{1122} + v^1 v^1_2 v^1_{2222} - v^1 v^1_{22} v^1_{112} - v^1 v^1_{22} v^1_{222}\big) v^1_{12222} + \\
&\big(-v^1 v^1_2 v^1_{1112} - v^1_2 v^1_{0112} - v^1 v^1_2 v^1_{1222} - v^1_2 v^1_{0222} + \big(v^1 v^1_{22} - 2(v^1_2)^2\big) v^1_{111} + \\
&\;2v^1_2 v^1_1 v^1_{112} + v^1_{22} v^1_{011} + \big(v^1 v^1_{22} - 2(v^1_2)^2\big) v^1_{122} + 2v^1_2 v^1_1 v^1_{222} + v^1_{22} v^1_{022} + \\
&\;\big(v^1_2 v^1_{12} - v^1_1 v^1_{22}\big) v^1_{11} + v^1_2 v^1_{22} v^1_{12} - v^1_1 (v^1_{22})^2\big) v^1_{22222} + \\
&\big(v^1_2 v^1_{1122} + v^1_2 v^1_{2222} - v^1_{22} v^1_{112} - v^1_{22} v^1_{222}\big) v^1_{02222} + \\
&\big(3(v^1_2)^2 v^1_{1122} + 3(v^1_2)^2 v^1_{2222} + \big(v^1 v^1_{222} - 3v^1_2 v^1_{22}\big) v^1_{112} + \\
&\;v^1 (v^1_{222})^2 - 3v^1_2 v^1_{22} v^1_{222}\big) v^1_{1112} - 3v^1_2 v^1_1 (v^1_{1122})^2 + \\
&\big(3(v^1_2)^2 v^1_{1222} - 6v^1_2 v^1_1 v^1_{2222} + \big(-v^1 v^1_{222} + 3v^1_2 v^1_{22}\big) v^1_{111} + \\
&\;\big(-3v^1_2 v^1_{12} + 3v^1_1 v^1_{22}\big) v^1_{112} - v^1_{222} v^1_{011} + \big(-v^1 v^1_{222} - v^1_2 v^1_{11} + 2v^1_2 v^1_{22}\big) v^1_{122} + \\
&\;\big(-v^1_{022} + v^1_1 v^1_{11} - 3v^1_2 v^1_{12} + 4v^1_1 v^1_{22}\big) v^1_{222}\big) v^1_{1122} + \big(v^1_{222} v^1_{112} + (v^1_{222})^2\big) v^1_{0112} + \\
&\big(3(v^1_2)^2 v^1_{2222} + \big(v^1 v^1_{222} - 3v^1_2 v^1_{22}\big) v^1_{112} + v^1 (v^1_{222})^2 - 3v^1_2 v^1_{22} v^1_{222}\big) v^1_{1222} - \\
&3v^1_2 v^1_1 (v^1_{2222})^2 + \big(\big(-v^1 v^1_{222} + 3v^1_2 v^1_{22}\big) v^1_{111} + \big(-3v^1_2 v^1_{12} + 3v^1_1 v^1_{22}\big) v^1_{112} - \\
&\;v^1_{222} v^1_{011} + \big(-v^1 v^1_{222} - v^1_2 v^1_{11} + 2v^1_2 v^1_{22}\big) v^1_{122} + \\
&\;\big(-v^1_{022} + v^1_1 v^1_{11} - 3v^1_2 v^1_{12} + 4v^1_1 v^1_{22}\big) v^1_{222}\big) v^1_{2222} + \big(v^1_{222} v^1_{112} + (v^1_{222})^2\big) v^1_{0222} + \\
&\big(\big(2v^1_2 v^1_{222} - 3(v^1_{22})^2\big) v^1_{112} + 2v^1_2 (v^1_{222})^2 - 3(v^1_{22})^2 v^1_{222}\big) v^1_{111} + \\
&\big(-2v^1_1 v^1_{222} + 3v^1_{22} v^1_{12}\big)(v^1_{112})^2 + \big(\big(2v^1_2 v^1_{222} + v^1_{22} v^1_{11} - 2(v^1_{22})^2\big) v^1_{122} - \\
&\;4v^1_1 (v^1_{222})^2 + \big(-v^1_{12} v^1_{11} + 5v^1_{22} v^1_{12}\big) v^1_{222}\big) v^1_{112} +
\end{aligned}
$$

$$\big(2v_2^1(v_{222}^1)\big)^2 + \big(v_{22}^1 v_{11}^1 - 2(v_{22}^1)^2\big)v_{222}^1\big)v_{122}^1 - 2v_1^1(v_{222}^1)\big)^3 +$$
$$\big(-v_{12}^1 v_{11}^1 + 2v_{22}^1 v_{12}^1\big)(v_{222}^1)\big)^2 = 0$$

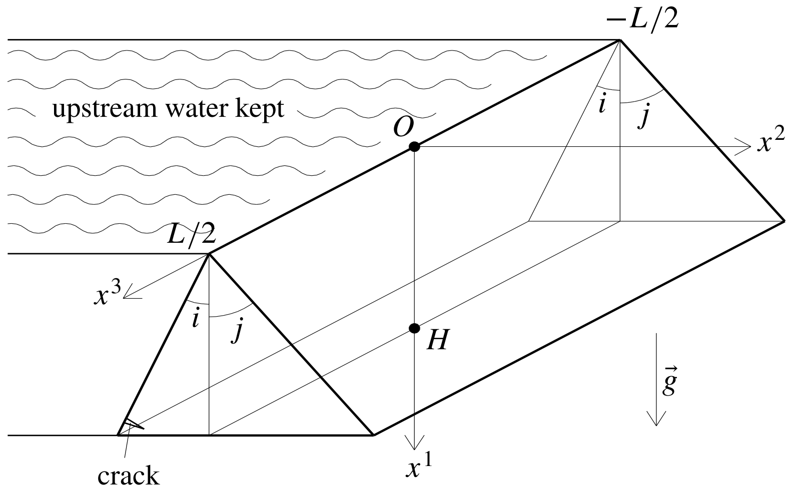
A similar PD equation could be obtained for v^2 but the decoupling PD equation for p should be even more difficult to obtain as it should be of degree higher than 5. *The case of a 3-dimensional flow is completely open*!

It follows from all these examples that, not only we should need a criterion for testing the formal definition of controllability we have given, at least on linear systems, but also, at the opposite, to have a criterion for knowing whether any observable is constrained by at least one PD equation ... without having to exhibit such a PD equation by considering each control variable seperately as we did in the previous example with v^1, v^2 and p.

