Séminaire de Mathématiques Appliquées du CERMICS

École des Ponts
ParisTech

Non-convexity measures

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5 mars 2020

Non-convexity measures

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Motivation

Multistage, non-convex, stochastic optimization

- Costly calculation of Lagrangian cuts in SDDiP.
- Addition of binary variables to represent Lipschitz or Augmented Lagrangian cuts in SLDP makes iterations take more and more time.

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Multistage, non-convex, stochastic optimization

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Table: A 4d, 8-stage, non-convex problem

	SB	SDDiP	SLDP
# cuts	100	100	250
LB	4.676	4.589	7.883
time (s)	7.3	4100	225

Motivation

Multistage, non-convex, stochastic optimization

- Costly calculation of Lagrangian cuts in SDDiP.
- Addition of binary variables to represent Lipschitz or Augmented Lagrangian cuts in SLDP makes iterations take more and more time.
- The complexity increase is not so dramatic in SDDiP, but still noticeable.

Table: A 4d, 8-stage, non-convex problem

	SB	SDDiP	SLDP
# cuts	100	100	250
LB	4.676	4.589	7.883
time (s)	7.3	4100	225
First iteration (s)	0.04	29	0.05
Last iteration (s)	0.10	150	1.7

Two-stage problems

Convex case

A two-stage **linear program** is

$$\begin{aligned} \min_x \quad & c_1^\top x + Q(x) \\ \text{s.t.} \quad & A_1 x = b_1 \\ & x \geqslant 0, \end{aligned}$$

where the cost-to-go function is

$$\begin{aligned} Q(x) = \quad \min_y \quad & c^\top y \\ \text{s.t.} \quad & Ax + By = b \\ & y \geqslant 0. \end{aligned}$$

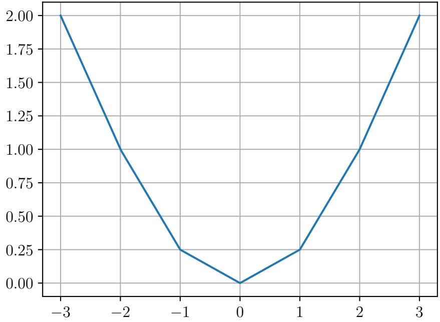

Figure: Example of cost-to-go.

Two-stage problems

Non-convex case

A two-stage **mixed integer linear program** is

$$\begin{aligned} \min_x \quad & c_1^\top x + Q(x) \\ \text{s.t.} \quad & A_1 x = b_1 \\ & x \geqslant 0, \end{aligned}$$

where the cost-to-go function is

$$Q(x) = \begin{aligned} \min_y \quad & c^\top y \\ \text{s.t.} \quad & Ax + By = b \\ & y \geqslant 0 \\ & y \in \mathbb{R}^n \times \mathbb{Z}^k. \end{aligned}$$

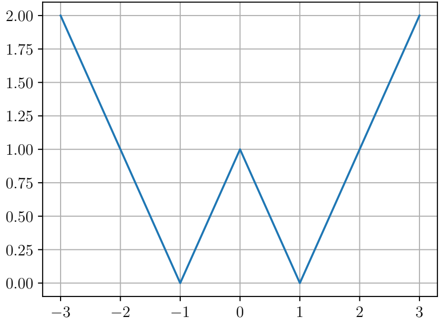

Figure: Example of cost-to-go.

Two-stage problems

Stochastic, non-convex case

Cost-to-go Q: **random function.**

- Uncertainty $\xi = (c^\top, A, B, b)$:

$$Q(x,\xi) = \begin{array}{ll} \min\limits_{y} & c^\top y \\ \text{s.t.} & Ax + By = b \\ & y \geqslant 0 \\ & y \in \mathbb{R}^n \times \mathbb{Z}^k. \end{array}$$

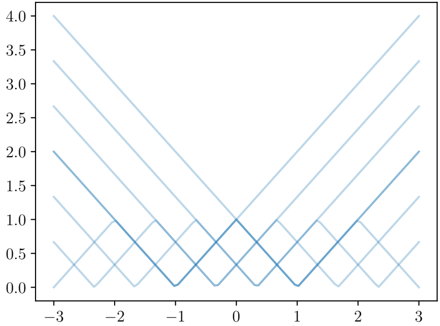

Figure: Random cost-to-go.

Two-stage problems

Stochastic, non-convex case

Cost-to-go Q: **random function.**

- First stage:

$$\begin{aligned} \min_x \quad & c_1^\top x + \mathbb{E}\left[Q(x,\xi)\right] \\ \text{s.t.} \quad & A_1 x = b_1 \\ & x \geqslant 0. \end{aligned}$$

- Uncertainty $\xi = (c^\top, A, B, b)$:

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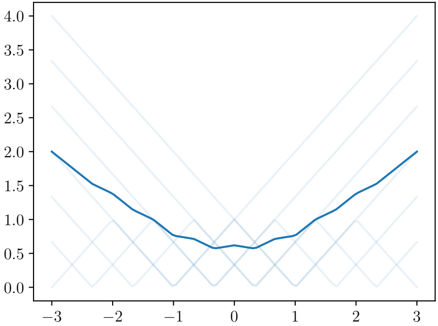
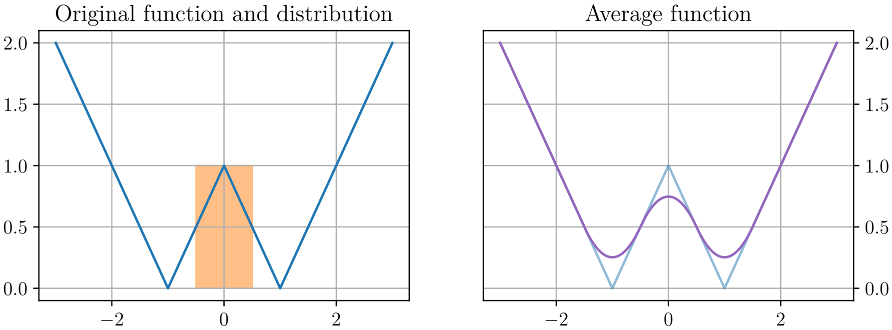


Figure: Expected cost-to-go.

Non-convexity reduction

Uniform noises



Non-convexity reduction

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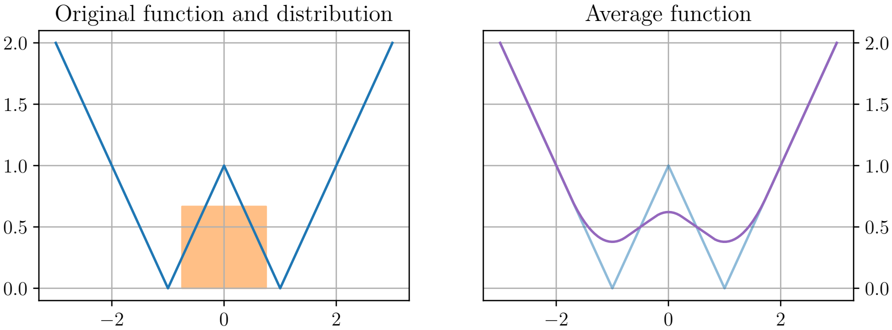

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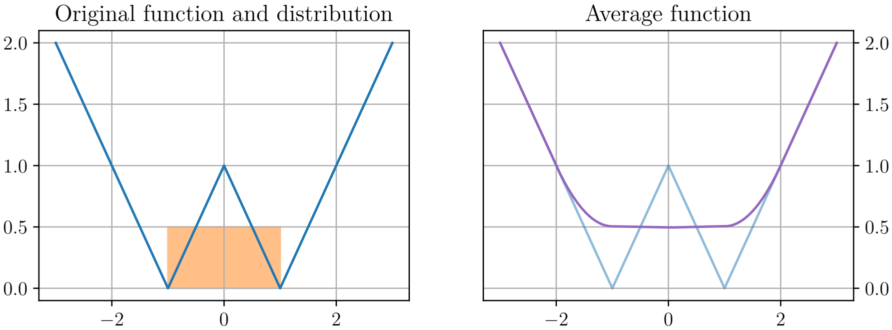

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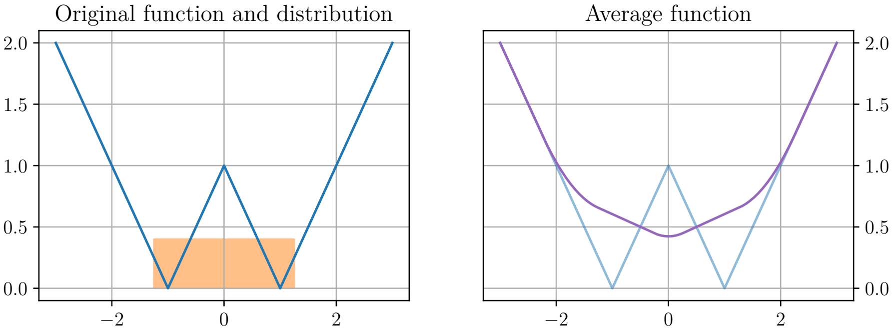

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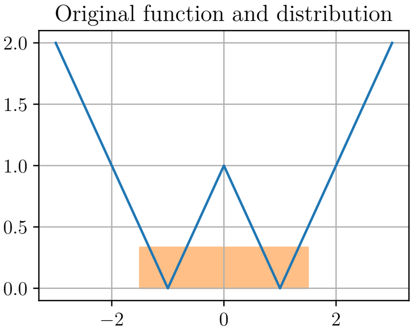

