Exact Monte Carlo methods
Application: Pricing of continuous Asian options

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Outline

1. Exact computation of expectations
2. Application: the pricing of continuous Asian options
3. Numerical results
4. Conclusion
In many application, especially finance, we are faced with the problem of computing

\[ C_0 = \mathbb{E}(f(X_T)) \]

where \( f \) is a given function \( X \) is the solution of a one-dimensional SDE:

\[ dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t; \quad X_0 = x \]

Monte Carlo methods for doing the job in case \( \mathcal{L}(X_T) \) is unknown:

- Approximate \( X_T \) by a discretized process (Euler, Milstein ...).
- Do rejection sampling (exact algorithm of Beskos and al. [1]).
- Construct unbiased estimators, ideally simple to simulate from and with low variance.

Advantage of an exact method: no discretization bias
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Assume that we can find a simulable process $Z$ such that, by a change of measure,

$$
C_0 = \mathbb{E} \left( f(Z_T) \psi(Z_T) \exp \left[ - \int_0^T \phi(Z_t) dt \right] \right)
$$

(1)

where $\psi$ and $\phi$ are two explicit functions.

Early nineties: Wagner [18] constructs several unbiased estimators of such an expectation by expanding the exponential term in a power series.

Recently: Beskos and al. [2] and Fearnhead and al. [5] extend his idea to get the Poisson estimator and the generalized Poisson estimator.
The unbiased estimator (U.E)

Suppose that,

- conditionally on \( Z = (Z_t)_{t \in [0,T]} \),
  - \( N \sim p_Z \) with \( p_Z \) a positive probability measure on \( \mathbb{N} \).
  - \((V_i)_{i \in \mathbb{N}^*} \overset{i.i.d.}{\sim} q_Z\) with \( q_Z \) a positive probability density on \([0, T]\).
  - \( N \) and \((V_i)_{i \in \mathbb{N}^*}\) are independent and \( c_Z \in \mathbb{R} \).

- (I.C) \( \mathbb{E} \left( |f(Z_T)\psi(Z_T)| e^{-c_ZT} \exp \left[ \int_0^T |c_Z - \phi(Z_t)| \, dt \right] \right) < \infty \)

Lemma

\[
f(Z_T)\psi(Z_T)e^{-c_ZT} \frac{1}{p_Z(N)N!} \prod_{i=1}^N \frac{c_Z - \phi(Z_{V_i})}{q_Z(V_i)} \quad \text{is an unbiased estimator of } C_0.
\]
Proof:

\[
\Delta(Z) := \mathbb{E}\left(f(Z_T)\psi(Z_T)e^{-cZT}\frac{1}{p_Z(N)N!}\prod_{i=1}^{N} \frac{cZ - \phi(Z_{V_i})}{q_Z(V_i)} \mid Z\right)
\]

\[
= f(Z_T)\psi(Z_T)e^{-cZT}\sum_{n=0}^{+\infty} \frac{\left(\int_{0}^{T} cZ - \phi(Z_t)dt\right)^n}{p_Z(n)n!} p_Z(n)
\]

\[
= f(Z_T)\psi(Z_T)\exp\left(-\int_{0}^{T} \phi(Z_t)dt\right).
\]

So, we can compute \(C_0\) by a simple Monte Carlo:

\[
C_0 \approx \frac{1}{n} \sum_{i=1}^{n} f(Z_T^i)\psi(Z_T^i)e^{-cZT}\frac{1}{p_Z(N^i)N^i!}\prod_{j=1}^{N^i} \frac{cZ - \phi(Z_{V_{ij}}^i)}{q_Z(V_{ij}^i)}.
\]
Levers for variance reduction

- Importance sampling: the choice of the process $Z$ is decisive.
- The choice of the parameters $p_Z$ and $q_Z$.

