Probabilistic Numerical Methods 2024–2025

Lecture 4: The Stochastic Integral

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The motivation of this lecture and the next one is to construct a differential calculus aimed at describing the (infinitesimal) evolution of quantities of the form $\Phi(B_t)$ where $(B_t)_{t\geq 0}$ is a Brownian motion and $\Phi : \mathbb{R} \to \mathbb{R}$ is smooth. Since the trajectories of the Brownian motion are not differentiable, the usual chain rule does not apply. Today we construct the first part of this differential calculus: the stochastic integral. It is partially inspired by the construction of the Stieltjes integral for functions of bounded variation.

Introductory exercise: let $T > 0$ and $\sigma = \{t_0, \ldots, t_n\}$ such that $0 = t_0 < t_1 < \cdots < t_n = T$. We write $\|\sigma\| = \max_{0\leq i\leq n-1} t_{i+1} - t_i$. Show that $\lim_{\|\sigma\|\to 0} \sum_{i=0}^{n-1}(B_{t_{i+1}} - B_{t_i})^2 = T$ in $\mathbf{L}^2$.

1 Functions of bounded variation and the Stieltjes integral

For $g : [0, T] \to \mathbb{R}$ such that $g(0) = 0$, we define $\mathrm{TV}(g) = \sup_\sigma \sum_{i=0}^{n-1} |g(t_{i+1}) - g(t_i)|$, and say that g is of bounded variation (BV) if $\mathrm{TV}(g) < \infty$. It turns out that g is BV if and only if $g = g_+ - g_-$ with nondecreasing functions $g_\pm : [0, T] \to \mathbb{R}$. In this case, g has left- and right limits everywhere, and the right-continuous, left-limited version of g is the cumulative distribution function of some bounded signed measure μ on $[0, T]$.

A typical example is when g is differentiable, then μ is the measure with density $g'(t)$ with respect to the Lebesgue measure on $[0, T]$. In particular, $\mathrm{TV}(g) = \int_0^T |g'(t)|\mathrm{d}t$ and one can take $g_\pm(t) = \int_0^t [g'(s)]_\pm \mathrm{d}s$.

Now let g be BV and $h : [0, T] \to \mathbb{R}$ be a continuous function. For any piecewise constant function h^n, of the form $\sum_{i=0}^{n-1} \xi_i \mathbb{1}_{[t_i,t_{i+1})}(t)$ on some subdivision σ, which is such that $\sup_{t\in[0,T]} |h^n(t) - h(t)| \to 0$, the quantity $\sum_{i=0}^{n-1} \xi_i(g(t_{i+1}) - g(t_i))$ converges, when $\|\sigma\| \to 0$, to a limit which does not depend on the choice of h^n. This is the Stieltjes integral of f with respect to g, it is denoted by $\int_0^T h(t)\mathrm{d}g(t)$.

It is related with:

- the Riemann integral because in the case where g is C^1 then $\int_0^T h(t)\mathrm{d}g(t) = \int_0^T h(t)g'(t)\mathrm{d}t$;
- the Lebesgue integral because $\int_0^T h(t)\mathrm{d}g(t) = \int_0^T h(t)\mu(\mathrm{d}t)$.

2 Construction of the stochastic integral

We mostly follow Section 10.1 of the 2023/2024 notes.

We now fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all negligible sets, and let $(B_t)_{t\geq 0}$ an $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion. The goal is to construct the stochastic integral

$$\int_0^T H_t \mathrm{d}B_t$$

for a large enough class of processes $(H_t)_{t\geq 0}$.

2.1 Construction for piecewise constant integrands

Construction for processes of the form $H_t^n = \sum_{i=0}^{n-1} \xi_i \mathbb{1}_{[t_i,t_{i+1})}(t)$.

Problem of limit $n \to +\infty$ for two approximations of $H_t = B_t$.