It took us eight years, from 1986 to 1994, to find such a criterion for testing *effectively* the torsion-free condition for PD control systems. It is only during a visit to Moscow in autumn 1995 that Prof. Palamodov from Moscow University brought our attention to his book [154, 155] studying PD operators with constant coefficients and to the extension done a few years later by M. Kashiwara for arbitrary coefficients in his thesis, unhappily only published in Japanese [111]. By a pure coincidence, the translation of this thesis into English as a "*Memoir of the French Mathematical Society*" just appeared at the same moment and motivated our study of differential modules in a more abstract way.

Nevertheless, we find an interesting matter, at this point, to explain how we arrived *totally independently* at similar results based on *duality techniques*.

Let us forget about control theory for a few pages and consider a typical problem of continuum mechanics, say the building of a dam for fixing the ideas in the case of 2-dimensional elasticity. We want to cut a river of width L with a dam of height H with upstream surface and downstream surface making respectively an angle i and j with the vertical. We set $\alpha = \mathrm{tg}\, i$, $\beta = \mathrm{tg}\, j$, we call π the mass per unit volume of concrete and ρ that of water containing a certain amount of mud. From the summit and center O of the dam, we fix a downwards vertical axis Ox^1, a downstream horizontal axis Ox^2 and complete by an axis Ox^3 in order to have a standard Cartesian frame with coordinates (x^1, x^2, x^3). We suppose that, away from the two ends of the dam, concrete only moves parallel to the plane x^1Ox^2 with displacement $y^1 = x^1 + \xi^1(x^1, x^2)$, $y^2 = x^2 + \xi^2(x^1, x^2)$ where $|\vec{\xi}|$ (a few centimeters) is small compared to L (about 100m) and H (about 25m). We shall look for the condition of selfstability of the dam with respect to crack damage, that is we want that a crack perpendicular to the surface appearing on the upstream bottom side of the dam when the water is at the upper level, does not tend to open under the pure hydrostatic pressure $p = \rho g x^1$ of the water kept upstream, but, on the contrary, could have a tendency to close because of the shape and weight of the dam described below, where $\vec{g} = (g, 0, 0)$ is gravity.



The main geometric ingredient in elasticity theory is the *small strain tensor* ε with components $\varepsilon_{ij} = \frac{1}{2}(\partial_i \xi_j + \partial_j \xi_i)$ where one uses the euclidean metric $\omega = (\omega_{ij} = \omega_{ji})$ with $\omega_{ij} = 1$ if $i = j$, 0 if $i \neq j$ in order to set $\xi_i = \omega_{ir}\xi^r$ in the n-dimensional space $\mathbb{R}^n$. In our case $n = 2$ and we have only three linearly independent components:

$$\varepsilon_{11} = \partial_1 \xi_1, \qquad \varepsilon_{12} = \frac{1}{2}(\partial_1 \xi_2 + \partial_2 \xi_1), \qquad \varepsilon_{22} = \partial_2 \xi_2.$$

In the general case, one can prove that there are $n^2(n^2 - 1)/12$ second order compatibility conditions between the ε_{ij} (see [162, 163] for details), that is for $n = 2$ the only condition:

$$\partial_{11}\varepsilon_{22} + \partial_{22}\varepsilon_{11} - 2\partial_{12}\varepsilon_{12} = 0$$

that can be easily checked.

Like any field theory, continuum mechanics studies the relations existing between *geometry* (strain) and *physics* (stress) by exhibiting a phenomenological *constitutive law* known as the *Hooke law* and respecting the tensorial nature of the quantities involved. More precisely, the *stress tensor* σ with symmetric components $\sigma^{ij} = \sigma^{ji}$ satisfies the static equilibrium condition $\partial_i \sigma^{ij} = f^j$ where f^j only depends on the volume density of forces acting on the medium, here gravity. In our case, inside concrete we have the two PD equations:

$$\partial_1 \sigma^{11} + \partial_2 \sigma^{21} = -\pi g, \qquad \partial_1 \sigma^{12} + \partial_2 \sigma^{22} = 0.$$

The general linear stress-strain relation introduces the two experimental *Lamé constants* (λ, μ) through the tensorial relation:

$$\sigma_{ij} = \lambda \operatorname{tr} \varepsilon \, \omega_{ij} + 2\mu\varepsilon_{ij}$$

where $\sigma_{ij} = \omega_{ir}\omega_{js}\sigma^{rs}$ and $\operatorname{tr} \varepsilon = \varepsilon_{11} + \cdots + \varepsilon_{nn}$. Here, we have:

$$\begin{aligned}
\sigma_{11} &= (\lambda + 2\mu)\varepsilon_{11} + \lambda\varepsilon_{22} \\
\sigma_{12} &= \sigma_{21} = 2\mu\varepsilon_{12} \\
\sigma_{22} &= \lambda\varepsilon_{11} + (\lambda + 2\mu)\varepsilon_{22}.
\end{aligned}$$

Taking the trace of the constitutive law:

$$\operatorname{tr} \sigma = (n\lambda + 2\mu) \operatorname{tr} \varepsilon$$

and inverting it as:

$$2\mu\varepsilon_{ij} = \sigma_{ij} - \frac{\lambda}{n\lambda + 2\mu} \operatorname{tr} \sigma \, \omega_{ij}$$

we discover that σ must not only satisfy the n equilibrium PD equations but *also* the $n^2(n^2 - 1)/12$ second order PD equations coming from the compatibility conditions after the previous substitution. In our case, the main technical difficulty is that we have $n = 2$ though we have to set $n = 3$ in the trace formula. Indeed, we notice that $\varepsilon_{33} = 0$ by assumption but that σ^{33} may not be equal to zero because, under its weight, the dam has a tendency to "flow" and thus to "push" against the two sides as we have indeed $\sigma_{33} = \lambda(\varepsilon_{11} + \varepsilon_{22})$. After an easy calculation left to the reader as a tedious but straightforward exercise, we check that not only σ must satisfy the two static equilibrium PD equations but must also satisfy the following second order Beltrami PD equation:

$$\partial_{11}\sigma_{22} + \partial_{22}\sigma_{11} - 2\partial_{12}\sigma_{12} = 0 \quad \Leftrightarrow \quad \Delta(\sigma_{11} + \sigma_{22}) = 0$$

which, surprisingly, does not depend on the Lamé constants.