A simple case: 

$$e^{\int_0^T g(t)dt} = \mathbb{E} \left( \frac{1}{p(N)N!} \prod_{i=1}^{N} \frac{g(V_i)}{q(V_i)} \right)$$

where $g : [0, T] \rightarrow \mathbb{R}$.

$$q_{opt}(t) = \frac{|g(t)|}{\int_0^T |g(t)|dt} \mathbb{1}_{[0,T]}(t) \text{ and } p_{opt}(n) = \left( \int_0^T |g(t)|dt \right)^n n! e^{-\int_0^T |g(t)|dt}$$

- Shifting: the parameter $c_Z$.
How to obtain such a process $Z$? (Beskos and al. [1])

- Without loss of generality, suppose that $X$ is solution of

$$\begin{cases} 
  dX_t &= a(X_t)dt + dW_t \\
  X_0 &= x.
\end{cases}$$

(If $X$ has a non-constant diffusion coefficient $\sigma(.)$, make the change of variables $Y_t = \eta(X_t)$ with $\eta(x) = \int_x^x \frac{1}{\sigma(u)} du$)

- Denote by $(W_t^x)_{t\in[0,T]}$ the process $(W_t + x)_{t\in[0,T]}$.

**Assumption 1:**

$$L_t = \exp\left[\int_0^t a(W_u^x) dW_u^x - \frac{1}{2} \int_0^t a^2(W_u^x) du\right] \text{ is a martingale.}$$
How to obtain such a process $Z$? (Beskos and al. [1])

- **Assumption 2**: $a$ is continuously differentiable.

Denote by $A$ the primitive of the drift $a$. By Itô’s lemma, one gets

$$L_T = \exp \left[ A(W_T^x) - A(x) - \frac{1}{2} \int_0^T a^2(W_t^x) + a'(W_t^x) dt \right].$$

- Consider the process $(Z_t)_{t \in [0,T]}$ distributed according to

$$Q_Z = \int_\mathbb{R} \mathcal{L}((W_t^x)_{t \in [0,T]} | W_T^x = y) \rho(y) dy.$$

where $\rho$ is a positive density on the real line (another lever for variance reduction). By the Girsanov theorem,

$$C_0 = \mathbb{E} \left( f(Z_T) \psi(Z_T) \exp \left[ - \int_0^T \phi(Z_t) dt \right] \right) \quad (3)$$

where $\psi : z \mapsto \frac{e^{A(z)-A(x)} - \frac{(z-x)^2}{2T}}{\sqrt{2\pi \rho(z)}}$ and $\phi : z \mapsto \frac{a^2(z)+a'(z)}{2}$. 
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In the Black & Scholes framework, under the risk-neutral measure

\[
\frac{dS_t}{S_t} = (r - \delta) dt + \sigma dW_t
\]

so \( S_t = S_0 e^{\sigma W_t + \gamma t} \) where \( \gamma = r - \delta - \frac{\sigma^2}{2} \).

Price of a continuous Asian option with pay-off \( f \):

\[
C_0 = \mathbb{E} \left( e^{-rT} f \left( \frac{1}{T} \int_0^T S_u \right) \right)
\]

\( \Rightarrow \) No simple closed form solution

Numerical methods:

- Analytical approximations (Turnbull and Wakeman [15], Vorst [17], Levy [11] and more recently Lord [12]).
- PDE methods (see Vecer [16], Rogers and Shi [13], Ingersoll [8], Lelievre and Dubois [4]).
- Monte Carlo simulation methods (see Kemna and Vorst [9], Broadie and Glasserman [3], Fu and al. [6], Lapeyre and Temam [10]).
- Laplace transform inversion methods (see Geman and Yor [7]).
A priori, a two dimensional problem: let $\overline{S}_t = \frac{1}{t} \int_0^t S_u du$, then

\[
\begin{cases}
    dS_t &= S_t((r - \delta)dt + \sigma dW_t) \\
    d\overline{S}_t &= (-\frac{1}{t}\overline{S}_t + \frac{S_t}{t})dt
\end{cases}
\]