2.2 The Itô convention

Progressively measurable processes. Notation $\mathbf{\Lambda}^2([0,T])$ and $\mathbf{\Lambda}_0^2([0,T])$. Characterisation of elements of $\mathbf{\Lambda}_0^2([0,T])$.

Construction of the stochastic integral by extension of the isometry from $\mathbf{\Lambda}_0^2([0,T])$ to $\mathbf{\Lambda}^2([0,T])$.

Application: computation of $\int_0^T B_t \mathrm{d}B_t$.

Properties: Chasles relation, linearity, identities for mean and variance.

Proposition 2.1 (Stochastic integral as a process)**.** *For any $(H_t)_{t\in[0,T]} \in \mathbf{\Lambda}^2([0,T])$, the process $(X_t)_{t\in[0,T]}$ defined by*

$$X_t = \int_0^t H_s \mathrm{d}B_s$$

is adapted and has an almost surely continuous modification, which is therefore progressively measurable.

See the proof in Appendix A.

Exercise: compute $\mathbb{E}[\int_0^T H_t \mathrm{d}B_t \int_0^T H'_t \mathrm{d}B_t]$.

2.3 Extension and localisation

The goal is now to remove the $\mathbf{L}^2(\mathbb{P})$ condition on $(H_t)_{t\geq 0}$. Definition of $\mathbf{\Lambda}_{\mathrm{loc}}$. Introduction of the stopping time τ_M and extension of the stochastic integral by localisation.

2.4 Exercises

Probability of exit from a strip.

The Wiener integral.

A Proof of Proposition 2.1

It is important to keep in mind that the stochastic integral on $[0,T]$ is constructed by a limiting procedure in $\mathbf{L}^2(\mathbb{P})$, and therefore it is only defined up to a negligible subset, which depends on T. So the first step is to realise that for any $t \in [0,T]$, we have

$$X_t := \int_0^t H_s \mathrm{d}B_s = \int_0^T \mathbb{1}_{\{s<t\}} H_s \mathrm{d}B_s, \qquad \text{almost surely,}$$

where the first stochastic integral is constructed on $[0,t]$ by approximation of the integrand $(H_s)_{s\in[0,t]}$, while the second stochastic integral is constructed on $[0,T]$ by approximation of the integrand $(\mathbb{1}_{\{s<t\}} H_s)_{s\in[0,T]}$. But it is clear that if $(H_s^n)_{s\in[0,t]}$ is a sequence of elements of $\mathbf{\Lambda}_0^2([0,t])$ such that $\|H^n - H\|_{\mathbf{\Lambda}^2([0,t])} \to 0$, then $(\mathbb{1}_{\{s<t\}} H_s^n)_{s\in[0,T]}$ is a sequence of elements of $\mathbf{\Lambda}_0^2([0,T])$ such that $\|\mathbb{1}_{\{\cdot<t\}} H^n - \mathbb{1}_{\{\cdot<t\}} H\|_{\mathbf{\Lambda}^2([0,T])} \to 0$, and moreover

$$\int_0^t H_s^n \mathrm{d}B_s = \int_0^T \mathbb{1}_{\{s<t\}} H_s^n \mathrm{d}B_s.$$

Since the left- and right-hand sides respectively converge to $\int_0^t H_s \mathrm{d}B_s$ and $\int_0^T \mathbb{1}_{\{s<t\}} H_s \mathrm{d}B_s$, in $\mathbf{L}^2(\mathbb{P})$, we deduce that these quantities coincide almost surely (but on an almost sure event which depends on t). In other words, the processes $(X_t)_{t\in[0,T]}$ and $(\int_0^T \mathbb{1}_{\{s<t\}} H_s \mathrm{d}B_s)_{t\in[0,T]}$ are modification of each other.