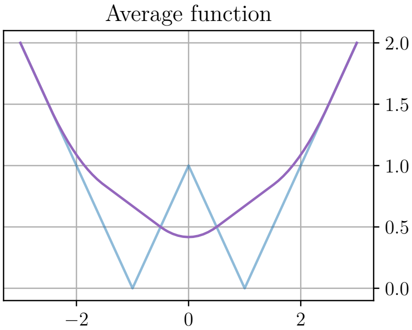

Non-convexity reduction

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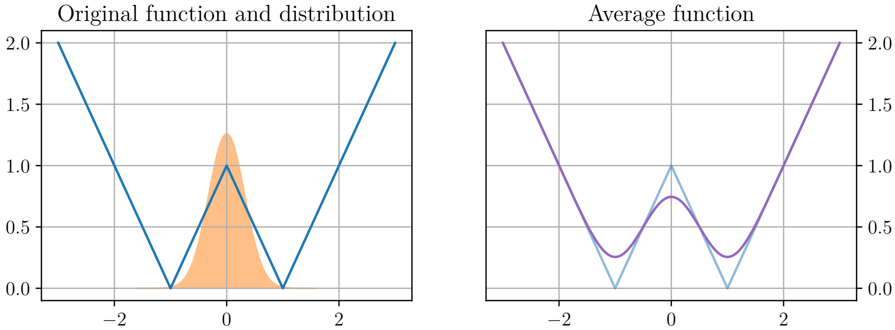

Non-convexity reduction

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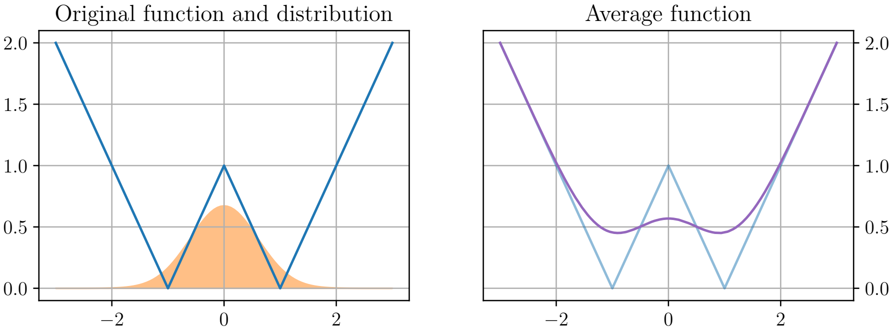

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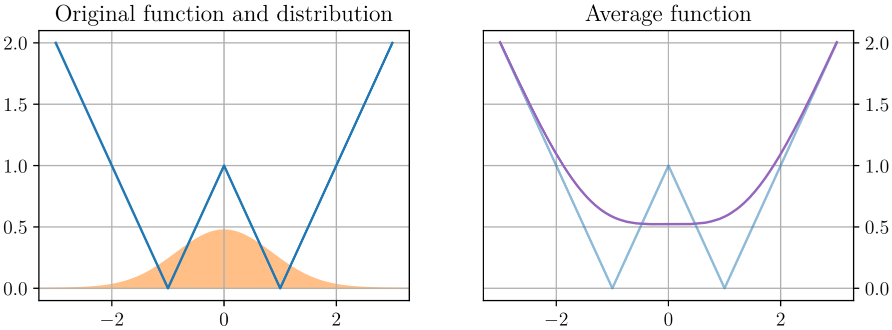

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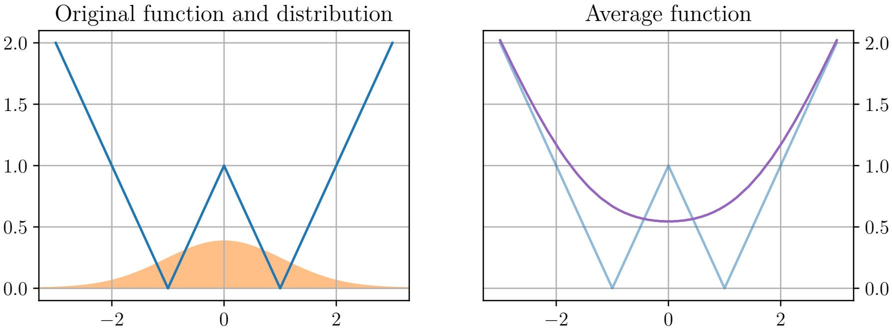
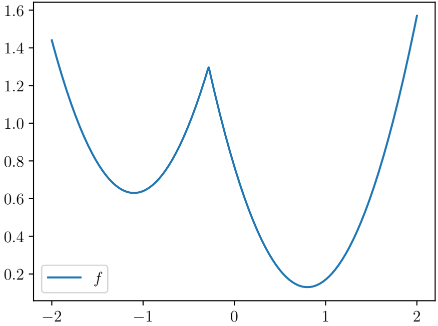


Measuring the non-convexity of a function

Convex relaxation

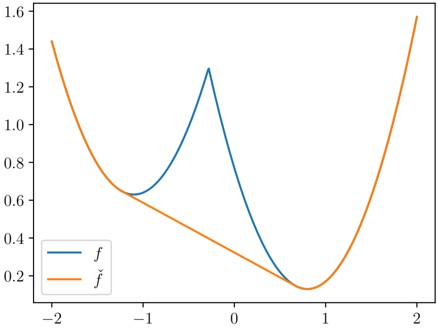
- Let f be a function.



Measuring the non-convexity of a function

Convex relaxation

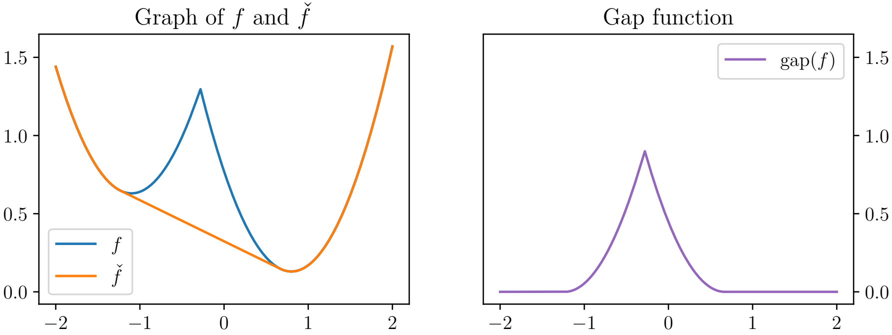
- Let f be a function.
- Its **convex relaxation** $\check{f}$ is the largest convex function everywhere less than f.



Measuring the non-convexity of a function

Gap function

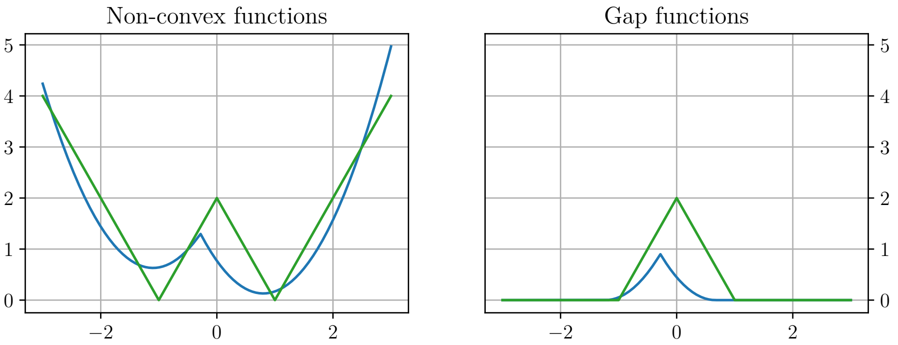
- $\mathrm{gap}(f) = f - \check{f}$;
- The gap is identically zero if and only if f is convex,
- $\mathrm{gap}(f)$ is always non-negative;



Measuring the non-convexity of a function
Gap comparisons

- We can say that f **is less non-convex than** g if

$$\mathrm{gap}(f) \leqslant \mathrm{gap}(g).$$



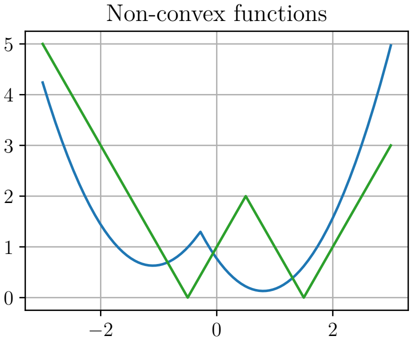
Measuring the non-convexity of a function

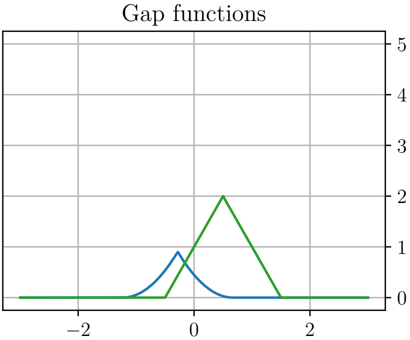
Gap comparisons

- We can say that f **is less non-convex than** g if

$$\mathrm{gap}(f) \leqslant \mathrm{gap}(g).$$

- But their gaps may not be comparable...





Measuring the non-convexity of a function

Monotone norms

- Project on $\mathbb{R}$ using a norm $\|\cdot\|$.
- Requirement: preserve gap comparisons,

$$\mathrm{gap}(f) \leqslant \mathrm{gap}(g) \implies \|\mathrm{gap}(f)\| \leqslant \|\mathrm{gap}(g)\| \,.$$

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Definition

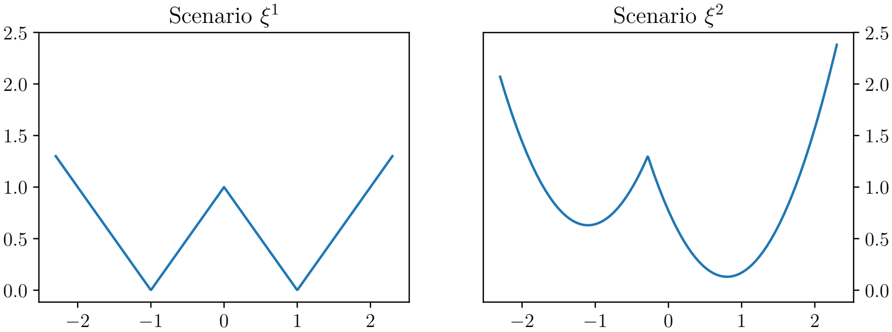
A function norm $\|\cdot\|$ is **monotone** if for all $g, h \geqslant 0$,

$$g \leqslant h \implies \|g\| \leqslant \|h\| \,.$$

Examples: The uniform norm, all p-norms...