But, with a suitable change of variables (Rogers and Shi [13]), we can reduce the dimension to one:

$$
\xi_t = S_0 \int_0^t e^{\sigma(W_t-W_u) + \gamma(t-u)} du
$$

$$
\xi_T = \int_0^T S_0 e^{\sigma(W_T-W_{T-S}) + \gamma S} ds \text{ and } \int_0^T S_u du \text{ have the same law so}
$$

$$
C_0 = \mathbb{E} \left( e^{-rT} f\left( \frac{1}{T} \xi_T \right) \right)
$$

where $\left( \xi_t \right)_{t \in [0,T]}$ is solution of

\[
\begin{cases}
    d\xi_t &= S_0 dt + \xi_t \left( \sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt \right) \\
    \xi_0 &= 0
\end{cases}
\]
Need for a new change of variables:

\[ \xi_t = \frac{1}{t} S_0 \int_0^t e^{\sigma (W_t - W_u) + \gamma (t - u)} du \]

\[ \xi_T = \frac{1}{T} \int_0^T S_0 e^{\sigma (W_T - W_T - s) + \gamma s} ds \text{ and } \frac{1}{T} \int_0^T S_u du \text{ have the same law so } \]

\[ C_0 = \mathbb{E} \left( e^{-rT} f(\xi_T) \right) \]

where \((\xi_t)_{t \in [0,T]}\) is solution of

\[
\begin{cases}
    d\xi_t = \frac{\xi_0 - \xi_t}{t} dt + \xi_t \left( \sigma dW_t + (\gamma + \frac{\sigma^2}{2}) dt \right) \\
    \xi_0 = S_0
\end{cases}
\]
\[ X_t = \log\left( \frac{\xi_t}{\xi_0} \right) \Rightarrow \left\{ \begin{array}{c}
 dX_t = \sigma dW_t + \gamma dt + \frac{e^{-X_t} - 1}{t} dt \\
 X_0 = 0.
\end{array} \right. \] (4)

**Difficulty**: singularity of the drift term for \( t \to 0 \) prevents \( X \) from having an a.c law w.r.t the law of \( W \). Therefore, we consider

\[ dZ_t = \sigma dW_t + \gamma dt - \frac{Z_t}{t} \, dt; \ Z_0 = X_0 = 0. \] (5)

**Lemma**

*Existence and strong uniqueness hold for (4) and (5). Moreover,*

\[ Z_t = \frac{\sigma}{t} \int_0^t s \, dW_s + \frac{\gamma}{2} t \text{ is a solution of (5).} \]

\((Z_t)_{t \in [0,T]}\) is a Gaussian process and \( Z_T \sim \mathcal{N}\left(\frac{\gamma}{2} T, \frac{\sigma^2}{3} T\right)\).
Proposition

\[ L_t = \exp \left[ \int_0^t \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} dW_s - \frac{1}{2} \int_0^t \left( \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} \right)^2 ds \right] \]

is a martingale and hence \( C_0 = \mathbb{E} \left( e^{-rT} f(S_0 e^{Z_T}) L_T \right) \)

**Proof:** By the L.I.L of the Brownian motion, we show that

\[ \forall \epsilon > 0, \text{ there exists a (random) neighborhood of } t = 0 \text{ for which} \]

\[ |Z_t| \leq ct^{1/2 - \epsilon} \text{ and } |X_t| \leq ct^{1/2 - \epsilon} \]

hence, almost surely,

\[ \int_0^t \left( \frac{e^{-Z_s} - 1 + Z_s}{\sigma_s} \right)^2 ds < \infty \text{ and } \int_0^t \left( \frac{e^{-X_s} - 1 + X_s}{\sigma_s} \right)^2 ds < \infty. \]

