Nonlinear finite-element approximations and a posteriori error analysis for complex porous media flows

Flore Nabet*

joint work with Clément Cancès† and Martin Vohralík‡

* CMAP, Ecole polytechnique

† Inria Lille – Nord Europe

‡ Inria Paris

Applied Mathematics Seminar, CERMICS

February 27th, 2020

MOTIVATIONS

SCOPE:

- Oil engineering.
- Carbon dioxide storage.
- Nuclear waste repository management.

MODELS CONSTRAINTS:

- Degenerate parabolic systems of equations.
- Possibly highly anisotropic.
- Meshes prescribed by geological data.

EXISTING NUMERICAL METHODS:

1. Isotropic tensor with Delaunay meshes: monotonicity arguments
   $\rightsquigarrow$ Decay of the physical energy, Maximum principle, $\cdots$
2. Anisotropic tensor or general grids: loss of monotonicity
   $\rightsquigarrow$ Use of Kirchhoff transform.

SPECIFICATIONS FOR THE NUMERICAL METHOD

EFFICIENT NUMERICAL METHOD:

- easy to implement;
- affordable computational cost;
- convergence reasonably fast.

ROBUST W.R.T. TO THE DATA:

- degenerate diffusion operators;
- (strong) anisotropy;
- general grids.

PRESERVATION AT THE DISCRETE LEVEL OF CRUCIAL FEATURES:

- conservation of mass;
- decay of non-quadratic energy.

NO KIRCHHOFF TRANSFORM IN THE SCHEME

⇒ Nonlinear numerical method

(Cancès, Guichard, JFoCM, '16),(Cancès, Chainais-Hillairet, Krell, CMAM, '17)

THE NONLINEAR FOKKER PLANCK EQUATION

NONLINEAR DEGENERATE PARABOLIC EQUATION:

- Contains the main difficulties arising in porous media flows:
  - anisotropic diffusion tensor,
  - degeneracy,
  - general meshes.
- Keystone for the approximation of more complex problem.

THE STUDIED PROBLEM: Find $u : (0, t_\mathrm{f}) \times \Omega \to \mathbb{R}$ such that

$$\begin{cases} \partial_t u - \operatorname{div}\left(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi)\right) = 0 & \text{in } (0, t_\mathrm{f}) \times \Omega, \\ \eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) \cdot \mathbf{n} = 0 & \text{on } (0, t_\mathrm{f}) \times \partial\Omega, \\ u_{|t=0} = u^0 & \text{in } \Omega, \end{cases}$$

where

- $\eta(u) \geq 0$ such that $\eta(0) = 0$, typically $\eta(u) = u$;
- p increasing, p may be singular $(p(u) = \log(u))$;
- $\boldsymbol{\Lambda}$ a symmetric tensor field;
- Ψ a given potential.

THE NONLINEAR FOKKER PLANCK EQUATION

NONLINEAR DEGENERATE PARABOLIC EQUATION:

- Contains the main difficulties arising in porous media flows:
  - anisotropic diffusion tensor,
  - degeneracy,
  - general meshes.
- Keystone for the approximation of more complex problem.

THE STUDIED PROBLEM: Find $u : (0, t_\mathrm{f}) \times \Omega \to \mathbb{R}$ such that $d = 2$ or $d = 3$

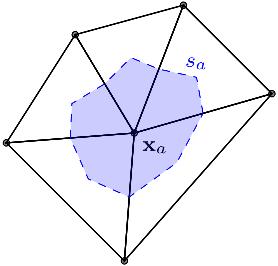
$$\begin{cases} \partial_t u - \operatorname{div}\left(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi)\right) = f_\mathrm{inj} - \eta(u^+) f_\mathrm{out} \quad \text{in } (0, t_\mathrm{f}) \times \Omega, \\ \eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) \cdot \mathbf{n} = 0 \quad \text{on } (0, t_\mathrm{f}) \times \Sigma_\mathrm{N}, \\ p(u) = p_\mathrm{D} \quad \text{on } (0, t_\mathrm{f}) \times \Sigma_\mathrm{D}, \\ u_{|t=0} = u^0 \quad \text{in } \Omega, \end{cases}$$

where

- $\eta(u) \geq 0$ such that $\eta(0) = 0$, typically $\eta(u) = u$;
- p increasing, p may be singular $(p(u) = \log(u))$;
- $\boldsymbol{\Lambda}$ a symmetric tensor field;
- Ψ a given potential.

THE DISCRETE SPACE SETTING

SPACE DISCRETIZATION: $\mathbb{P}_1$ Finite-Element with mass lumping



- $\mathcal{T}$: triangles mesh, $T \in \mathcal{T}$
- $\mathcal{V}$: vertices, $a \in \mathcal{V}$
- $h_{\mathcal{T}}$: mesh size

- V_h: usual $\mathbb{P}_1$ FE space
- FE $\mathbb{P}_1$ basis: $(\phi_a)_{a\in\mathcal{V}}$

PIECEWISE LINEAR RECONSTRUCTION: $\forall u \in \mathcal{C}(\overline{\Omega})$,

$$\pi_1 u(\mathbf{x}) = \sum_{a\in\mathcal{V}} u(\mathbf{x}_a)\phi_a(\mathbf{x}) \quad \Longrightarrow \quad u_h = \pi_1 u \in V_h.$$

PIECEWISE CONSTANT RECONSTRUCTION: $\forall u \in \mathcal{C}(\overline{\Omega})$,

$$\pi_0 u(\mathbf{x}) = \sum_{a\in\mathcal{V}} u(\mathbf{x}_a)\mathbf{1}_{s_a}(\mathbf{x}) \quad \text{and} \quad \pi_0 u_h = \overline{u}_h,\ \forall u_h \in V_h.$$

THE FULLY DISCRETE SCHEME

TIME DISCRETIZATION:

- Discretization of $[0, t_\mathrm{f}]$: $0 = t_0 < t_1 < \cdots < t_\mathrm{N} = t_\mathrm{f}$.
- Time step $\tau = \max\limits_{1 \le n \le N} \tau_n$ with $\tau_n = t_n - t_{n-1}$ for $1 \le n \le N$.
- $u_h^n = (u_a^n)_{a \in \mathcal{V}}$ unknown at time t^n.

$\mathbb{P}_1$ FINITE-ELEMENT SCHEME

Let $u_h^{n-1} \in V_h$, we look for $u_h^n \in V_h$ such that for any $v_h \in V_h$,

$$\boxed{\int_\Omega \frac{\overline{u}_h^n - \overline{u}_h^{n-1}}{\tau_n} \overline{v}_h + \int_\Omega \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla} v_h = 0,}$$

with

$$\mathbf{\Lambda}_h(\mathbf{x}) = \frac{1}{|T|} \int_T \mathbf{\Lambda}(\mathbf{x}) \mathrm{d}\mathbf{x}, \quad \text{if } \mathbf{x} \in T;$$

and

$$\eta_h^n = \pi_1 \eta(u_h^n), \quad p_h^n = \pi_1 p(u_h^n), \quad \Psi_h = \pi_1 \Psi.$$

PROPERTIES OF THE SCHEME

- Mass conservation and energy dissipation;
- Positivity of solutions if $p(0) = -\infty$;
- Existence of a discrete solution.