The reason for looking after σ is that the boundary conditions on the dam can be easily described in terms of the pressure on the surface of the dam. The "trick" will be to set:

$$\overline{\sigma}^{11} = \sigma^{11} + \pi g x^1, \qquad \overline{\sigma}^{12} = \sigma^{12}, \qquad \overline{\sigma}^{22} = \sigma^{22}$$

in order to have simply:

$$\partial_1\overline{\sigma}^{11} + \partial_2\overline{\sigma}^{21} = 0, \qquad \partial_1\overline{\sigma}^{12} + \partial_2\overline{\sigma}^{22} = 0.$$

We can rewrite the first PD equation as a curl-like PD equation $\partial_1\overline{\sigma}^{11} - \partial_2(-\overline{\sigma}^{21}) = 0$ and thus we can always find a function φ such that:

$$\overline{\sigma}^{11} = \partial_2\varphi, \qquad \overline{\sigma}^{21} = -\partial_1\varphi.$$

We can proceed in the same way for the second PD equation with a function ψ and obtain:

$$\overline{\sigma}^{22} = \partial_1\psi, \qquad \overline{\sigma}^{12} = -\partial_2\psi.$$

As $\overline{\sigma}^{12} = \overline{\sigma}^{21}$, we must have the curl-like condition:

$$\partial_1\varphi - \partial_2\psi = 0$$

and thus one can always find a function ϕ, also called *Airy function*, such that $\varphi = \partial_2\phi$, $\psi = \partial_1\phi$. We therefore obtain the following parametrization:

$$\sigma^{11} = \partial_{22}\phi - \pi g x^1, \qquad \sigma^{12} = -\partial_{12}\phi, \qquad \sigma^{22} = \partial_{11}\phi.$$

Substituting in the Beltrami PD equation we obtain at once the 4$^{\text{th}}$ order PD equation:

$$\Delta\Delta\phi = 0$$

that is ϕ is a biharmonic function. The simplest such function is the 3$^{\text{rd}}$ degree homogeneous polynomial:

$$\phi = \frac{A}{6}\left(x^1\right)^3 + \frac{B}{6}\left(x^2\right)^3 + \frac{C}{2}x^1\left(x^2\right)^2 + \frac{D}{2}\left(x^1\right)^2 x^2.$$

The boundary conditions allow to determine uniquely A, B, C, D and we have for example:

$$A = \frac{-2\alpha^2\beta^2}{(\alpha+\beta)^2}\pi g + \frac{\beta^2(2\alpha^2\beta - 3\alpha - \beta)}{(\alpha+\beta)^3}\rho g.$$

The value of σ thus found shows that the bottom crack has a tendency to close by itself if and only if we have [51, 52]:

$$\beta > \frac{1}{2(\pi-\rho)}\left(\sqrt{\alpha^2\pi^2 + 4\rho(\pi-\rho)} - \alpha\pi\right).$$

Now the volume of the dam is $LH(\alpha+\beta)/2$ and the cost is minimized if and only if $\alpha+\beta$ is minimized under the previous inequality, that is:

$$\alpha = 0, \qquad \beta > \sqrt{\frac{\rho}{\pi-\rho}}$$

or equivalently, in standard situations:

$$i = 0, \qquad j > 40^{\circ}.$$

Such a result explains why a dam made with concrete is always vertical on the side in contact with water.

We now point out the surprising link existing between such a PD mechanical example and controllability.

Let us consider the differential operator $\mathcal{D}$ such that $\mathcal{D}\xi = \varepsilon$ and let us determine its formal adjoint. For $n = 2$, on a purely mathematical side, we have three linearly independent PD equations with second members $(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})$ and we should like to contract these with three test functions, say $(\sigma^{11}, \sigma^{12}, \sigma^{22})$ in order to obtain the scalar $\sigma^{11}\varepsilon_{11} + \sigma^{12}\varepsilon_{12} + \sigma^{22}\varepsilon_{22}$. However, if we want to preserve the tensorial meaning of ε and σ, we should like to write down a tensor contraction of stress with strain such as $\sigma^{ij}\varepsilon_{ij}$ with Einstein summation and obtain thus $\sigma^{11}\varepsilon_{11} + 2\sigma^{12}\varepsilon_{12} + \sigma^{22}\varepsilon_{22}$ for $n = 2$. This result "*tends to prove*" that a good choice for dual test functions is rather $(\sigma^{11}, 2\sigma^{12}, \sigma^{22})$ and we obtain therefore:

$$\begin{aligned}\sigma^{11}\varepsilon_{11} + 2\sigma^{12}\varepsilon_{12} + \sigma^{22}\varepsilon_{22} &= \sigma^{11}\partial_1\xi_1 + \sigma^{12}(\partial_1\xi_2 + \partial_2\xi_1) + \sigma^{22}\partial_2\xi_2 \\ &= -\big[(\partial_1\sigma^{11} + \partial_2\sigma^{21})\xi_1 + (\partial_1\sigma^{12} + \partial_2\sigma^{22})\xi_2\big] \\ &\quad + \partial_1(\sigma^{11}\xi_1 + \sigma^{12}\xi_2) + \partial_2(\sigma^{12}\xi_1 + \sigma^{22}\xi_2).\end{aligned}$$

More generally, we have:

$$\sigma^{ij}\frac{1}{2}(\partial_i\xi_j + \partial_j\xi_i) = \sigma^{ij}\partial_i\xi_j = -(\partial_i\sigma^{ij})\xi_j + \partial_i(\sigma^{ij}\xi_j).$$

With such a definition of duality we discover that the adjoint $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ is nothing else but the operator defining the stress equations. Now, the compatibility condition of $\mathcal{D}$ is the second order operator $\mathcal{D}_1$ determined by the $n^2(n^2 - 1)/12$ compatibility conditions for the strain tensor ε. For $n = 2$, we have $\mathcal{D}_1 : \varepsilon \to \partial_{11}\varepsilon_{22} + \partial_{22}\varepsilon_{11} - 2\partial_{12}\varepsilon_{12}$ and the formal adjoint $\widetilde{\mathcal{D}_1}$ of $\mathcal{D}_1$ is easily determined by means of the relation:

$$\begin{aligned}\lambda(\partial_{11}\varepsilon_{22} + \partial_{22}\varepsilon_{11} - 2\partial_{12}\varepsilon_{12}) = {} & (\partial_{22}\lambda)\varepsilon_{11} - 2(\partial_{12}\lambda)\varepsilon_{12} + (\partial_{11}\lambda)\varepsilon_{22} + \\ & \partial_1\big(\lambda\partial_1\varepsilon_{22} - (\partial_1\lambda)\varepsilon_{22} - 2\lambda\partial_2\varepsilon_{12}\big) + \\ & \partial_2\big(\lambda\partial_2\varepsilon_{11} - (\partial_2\lambda)\varepsilon_{11} + 2(\partial_1\lambda)\varepsilon_{12}\big).\end{aligned}$$