A bit of notation

- Random variable: ξ.
- Random function: $Q(x, \xi)$.



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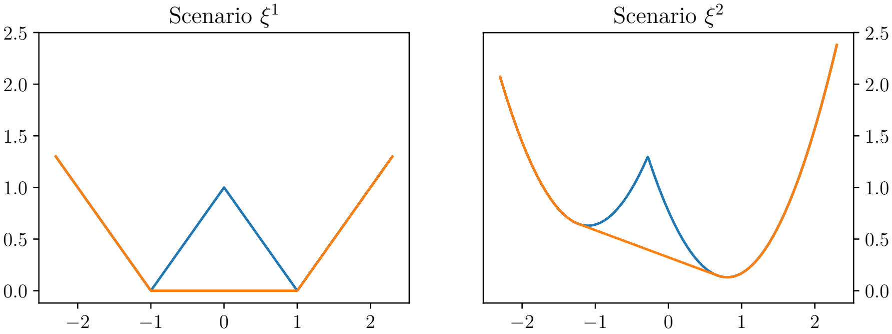
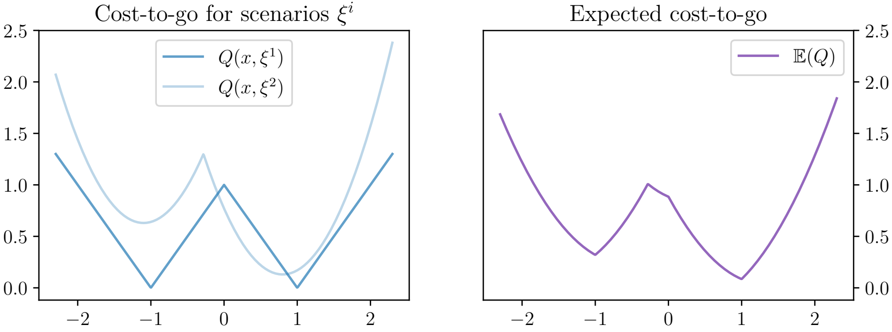


A bit of notation

- Random variable: ξ.
- Random function: $Q(x, \xi)$.
- Convex relaxation $\check{Q}$: Defined for each realization $Q(\cdot, \xi)$.
- Average function: $\mathbb{E}\left[Q\right] = x \mapsto \mathbb{E}^{\xi}\left[Q(x, \xi)\right]$



Measuring the non-convexity of a function

Main inequality

Let Q be a random function (such as a cost-to-go function, for example)

- By definition: $\check{Q} \leqslant Q$.

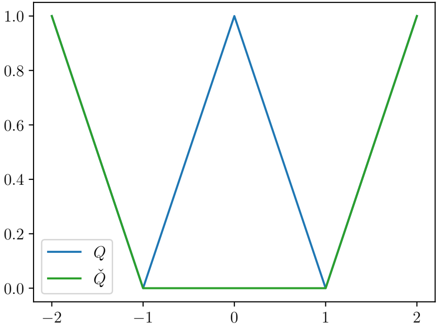


Figure: A realization of Q.

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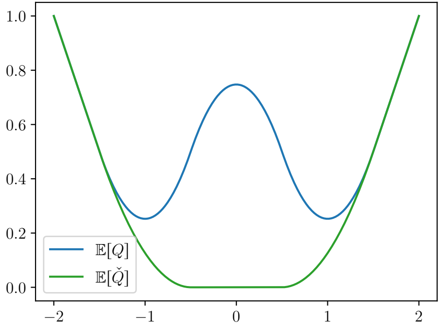

Figure: Average function.

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- And then:

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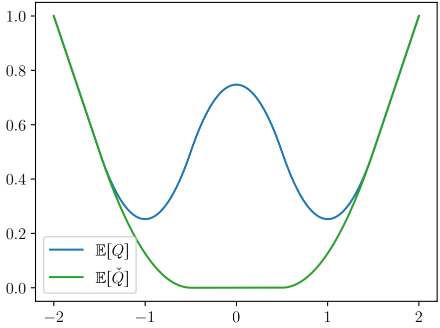

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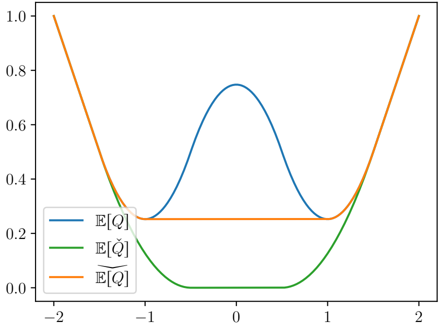


Figure: Average function.

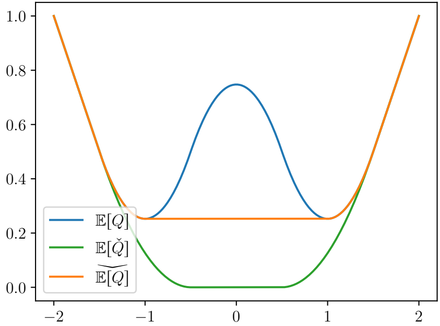
Measuring the non-convexity of a function

Gap inequality

$\mathbb{E}[\check{Q}]$ and $\widetilde{\mathbb{E}[Q]}$: underapproximations to $\mathbb{E}[Q]$.

- $\mathbb{E}[\check{Q}] \leqslant \widetilde{\mathbb{E}[Q]} \leqslant \mathbb{E}[Q]$
- Rewriting:

$$\mathbb{E}[Q] - \widetilde{\mathbb{E}[Q]} \leqslant \mathbb{E}[Q] - \mathbb{E}[\check{Q}]$$



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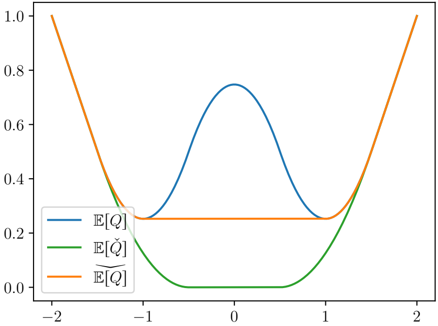

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Measuring the non-convexity of a function

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- So:

$$\operatorname{gap}(\mathbb{E}\left[Q\right]) \leqslant \mathbb{E}\left[\operatorname{gap}(Q)\right].$$

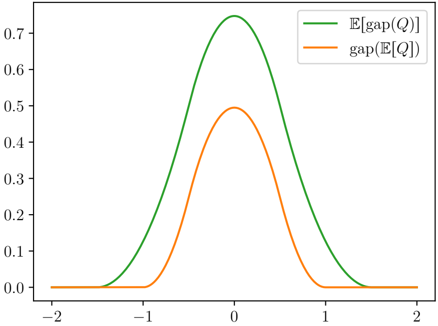

Figure: Gap functions.

Measuring the non-convexity of a function

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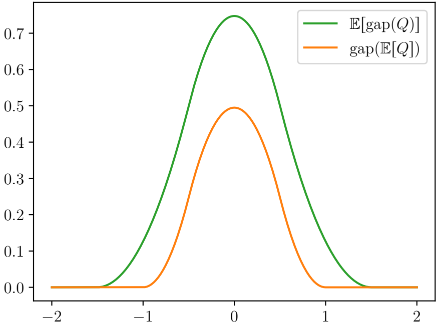

- Monotone norm:

$$\|\operatorname{gap}(\mathbb{E}[Q])\| \leqslant \|\mathbb{E}[\operatorname{gap}(Q)]\| \leqslant \mathbb{E}\|\operatorname{gap}(Q)\|\,.$$

Back to optimization

- Common case: cost-to-go is only uncertain on stage transition,

$$
\begin{aligned}
Q(x,\xi) = \min_{y} \quad & c^\top y \\
\text{s.t.} \quad & Ay = T(x-\xi) + b \\
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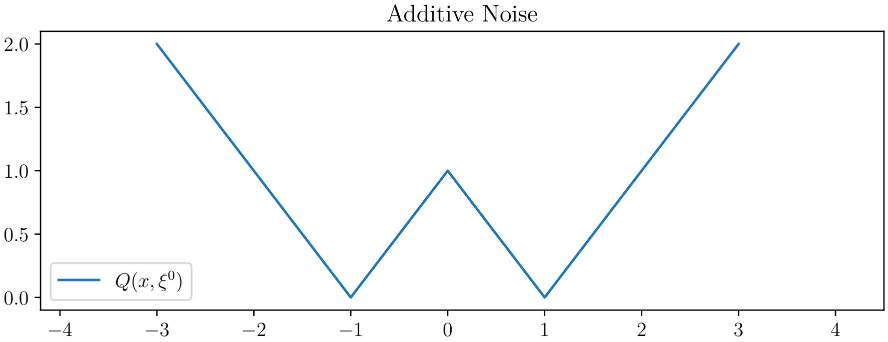
- Additive noise: $Q(x, \xi) = f(x - \xi)$.