CONVERGENCE ANALYSIS

$$\partial_t u - \operatorname{div}\left(\eta(u)\mathbf{\Lambda}\mathbf{\nabla}\left(p(u)+\Psi\right)\right) = 0.$$

KIRCHHOFF TRANSFORMS:

$$\boxed{\phi(u) = \int^u \eta(s)p'(s)\mathrm{d}s, \qquad \xi(u) = \int^u \sqrt{\eta(s)}p'(s)\mathrm{d}s, \quad u \geq 0.}$$

- If $p(u)$ is regular enough,

$$\eta(u)\mathbf{\nabla}p(u) = \mathbf{\nabla}\phi(u), \qquad \eta(u)|\mathbf{\nabla}p(u)|^2 = |\mathbf{\nabla}\xi(u)|^2.$$

- The nonlinear Fokker-Planck equation rewrites,

$$\partial_t u - \operatorname{div}\left(\mathbf{\Lambda}\mathbf{\nabla}\phi(u) + \eta(u)\mathbf{\Lambda}\mathbf{\nabla}\Psi\right) = 0.$$

- $\phi(u) \in L^2((0,t_{\mathrm{f}}); H^1(\Omega))$ is well defined
  $\rightsquigarrow$ existence and uniqueness of weak solutions. **(Alt-Luckhaus '83, Otto '96)**

- Monotone operator $\rightsquigarrow$ the implicit Euler scheme converges.

CONVERGENCE ANALYSIS

CONVERGENCE THEOREM:

DEFINITION (WEAK SOLUTION)

A measurable function u is a weak solution if

- $u, \eta(u) \in L^\infty(0, t_\mathrm{f}; L^1(\Omega))$,
- $\xi(u) \in L^2(0, t_\mathrm{f}; H^1(\Omega))$,
- *and for any* $\varphi \in \mathcal{C}_c^\infty([0, t_\mathrm{f}[\times\overline{\Omega})$,

$$\int_0^{t_\mathrm{f}} \int_\Omega u \partial_t \varphi + \int_\Omega u^0 \varphi(0, \cdot) - \int_0^{t_\mathrm{f}} \int_\Omega (\boldsymbol{\nabla}\phi(u) + \eta(u)\boldsymbol{\nabla}\Psi) \cdot \boldsymbol{\Lambda}\boldsymbol{\nabla}\varphi = 0.$$

Time reconstruction: $u_{h\tau}(t, \cdot) = u_h^n$ on $(t_{n-1}, t_n]$.

THEOREM (CONVERGENCE)

For any $u^0 : \Omega \to \mathbb{R}^+$ be such that $E(u^0) < +\infty$, there exists a weak solution u such that, up to a subsequence,

$$\overline{u}_{h\tau} \xrightarrow[h_\mathcal{T}, \tau \to 0]{} u \quad \textit{strongly in } L^1((0, t_\mathrm{f}) \times \Omega).$$

SKETCH OF PROOF:

1. Compactness of the family $\overline{u}_{h\tau}$.
2. Identification of the limit of a subsequence as a weak solution.

CONVERGENCE ANALYSIS

COMPACTNESS OF THE FAMILY $\overline{u}_{h\tau}$

$$\boxed{\int_0^{t_{\mathrm{f}}} \int_\Omega \mathbf{\Lambda}_h \boldsymbol{\nabla} \xi_{h\tau} \cdot \boldsymbol{\nabla} \xi_{h\tau} \leq C}$$

CONVERGENCE ANALYSIS

COMPACTNESS OF THE FAMILY $\overline{u}_{h\tau}$

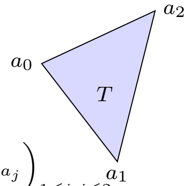
$$\sum_{n=0}^{N-1} \tau_n \sum_{T\in\mathcal{T}} \int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\xi_h^n \; = \; \int_0^{t_{\mathrm{f}}} \int_\Omega \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_{h\tau} \cdot \boldsymbol{\nabla}\xi_{h\tau} \leq C$$

CONVERGENCE ANALYSIS

COMPACTNESS OF THE FAMILY $\overline{u}_{h\tau}$

$$\sum_{n=0}^{N-1} \tau_n \sum_{T\in\mathcal{T}} \boxed{\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\xi_h^n} = \int_0^{t_\mathrm{f}} \int_\Omega \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_{h\tau} \cdot \boldsymbol{\nabla}\xi_{h\tau} \leq C$$

- Discretization of the diffusion operator:



$$\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} v_h = \boldsymbol{\delta}_T v \cdot \mathbf{A}^T \boldsymbol{\delta}_T u$$

with

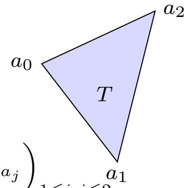
$$\boldsymbol{\delta}_T v = \begin{pmatrix} v_{a_1} - v_{a_0} \\ v_{a_2} - v_{a_0} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^T = \left(\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\phi_{a_i} \cdot \boldsymbol{\nabla}\phi_{a_j} \right)_{1\leq i,j\leq 2}$$

CONVERGENCE ANALYSIS

COMPACTNESS OF THE FAMILY $\overline{u}_{h\tau}$

$$\sum_{n=0}^{N-1} \tau_n \sum_{T\in\mathcal{T}} \boxed{\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\xi_h^n} = \int_0^{t_\mathrm{f}} \int_\Omega \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_{h\tau} \cdot \boldsymbol{\nabla}\xi_{h\tau} \leq C$$

- Discretization of the diffusion operator:



$$\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla} u_h \cdot \boldsymbol{\nabla} v_h = \boldsymbol{\delta}_T v \cdot \mathbf{A}^T \boldsymbol{\delta}_T u$$

with

$$\boldsymbol{\delta}_T v = \begin{pmatrix} v_{a_1} - v_{a_0} \\ v_{a_2} - v_{a_0} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^T = \left(\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\phi_{a_i} \cdot \boldsymbol{\nabla}\phi_{a_j} \right)_{1\leq i,j\leq 2}$$

- Definition of ξ: $\xi(u) = \int^u \sqrt{\eta(s)}p'(s)\mathrm{d}s$

$$\left(\xi(u_{a_i}^n) - \xi(u_{a_0}^n)\right)^2 \leq \left(\max_T \eta\right) \left(p(u_{a_i}^n) - p(u_{a_0}^n)\right)^2.$$

$\mathbb{P}_0$ reconstruction on each triangle:

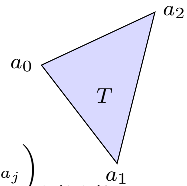
$$\eta_T^n = \frac{1}{3}\sum_{i=0}^{2} \eta(u_{a_i}^n) \implies \left(\xi(u_{a_i}^n) - \xi(u_{a_0}^n)\right)^2 \leq 3\eta_T^n \left(p(u_{a_i}^n) - p(u_{a_0}^n)\right)^2.$$

CONVERGENCE ANALYSIS

COMPACTNESS OF THE FAMILY $\overline{u}_{h\tau}$

$$\sum_{n=0}^{N-1} \tau_n \sum_{T\in\mathcal{T}} \boxed{\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\xi_h^n} = \int_0^{t_\mathrm{f}} \int_\Omega \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_{h\tau} \cdot \boldsymbol{\nabla}\xi_{h\tau} \le C$$

- Discretization of the diffusion operator:



$$\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}u_h \cdot \boldsymbol{\nabla}v_h = \boldsymbol{\delta}_T v \cdot \mathbf{A}^T \boldsymbol{\delta}_T u$$

with

$$\boldsymbol{\delta}_T v = \begin{pmatrix} v_{a_1} - v_{a_0} \\ v_{a_2} - v_{a_0} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^T = \left(\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\phi_{a_i} \cdot \boldsymbol{\nabla}\phi_{a_j} \right)_{1\le i,j\le 2}$$