Caring again about the factor 2 introduced above for simplifying reasons, we obtain through this duality, the formal adjoint $\widetilde{\mathcal{D}_1}$ in the unexpected form:

$$\widetilde{\mathcal{D}_1} : \lambda \to \begin{cases}\sigma^{11} = \partial_{22}\lambda \\ \sigma^{12} = -\partial_{12}\lambda \\ \sigma^{22} = \partial_{11}\lambda\end{cases}$$

that is to say *exactly the parametrization of the stress by means of the Airy function* that we have just found in a more direct way. Now, we have of course $\mathcal{D}_1 \circ \mathcal{D} \equiv 0$ and thus $\widetilde{\mathcal{D}} \circ \widetilde{\mathcal{D}_1} \equiv 0$, that is $\widetilde{\mathcal{D}}$ is *among* the compatibility conditions of $\widetilde{\mathcal{D}_1}$. The fact that, in our case $\widetilde{\mathcal{D}}$ *exactly* generates these compatibility conditions is a pure chance as shown by the following example where we have successively:

$$\begin{aligned}
&\mathcal{D} \ : \xi \to \partial_{22}\xi = \eta^2, \quad \partial_{12}\xi = \eta^1 \\
&\mathcal{D}_1 : (\eta^1, \eta^2) \to \partial_1\eta^2 - \partial_2\eta^1 = \zeta \\
&\widetilde{\mathcal{D}_1} : \lambda \to \partial_2\lambda = \mu_1, \quad -\partial_1\lambda = \mu_2.
\end{aligned}$$

Hence $\widetilde{\mathcal{D}_1}$ has a first order compatibility condition $\partial_1\mu_1 + \partial_2\mu_2 = 0$ while we have:

$$\widetilde{\mathcal{D}} : (\mu_1, \mu_2) \to \partial_{12}\mu_1 + \partial_{22}\mu_1.$$

As another simple example, known by everybody, we may quote the chain of 'grad', 'curl', 'div' operators which is also called a *differential sequence* or *complex* because the composition of two successive operators is zero, namely in this case because $\mathrm{curl} \circ \mathrm{grad} \equiv 0$, $\mathrm{div} \circ \mathrm{curl} \equiv 0$. Moreover, one checks easily that 'curl' generates the compatibility conditions for 'grad' while 'div' generates the compatibility conditions for 'curl' and the adjoint operators are respectively 'div', 'curl', 'grad', that is the same operators but in reverse order, having therefore the same properties with respect to the corresponding compatibility conditions.

As we shall see in the second part of the book, these simple example will lead us to a few important results.

The first one is a criterion for testing the controllability of a PD control system in 5 steps:

1. Start with the given control system and denote by η all the control variables in such a way that the control system becomes described by $\mathcal{D}_1\eta = 0$.
2. Construct the formal adjoint $\widetilde{\mathcal{D}_1}$ of $\mathcal{D}_1$.
3. Look for all the compatibility conditions of the linear inhomogeneous system $\widetilde{\mathcal{D}_1}\lambda = \mu$ by exhibiting a generating set in the form $\widetilde{\mathcal{D}}\mu = 0$.
4. Construct the formal adjoint $\mathcal{D}$ of $\widetilde{\mathcal{D}}$ while using $\widetilde{\widetilde{\mathcal{D}}} = \mathcal{D}$.
5. Look for all the compatibility conditions of the linear inhomogeneous system $\mathcal{D}\xi = \eta$ by exhibiting a generating set in the form $\mathcal{D}_1'\eta = 0$.

As we have $\widetilde{\mathcal{D}} \circ \widetilde{\mathcal{D}_1} \equiv 0$, we get $\mathcal{D}_1 \circ \mathcal{D} \equiv 0$ and $\mathcal{D}_1\eta = 0$ is surely *among* the compatibility conditions already found though we have only in general the symbolic inequality $\mathcal{D}_1 \leqslant \mathcal{D}_1'$.

Theorem. *The differential operator* $\mathcal{D}_1$ *is* controllable *if and only if* $\mathcal{D}'_1 = \mathcal{D}_1$.

Corollary 1. The differential operator $\mathcal{D}_1$ is *controllable* if and only if it is *parametrizable* by an operator $\mathcal{D}$, that is $\mathcal{D}_1\eta = 0$ generates all the compatibility conditions of the linear inhomogeneous system $\mathcal{D}\xi = \eta$.

Corollary 2. When $n = 1$, a *surjective* differential operator $\mathcal{D}_1$ is *controllable* if and only if $\widetilde{\mathcal{D}_1}$ is *injective*.

In particular we understand that the rank inequality/equality for the controllability matrix existing in the Kalman test is nothing else but a very special case of the inequality/equality for operators existing in the previous test. Also, as a byproduct of these results, *the study of controllability cannot be separated from the study of formal integrability*. Such a study can lead to *formal obstructions* to the controllability, expressed by various "*trees*" of possibilities as the tricky next example will immediately show out [168].

Consider the following OD control system:

$$\mathcal{D}_1 : (u, y) \to \ddot{y} + \alpha(t)\dot{y} + \dot{\alpha}(t)y + \ddot{u} - \beta u = 0$$

where $\alpha(t)$ is a time varying coefficient and β is a constant coefficient. We shall look for the condition on $\alpha(t)$ and β for the above control system to be controllable or not. For this, we construct the adjoint of $\mathcal{D}_1$:

$$\widetilde{\mathcal{D}_1} : \lambda \to \ddot{\lambda} - \alpha(t)\dot{\lambda} = \mu_1, \qquad \ddot{\lambda} - \beta\lambda = \mu_2$$

and investigate whether $\widetilde{\mathcal{D}_1}$ is injective or not. Setting $\mu_1 = 0$, $\mu_2 = 0$ we shall bring the resulting system:

$$\ddot{\lambda} - \beta\lambda = 0, \qquad \alpha(t)\dot{\lambda} - \beta\lambda = 0$$

to formal integrability by looking for all the OD equations of order 0, 1, 2 respectively.

