Probabilistic Numerical Methods 2024–2025

Lecture 6: Stochastic Differential Equations

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We fix integers $n, d \geq 1$, a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ provided with a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathcal{F}_0$ contains all negligible events, and $(B_t)_{t\geq 0}$ a d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion.

Given a time-interval $I = [0,T]$ or $I = [0,+\infty)$, we let $b : I \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : I \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be measurable functions which are bounded on bounded subsets of $I \times \mathbb{R}^n$.

We are interested in the Stochastic Differential Equation

$$\mathrm{d}X_t = b(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}B_t, \tag{SDE}$$

complemented with the initial condition

$$X_0 = \xi, \tag{IC}$$

where ξ is an $\mathcal{F}_0$-measurable random variable in $\mathbb{R}^n$.

The function b is called the *drift* of the SDE, and σ is the *dispersion matrix*. The $n \times n$ matrix

$$a(t,x) := \sigma(t,x)\sigma^\top(t,x)$$

is called the *diffusion* matrix.

1 Solution to (SDE)–(IC)

1.1 Notion of solution and associated differential operator

Definition 1.1 (Solution to (SDE)–(IC))**.** *A solution to (SDE)–(IC) is an n-dimensional Ito process such that, almost surely*[1]*,*

$$\forall t \in I, \quad \forall i \in \{1,\dots,n\}, \qquad X_t^i = \xi^i + \int_{s=0}^t b_i(s, X_s)\mathrm{d}s + \sum_{k=1}^d \int_{s=0}^t \sigma_{ik}(s, X_s)\mathrm{d}B_s^k.$$

An important object related with (SDE) is the differential operator L_t defined by, for all C^2 functions $\phi : \mathbb{R}^n \to \mathbb{R}$,

$$L_t\phi(x) = \sum_{i=1}^n b_i(t,x)\frac{\partial \phi}{\partial x_i}(x) + \frac{1}{2}\sum_{i,j=1}^n a_{ij}(t,x)\frac{\partial^2 \phi}{\partial x_i \partial x_j}(x).$$

The reason for the importance of this operator is that if $(X_t)_{t\in I}$ is a solution to (SDE), then when one wants to apply the Ito formula to $\phi(X_t)$, one gets

$$\mathrm{d}\phi(X_t) = L_t\phi(X_t)\mathrm{d}t + \sigma^\top(t, X_t)\nabla\phi(X_t)\cdot \mathrm{d}B_t.$$

[1] Throughout the chapter we systematically work with continuous versions of Ito processes.

1.2 Existence and uniqueness for globally Lipschitz continuous coefficients

Theorem 1.2 (Ito)**.** *Assume that there exists $K \geq 0$ such that:*

(i) for any $t \in I$, for any $x, y \in \mathbb{R}^n$, $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|$;

(ii) for any $t \in I$, for any $x \in \mathbb{R}^n$, $|b(t,x)| + |\sigma(t,x)| \leq K(1+|x|)$.

Then (SDE)–(IC) *admits a unique solution*[2].

Notice that if b and σ do not depend on t, then the condition (ii) is implied by (i) so it does not need to be checked.

The proof of Theorem 1.2 can be decomposed in 4 steps, see lecture notes for details.

1. If $|\xi| \in \mathbf{L}^2$ and $I = [0,T]$, then (SDE)–(IC) has a unique solution in $\mathbf{\Lambda}^2([0,T])$: this follows from a fixed point argument.
2. If $|\xi| \in \mathbf{L}^2$ and $I = [0,T]$, then in fact any solution to (SDE)–(IC) is in $\mathbf{\Lambda}^2([0,T])$: this is an a priori estimate, which follows from the Gronwall Lemma, and therefore proves the statement of the Theorem if $|\xi| \in \mathbf{L}^2$ and $I = [0,T]$.
3. If $|\xi|$ is no longer assumed to be in $\mathbf{L}^2$, one may still construct a solution as follows: the first two steps provide a collection of processes $\{(X_t^x)_{t\in I}, x \in \mathbb{R}^n\}$ which solve (SDE) with deterministic (and a fortiori $\mathbf{L}^2$) initial condition $X_0^x = x$. Then setting $X_t(\omega) := X_t^{\xi(\omega)}(\omega)$ yields a solution to (SDE)–(IC), and uniqueness follows from the Lipschitz condition.
4. If $I = [0, +\infty)$, then the extension of the construction is straightforward.

Example: Ornstein–Uhlenbeck process, explicit solution, law at time t, limit when $t \to +\infty$.

1.3 The case of locally Lipschitz continuous coefficients

Theorem 1.3 (Local existence and uniqueness)**.** *Let D be an open subset of $\mathbb{R}^n$, and assume that there exists $K_D \geq 0$ such that:*

(i) for any $t \in I$, for any $x, y \in D$, $|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K_D|x-y|$;

(ii) for any $t \in I$, for any $x \in D$, $|b(t,x)| + |\sigma(t,x)| \leq K_D(1+|x|)$.