- Definition of ξ and η_T^n: $\qquad \xi(u) = \int^u \sqrt{\eta(a)}p'(a)\mathrm{d}a$ and $\eta_T^n = \dfrac{1}{3}\sum_{i=0}^{2} \eta(u_{a_i}^n)$

$$\eta_T^n \left|\boldsymbol{\delta}_T p_h^n\right|^2 \ge \frac{1}{3} \left|\boldsymbol{\delta}_T \xi_h^n\right|^2 .$$

- Using $\langle \mathbf{y}, A\mathbf{y}\rangle \le \langle \mathbf{x}, A\mathbf{x}\rangle$, $\forall \mathbf{x},\mathbf{y}$ s.t. $|\mathbf{x}|^2 \ge \mathrm{Cond}_2(A)|\mathbf{y}|^2$ and since $\mathrm{Cond}(A^T) \le C$, **(Brenner-Masson, IJFV, '13)**

$$\int_T \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\xi_h^n \le C \int_T \eta_T^n \mathbf{\Lambda}_h \boldsymbol{\nabla}p_h^n \cdot \boldsymbol{\nabla}p_h^n.$$

Convergence analysis

Identification of the limit $\exists u$ s.t $\overline{u}_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} u$ strongly in $L^1((0,t_{\mathrm{f}})\times\Omega)$.

Convergence analysis

IDENTIFICATION OF THE LIMIT $\exists u$ s.t $\overline{u}_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} u$ strongly in $L^1((0,t_{\mathrm{f}})\times\Omega)$.

Convergence for the term $\widetilde{\eta}_h^n(\mathbf{x}) = \eta_T^n$ if $\mathbf{x}\in T$

$$\int_\Omega \widetilde{\eta}_h^n \boldsymbol{\Lambda}_h \boldsymbol{\nabla} p_h^n \cdot \boldsymbol{\nabla}\varphi_h^n = \int_\Omega \eta_h^n \boldsymbol{\Lambda}_h \boldsymbol{\nabla} p_h^n \cdot \boldsymbol{\nabla}\varphi_h^n, \quad \text{with } \varphi_h^n = \pi_1\varphi(t^n,\cdot).$$

- de la Vallée-Poussin theorem + Vitali's theorem:

$$\widetilde{\eta}_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \eta(u) \;\; \text{strongly in } L^1((0,t_{\mathrm{f}})\times\Omega),$$

BUT

$$\xcancel{\boldsymbol{\nabla} p_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \boldsymbol{\nabla} p(u) \;\; \text{in } L^\infty((0,t_{\mathrm{f}})\times\Omega).}$$

CONVERGENCE ANALYSIS

IDENTIFICATION OF THE LIMIT $\exists u$ s.t $\overline{u}_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} u$ strongly in $L^1((0,t_{\mathrm{f}})\times\Omega)$.

Convergence for the term $\widetilde{\eta}_h^n(\mathbf{x}) = \eta_T^n$ if $\mathbf{x}\in T$

$$\int_\Omega \widetilde{\eta}_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla} p_h^n \cdot \boldsymbol{\nabla}\varphi_h^n = \int_\Omega \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla} p_h^n \cdot \boldsymbol{\nabla}\varphi_h^n, \quad \text{with } \varphi_h^n = \pi_1\varphi(t^n,\cdot).$$

- de la Vallée-Poussin theorem + Vitali's theorem:

$$\widetilde{\eta}_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \eta(u) \ \text{ strongly in } L^1((0,t_{\mathrm{f}})\times\Omega),$$

BUT

$$\xcancel{\boldsymbol{\nabla} p_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \boldsymbol{\nabla} p(u) \ \text{ in } L^\infty((0,t_{\mathrm{f}})\times\Omega).}$$

- Use of Kirchhoff transform:

$$\boxed{\int_\Omega \widetilde{\eta}_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla} p_h^n \cdot \boldsymbol{\nabla}\varphi_h^n = \int_\Omega \sqrt{\widetilde{\eta}_h^n}\, \mathbf{\Lambda}_h \boldsymbol{\nabla}\xi_h^n \cdot \boldsymbol{\nabla}\varphi_h^n + R_h^n}$$

with $R_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} 0$ and

$$\sqrt{\widetilde{\eta}_{h\tau}} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \sqrt{\eta(u)} \text{ strongly in } L^2((0,t_{\mathrm{f}})\times\Omega),$$

$$\text{and } \boldsymbol{\nabla}\xi_{h\tau} \xrightarrow[h_{\mathcal{T}},\tau\to 0]{} \boldsymbol{\nabla}\xi(u) \text{ weakly in } L^2((0,t_{\mathrm{f}})\times\Omega).$$

A POSTERIORI ANALYSIS

EQUILIBRATED FLUX RECONSTRUCTION

- Continuous level:

$$\boldsymbol{\sigma} := -\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u)+\Psi) \in L^1((0,t_{\mathrm{f}});\mathbf{H}(\mathrm{div},\Omega)) \quad \text{and} \quad \mathrm{div}\boldsymbol{\sigma} = -\partial_t u.$$

- Discrete level: for any $n \geq 1$,

$$-\eta_h^n \boldsymbol{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \notin \mathbf{H}(\mathrm{div},\Omega) \quad \text{and} \quad \mathrm{div}\big(\eta_h^n \boldsymbol{\Lambda}_h \boldsymbol{\nabla}(p_h^n+\Psi_h)\big) \neq -\frac{u_h^n - u_h^{n-1}}{\tau_n}.$$

THEOREM

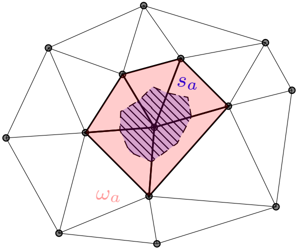
There exists $\boldsymbol{\sigma}_{h\tau} \in L^2((0,t_{\mathrm{f}});\mathbf{H}(\mathrm{div},\Omega))$ *such that*

$$\partial_t \hat{u}_{h\tau} + \mathrm{div}\boldsymbol{\sigma}_{h\tau} = 0 \quad \textit{and} \quad \boldsymbol{\sigma}_{h\tau}\cdot\mathbf{n} = 0 \textit{ on } (0,t_{\mathrm{f}})\times\partial\Omega,$$

where $\hat{u}_{h\tau}$ *is the piecewise affine in space and time approximation of* $u_{h\tau}$*.*

Scheme locally conservative

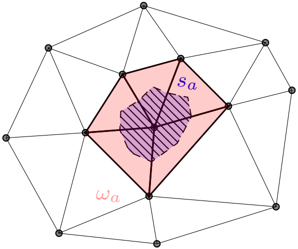
EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF



Scheme:

$$\int_\Omega \frac{\overline{u}_h^n - \overline{u}_h^{n-1}}{\tau_n}\overline{v}_h + \int_\Omega \eta_h^n \mathbf{\Lambda}_h \mathbf{\nabla}(p_h^n + \Psi_h) \cdot \mathbf{\nabla} v_h = 0,$$

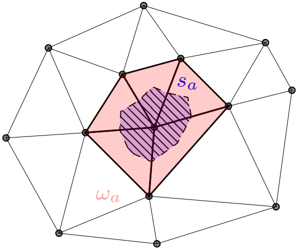
EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF



Scheme: $v_h = \phi_a$

$$\int_\Omega \frac{\overline{u}_h^n - \overline{u}_h^{n-1}}{\tau_n} \overline{\phi}_a + \int_\Omega \eta_h^n \mathbf{\Lambda}_h \mathbf{\nabla}(p_h^n + \Psi_h) \cdot \mathbf{\nabla} \phi_a = 0,$$

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF

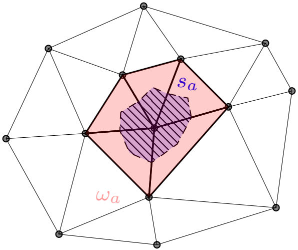


Scheme: $v_h = \phi_a$

$$\int_\Omega \frac{\overline{u}_h^n - \overline{u}_h^{n-1}}{\tau_n} \overline{\phi}_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a = 0,$$

1. $\mathrm{supp}(\phi_a) = \omega_a$.