1. If $\alpha(t) = 0$, we have $\ddot{\lambda} - \beta\lambda = 0$, $\beta\lambda = 0$.
   (a) If $\beta = 0$, we are left with $\ddot{\lambda} = 0$ and $\widetilde{\mathcal{D}_1}$ is not injective, that is $\mathcal{D}_1$ is not controllable with the constrained observable $z = y + u$ satisfying $\ddot{z} = 0$.
   (b) If $\beta \neq 0$, we have $\lambda = 0$ and $\widetilde{\mathcal{D}_1}$ is injective, that is $\mathcal{D}_1$ is controllable as can be checked directly by using the transfer function approach on the OD equation $\ddot{y} = -\ddot{u} + \beta u$.
2. If $\alpha(t) \neq 0$, by differentiating and substituting $\ddot{\lambda} = \beta\lambda$ we obtain the OD equation:

$$\beta(\dot{\alpha} + \alpha^2 - \beta)\lambda = 0$$

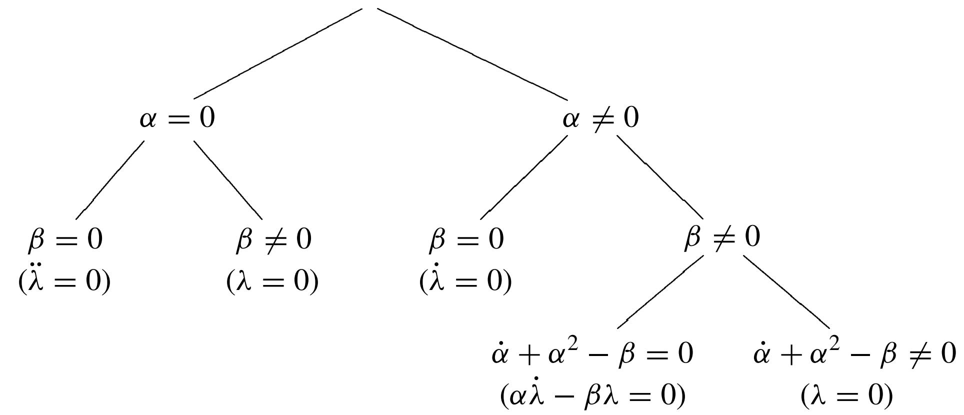
(a) If $\beta = 0$, we are left with $\dot{\lambda} = 0$ and $\widetilde{\mathcal{D}_1}$ is not injective, that is $\mathcal{D}_1$ is not controllable with the constrained observable $z = \dot{y} + \alpha y + \dot{u}$ satisfying $\dot{z} = 0$.

(b) If $\beta \neq 0$ then:

  (i) If $\dot{\alpha} + \alpha^2 - \beta = 0$, we are left with the single OD equation $\alpha\dot{\lambda} - \beta\lambda = 0$ and $\widetilde{\mathcal{D}_1}$ is not injective, that is $\mathcal{D}_1$ is not controllable with the constrained observable $z = \dot{y} + \dot{u} - \frac{\dot{\alpha}}{\alpha}(y + u) - \alpha u$ satisfying $\alpha\dot{z} + \beta z = 0$ (exercise).

  (ii) If $\dot{\alpha} + \alpha^2 - \beta \neq 0$, we are left with $\lambda = 0$ and $\widetilde{\mathcal{D}_1}$ is injective, that is to say $\mathcal{D}_1$ is controllable.

We recapitulate all these results in the following tree:



We notice that the control system is controllable for the leaf corresponding to the *most generic* situation $\alpha \neq 0$, $\beta \neq 0$, $\dot{\alpha} + \alpha^2 - \beta \neq 0$ as well as for the less generic one $\alpha = 0$, $\beta \neq 0$ and these two situations are *robust*, that is to say the property is kept under a small change of the corresponding parameters.

It does not seem that such trees appear in the control literature. Their extension to PD control systems will be done in the second part of the book and is much more difficult as the criterion involves *two* studies of formal integrability, namely the one of $\widetilde{\mathcal{D}_1}$ for determining $\widetilde{\mathcal{D}}$ *and* the one of $\mathcal{D}$ for determining $\mathcal{D}'_1$.

The second crucial result will establish the desired link between control theory and homological algebra within module theory. We have already seen the equivalence of the following objects:

1. linear system of PD equations.

2. linear differential operator $\mathcal{D}$.

3. differential module M.

Moreover we notice the relation $\widetilde{\widetilde{\mathcal{D}}} = \mathcal{D}$ and we may add the new object:

4. linear differential operator $\widetilde{\mathcal{D}} = \mathrm{ad}(\mathcal{D})$.

It is clear that the procedure followed in the criterion and allowing to obtain $\mathcal{D}_1'$ from $\mathcal{D}_1$, can be applied to any operator $\mathcal{D}$ and will provide an operator still denoted by $\mathcal{D}'$. However, we emphasize that the module determined by $\widetilde{\mathcal{D}}$ has nothing to do with the dual of M realized by considering the homomorphisms $\hom_D(M, D)$ over D from M to D as we shall stress out this point in the first part of the book. In fact, we shall see that the equality $\mathcal{D}' = \mathcal{D}$ can be interpreted through two equivalent results.

- The differential module determined by $\mathcal{D}$ is torsion-free.
- The first differential module of extensions of the differential module determined by $\widetilde{\mathcal{D}}$ and D is nul.

We have already explained the first interpretation. In fact, denoting by $t(M)$ the torsion submodule of M generated by all the torsion elements of M and setting $M' = M/t(M)$, then M' is the torsion-free module determined by $\mathcal{D}'$. We now explain the second interpretation.

Alternatively, if we do not want to give a name to the differential module determined by $\widetilde{\mathcal{D}}$, we may "turn around" the names in the criterion and start afresh with a differential operator $\mathcal{D}$ and the corresponding module M. If $\mathcal{D}_1$ describes the compatibility conditions of $\mathcal{D}$, the first module of extensions of M and D over D will be denoted by $\mathrm{ext}_D^1(M, D)$ and will, roughly speaking, describe the difference existing in general between $\widetilde{\mathcal{D}}$ and the compatibility conditions of $\widetilde{\mathcal{D}_1}$. More precisely, we shall see that $\mathrm{ext}_D^1(M, D)$ is the kernel of the projection of the differential module determined by $\widetilde{\mathcal{D}}$ onto the differential module determined by the compatibility conditions of $\widetilde{\mathcal{D}_1}$. Surprisingly, we shall also see that $\mathrm{ext}_D^1(M, D)$ will not depend on $\mathcal{D}$ and $\mathcal{D}_1$ but only on M. Similarly, we shall see that, in the criterion, the kernel of the projection of the differential module determined by $\mathcal{D}_1$ onto the differential module determined by $\mathcal{D}_1'$ is a differential module that does not depend on $\mathcal{D}$ or $\mathcal{D}_1$ as it is only the torsion submodule of the differential module determined by $\mathcal{D}_1$ as we just said.