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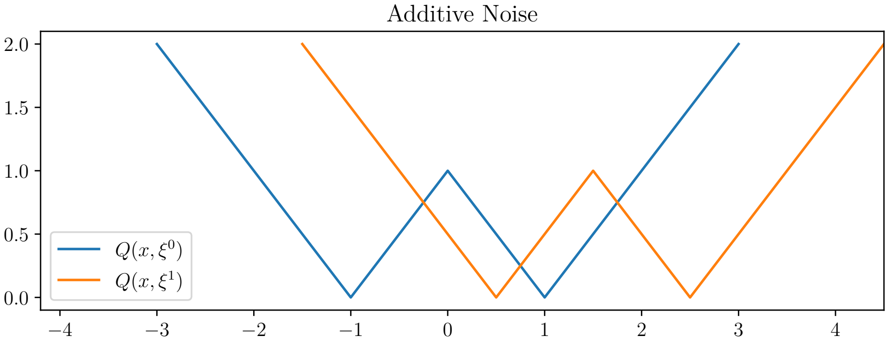

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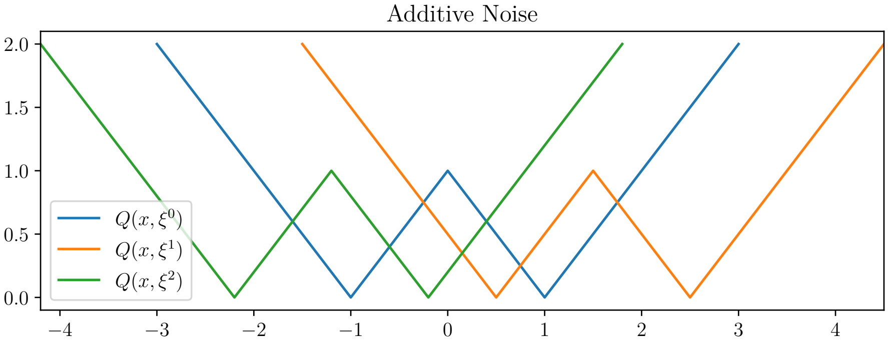

Additive noise

- Let $Q(x, \xi) = f(x - \xi)$.
- The **translation operator** τ_a shifts a function by a,

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Additive noise

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- The **translation operator** τ_a shifts a function by a,

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- A function norm is **translation invariant** if

$$\|f\| = \|\tau_a f\| .$$

Examples: Again, uniform norm and all p-norms.

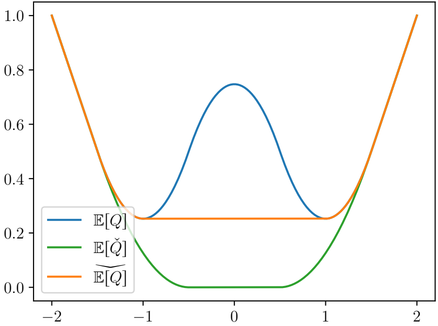
Additive noise

Inequality

- Let $Q(x, \xi) = \tau_\xi f(x) = f(x - \xi)$.
- $\|\cdot\|$: **monotone** and **translation invariant.**

- From the previous inequalities:

$$\begin{aligned}\|\mathrm{gap}(\mathbb{E}[Q])\| &\leqslant \|\mathbb{E}[\mathrm{gap}(Q)]\| \\ &\leqslant \mathbb{E}\|\mathrm{gap}(Q)\|\end{aligned}$$



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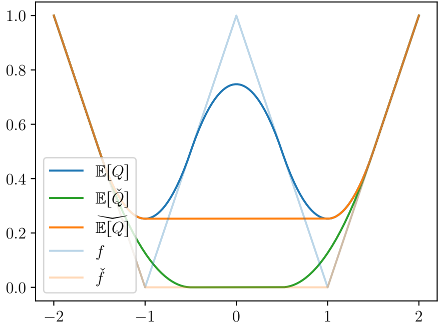

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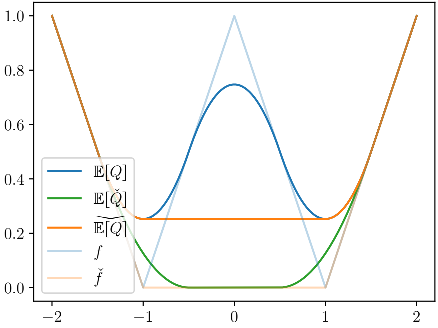

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&= \mathbb{E}\,\|\tau_\xi f - \widecheck{\tau_\xi f}\| \\
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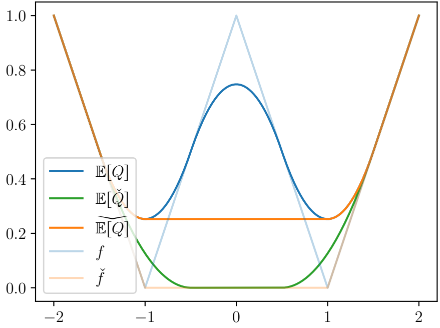

Additive noise

Inequality

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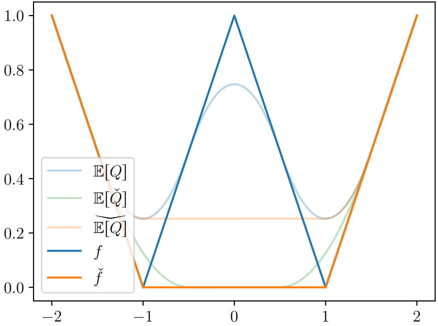

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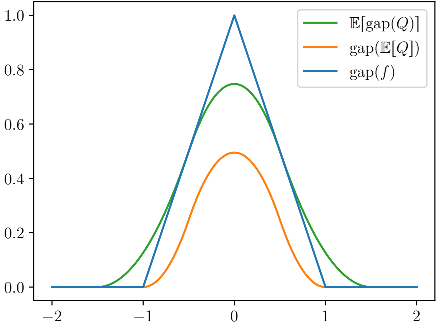

Additive noise

Uniform bounds

- Inequality for uniform norm:

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- κ is small if the distribution of ξ is "scattered enough".

Additive noise

Asymptotic behaviour

Theorem

If $\operatorname{gap}(f)$ *is* ***integrable*** *and* ξ_k *are random variables whose* ***densities*** μ_k *are bounded,*

$$\left\|\operatorname{gap}(\mathbb{E}\left[Q\right])\right\|_\infty \leqslant \left\|\mathbb{E}\left[\operatorname{gap}(Q)\right]\right\|_\infty \leqslant \left\|\mu_k\right\|_\infty \left\|\operatorname{gap}(f)\right\|_1 .$$

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$$\|\operatorname{gap}(\mathbb{E}[Q])\|_\infty \leqslant \|\mathbb{E}[\operatorname{gap}(Q)]\|_\infty \leqslant \|\mu_k\|_\infty \|\operatorname{gap}(f)\|_1 .$$

- This means that if

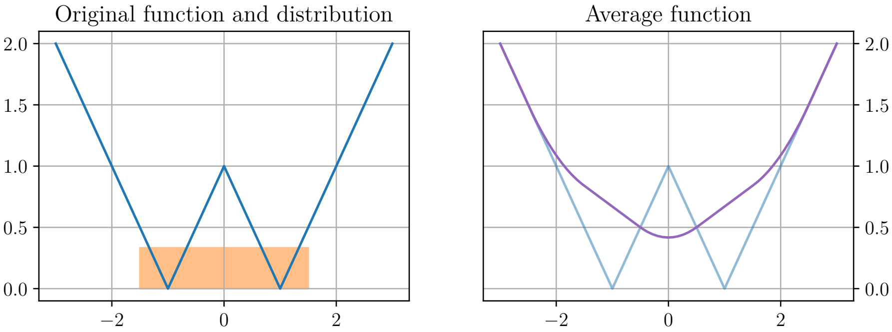
$$\|\mu_k\|_\infty \to 0,$$

the expected function $\mathbb{E}[Q]$ becomes **asymptotically** convex.

Additive noise

Asymptotic behaviour

- $\xi_h \sim U[-h, h]$
- $\|\mu_h\|_\infty = \frac{1}{2h} \to 0$



Additive noise
Asymptotic behaviour

- $\xi_h \sim N(m, \sigma^2)$
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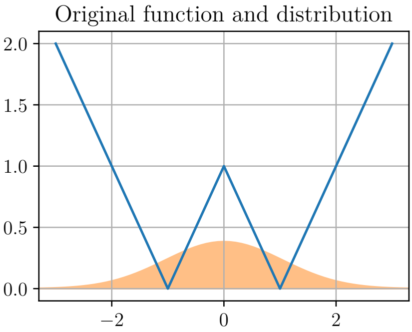

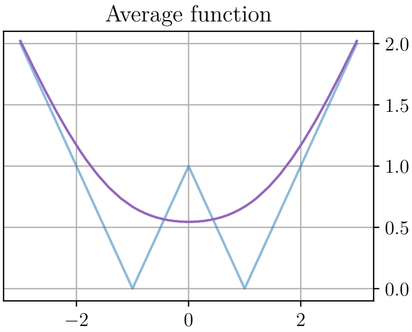

Another approach: second derivative

- Why the gap?
- For twice differentiable $f\colon (a,b) \to \mathbb{R}$,

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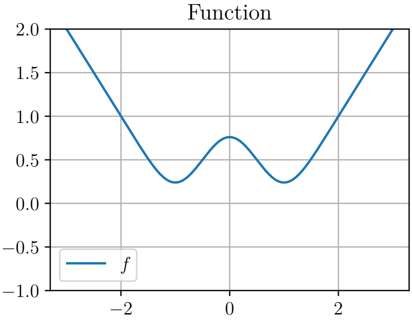

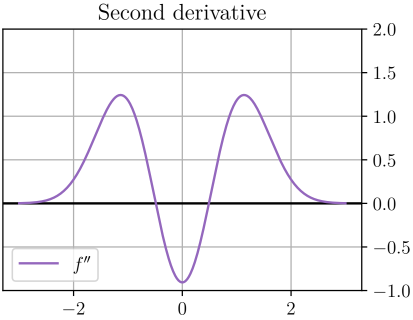

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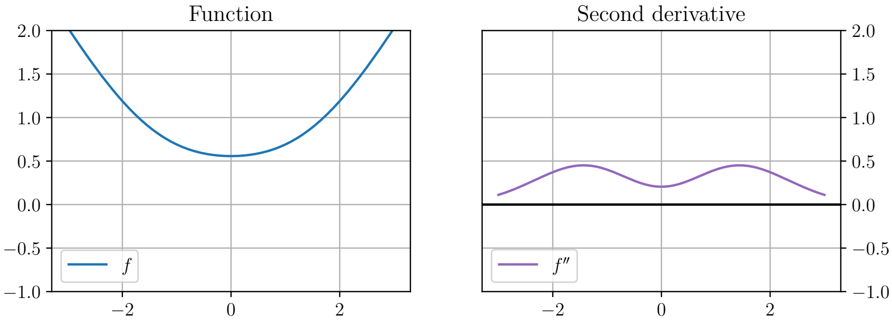

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Use the negative part of f'' to measure the non-convexity of f.