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF



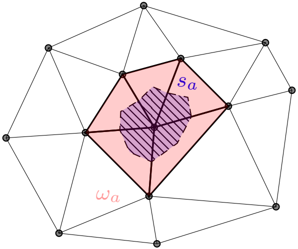
Scheme: $v_h = \phi_a$

$$\int_\Omega \frac{\overline{u}_h^n - \overline{u}_h^{n-1}}{\tau_n}\overline{\phi}_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a = 0,$$

1. $\mathrm{supp}(\phi_a) = \omega_a$.
2. $\overline{\phi}_a(\mathbf{x}) = \phi_a(\mathbf{x}_{a'})$ if $\mathbf{x} \in s_{a'} \implies \overline{\phi}_a = 1_{s_a}$:

$$\int_\Omega \overline{u}_h \overline{\phi}_a = \int_{s_a} \overline{u}_h = |s_a| u_a = \int_{\omega_a} u_a \phi_a.$$

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF



Scheme: $v_h = \phi_a$

$$\int_{\omega_a} \frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \mathbf{\nabla}(p_h^n + \Psi_h) \cdot \mathbf{\nabla} \phi_a = 0,$$

1. $\mathrm{supp}(\phi_a) = \omega_a$.
2. $\overline{\phi}_a(\mathbf{x}) = \phi_a(\mathbf{x}_{a'})$ if $\mathbf{x} \in s_{a'} \quad \Longrightarrow \quad \overline{\phi}_a = 1_{s_a}$:

$$\int_\Omega \overline{u}_h \overline{\phi}_a = \int_{s_a} \overline{u}_h = |s_a| u_a = \int_{\omega_a} u_a \phi_a.$$

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF

- Hat-function orthogonality: For any $a \in \mathcal{V}$,

$$\int_{\omega_a} \frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla} \phi_a = 0.$$

- Definition of $\boldsymbol{\sigma}_h^n$:

$$\boldsymbol{\sigma}_h^n = \sum_{a \in \mathcal{V}} \boldsymbol{\sigma}_{h,a}^n,$$

where we look for $\boldsymbol{\sigma}_{h,a}^n \in \mathbf{RTN}_1^0(\omega_a)$ solution of the minimization problem:

$$\boldsymbol{\sigma}_{h,a}^n = \arg\min_{\mathbf{v}_h \in \mathbf{RTN}_1^0(\omega_a)} \|\phi_a \eta_h^n \boldsymbol{\nabla}(p_h^n + \Psi_h) + \mathbf{v}_h\|_{\omega_a}$$

under the constraint

$$\mathrm{div} \mathbf{v}_h = -\Pi_{\mathbb{P}_1^*(\omega_a)} \left[\frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + (\eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h)) \cdot \boldsymbol{\nabla} \phi_a \right].$$

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF

- Hat-function orthogonality: For any $a \in \mathcal{V}$,

$$\int_{\omega_a} \frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a = 0.$$

- Definition of $\boldsymbol{\sigma}_h^n$:

$$\boldsymbol{\sigma}_h^n = \sum_{a \in \mathcal{V}} \boldsymbol{\sigma}_{h,a}^n,$$

where we look for $\boldsymbol{\sigma}_{h,a}^n \in \mathbf{RTN}_1^0(\omega_a)$ and $r_h^a \in \mathbb{P}_1^*(\omega_a)$ are solution to

$$\int_{\omega_a} \boldsymbol{\sigma}_{h,a}^n \mathbf{v}_h - \int_{\omega_a} \mathrm{div}\mathbf{v}_h r_h^a = -\int_{\omega_a} \phi_a \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h)\mathbf{v}_h, \ \forall \mathbf{v}_h \in \mathbf{RTN}_1^0(\omega_a)$$

$$\int_{\omega_a} \mathrm{div}\boldsymbol{\sigma}_{h,a}^n q_h = -\int_{\omega_a} \left[\frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a \right] q_h, \ \forall q_h \in \mathbb{P}_1^*(\omega_a).$$

EQUILIBRATED FLUX RECONSTRUCTION: SKETCH OF PROOF

- Hat-function orthogonality: For any $a \in \mathcal{V}$,

$$\int_{\omega_a} \frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \int_{\omega_a} \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a = 0.$$

- Definition of $\boldsymbol{\sigma}_h^n$:

$$\boldsymbol{\sigma}_h^n = \sum_{a \in \mathcal{V}} \boldsymbol{\sigma}_{h,a}^n,$$

where we look for $\boldsymbol{\sigma}_{h,a}^n \in \mathbf{RTN}_1^0(\omega_a)$ and $r_h^a \in \mathbb{P}_1^*(\omega_a)$ are solution to

$$\int_{\omega_a} \boldsymbol{\sigma}_{h,a}^n \mathbf{v}_h - \int_{\omega_a} \operatorname{div} \mathbf{v}_h r_h^a = -\int_{\omega_a} \phi_a \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \mathbf{v}_h, \ \forall \mathbf{v}_h \in \mathbf{RTN}_1^0(\omega_a)$$

$$\underbrace{\int_{\omega_a} \operatorname{div} \boldsymbol{\sigma}_{h,a}^n q_h}_{=0} = -\underbrace{\int_{\omega_a} \left[\frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a + \eta_h^n \mathbf{\Lambda}_h \boldsymbol{\nabla}(p_h^n + \Psi_h) \cdot \boldsymbol{\nabla}\phi_a \right]}_{=0} q_h, \ \forall q_h \in \mathbb{P}_1^*(\omega_a).$$

- $\boldsymbol{\sigma}_h^n \in H(\operatorname{div}, \Omega)$.
- $\boldsymbol{\sigma}_{h,a}^n \cdot \mathbf{n}_{\omega_a} = 0$ on $\partial\omega_a$ and hat-function orthogonality $\Longrightarrow$ $q_h \in \mathbb{P}_1(\omega_a)$.
- Let $T \in \mathcal{T}$ and $q_h \in \mathbb{P}_1(T)$,

$$\int_T \operatorname{div} \boldsymbol{\sigma}_h^n q_h = -\sum_{a \in \mathcal{V}_T} \int_T \left[\frac{u_a^n - u_a^{n-1}}{\tau_n} \phi_a \right] q_h = -\int_T \frac{u_h^n - u_h^{n-1}}{\tau_n} q_h.$$