All these results can be extended by considering a *differential resolution* of the set of solutions or kernel of $\mathcal{D}$ in the following fashion. Starting with $\mathcal{D}$, we construct its compatibility conditions, say $\mathcal{D}_1$. Then we construct the compatibility conditions $\mathcal{D}_2$ of $\mathcal{D}_1$ and so on. The reason for indexing the $\mathcal{D}_r$ in this way depends on the possibility to relate the various $\mathcal{D}_r$ to a specific first order operator introduced by D.C. Spencer and called the *Spencer operator*. Such a construction of successive compatibility conditions eventually stops after a finite number of steps equal to the number n of independent variables and the corresponding differential sequence will be called *Janet sequence* after results found by M. Janet in 1920 [90] from the work of D. Hilbert on syzygies. As before, we may construct the *adjoint sequence* made by the operators $\widetilde{\mathcal{D}}, \widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2, \ldots$ in reverse order and we shall prove that the difference existing between $\widetilde{\mathcal{D}}_{r-1}$ and the compatibility conditions of $\widetilde{\mathcal{D}}_r$ does not depend on the resolution, that is on $\mathcal{D}$ and the $\mathcal{D}_r$, but only on the kernel of $\mathcal{D}$ or, equivalently, on the differential module determined by $\mathcal{D}$. Such a difference will therefore give rise as before to a differential module called the r-extension of M and D over D and denoted by $\mathrm{ext}_D^r(M, D)$. We now explain how to use such differential modules in control theory by means of a classification theory of modules that will refine the one brought before by the torsion-free condition. It is essential to notice that *such a classification could not even be imagined without homological algebra, even within the framework of the formal theory of systems of PD equations*.

We shall try to understand part of this classification on a few simple examples. By a slight abuse of language, we shall sometimes use the same qualifying term for an operator and for the differential module it determines.

Let us start again with the previous 'grad', 'curl', 'div' operators on the cartesian space $\mathbb{R}^3$. We already know that, among modules, we can consider the subclass made by torsion-free modules. Accordingly, among differential operators, we can consider the subclass made by those which admit a parametrization. In our case, 'grad' does not admit a parametrization while 'curl' and 'div' are respectively parametrized by 'grad' and 'curl'. Hence both 'curl' and 'div' can be parametrized but only 'div' can be parametrized by a differential operator which can also be parametrized. The corresponding torsion-free modules will be called *reflexive modules*.

Among reflexive modules, we shall find *projective modules* which play a central role in homological algebra, despite the fact that their definition is far from being intuitive. To understand it, we proceed by first defining *free modules*, saying that an A-module is free if it is isomorphic to a finite power A^r for a certain positive integer r. Next, we define the *direct sum* $M \oplus M'$ of two modules M and M' as the set of all couples (x, x') with $x \in M, x' \in M'$. Then a module M is called projective if one can find a module M' such that the direct sum $M \oplus M'$ is a free module. Of course a free module is projective and a projective module is torsion-free because, if $ax = 0$, we obtain $a(x, 0) = (0, 0)$ and this is impossible because a free module is trivially torsion-free as we can only use non zerodivisors. We shall prove that the differential module determined by a surjective differential operator, that is to say an operator with no compatibility conditions, is projective if and only if its formal adjoint is injective, as a generalization of Corollary 2.

The coherence of this last result with controllability lies in two technical results for modules over specific rings.

First of all, if $A = k[\chi]$ is the polynomial ring in one indeterminate over a field k, a situation we have already met in OD control theory for systems with constant coefficients, then A if a *principal ideal domain*, that is it has no zero divisor and every ideal is generated by a single polynomial, and a torsion-free module is free. Hence, in this case, *all the preceding classification collapses* and this is the main reason for which many results of OD control theory are based on pure coincidences which are no longer true in PD control theory.

The second result is the famous solution given in 1976 independently by D. Quillen and A. Suslin of a conjecture proposed by J.P. Serre in 1950 (see [119, 125, 181, 222] for more details). It tells that, if $A = k[\chi_1, \dots, \chi_n]$ is a polynomial ring in $n \geqslant 2$ indeterminates over a field k, then every projective A-module is free. It follows that examples of differential projective modules which are not free and come from differential operators, can only be obtained by considering PD operators with non constant coefficients. Such a situation has never been considered for applications because the formal theory of systems of PD equations is *far more difficult* than the formal theory of systems of OD equations and the reader must always keep in mind that the torsion-free interpretation of controllability has just been produced ten years ago.

Taking into account the previous results, we may find interesting to push on to the right our chain of successive inclusions. For this we denote by S the multiplicative set of non zerodivisors in A and we introduce the *annihilator* along the two following definitions:

$$\begin{aligned} &\mathrm{ann}_A(x) = \{a \in A \mid ax = 0\} && \forall x \in M \\ &\mathrm{ann}_A(M) = \{a \in A \mid ax = 0, \forall x \in M\}. \end{aligned}$$

We obtain at once the relations:

$$\begin{aligned} &M \text{ torsion-free } \Leftrightarrow \mathrm{ann}_A(x) \cap S = \emptyset, && \forall x \in M \\ &\mathrm{ann}_A(M) \subseteq \mathrm{ann}_A(x) && \forall x \in M. \end{aligned}$$

Hence we may state the three following definitions with their corresponding inclusions whenever A is an integral domain, that is A has no zero divisor, or equivalently $S = A - \{0\}$:

$$\begin{array}{rcll} M \textit{ torsion-free } A\text{-module} & \Leftrightarrow & \mathrm{ann}_A(x) = \mathrm{ann}_A(M) = 0, & \forall 0 \neq x \in M \\ \cup\!\shortmid & & & \\ M \textit{ prime } A\text{-module} & \Leftrightarrow & \mathrm{ann}_A(x) = \mathrm{ann}_A(M) = \mathfrak{p}, & \forall 0 \neq x \in M \\ \cup\!\shortmid & & & \\ M \textit{ primary } A\text{-module} & \Leftrightarrow & \sqrt{\mathrm{ann}_A(x)} = \sqrt{\mathrm{ann}_A(M)} = \mathfrak{p}, & \forall 0 \neq x \in M \end{array}$$

where $\mathfrak{p}$ is a prime ideal and the *root* or *radical* of an ideal is the biggest ideal made by all the elements having a power in the given ideal, in a way coherent with Hilbert theorem of zeros.