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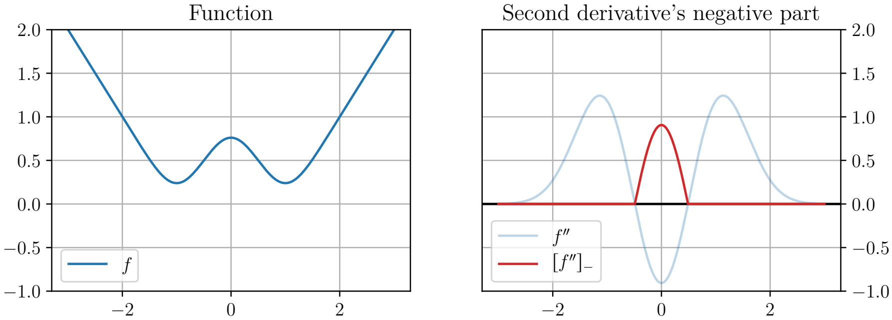

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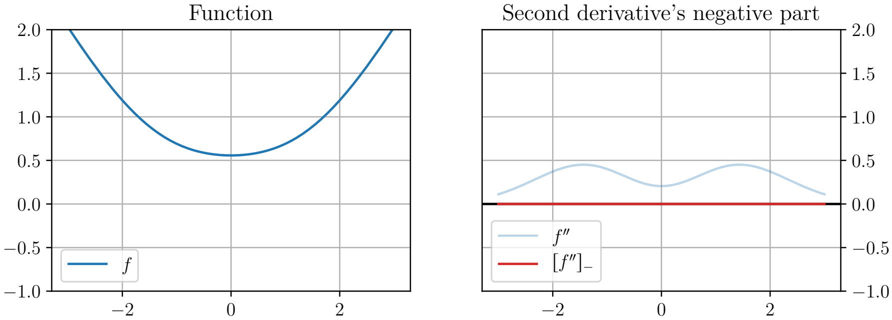

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Properties:

- $[f'']_-$ is identically zero if and only if f is convex,
- $[f'']_-$ is always non-negative.

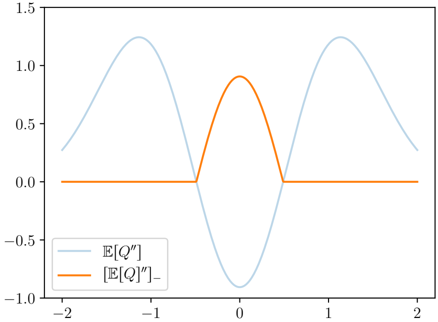
Another approach: second derivative

Main inequality

Let Q be a smooth random function,

- Exchanging expected value and derivative:

$$\mathbb{E}\left[Q\right]'' = \mathbb{E}\left[Q''\right].$$



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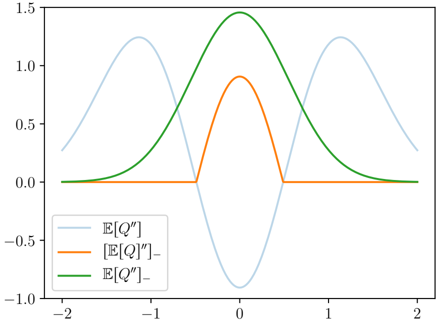
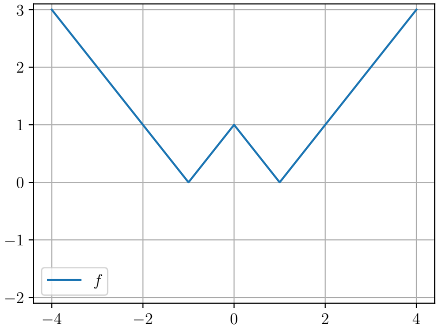


Derivative of cost-to-go

- The optimal value function of a mixed integer program is only **piecewise linear**.



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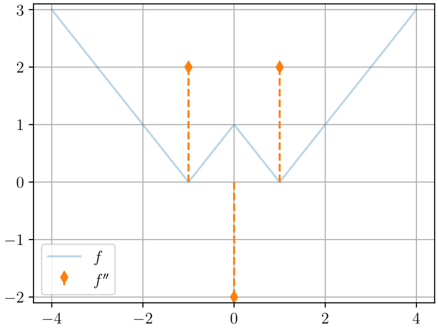

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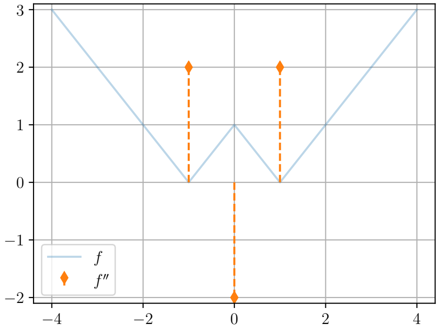

- More: f'' is a non-negative measure $\iff$ f is a convex function.

Another approach: second derivative

Decomposition of measures

Hahn-Jordan decomposition

Every (signed) measure μ can be uniquely decomposed as the difference of two non-negative and mutually singular measures:

$$\mu = [\mu]_+ - [\mu]_-$$

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Let $f : (a, b) \to \mathbb{R}$ be a continuous function. It is convex if and only if

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Another approach: second derivative

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Let Q be a random continuous function.

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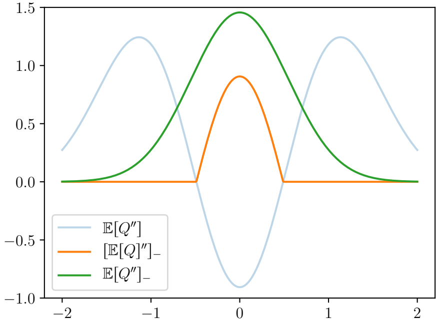

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Additive Noise

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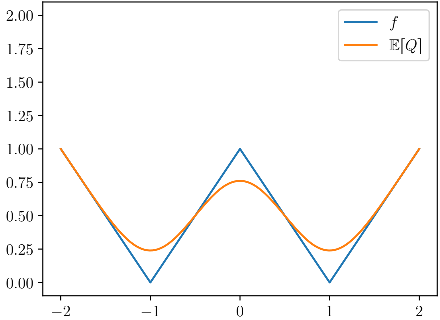

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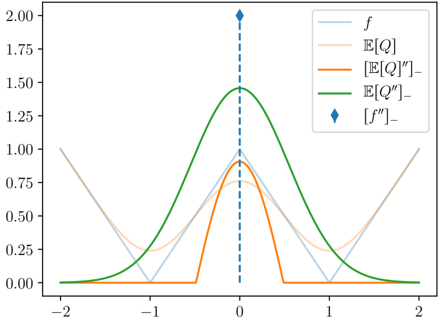

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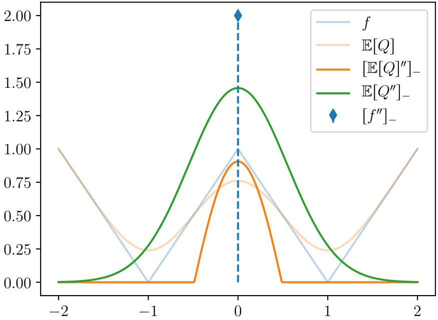

Similarities between $\mathrm{gap}(f)$ and $[f'']_-$

Inequalities for $\mathrm{gap}(f)$:

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Convex cones

Definition

A **pointed convex cone** K is a set that

- Contains all rays:

$$\forall \lambda \geqslant 0,\, x \in K \implies \lambda x \in K.$$

- Is convex:

$$x,\, y \in K, \theta \in [0,1] \implies \theta x + (1-\theta) y \in K,$$

- Contains no lines:

$$x \in K \text{ and } -x \in K \implies x = 0.$$

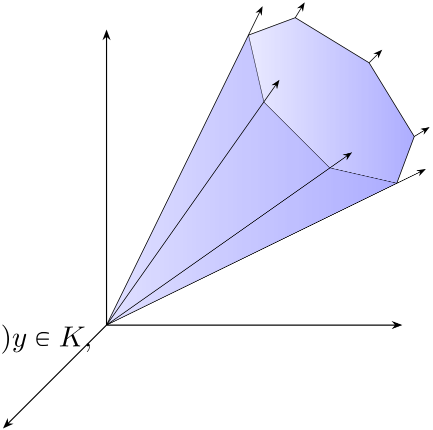

Figure: Example of convex cone.

Convexity with respect to a cone

Definition

Every proper convex cone K induces a **partial order** $\leqslant_K$ compatible with the linear structure and given by

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A function $f\colon X \to Y$ is **convex with respect to a cone** $K \subset Y$ if its domain is a convex set and for all $\lambda \in [0, 1]$,

$$f(\lambda x + (1-\lambda)y) \leqslant_K \lambda f(x) + (1-\lambda) f(y).$$

Non-convexity measures

Definition

A **non-convexity measure** on a set X of functions is an operator $\mathcal{M}\colon X \to Y$ whose codomain has a conic order $\leqslant_K$ such that

- $\mathcal{M}(f) = 0 \iff f$ is convex;
- $\mathcal{M}(f) \geqslant_K 0, \forall f$;
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$\mathcal{M} = \mathrm{gap}$ with K the cone of non-negative functions:

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If $\mathcal{M}(f)$ is integrable,

$$\|\mathbb{E}[\mathcal{M}(Q)]\|_\infty \leqslant \|\mu\|_\infty \|\mathcal{M}(f)\|_1$$

where μ is the probability density of the random variable.

Other examples

Hessian's smallest eigenvalue

- Second derivative for $\mathbb{R}^n$,
- f is convex $\iff$ its Hessian $D^2 f$ is positive semi-definite.