A POSTERIORI ANALYSIS

BOUND ON THE RESIDUAL

$$X = \left\{ \varphi \in C([0,t_\mathrm{f}];\overline{\Omega}) \middle| \varphi(t_\mathrm{f},\cdot) \equiv 0, \partial_t \varphi \in L^1((0,t_\mathrm{f});L^\infty(\Omega)), \boldsymbol{\nabla}\varphi \in L^\infty((0,t_\mathrm{f})\times\Omega)^2 \right\}.$$

$$\|\varphi\|_X = \|\boldsymbol{\nabla}\varphi\|_{L^\infty((0,t_\mathrm{f})\times\Omega)} + \int_0^{t_\mathrm{f}} \|\partial_t\varphi\|_{L^\infty(\Omega)}.$$

DEFINITION

- The <u>residual</u> $R(v) \in X'$ is s.t. for any $\varphi \in X$,

$$\langle R(v), \varphi \rangle_{X',X} = \int_0^{t_\mathrm{f}} \int_\Omega v \partial_t \varphi + \int_\Omega u^0 \varphi(0,\cdot) - \int_0^{t_\mathrm{f}} \int_\Omega (\boldsymbol{\nabla}\phi(v) + \eta(v)\boldsymbol{\nabla}\Psi) \cdot \boldsymbol{\Lambda}\boldsymbol{\nabla}\varphi.$$

- The <u>error measure</u> $\mathcal{J}(\hat{u}_{h\tau})$ is defined by,

$$\mathcal{J}(\hat{u}_{h\tau}) = \sup_{\varphi \in X, \|\varphi\|_X = 1} \langle R(\hat{u}_{h\tau}), \varphi \rangle_{X',X} \, .$$

THEOREM (GUARANTEED UPPER BOUND)

Let $u \in X$ be a weak solution to the continuous problem and let $u_{h\tau}$ be an approximate solution to the numerical scheme, then

$$\mathcal{J}(\hat{u}_{h\tau}) \leq \underbrace{\int_0^{t_\mathrm{f}} \|\boldsymbol{\Lambda}\left(\boldsymbol{\nabla}\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\boldsymbol{\nabla}\Psi\right) + \boldsymbol{\sigma}_{h\tau}\|_{L^1(\Omega)}}_{\eta_\mathrm{F}} + \underbrace{\|\hat{u}_{h\tau}(0,\cdot) - u_0\|_{L^1(\Omega)}}_{\eta_\mathrm{IC}}$$

A POSTERIORI ANALYSIS

BOUND ON THE RESIDUAL: SKETCH OF PROOF

Let $\varphi \in X$ be s.t. $\|\varphi\|_X = 1$.

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi \rangle_{X',X} =& \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi + \int_\Omega u^0 \varphi(0,\cdot) \\ & - \int_0^{t_\mathrm{f}} \int_\Omega \left(\boldsymbol{\nabla} \phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau}) \boldsymbol{\nabla} \Psi \right) \cdot \boldsymbol{\Lambda} \boldsymbol{\nabla} \varphi\end{aligned}$$

- Green's identity:

$$\int_0^{t_\mathrm{f}} \int_\Omega \mathrm{div} \boldsymbol{\sigma}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \boldsymbol{\sigma}_{h\tau} \cdot \boldsymbol{\nabla} \varphi = 0.$$

- Integration by parts:

$$\int_0^{t_\mathrm{f}} \int_\Omega \partial_t \hat{u}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi = - \int_\Omega \hat{u}_{h\tau}(0,\cdot) \varphi(0,\cdot).$$

BOUND ON THE RESIDUAL: SKETCH OF PROOF

Let $\varphi \in X$ be s.t. $\|\varphi\|_X = 1$.

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = &\int_0^{t_{\mathrm{f}}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi + \int_\Omega u^0 \varphi(0,\cdot) \\ &- \int_0^{t_{\mathrm{f}}} \int_\Omega (\boldsymbol{\nabla} \phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau}) \boldsymbol{\nabla} \Psi) \cdot \boldsymbol{\Lambda} \boldsymbol{\nabla} \varphi\end{aligned}$$

- Green's identity:

$$\int_0^{t_{\mathrm{f}}} \int_\Omega \mathrm{div} \boldsymbol{\sigma}_{h\tau} \varphi + \int_0^{t_{\mathrm{f}}} \int_\Omega \boldsymbol{\sigma}_{h\tau} \cdot \boldsymbol{\nabla} \varphi = 0.$$

- Integration by parts:

$$\int_0^{t_{\mathrm{f}}} \int_\Omega \partial_t \hat{u}_{h\tau} \varphi + \int_0^{t_{\mathrm{f}}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi = - \int_\Omega \hat{u}_{h\tau}(0,\cdot) \varphi(0,\cdot).$$

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = &\int_0^{t_{\mathrm{f}}} \int_\Omega (-\partial_t \hat{u}_{h\tau} - \mathrm{div} \boldsymbol{\sigma}_{h\tau}) \varphi + \int_\Omega \left(u^0 - \hat{u}_{h\tau}(0,\cdot)\right) \varphi(0,\cdot) \\ &- \int_0^{t_{\mathrm{f}}} \int_\Omega (\boldsymbol{\Lambda} \boldsymbol{\nabla} \phi(\hat{u}_{h\tau}) + \boldsymbol{\Lambda} \eta(\hat{u}_{h\tau}) \boldsymbol{\nabla} \Psi + \boldsymbol{\sigma}_{h\tau}) \cdot \boldsymbol{\nabla} \varphi.\end{aligned}$$

BOUND ON THE RESIDUAL: SKETCH OF PROOF

Let $\varphi \in X$ be s.t. $\|\varphi\|_X = 1$.

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = & \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi + \int_\Omega u^0 \varphi(0,\cdot) \\ & - \int_0^{t_\mathrm{f}} \int_\Omega \left(\boldsymbol{\nabla}\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\boldsymbol{\nabla}\Psi\right) \cdot \boldsymbol{\Lambda}\boldsymbol{\nabla}\varphi\end{aligned}$$

- Green's identity:

$$\int_0^{t_\mathrm{f}} \int_\Omega \mathrm{div}\boldsymbol{\sigma}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \boldsymbol{\sigma}_{h\tau} \cdot \boldsymbol{\nabla}\varphi = 0.$$

- Integration by parts:

$$\int_0^{t_\mathrm{f}} \int_\Omega \partial_t \hat{u}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi = - \int_\Omega \hat{u}_{h\tau}(0,\cdot)\varphi(0,\cdot).$$

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = & \int_0^{t_\mathrm{f}} \int_\Omega \left(-\partial_t \hat{u}_{h\tau} - \mathrm{div}\boldsymbol{\sigma}_{h\tau}\right) \varphi + \int_\Omega \left(u^0 - \hat{u}_{h\tau}(0,\cdot)\right) \underbrace{\varphi(0,\cdot)}_{= - \int_0^{t_\mathrm{f}} \partial_t \varphi} \\ & - \int_0^{t_\mathrm{f}} \int_\Omega \left(\boldsymbol{\Lambda}\boldsymbol{\nabla}\phi(\hat{u}_{h\tau}) + \boldsymbol{\Lambda}\eta(\hat{u}_{h\tau})\boldsymbol{\nabla}\Psi + \boldsymbol{\sigma}_{h\tau}\right) \cdot \boldsymbol{\nabla}\varphi.\end{aligned}$$

BOUND ON THE RESIDUAL: SKETCH OF PROOF

Let $\varphi \in X$ be s.t. $\|\varphi\|_X = 1$.