For example, the torsion module determined by the PD equation $\ddot{y} - y = 0$ is not prime as we have

$$\ddot{y} - y \equiv \Big(\frac{d}{dt} - 1\Big)\Big(\frac{d}{dt} + 1\Big)y \equiv \Big(\frac{d}{dt} + 1\Big)\Big(\frac{d}{dt} - 1\Big)y = 0$$

without having $\dot{y} - y = 0$ or $\dot{y} + y = 0$. However the torsion module determined by the two OD equations $\dot{y}^1 - y^1 = 0$, $\dot{y}^2 - y^2 = 0$ is prime and the torsion module determined by the second order OD equation $\ddot{y} + 2\dot{y} + y = 0$ is primary but not prime because:

$$\ddot{y} + 2\dot{y} + y \equiv \Big(\frac{d}{dt} + 1\Big)\Big(\frac{d}{dt} + 1\Big)y = 0$$

and $\dot{y} + y \neq 0$ is killed by $\frac{d}{dt} + 1$ while y is therefore killed by $(\frac{d}{dt} + 1)^2$.

We recapitulate the whole classification so far obtained for modules over integral domains:

$$\begin{array}{c}
\text{FREE} \\
\cap \\
\text{PROJECTIVE} \\
\cap \\
\text{REFLEXIVE} \\
\cap \\
\text{TORSION-FREE} \\
\cap \\
\text{PRIME} \\
\cap \\
\text{PRIMARY.}
\end{array}$$

We shall also see that every module can be embedded into a direct sum of primary modules. Indeed we may call a submodule of a given module, a *primary* (*prime*) *submodule* whenever the corresponding residual module is a primary (prime) module, and it is known that every submodule can be written as the intersection of primary submodules. The above result is thus obtained by residue and we warn the reader about the confusion existing in the literature between similar names existing for submodules and modules. Most of these results will also be extended to the case of non commutative rings arising from the use of PD operators with non constant coefficients.

We do believe that the preceding classification, which is only strict in the PD situation with variable coefficients, will soon appear in applications exactly like we have shown that controllability was undoubtedly related to the torsion-free class. We have already claimed in [162, 163] that the difficulty of differential algebra is lying more on the PD equation side than on the algebra side. The main problem with differential modules is that the difficulty is equally lying on both sides!

We now turn to a completely different aspect of control theory which makes the difference between a PD control system and a system of PD equations. Indeed, up to now, as we have only focused on controllability and proved that it was a built-in property of the control system, we have not paid attention to the separation of the control variables between input and output variables. Similarly, we have seen that new variables, called *internal variables* such as the state variables, could be used to obtain a simpler description of the control system, for example by bringing high order OD equations to first order ones. Hence, combining the input and output variables into *external variables*, we may have a system of OD equations or PD equations mixing both internal and external variables that must be separated at a certain stage, eventually through differential elimination theory. A similar behaviour may be adopted when matching systems, the output of one becoming the input of the next one, and we may also eliminate the input to look for equations among the outputs alone.

For example, in our basic system:

$$\dot{y}^1 = y^2 + u, \qquad \dot{y}^2 = ay^1 + u$$

by substitution, we obtain the single OD equation:

$$\dot{y}^2 - \dot{y}^1 + y^2 - ay^1 = 0$$

which no longer contains the input u for any value of the parameter a.

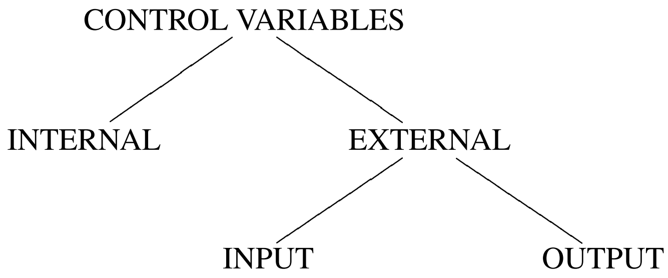
In the nonlinear algebraic framework, translating this approach in terms of differential fields, we shall get a "big" differential field, called *input/output differential field* and various intermediate fields such as the *input differential field* and the *output differential field* in the following picture of extensions:

$$\begin{tikzcd}
 & \underset{N}{\text{IN/OUT}} & \\
\text{IN } L \arrow[ur] & & M \text{ OUT} \arrow[ul] \\
 & \overset{K}{\text{GROUND}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Similarly, in the linear framework, we shall get a "big" *input/output differential module* and various differential submodules such as the *input differential module* and the *output differential module*.

In the above example, the output differential module determined by the single OD equation in (y^1, y^2) can be considered (exercise) as a submodule of the input/output differential module determined by the two OD equations in (y^1, y^2, u).

We summarize in the diagram below all the control variables that can be used:



The following linear SISO example, that we shall treat with more details through the second part of this book, furnishes a good illustration of the kind of differential elimination involved in the OD framework. Let (x^1, x^2, x^3) be the internal variables and $(w^1 = y, w^2 = u)$ be the external variable in the following linear OD system:

$$\begin{cases} \dot{x}^1 + x^2 + w^2 = 0 \\ \dot{x}^1 + \dot{x}^2 + x^1 + w^1 + w^2 = 0 \\ \dot{x}^3 + x^1 = 0 \\ \dot{x}^1 + x^2 + x^3 + w^1 + w^2 = 0 \end{cases}$$

of the general form $A\dot{x} + Bx + Cw = 0$ where A is a 4×3 matrix, B is a 4×3 matrix and C is a 4×2 matrix. From the first and second OD equations we obtain:

$$\dot{x}^2 + x^1 - x^2 + w^1 = 0, \qquad \dot{x}^1 + x^2 + w^2 = 0$$

while, from the last OD equation, we get:

$$\dot{x}^2 + x^1 - x^2 - x^3 = 0$$

and x^3 is thus a superfluous internal variable. Accordingly, the internal/external differential module only depends on (x^1, x^2) related by the single OD equation:

$$\ddot{x}^2 + \dot{x}^1 - \dot{x}^2 + x^1 = 0$$

and is equal to the input/output differential module as we easily get (exercise)

$$x^1 = \dot{w}^1, \qquad x^2 = -\ddot{w}^1 - w^2.$$

Hence, to know whether the control system is controllable or not, we have just to find the adjoint of the last second order OD equation, even though the variables involved are *not* the external variables. Multiplying by λ ad integrating by part, we obtain:

$$\left.\begin{cases} x^1 \rightarrow -\dot{\lambda} + \lambda = 0 \\ x^2 \rightarrow \ddot{\lambda} + \dot{\lambda} = 0 \end{cases}\right\} \Rightarrow \lambda = 0$$

and this adjoint is injective, therefore showing that we have indeed a torsion-free differential module. A direct approach should lead to the third order OD equation (exercise):

$$\dddot{w}^1 - \ddot{w}^1 - \dot{w}^1 + \dot{w}^2 - w^2 = 0$$

by eliminating (x^1, x^2). We let the reader find himself directly (exercise) that the corresponding differential module is indeed torsion-free with a third order parametrization:

$$w^1 = \dot{\xi} - \xi, \qquad w^2 = -\dddot{\xi} + \ddot{\xi} + \dot{\xi} + \xi.$$