Non-convexity measure

Negative part of smallest eigenvalue of $D^2 f$,

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- Min-max theorem:

$$\lambda_1(D^2 f) = \inf_{\|v\|_2=1} v^t (D^2 f) v.$$

Risk-averse convexification

Coherent risk measures

- Replace $\mathbb{E}$ in all formulas by a **risk measure**.

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- Replace $\mathbb{E}$ in all formulas by a **risk measure**.

Definition

A **coherent risk measure** is a function ρ satisfying:

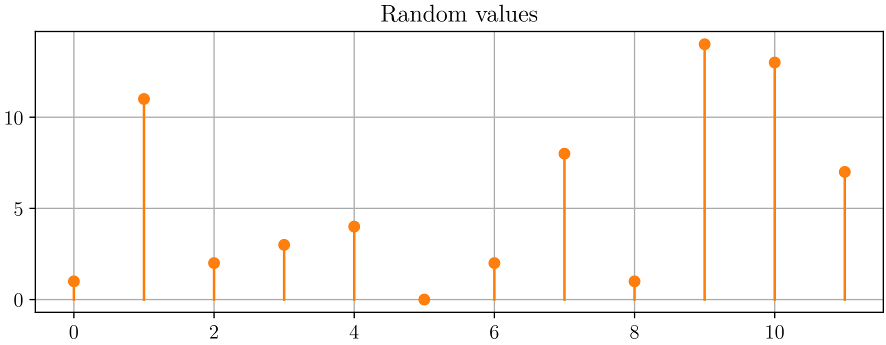
- Monotonicity: $X \leqslant Y \implies \rho(X) \leqslant \rho(Y)$;
- Translation equivariance: for all $a \in \mathbb{R}$, $\rho(X + a) = \rho(X) + a$;
- Convexity: for all $\lambda \in [0, 1]$,

$$\rho(\lambda X + (1 - \lambda)Y) \leqslant \lambda\rho(X) + (1 - \lambda)\rho(Y);$$

- Positive homogeneity: for all $t \geqslant 0$, $\rho(tX) = t\rho(X)$.

Coherent risk measures

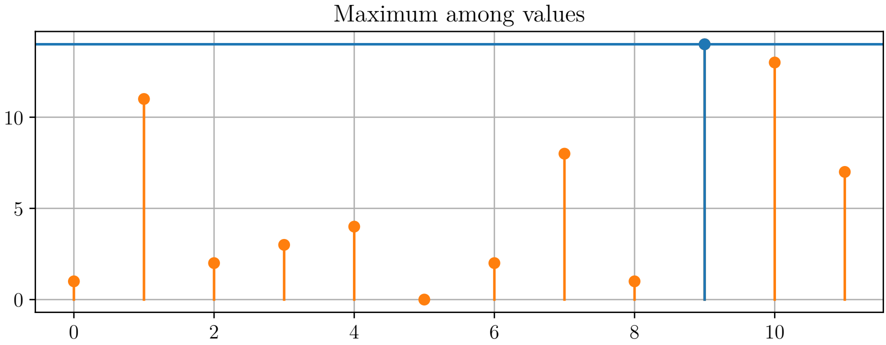
Examples of coherent risk measures:



Coherent risk measures

Examples of coherent risk measures:

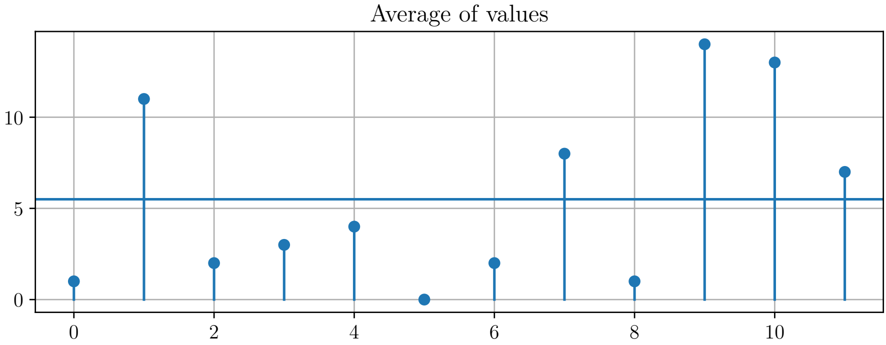
- Maximum value: $\sup X$;



Coherent risk measures

Examples of coherent risk measures:

- Maximum value: $\sup X$;
- Expected value: $\mathbb{E}[X]$;



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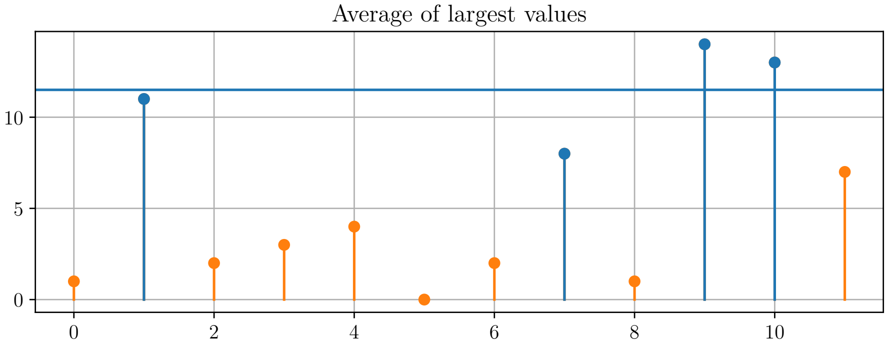


Coherent risk measures

Examples of coherent risk measures:

- Maximum value: $\sup X$;
- Expected value: $\mathbb{E}[X]$;
- Expectation of largest values;

Dual representation

A coherent risk measure can be written as

$$\rho(X) = \sup_{\mu \in \mathcal{P}} \mathbb{E}^{\mu}[X]$$

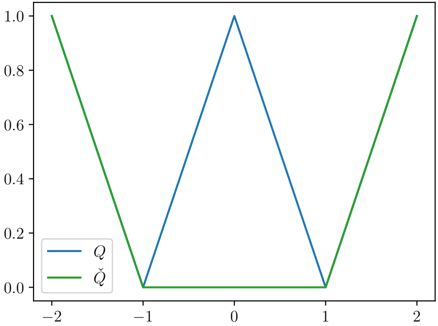
for a given family of probabilities $\mathcal{P}$.

Risk-averse convexification

Main inequality

Let Q be a random function, ρ a coherent risk measure.

- By definition: $\check{Q} \leqslant Q$.



Risk-averse convexification

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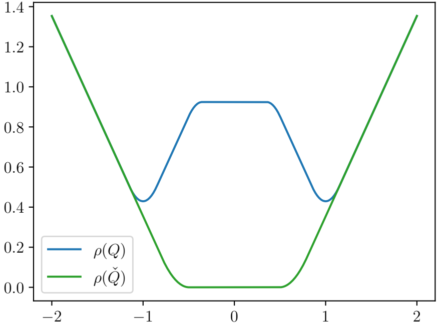

Risk-averse convexification

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Let Q be a random function, ρ a coherent risk measure.

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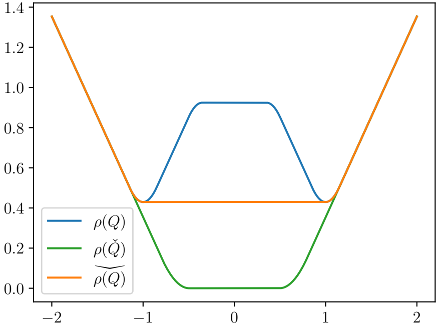
- Convex relaxation:

$$\rho(\check{Q}) \leqslant \overbrace{\rho(Q)} \leqslant \rho(Q).$$



Risk-averse convexification

Gap inequality

$\rho(\check{Q})$ and $\overbrace{\rho(Q)}$: underapproximations to $\rho(Q)$.

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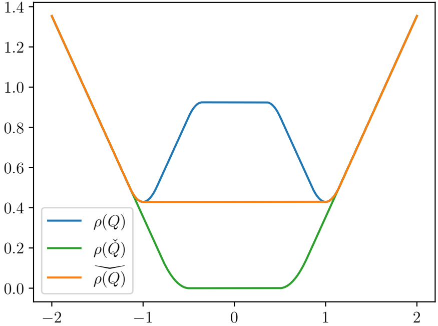

Risk-averse convexification

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- Then:

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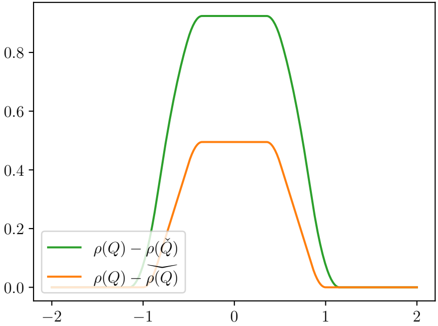


Risk-averse convexification

Gap inequality

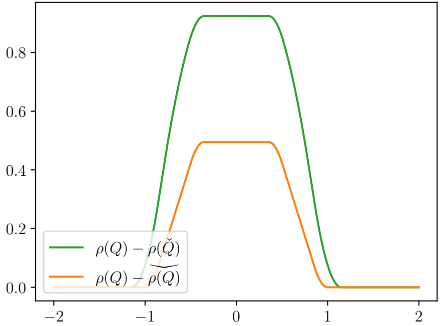
$\rho(\check{Q})$ and $\widetilde{\rho(Q)}$: underapproximations to $\rho(Q)$.