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = & \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi + \int_\Omega u^0 \varphi(0,\cdot) \\ & - \int_0^{t_\mathrm{f}} \int_\Omega (\boldsymbol{\nabla} \phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau}) \boldsymbol{\nabla} \Psi) \cdot \boldsymbol{\Lambda} \boldsymbol{\nabla} \varphi\end{aligned}$$

- Green's identity:

$$\int_0^{t_\mathrm{f}} \int_\Omega \mathrm{div} \boldsymbol{\sigma}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \boldsymbol{\sigma}_{h\tau} \cdot \boldsymbol{\nabla} \varphi = 0.$$

- Integration by parts:

$$\int_0^{t_\mathrm{f}} \int_\Omega \partial_t \hat{u}_{h\tau} \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \hat{u}_{h\tau} \partial_t \varphi = - \int_\Omega \hat{u}_{h\tau}(0,\cdot) \varphi(0,\cdot).$$

$$\begin{aligned}\langle R(\hat{u}_{h\tau}), \varphi\rangle_{X',X} = & - \int_0^{t_\mathrm{f}} \int_\Omega \Bigg(\underbrace{\partial_t \hat{u}_{h\tau} + \mathrm{div} \boldsymbol{\sigma}_{h\tau}}_{=0} \Bigg) \varphi + \int_0^{t_\mathrm{f}} \int_\Omega \left(\hat{u}_{h\tau}(0,\cdot) - u^0 \right) \partial_t \varphi \\ & - \int_0^{t_\mathrm{f}} \int_\Omega (\boldsymbol{\Lambda} \boldsymbol{\nabla} \phi(\hat{u}_{h\tau}) + \boldsymbol{\Lambda} \eta(\hat{u}_{h\tau}) \boldsymbol{\nabla} \Psi + \boldsymbol{\sigma}_{h\tau}) \cdot \boldsymbol{\nabla} \varphi.\end{aligned}$$

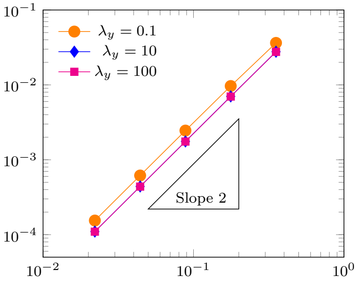
NUMERICAL RESULTS

EQUATION: $\eta(u) = u$ and $\boxed{p(u) = \log(u)}$

$$\partial_t u - \mathrm{div}(u\mathbf{\Lambda}\nabla(\log u - x)) = \partial_t u - \mathrm{div}(\mathbf{\Lambda}(\nabla u - \mathbf{e}_x)) = 0 \quad \text{with } \mathbf{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_y \end{pmatrix}.$$

EXACT SOLUTION:

$$u(t,(x,y)) = e^{-(\pi^2+\frac{1}{4})t+\frac{x}{2}} \left(\pi\cos(\pi x) + \frac{1}{2}\sin(\pi x)\right) + \pi e^{x-\frac{1}{2}}.$$



L^2 error

$\lambda_y = 0.1$	$\lambda_y = 10$	$\lambda_y = 100$
0.434391	0.456011	0.466597
0.128222	0.132656	0.135209
0.0339114	0.0349284	0.0354816
0.00865144	0.00889535	0.009028
0.00218037	0.00224013	0.00227289

Minimum of u

$\boxed{\text{Preserving of the positivity}}$

NUMERICAL RESULTS

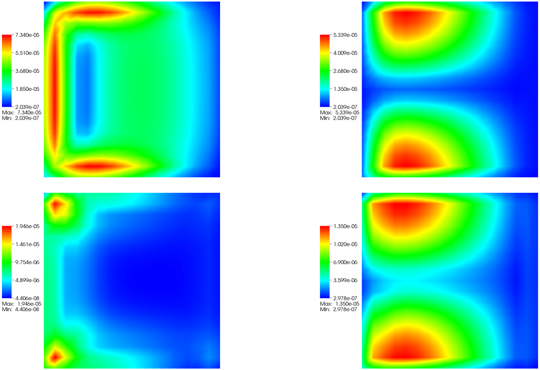
COMPARISON BETWEEN

- Actual error distribution

$$\int_{t^{n-1}}^{t^n} \|\mathbf{\Lambda}(\nabla\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\nabla\Psi) - \mathbf{\Lambda}(\nabla\phi(u) + \eta(u)\nabla\Psi)\|_{L^1(T)} \,.$$

- Predicted error distribution (both in time and in space)

$$\eta^n_{\mathrm{F},T} := \int_{t^{n-1}}^{t^n} \|\mathbf{\Lambda}(\nabla\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\nabla\Psi) + \boldsymbol{\sigma}_{h\tau}\|_{L^1(T)} \,.$$



Actual error

Predicted error

$\lambda_y = 1$

$\lambda_y = 100$

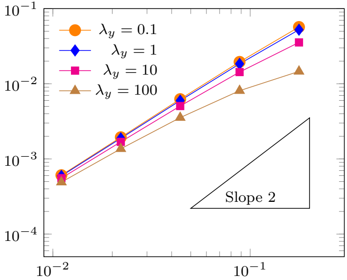
NUMERICAL RESULTS

EQUATION: $\eta(u) = u$ and $\boxed{p(u) = 2u}$.

$$\partial_t u - \mathrm{div}(u\boldsymbol{\Lambda}\boldsymbol{\nabla}(2u - x)) = \partial_t u - \mathrm{div}(\boldsymbol{\Lambda}(\boldsymbol{\nabla} u^2 - \mathbf{e}_x)) = 0 \quad \text{with } \boldsymbol{\Lambda} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda_y \end{pmatrix}.$$

EXACT SOLUTION:

$$u(t,(x,y)) = \max(3t - x, 0) \quad \text{with Dirichlet BC.}$$



Error L^2

$\lambda_y = 0.1$	$\lambda_y = 1$	$\lambda_y = 10$	$\lambda_y = 100$
1.680	1.644	1.428	0.930
1.721	1.698	1.569	1.244
1.722	1.704	1.620	1.408
1.696	1.681	1.622	1.492

Convergence rate

Loss of the positivity

NUMERICAL RESULTS

COMPARISON BETWEEN

- Actual error distribution

$$\int_{t^{n-1}}^{t^n} \left\| \boldsymbol{\Lambda}(\boldsymbol{\nabla}\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\boldsymbol{\nabla}\Psi) - \boldsymbol{\Lambda}(\boldsymbol{\nabla}\phi(u) + \eta(u)\boldsymbol{\nabla}\Psi) \right\|_{L^1(T)} .$$

- Predicted error distribution (both in time and in space)

$$\eta_{\mathrm{F},T}^{n} := \int_{t^{n-1}}^{t^n} \left\| \boldsymbol{\Lambda}(\boldsymbol{\nabla}\phi(\hat{u}_{h\tau}) + \eta(\hat{u}_{h\tau})\boldsymbol{\nabla}\Psi) + \boldsymbol{\sigma}_{h\tau} \right\|_{L^1(T)} .$$

- Predicted error distribution in space only

$$\int_{t^{n-1}}^{t^n} \left\| \boldsymbol{\Lambda}(\boldsymbol{\nabla}\phi(u_{h\tau}) + \eta(u_{h\tau})\boldsymbol{\nabla}\Psi) + \boldsymbol{\sigma}_{h\tau} \right\|_{L^1(T)} .$$

where

- $u_{h\tau}$: solution of the numerical scheme;
- $\hat{u}_{h\tau}$: piecewise affine reconstruction in time and space of $u_{h\tau}$.