Existence and strong uniqueness of the SDEs (4) and (5) permits to conclude (see Rydberg [14]).
Let \( A(t, z) = \frac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t} \). By Itô’s lemma

\[
A(T, Z_T) = \int_0^T \frac{e^{-Z_t} - 1 + Z_t}{\sigma^2 t} dZ_t - \int_0^T 1 - Z_t + \frac{Z_t^2}{2} - e^{-Z_t} \frac{1}{\sigma^2 t^2} dt + \int_0^T 1 - e^{-Z_t} \frac{1}{2t} dt.
\]

Finally, \( C_0 = \mathbb{E} \left( e^{-rT} f(S_0 e^{Z_T}) e^{A(T, Z_T)} \exp \left[ \int_0^T \phi(t, Z_t) dt \right] \right) \) with

\[
\phi(t, z) = \frac{1 - z + \frac{z^2}{2} - e^{-z}}{\sigma^2 t^2} - \frac{1 - e^{-z}}{2t} - \frac{e^{-z} - 1 + z}{\sigma^2 t} \left( \frac{e^{-z} - 1 - z}{2t} + \gamma \right).
\]
A first conjecture

In order to be able to deal with both calls and puts, we need the following integrability condition (I.C)

**Conjecture**

\[
\mathbb{E} \left( e^{A(T,Z_T)} - rT (e^{Z_T} + 1) e^{\int_0^T |\phi(t,Z_t)| \, dt} \right) < \infty.
\]

For a call, this implies that

\[
C_0 = \mathbb{E} \left( e^{A(T,Z_T)} - rT (S_0 e^{Z_T} - K) + \frac{1}{p(N) \, N!} \prod_{i=1}^N \frac{\phi(U_i, Z_{U_i})}{q(U_i)} \right)
\]

with well chosen probability distributions \( p \) and \( q \).
Choice of the distributions $p$ and $q$

We need square integrability in order to construct confidence intervals.

$$\mathbb{E} \left( e^{2A(T,Z_T)-2rTf^2(S_0e^{Z_T})} \left( \frac{\int_0^T \phi^2(t,Z_t) \frac{1}{q(t)} dt}{p(N)^2 (N!)^2} \right)^N \right) < \infty \ ?$$

False for the naive choice of a uniform distribution for $q$:

**Lemma**

\[ \forall \epsilon > 0, \text{ we have a.s. } \phi(t, Z_t) - \frac{2Z_t^3}{3\sigma^2 t^2} + \frac{Z_t}{2t} = O(t^{-\epsilon}). \]

Therefore, we have

$$\int_0^T \frac{\phi^2(t,Z_t)}{t^a} dt < \infty \text{ a.s. if and only if } a < 0.$$
Variance reduction

With \( p = P(c_p T) \) and \( q(t) = \frac{1}{2\sqrt{t} \sqrt{T}} \mathbb{1}_{[0,T]}(t) \) (since \( \phi \sim \frac{1}{\sqrt{t}} \) near 0), our estimator writes

\[
\delta = \frac{1}{m} \sum_{j=1}^{m} e^{A(T,Z^j_T)-rT} (S_0 e^{Z^j_T} - K) + e^{c_p T - c_{Zj} T} \prod_{i=1}^{N^j} 2 \sqrt{U^j_i} \left( c_{Zj} - \phi(U^j_i, Z^j_{U^j_i}) \right) / c_p \sqrt{T}
\]

- **Conditioning**: for every simulated trajectory \( Z^j \), we compute

\[
\frac{1}{n} \sum_{k=1}^{n} \prod_{i=1}^{N^j_k} 2 \sqrt{U^j_{i,k}} \left( c_{Zj} - \phi(U^j_{i,k}, Z^j_{U^j_{i,k}}) \right) / c_p \sqrt{T}
\]

instead of

\[
\prod_{i=1}^{N^j} 2 \sqrt{U^j_i} \left( c_{Zj} - \phi(U^j_i, Z^j_{U^j_i}) \right) / c_p \sqrt{T}
\]