- $\rho(\check{Q}) \leqslant \widetilde{\rho(Q)} \leqslant \rho(Q)$
- Then:

$$\rho(Q) - \widetilde{\rho(Q)} \leqslant \rho(Q) - \rho(\check{Q}).$$

- Subadditivity:

$$\begin{aligned}\rho(Q) &= \rho(Q + \check{Q} - \check{Q}) \\ &\leqslant \rho(Q - \check{Q}) + \rho(\check{Q}).\end{aligned}$$



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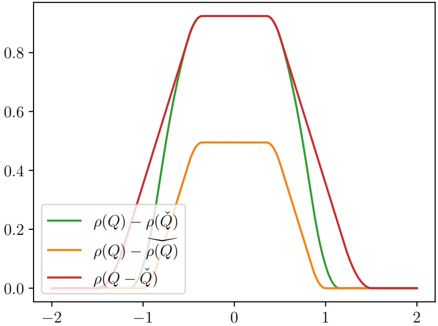

- Putting it all together:

$$\rho(Q) - \widetilde{\rho(Q)} \leqslant \rho(Q) - \rho(\check{Q}) \leqslant \rho(Q - \check{Q}).$$

Question

When the uncertainty is additive,

$$Q(x,\xi) = f(x-\xi) = \tau_\xi f(x),$$

do the results regarding translation invariant norms still hold?

Risk-averse convexification

Additive Noise

Question

When the uncertainty is additive,

$$Q(x, \xi) = f(x - \xi) = \tau_\xi f(x),$$

do the results regarding translation invariant norms still hold?

Answer

Not in general...

Additive noise

Counterexample

- Expected value: (always)

$$\|\mathrm{gap}(\mathbb{E}\left[Q\right])\| \leqslant \|\mathbb{E}\left[\mathrm{gap}(Q)\right]\| \leqslant \|\mathrm{gap}(f)\|\,.$$

- Risk measures: (always)

$$\|\mathrm{gap}(\rho(Q))\| \leqslant \|\rho(\mathrm{gap}(Q))\|\,.$$

Additive noise

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- Risk measures: (always)

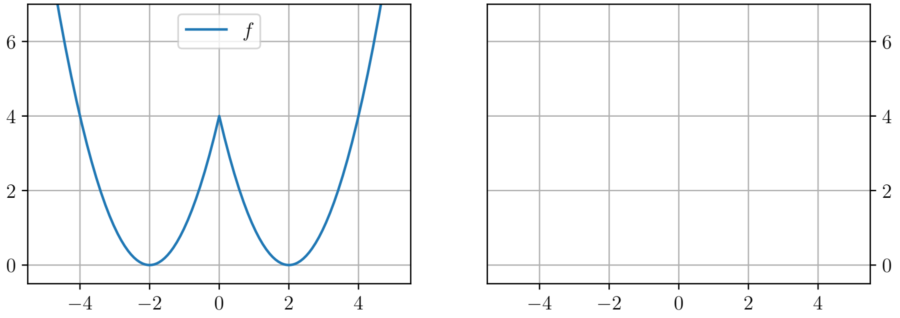
$$\|\mathrm{gap}(\rho(Q))\| \leqslant \|\rho(\mathrm{gap}(Q))\|\,.$$

- Risk measures: (possible)

$$\|\mathrm{gap}(\rho(Q))\| \qquad\qquad \geqslant \|\mathrm{gap}(f)\|\,.$$

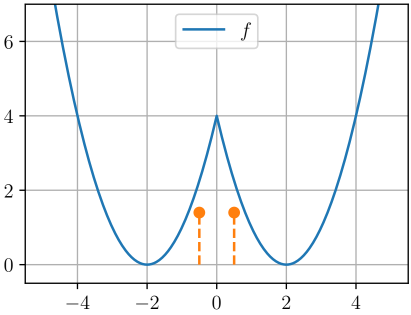
Additive noise

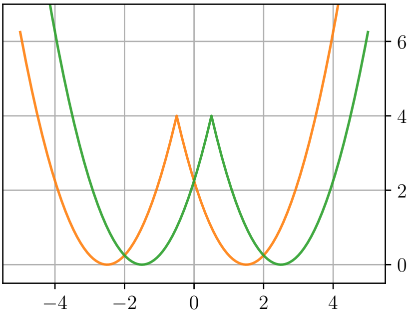
Counterexample



Additive noise

Counterexample





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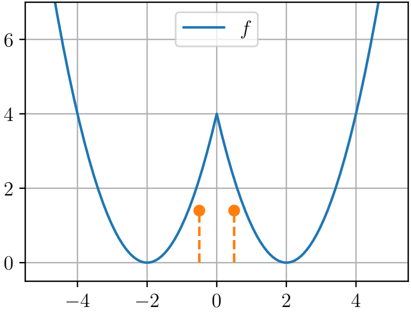

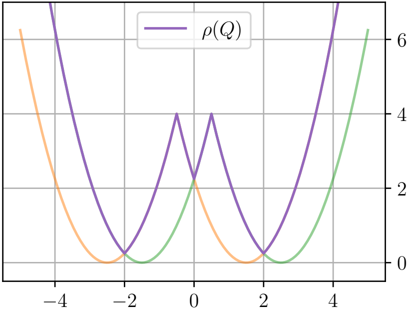

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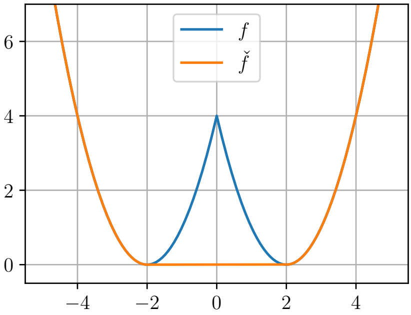

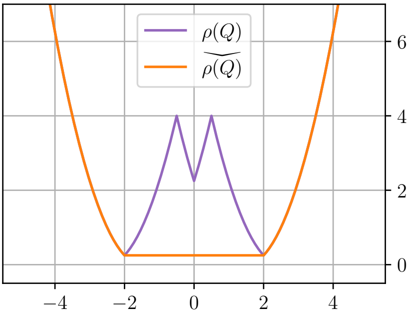

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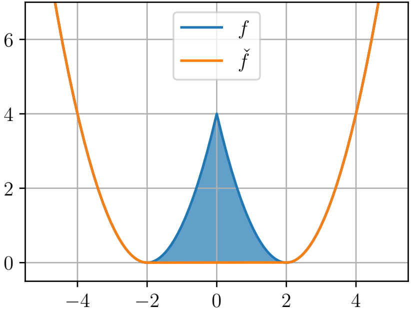
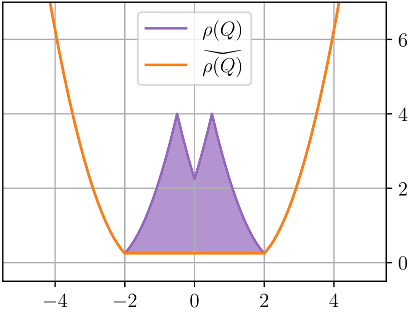


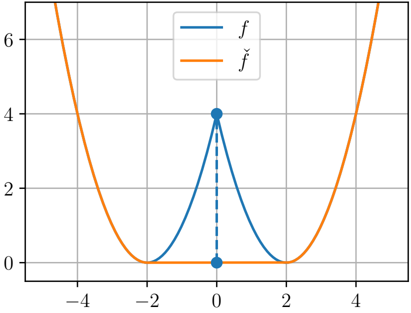


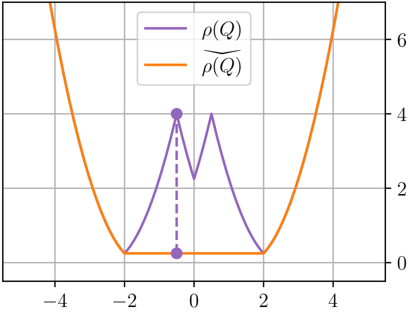
- Filled areas:

$$\|\text{gap}(f)\|_1 = \frac{16}{3}, \qquad \|\text{gap}(\rho(Q))\|_1 = \frac{22}{3}$$

Additive noise

Counterexample





- Filled areas and max heights:

$$\|\text{gap}(f)\|_1 = \frac{16}{3}, \qquad \|\text{gap}(\rho(Q))\|_1 = \frac{22}{3}$$

$$\|\text{gap}(f)\|_\infty = 4, \qquad \|\text{gap}(\rho(Q))\|_\infty = 3.75$$

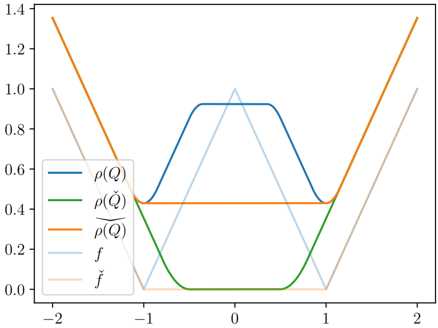
Additive Noise

Uniform Norm

Let $Q(x,\xi) = f(x-\xi) = \tau_\xi f(x)$.

- Always valid inequalities:

$$\rho(Q) - \widetilde{\rho(Q)} \leqslant \rho(Q) - \rho(\check{Q}) \leqslant \rho(Q - \check{Q}).$$



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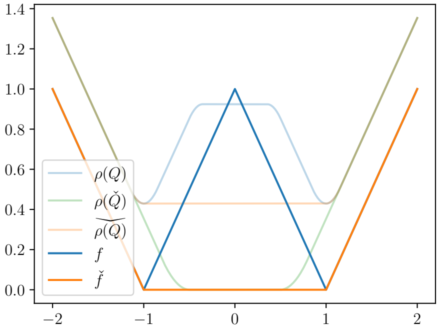

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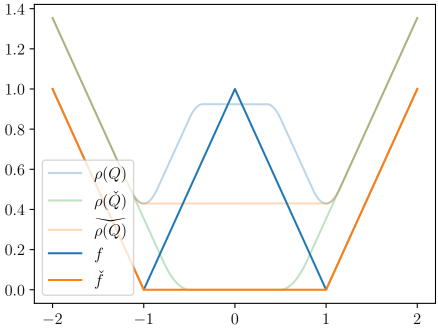

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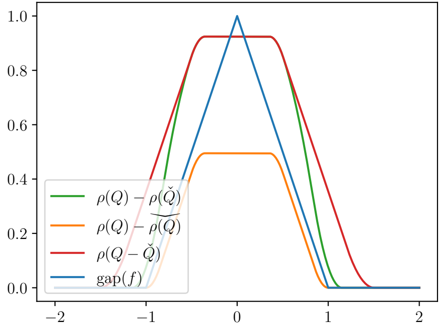

Approximation by cuts

Stochastic programs

What we want

Approximate $\mathbb{E}[Q]$ by cuts.