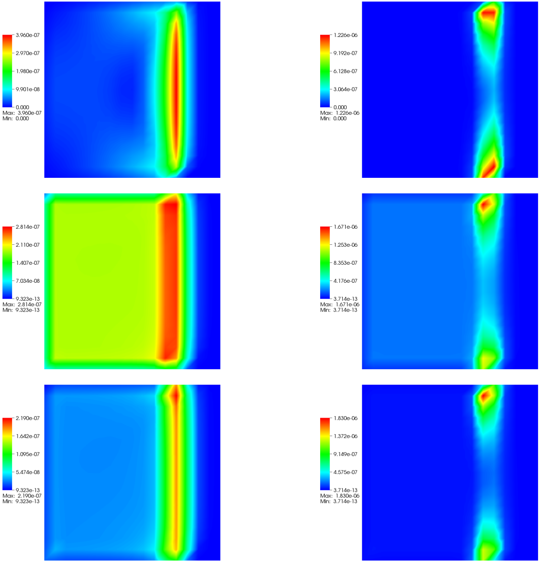
NUMERICAL RESULTS

COMPARISON BETWEEN

Actual error

Predicted error (time and space)

Predicted error (space)



$\lambda_y = 1$ $\lambda_y = 100$

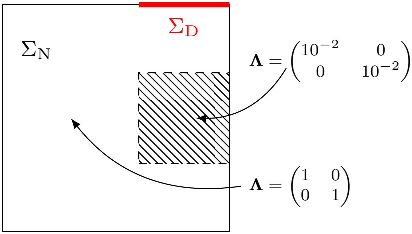
NUMERICAL RESULTS

BARRIER

$$\partial_t u - \mathrm{div}(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) = 0$$

with

- Dirichlet Boundary conditions $u = 1$ on Σ_{D};
- homogeneous Neumann Boundary conditions on Σ_{N};
- $\eta(u) = |u|$, $p(u) = u$, $\Psi(x, y) = y$;
- adaptive time-step strategy.



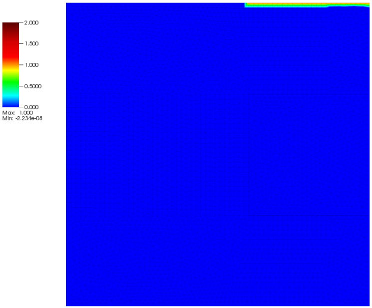
Numerical results

Barrier

$$\partial_t u - \operatorname{div}(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) = 0$$

with

- Dirichlet Boundary conditions $u = 1$ on Σ_{D};
- homogeneous Neumann Boundary conditions on Σ_{N};
- $\eta(u) = |u|$, $p(u) = u$, $\Psi(x, y) = y$;
- adaptive time-step strategy.



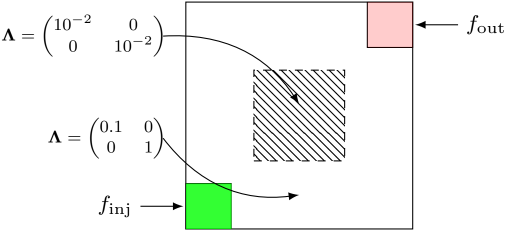
Numerical results

Quarter Five Spot

$$\partial_t u - \operatorname{div}(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) = f_{\text{inj}} - \eta(u^+) f_{\text{out}}$$

with

- homogeneous Neumann Boundary conditions;
- $\eta(u) = u^2$, $p(u) = 2u$, $\Psi(x, y) = -x$;
- adaptive time-step strategy.



$$\boldsymbol{\Lambda} = \begin{pmatrix} 10^{-2} & 0 \\ 0 & 10^{-2} \end{pmatrix}$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} 0.1 & 0 \\ 0 & 1 \end{pmatrix}$$

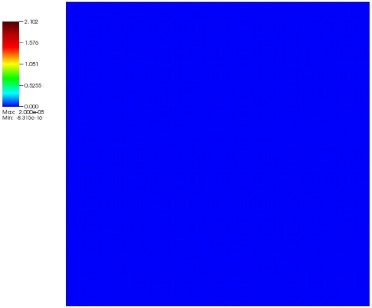
Numerical results

Quarter Five Spot

$$\partial_t u - \operatorname{div}(\eta(u)\boldsymbol{\Lambda}\boldsymbol{\nabla}(p(u) + \Psi) = f_{\text{inj}} - \eta(u^+) f_{\text{out}}$$

with

- homogeneous Neumann Boundary conditions;
- $\eta(u) = u^2$, $p(u) = 2u$, $\Psi(x, y) = -x$;
- adaptive time-step strategy.



NUMERICAL RESULTS

LINEARIZATION ADAPTIVE STOPPING CRITERIA

- Ω unit disk, $R = 2.6$;
- Mesh size ~ 0.16;
- $\eta(u) = u$, $p(u) = \dfrac{m}{m-1} u^{m-1}$, $m = 4$;
- $\boldsymbol{\Lambda} = I_d$;
- Exact solution:

$$u(t,(x,y)) = \left(\frac{1}{t+1} \left(\left[1 - \frac{m-1}{4m^2} \frac{x^2+y^2}{(t+1)^m} \right]^+ \right)^{\frac{m}{m-1}} \right)^{\frac{1}{m}},$$

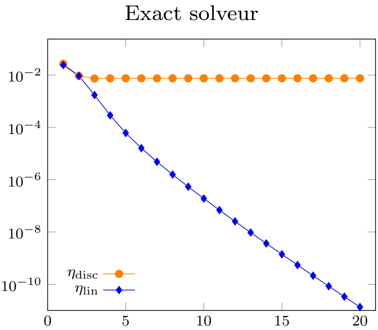
  with Dirichlet BC
- $\tau = 0.01$, $t_0 = 0$, $t_\mathrm{f} = 0.1$
- Stopping criterion for the nonlinear solveur: $\eta_\mathrm{lin} \leq \gamma \eta_\mathrm{disc}$

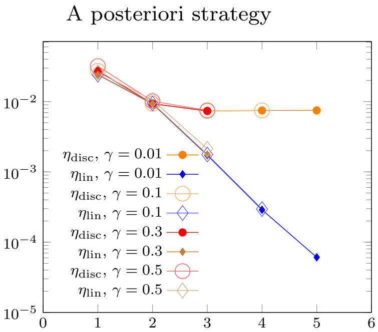
NUMERICAL RESULTS

LINEARIZATION ADAPTIVE STOPPING CRITERIA

- Stopping criterion for the nonlinear solveur: $\eta_{\mathrm{lin}} \leq \gamma \eta_{\mathrm{disc}}$

Time	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1	Cumulated iterations
Exact solver	22	20	18	18	16	15	15	16	15	15	170
$\gamma = 0.01$	7	5	5	4	4	4	4	4	4	3	44
$\gamma = 0.1$	6	4	3	3	3	3	3	2	2	2	31
$\gamma = 0.3$	6	3	3	3	2	2	2	2	2	2	27
$\gamma = 0.5$	5	3	2	2	2	2	1	2	1	1	21