To check that $(X_t)_{t\in[0,T]}$ is adapted, we fix t and notice that by construction, X_t is the limit, in $\mathbf{L}^2(\mathbb{P})$, of a sequence of $\mathcal{F}_t$-measurable random variables X_t^n, therefore X_t is $\mathcal{F}_t$-measurable (in fact, since $\mathcal{F}_0 \subset \mathcal{F}_t$ contains all negligible sets, any random variable $\widetilde{X}_t$ which coincides almost surely with X_t is $\mathcal{F}_t$-measurable).

We now construct an almost surely continuous modification of $(X_t)_{t\in[0,T]}$. To proceed, we let $(H_s^n)_{s\in[0,T]}$ be a sequence of elements of $\mathbf{\Lambda}_0^2([0,T])$ which converges to $(H_s)_{s\in[0,T]}$ in $\mathbf{\Lambda}^2([0,T])$. Since we are working with an almost surely continuous modification of the Brownian motion $(B_s)_{s\in[0,T]}$, the process $(X_t^n)_{t\in[0,T]}$ defined by

$$X_t^n = \int_0^T \mathbb{1}_{\{s<t\}} H_s^n \mathrm{d}B_s$$

is almost surely continuous (recall that since $(\mathbb{1}_{\{s<t\}} H_s^n)_{s\in[0,T]} \in \mathbf{\Lambda}_0^2([0,T])$, this construction is elementary). As a consequence, it may be viewed as a random variable in the space $C([0,T])$ of continuous trajectories, endowed with the Borel σ-field associated with the sup norm.

Moreover, each process $(X_t^n)_{t\in[0,T]}$ is easily seen to be a martingale. As a consequence, for $n, m \geq 1$, Doob's maximal inequality (Karatzas and Shreve, Section 1.3) yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t^n - X_t^m|^2\right] \leq \mathbb{E}\left[|X_T^n - X_T^m|^2\right] = \|H^n - H^m\|^2_{\mathbf{\Lambda}^2([0,T])}.$$

We deduce that if $(H^n)_{n\geq 1}$ is a Cauchy sequence in $\mathbf{\Lambda}^2([0,T])$, then $(X^n)_{n\geq 1}$ is a Cauchy sequence in $\mathbf{L}^2(\Omega; C([0,T]))$. Since this space is complete, there exists $(\widetilde{X}_t)_{t\in[0,T]} \in \mathbf{L}^2(\Omega; C([0,T]))$ such that $\mathbb{E}[\sup_{t\in[0,T]} |X_t^n - \widetilde{X}_t|^2] \to 0$. By construction, $(\widetilde{X}_t)_{t\in[0,T]}$ is almost surely continuous, but on the other hand, for any $t \in [0,T]$ it is such that $X_t = \widetilde{X}_t$, almost surely. This completes the proof.

Remark A.1. *A generalisation of the Doob maximal inequality for the (almost surely continuous modification of the) stochastic integral writes as follows: for any $p \geq 1$, there exist absolute constants $0 < c_p \leq C_p < \infty$ such that, for any $T > 0$, for any $(H_t)_{t\in[0,T]} \in \mathbf{\Lambda}^2([0,T])$,*

$$c_p \mathbb{E}\left[\left(\int_0^T H_t^2 \mathrm{d}t\right)^{p/2}\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t H_s \mathrm{d}B_s\right|^p\right] \leq C_p \mathbb{E}\left[\left(\int_0^T H_t^2 \mathrm{d}t\right)^{p/2}\right]$$

This statement is called the Burkhölder–Davis–Gundy inequality (Karatzas and Shreve, Section 3.3).

Remark A.2. *Now that we know that $(X_t)_{t\in[0,T]}$ has a progressively measurable modification, given a stopping time τ and $T > 0$ we may define the random variable $(X_{T\wedge\tau})(\omega) := X_{T\wedge\tau(\omega)}(\omega)$. Then by similar arguments as in the proof above, we may check that*

$$X_{T\wedge\tau} = \int_0^T \mathbb{1}_{\{s<\tau\}} H_s \mathrm{d}B_s, \qquad \textit{almost surely,}$$

which is useful in localisation procedures.