- **Control variate**: we can use \( e^{-rT} (S_0 e^{Z_T} - K) \) as a control variate since \( Z_T \sim \mathcal{N}(\frac{\gamma}{2} T, \frac{\sigma^2}{3} T) \).
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Numerical test of the conjectures

\[ \frac{1}{m} \sum_{j=1}^{m} e^{A(T, Z_j^T)} - rT (S_0 e^{Z_j^T} + 1) e^{c_p T} \prod_{i=1}^{N_j} 2 \sqrt{U_i^j} \frac{|\phi(U_i^j, Z_j^U)|}{c_p \sqrt{T}} \]  

(6)

**FIG.**: Evaluation of (6) with respect to the number \( m \) of simulations
Numerical test of the conjectures

\[
\frac{1}{m} \sum_{j=1}^{m} e^{2A(T,Z^j_T)} - 2rT \left( S_0 e^{Z^j_T} + 1 \right)^2 e^{2c_p T} \prod_{i=1}^{N^j} 4U^j_i \frac{\phi^2(U^j_i, Z^j_i)}{c^2_p T} \tag{7}
\]

**Fig.:** Evaluation of (7) with respect to the number \( m \) of simulations
Comparison with a standard Monte Carlo method

Set of parameters:
- \( S_0 = K = 100 \)
- \( r = 0.1 \)
- \( \sigma = 0.2 \)
- \( T = 0 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>Price</th>
<th>L.C.I at 95%</th>
<th>N</th>
<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.C.E std</td>
<td>7.035</td>
<td>0.035</td>
<td>2.10^5</td>
<td>( \sim 1s )</td>
</tr>
<tr>
<td>E.C.E opt</td>
<td>7.043</td>
<td>0.005</td>
<td>2.10^5</td>
<td>( \sim 1s )</td>
</tr>
<tr>
<td>MC std (Trap)</td>
<td>7.051</td>
<td>0.053</td>
<td>10^5</td>
<td>( \sim 1s )</td>
</tr>
<tr>
<td>MC opt (Trap+KV)</td>
<td>7.041</td>
<td>0.002</td>
<td>10^5</td>
<td>( \sim 1s )</td>
</tr>
</tbody>
</table>

Tab.: Asian call price with different MC methods. For the standard E.C.E (without variance reduction), we took \( c_p = 1 \). For the optimized E.C.E, we took \( c_p = c_T = \frac{1}{2T} \) and \( n = 5 \). For MC, the number of time steps is 20.
Comparison with a standard Monte Carlo method

Set of parameters:
- \( S_0 = K = 100 \)
- \( r = 0.1 \)
- \( \sigma = 0.2 \)
- \( T = 0 \)

<table>
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<th>CPU</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.C.E opt</td>
<td>7.0404</td>
<td>0.0001</td>
<td>( 7 \times 10^6 )</td>
<td>( \sim 15 \text{s} )</td>
</tr>
<tr>
<td>MC opt (Trap+KV)</td>
<td>7.0401</td>
<td>0.0008</td>
<td>( 10^6 )</td>
<td>( \sim 15 \text{s} )</td>
</tr>
</tbody>
</table>

**Table:** Asian call price with different MC methods. For the optimized E.C.E, we took \( c_p = c_T = \frac{1}{2T} \) and \( n = 5 \). For Trap+KV, the number of time steps is 100.

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Conclusion

Pros

- A MC price of an Asian option that is not prone to discretization bias.  
  \[ \rightarrow \text{A reliable benchmark} \]

- A competitive method if high precision is required.

- A competitive MC method for pricing Asian like options with pay-off  
  \[ \alpha S_T + \beta \int_0^T S_u du, \quad \alpha \neq 0. \]

Cons

- Less competitive than an optimized Monte Carlo (Lapeyre and Temam [10]) for usual precision levels (any other possible variance reduction method?).

- No theoretical justification of the integrability conjectures.
Thank you!
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