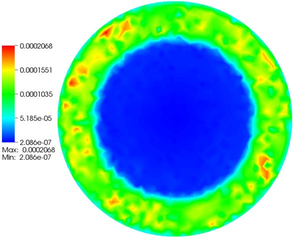

Exact solveur


A posteriori strategy

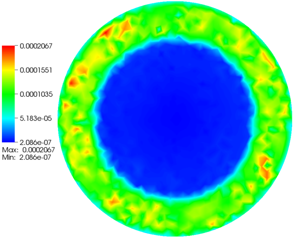
t = 0.02

Numerical results

Total errors

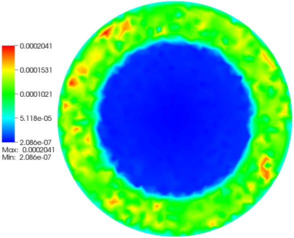


Exact solver

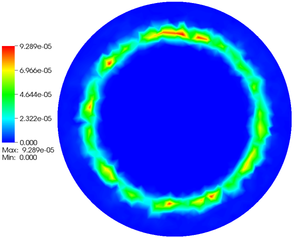


Adaptive strategy, $\gamma = 0.5$

Distribution of the error components, $\gamma = 0.5$



Discretization error



Linearization error

MORE COMPLEX PHYSICS: INCOMPRESSIBLE TWO PHASE FLOWS

Find saturations $s_\mathrm{o}, s_\mathrm{w} : (0, t_\mathrm{f}) \times \Omega \to [0, 1]$ and pressures $p_\mathrm{o}, p_\mathrm{w} : (0, t_\mathrm{f}) \times \Omega \to \mathbb{R}$ such that

$$\begin{aligned}
\partial_t s_\mathrm{o} - \mathrm{div}(\eta_\mathrm{o}(s_\mathrm{o})\boldsymbol{\nabla} p_\mathrm{o}) &= f_\mathrm{o}(s_\mathrm{o}), && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
\partial_t s_\mathrm{w} - \mathrm{div}(\eta_\mathrm{w}(s_\mathrm{w})\boldsymbol{\nabla} p_\mathrm{w}) &= f_\mathrm{w}(s_\mathrm{w}), && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
s_\mathrm{o} + s_\mathrm{w} &= 1, && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
p_\mathrm{o} - p_\mathrm{w} &= \pi(s_\mathrm{o}), && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
\eta_\mathrm{o}(s_\mathrm{o})\boldsymbol{\nabla} p_\mathrm{o} \cdot \mathbf{n} &= 0, && \text{on } (0, t_\mathrm{f}) \times \partial\Omega;\\
\eta_\mathrm{w}(s_\mathrm{w})\boldsymbol{\nabla} p_\mathrm{w} \cdot \mathbf{n} &= 0, && \text{on } (0, t_\mathrm{f}) \times \partial\Omega;\\
\int_\Omega p_\mathrm{w} &= 0, && \text{on } (0, t_\mathrm{f});
\end{aligned}$$

GOING FURTHER

MORE COMPLEX PHYSICS: INCOMPRESSIBLE TWO PHASE FLOWS

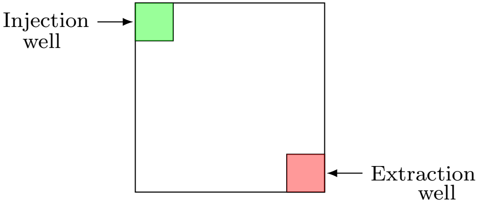
Find the saturation $s_{\mathrm{o}} : (0, t_{\mathrm{f}}) \times \Omega \to [0, 1]$ and the pressure $p_{\mathrm{w}} : (0, t_{\mathrm{f}}) \times \Omega \to \mathbb{R}$ such that,

$$
\begin{aligned}
\partial_t s_{\mathrm{o}} - \operatorname{div}(\eta_{\mathrm{o}}(s_{\mathrm{o}}) \boldsymbol{\nabla}(p_{\mathrm{w}} + \pi(s_{\mathrm{o}}))) &= f_{\mathrm{o}}(s_{\mathrm{o}}), && \text{in } (0, t_{\mathrm{f}}) \times \Omega; \\
-\partial_t s_{\mathrm{o}} - \operatorname{div}(\eta_{\mathrm{w}}(1 - s_{\mathrm{o}}) \boldsymbol{\nabla} p_{\mathrm{w}}) &= f_{\mathrm{w}}(1 - s_{\mathrm{o}}), && \text{in } (0, t_{\mathrm{f}}) \times \Omega; \\
\eta_{\mathrm{o}}(s_{\mathrm{o}}) \boldsymbol{\nabla}(p_{\mathrm{w}} + \pi(s_{\mathrm{o}})) \cdot \mathbf{n} &= 0, && \text{on } (0, t_{\mathrm{f}}) \times \partial\Omega; \\
\eta_{\mathrm{w}}(1 - s_{\mathrm{o}}) \boldsymbol{\nabla} p_{\mathrm{w}} \cdot \mathbf{n} &= 0, && \text{on } (0, t_{\mathrm{f}}) \times \partial\Omega; \\
\int_{\Omega} p_{\mathrm{w}} &= 0, && \text{on } (0, t_{\mathrm{f}}).
\end{aligned}
$$

MORE COMPLEX PHYSICS: INCOMPRESSIBLE TWO PHASE FLOWS

Find the saturation $s_\mathrm{o} : (0, t_\mathrm{f}) \times \Omega \to [0, 1]$ and the pressure $p_\mathrm{w} : (0, t_\mathrm{f}) \times \Omega \to \mathbb{R}$ such that,

$$\begin{aligned}
\partial_t s_\mathrm{o} - \mathrm{div}(\eta_\mathrm{o}(s_\mathrm{o})\boldsymbol{\nabla}(p_\mathrm{w} + \pi(s_\mathrm{o}))) &= f_\mathrm{o}(s_\mathrm{o}), && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
-\partial_t s_\mathrm{o} - \mathrm{div}(\eta_\mathrm{w}(1 - s_\mathrm{o})\boldsymbol{\nabla} p_\mathrm{w}) &= f_\mathrm{w}(1 - s_\mathrm{o}), && \text{in } (0, t_\mathrm{f}) \times \Omega;\\
\eta_\mathrm{o}(s_\mathrm{o})\boldsymbol{\nabla}(p_\mathrm{w} + \pi(s_\mathrm{o})) \cdot \mathbf{n} &= 0, && \text{on } (0, t_\mathrm{f}) \times \partial\Omega;\\
\eta_\mathrm{w}(1 - s_\mathrm{o})\boldsymbol{\nabla} p_\mathrm{w} \cdot \mathbf{n} &= 0, && \text{on } (0, t_\mathrm{f}) \times \partial\Omega;\\
\int_\Omega p_\mathrm{w} &= 0, && \text{on } (0, t_\mathrm{f}).
\end{aligned}$$



More complex physics: Incompressible two phase flows

Thank you for your attention

DEFINITION

A sequence $(f_n) \subset L^1(\Omega)$ is uniformly equi-integrable if

$$\forall \varepsilon > 0, \exists \alpha > 0 \text{ s.t. } \left[m(E) < \alpha \Rightarrow \int_E |f_n| \mathrm{d}x < \varepsilon, \forall n \right].$$

THEOREM (VITALI)

Let $(f_n)_n$ be a sequence of functions uniformly equi-initegrable, such that

$$f_n \xrightarrow[n \to +\infty]{} f \text{ a.e. in } \Omega.$$

Then,

$$f_n \xrightarrow[n \to +\infty]{} f \quad \text{strongly in } L^1(\Omega).$$

THEOREM (DE LA VALLÉE-POUSSIN)

The sequence $(f_n)_n$ is uniformly equi-initegrable if and only if there exists $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ with $\dfrac{\Gamma(x)}{x} \xrightarrow[x \to \infty]{} +\infty$ such that

$$\int_\Omega \Gamma(|f_n|) \mathrm{d}x \leq C,$$

for some $C > 0$.