Then there exists an Ito process $(X_t)_{t\in I}$ such that, letting $\tau_D := \inf\{t \in I : X_t \notin D\}$[3]*, we have, almost surely,*

$$\forall t < \tau_D, \qquad X_t = \xi + \int_{s=0}^t b(s, X_s)\mathrm{d}s + \int_{s=0}^t \sigma(s, X_s)\mathrm{d}B_s.$$

Moreover, if there exists another Ito process $(X'_t)_{t\in I}$ satisfying the same properties (with exit time from D denoted by τ'_D), then almost surely,

$$\tau_D = \tau'_D \qquad \textit{and} \qquad \forall t < \tau_D, \quad X_t = X'_t.$$

This theorem follows from the combination of Theorem 1.2 and Theorem 2.1, p. 102 in Friedman (SDEs vol. 1).

Assume for simplicity that $I = [0, +\infty)$ and that b and σ satisfy the assumptions of Theorem 1.3 on every open ball with radius M, namely $D = B(0, M)$. An important such case is when b and σ do not depend on t and are C^1 in x. Denoting by τ_M the corresponding exit time, we therefore have the existence and uniqueness of a solution up to the *explosion* time $\tau_* = \sup_M \tau_M$. There are situations in which τ_* is finite, so that X_t indeed exploses when t reaches τ_* (see the SDE $\mathrm{d}X_t = \frac{1}{2}\mathrm{e}^{2X_t}\mathrm{d}t + \mathrm{e}^{X_t}\mathrm{d}B_t$ in the exercise sheet). On the other hand, it is useful to have criteria ensuring that $\tau_* = \infty$, almost surely, so that existence and uniqueness of a (global-in-time) solution still holds even if the coefficients of the SDE are not globally Lipschitz continuous. An example of such a criterion is provided by the next statement.

[2] Uniqueness is understood here as: the continuous versions of any two solutions are indistinguishable.

[3] If $I = [0,T]$ and $X_t \in D$ for all $t \in [0,T]$, we set $\tau_D = T$.

Proposition 1.4 (Global existence by Lyapunov function). *Assume that $I = [0, +\infty)$ and that the assumptions of Theorem 1.3 hold on every open ball $D = B(0, M)$, with corresponding exit time denoted by τ_M. Assume moreover that there exists a C^2 function $\Phi : \mathbb{R}^n \to \mathbb{R}$ such that:*

(i) $\Phi \geq 0$ *and* $\lim_{|x|\to+\infty} \Phi(x) = +\infty$*;*

(ii) $\mathbb{E}[\Phi(\xi)] < +\infty$*;*

(iii) there exists $c \geq 0$ *such that for all* $t \geq 0$, $L_t\Phi(x) \leq c\Phi(x)$.

Then $\tau_* = \infty$*, almost surely (so* (SDE)–(IC) *has a unique global-in-time solution), and moreover we have the estimate*

$$\forall t \geq 0, \qquad \mathbb{E}[\phi(X_t)] \leq \mathrm{e}^{ct}\mathbb{E}[\Phi(\xi)].$$

Proof. Applying Ito's formula to $\Phi(X_t)\mathrm{e}^{-ct}$ for $t < \tau_M$, we get

$$\Phi\left(X_{t\wedge\tau_M}\right)\mathrm{e}^{-ct\wedge\tau_M} = \Phi(\xi) + \int_{s=0}^{t\wedge\tau_M} \mathrm{e}^{-cs}\left(L_s\Phi(X_s) - c\Phi(X_s)\right)\mathrm{d}s + \int_{s=0}^{t\wedge\tau_M} \mathrm{e}^{-cs}\sigma^\top(s, X_s)\nabla\phi(X_s)\cdot\mathrm{d}B_s.$$

Since $(s, x) \mapsto \mathrm{e}^{-cs}\sigma^\top(s, x)\nabla\phi(x)$ is bounded on the bounded set $[0, t] \times B(0, M)$, we deduce that the stochastic integral is integrable and has expectation 0. On the other hand, by (iii), the time integral is almost surely nonpositive. Therefore

$$\mathbb{E}\left[\Phi\left(X_{t\wedge\tau_M}\right)\mathrm{e}^{-ct\wedge\tau_M}\right] \leq \mathbb{E}[\Phi(\xi)],$$

and since $t \wedge \tau_M \leq t$ we deduce that

$$\mathbb{E}\left[\Phi\left(X_{t\wedge\tau_M}\right)\right] \leq \mathrm{e}^{ct}\mathbb{E}[\Phi(\xi)].$$

We now show that $\tau_M \to +\infty$, almost surely. Writing

$$\mathbb{E}\left[\Phi\left(X_{t\wedge\tau_M}\right)\right] \geq \mathbb{E}\left[\Phi\left(X_{\tau_M}\right)\mathbb{1}_{\{\tau_M\leq t\}}\right] \geq \inf_{|x|=M}\Phi(x)\mathbb{P}(\tau_M \leq t),$$

we deduce that

$$\mathbb{P}(\tau_M \leq t) \leq \frac{\mathrm{e}^{ct}\mathbb{E}[\Phi(\xi)]}{\inf_{|x|=M}\Phi(x)}.$$

Using (i) and (ii) we get that the right-hand side goes to 0 when $M \to +\infty$. This shows that $\tau_M \to +\infty$ and therefore that $\tau_* = \sup_M \tau_M = +\infty$, almost surely. The final estimate now follows from Fatou's Lemma. □

2 Discretisation

See lecture notes for details: Euler–Maruyama scheme, strong error, weak error, computation for the Ornstein–Uhlenbeck process.