The basic philosophy involved is to construct the various differential equations and modules involved in order to study them separately or through their respective interconnections. Surprisingly and apart from the recent references [28, 29], such a procedure/picture has never been used in OD control theory, *a fortiori* in PD control theory.

We conclude this introduction with an application of the duality technique to OD or PD *optimal control* and, more generally, to *constrained variational calculus*.

In the case of a linear constraint:

$$\dot{y} + ay - \dot{u} - u = 0$$

let us look for the extremum of the integral:

$$\int \frac{1}{2}(y^2 - u^2)dt.$$

Introducing the Lagrange multiplier λ, we are led to the variational condition:

$$\delta \int \Big[\frac{1}{2}(y^2 - u^2) + \lambda(\dot{y} + ay - \dot{u} - u)\Big]dt = 0$$

where now (u, y, λ) are independent variables. The corresponding Euler–Lagrange equations can be written in the form:

$$\begin{cases} u \to \dot{\lambda} - \lambda = u \\ y \to \dot{\lambda} - a\lambda = y \\ \lambda \to \dot{y} + ay - \dot{u} - u = 0 \end{cases}$$

and we want to eliminate λ among them.

- If $a = 1$, we are left with $y - u = 0$ only.
- If $a \neq 1$, we are left with the possibility to express λ uniquely as:

  $$\lambda = \frac{u - y}{a - 1}$$

  and obtain after substituting:

  $$\dot{y} - y - \dot{u} + au = 0.$$

  Substituting the constraint, we get:

  $$(a + 1)(y - u) = 0.$$

  (i) If $a \neq -1$ (for example $a = 0$) we must have

  $$y - u = 0 \quad \Rightarrow \quad u = 0, y = 0.$$

  (ii) If $a = -1$, there is no condition other than the constraint itself in order to have an extremum. The reason for that lies in the parametrization existing because the system is controllable. We let the reader find the parametrization (exercise):

  $$u = \dot{\xi} - \xi, \qquad y = \dot{\xi} + \xi$$

  and we obtain:

  $$\frac{1}{2}\left(y^2 - u^2\right) \;=\; 2\xi\dot{\xi} \;=\; \frac{d}{dt}\left(\xi^2\right).$$

Such an example points out the existence of very unusual trees related to the study of the formal integrability of the various systems and the study of controllability, through the formal adjoint, becomes part of that tree.

More precisely, in the last example, we have:

$$
\text{tree}\left\{
\begin{array}{ccccc}
 & & \diagup \quad \diagdown & & \\
 & a=1 & & a \neq 1 & \\
 & & \diagup & & \diagdown \\
 & & a=-1 & & a \neq -1
\end{array}\right.
\qquad
\begin{array}{c}
\textit{formal integrability} \\
\textit{of the adjoint} \\
\\
\textit{formal integrability} \\
\textit{of the Euler–} \\
\textit{Lagrange equations}
\end{array}
$$

Hence we discover that the way to eliminate λ is particularly simple when the control system (constraint) is controllable as we recognize the adjoint operator acting on λ, but now with a second member function of (u, y). This operator being injective, we may express λ uniquely as a linear differential combination of (u, y) and substitute it in the Euler–Lagrange equations. Equivalently, as we did at the end of the example, we may use a parametrization of the control variables and bring the variational problem with constraint to a variational problem without constraint. *In both cases, we need not separate the input from the output*. Otherwise, we have to look for the compatibility conditions of the adjoint operator (once more a problem of formal integrability) and substitute the second member. In this new framework, it is essential to notice that *the resulting computation has no more to do with trajectories*.

Considering now the nonlinear constraint:

$$u\dot{y} - \dot{u} = a.$$

If we want to look for the extremum of:

$$\int \frac{1}{2} y^2 dt$$

we are led to the variational problem:

$$\delta \int \Big[\frac{1}{2} y^2 + \lambda(u\dot{y} - \dot{u} - a)\Big] dt = 0$$

and the Euler–Lagrange equations become:

$$\begin{cases} u \rightarrow \dot{\lambda} + \dot{y}\lambda = 0 \\ y \rightarrow u\dot{\lambda} + \lambda\dot{u} = y \\ \lambda \rightarrow u\dot{y} - \dot{u} - a = 0. \end{cases}$$

We obtain at once $a\lambda = -y$.

- If $a = 0$, we are left with $y = 0$, $\dot{u} = 0$, $\dot{\lambda} = 0$.
- If $a \neq 0$, we are left with $(y + 1)\dot{y} = 0$ and thus $\dot{y} = 0$, $\dot{u} = -a$, $\dot{\lambda} = 0$.

To link this approach with differential algebra, we notice that the differential polynomial $P \equiv (y + 1)\dot{y}$ is such that $\dot{y}\frac{dP}{dt} - \ddot{y}P \equiv (\dot{y})^3$ and the root of the differential ideal generated by P is a prime differential ideal as it contains $\dot{y}$. It is also essential to point out that the operator in λ is just the adjoint of the linear operator obtained by linearizing the constraint.

This long introduction explains by itself the table of contents of the book.

In the first part of the book we present all the mathematical tools and their interconnections in a rather self-contained way as it is very difficult to touch many points through the extended specialized literature. In particular *we present for the first time a coherent parallel between the formal theory of differential operators and the formal theory of differential modules*. We repeat once more, that the simplest way for any reader to understand the novelty of this book is to open simultaneously Bourbaki's treatise, Palamodov's book and Kashiwara's thesis and to wonder how these works can interfere with control theory.

In the second part of this book, we collect all these results along the lines sketched in this introduction and we separate the study of control theory between the linear framework and the nonlinear framework, though the nonlinear aspect highly depends on the linear one.

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