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- Calculate a cut for each scenario,
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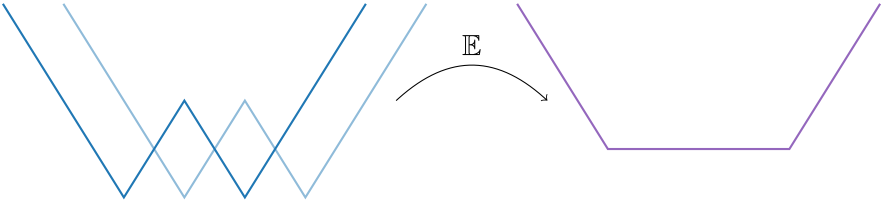

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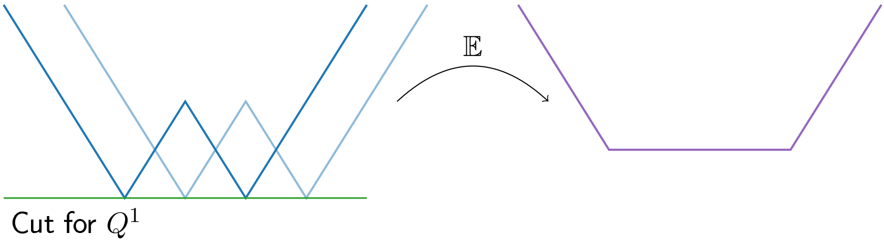

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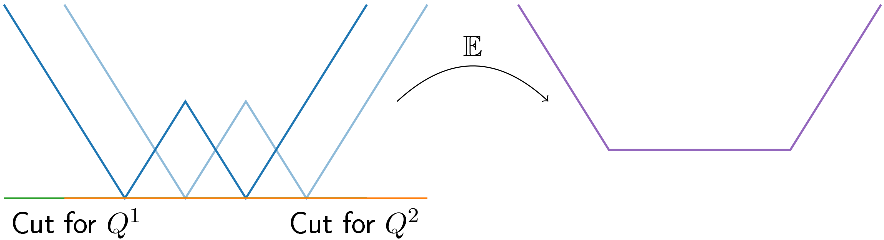

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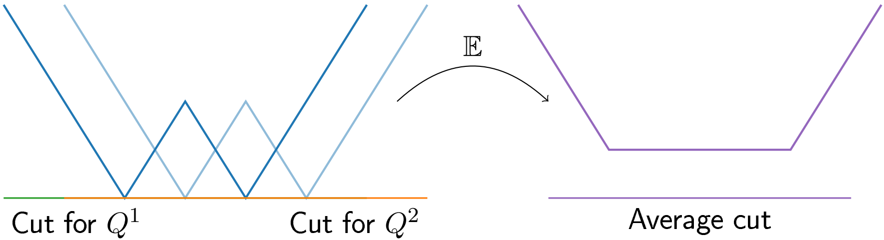

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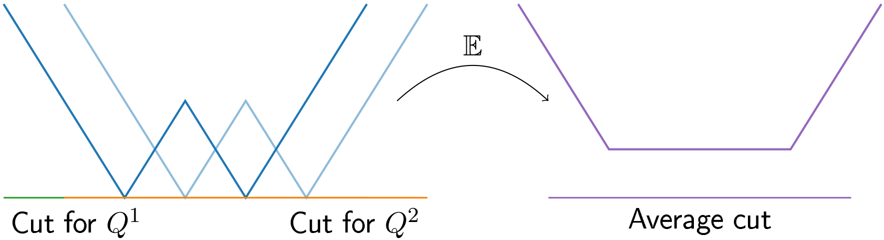
Stochastic programs

Question

Why was that cut not tight?



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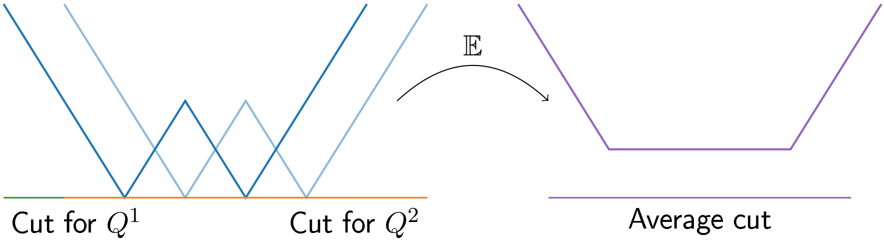

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- $\mathbb{E}\left[\check{Q}\right] \leqslant \overbrace{\mathbb{E}\left[Q\right]} \leqslant \mathbb{E}\left[Q\right]$.
- Cut for scenario ξ^i is only tight for $\check{Q}(\cdot, \xi^i)$.
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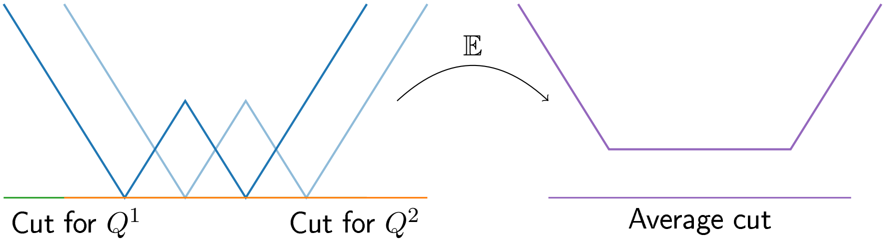

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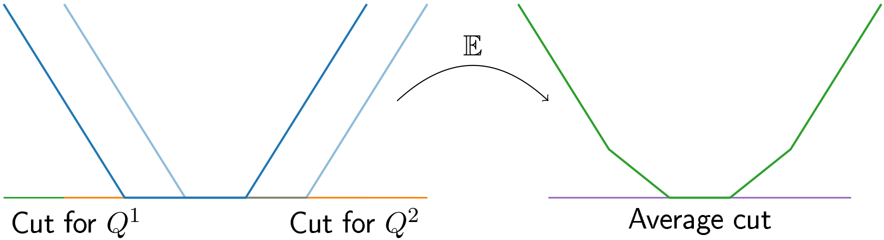

Linked formulation

Question

How to directly approximate $\widetilde{\mathbb{E}\left[Q\right]}$ by cuts?

Cost-to-go function:

$$\begin{aligned} Q(x,\xi) = \quad \min_{y} \quad & c^\top x \\ \text{s.t.} \quad & Ax + By = b \\ & y \geqslant 0 \\ & y \in \mathbb{R}^n \times \mathbb{Z}^k. \end{aligned}$$

Linked formulation

Question

How to directly approximate $\widetilde{\mathbb{E}[Q]}$ by cuts?

Finite number of scenarios:

$$
\begin{aligned}
Q(x, \xi^i) = \min_{y_i} \quad & c_i^\top y_i \\
\text{s.t.} \quad & A_i x + B_i y_i = b_i \\
& y_i \geqslant 0 \\
& y_i \in \mathbb{R}^n \times \mathbb{Z}^k.
\end{aligned}
$$

Linked formulation

Question

How to directly approximate $\widetilde{\mathbb{E}[Q]}$ by cuts?

Average over all scenarios:

$$\mathbb{E}[Q(x,\xi)] = \sum_{i=1}^{N} p_i \begin{array}{rl} \min\limits_{y_i} & c_i^\top y_i \\ \text{s.t.} & A_i x + B_i y_i = b_i \\ & y_i \geqslant 0 \\ & y_i \in \mathbb{R}^n \times \mathbb{Z}^k. \end{array}$$

Linked formulation

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How to directly approximate $\widetilde{\mathbb{E}\left[Q\right]}$ by cuts?

Build a linked program:

$$
\begin{aligned}
\mathbb{E}\left[Q(x,\xi)\right] = \min_{\substack{y_1,\dots,y_N \\ x_1,\dots,x_N}} \quad & \textstyle\sum_{i=1}^{N} p_i\, c_i^\top y_i \\
\text{s.t.} \quad & A_i x_i + B_i y_i = b_i, && \text{for } i = 1,\dots,N \\
& y_i \geqslant 0, && \text{for } i = 1,\dots,N \\
& y_i \in \mathbb{R}^n \times \mathbb{Z}^k, && \text{for } i = 1,\dots,N. \\
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\end{aligned}
$$

- Cuts for this problem can be tight for $\widetilde{\mathbb{E}\left[Q\right]}$!

Computational experiments

Table: A 4d, 8-stage, non-convex problem

	SB	SDDiP	SLDP	Relink
# cuts	100	100	250	250
LB	4.676	4.589	7.883	12.59
time (s)	7.3	4100	225	191
First iteration (s)	0.04	29	0.03	0.28
Last iteration (s)	0.10	150	1.7	0.67

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“Relinking” is very expensive for several scenarios: this problem has only 10 scenario branches per stage.

Hydrothermal planning

Computational experiments

- 12 stages;
- 2 connected hydro reservoirs;
- Binary operational constraints.

Table: Results for non-convex model with 2 subsystems.

	Cut types	
	Decomposed	Linked
Time (seconds)	759	16854
Iterations	500	500
Memory (GB)	0.418	1.089
Calculated cost (Bi R$)	10.410	11.170
Simulated cost (Bi R$)	12.227	12.209
Gap (%)	14.44	8.19

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Calculated cost (Bi R$)	10.410	11.170
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Hydrothermal planning

Computational experiments

- 12 stages;
- 2 connected hydro reservoirs;
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Future work

- Large-scale problems: linking only parts of the scenario tree.
- Applications to SDDiP or SLDP.

Thank you!