Fast rotating Bose Einstein condensates

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Outline

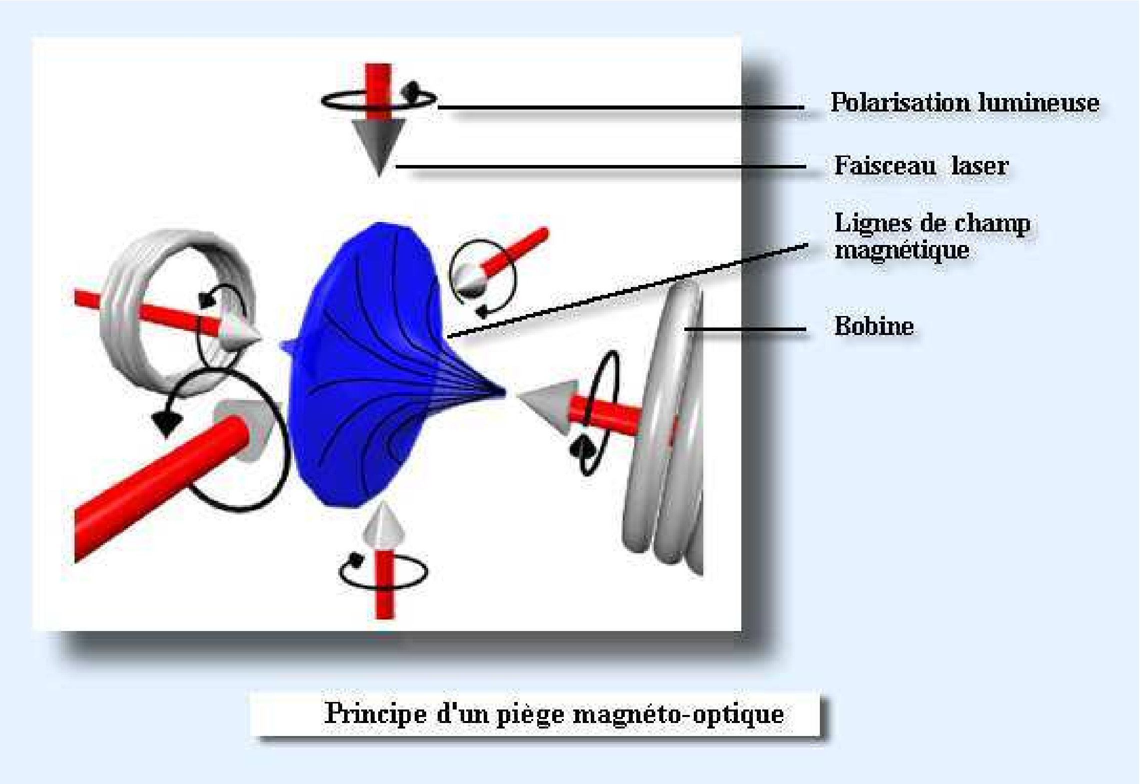
- Bose-Einstein condensates – rotation
- Model – Lowest Landau level
- Numerics
- Bargmann space and Bargmann transform
- Hypercontractivity and consequences.
- Number of vortices ?
- Approximation by polynomials.
- Asymptotic expansion as $h = \sqrt{1 - \Omega^2} \to 0$?

Bose-Einstein condensates

1925: prediction of Bose and Einstein that for a gas of non interacting particules, for $T < T_c$, a macroscopic fraction of the gas which is condensed in the ground state ($\overrightarrow{p} = 0$).
No experimental evidence.
1995: Bose Einstein condensation is achieved in atomic gases (Jila group, Rubidium)
2001: Nobel Prize (Cornell, Wiemann, Ketterle)
2003: Nobel Prize (Ginzburg, Abrikosov, Leggett).

A BEC is a quantum macroscopic object described by a macroscopic wave function (complex-valued function on $\mathbb{R}^3$).

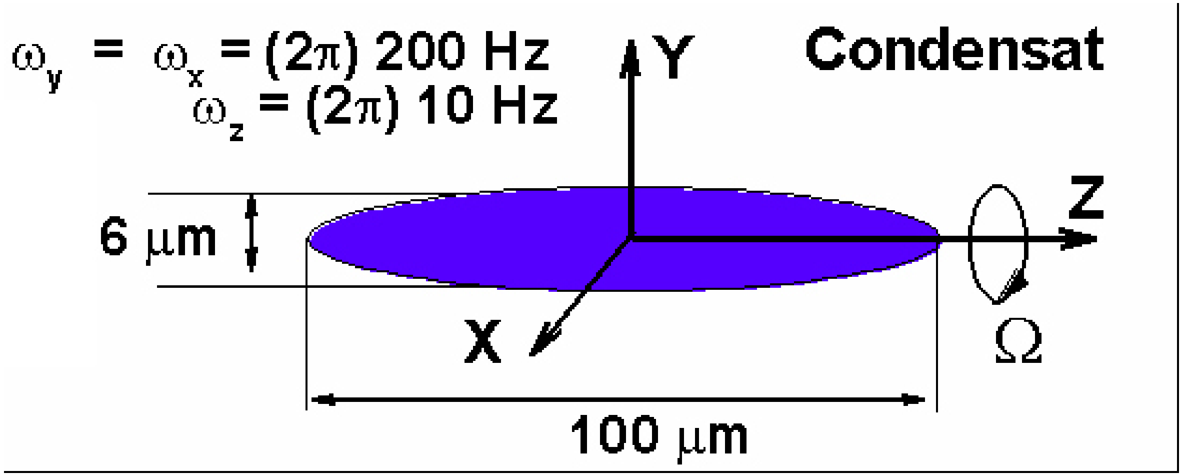
Bose-Einstein condensates



- laser cooling ($100\mu K, \quad 10^9$ atoms, $1cm^3$)
- magnetic trapping
- evaporative cooling: 10^7 atoms, temperature $\sim 90nK$

Bose-Einstein condensates

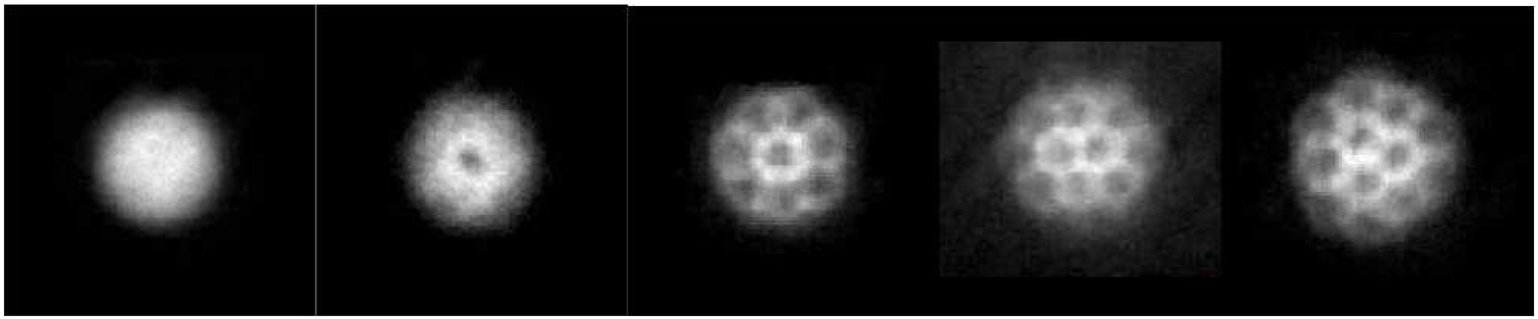
Harmonic trapping $\frac{1}{2}(\omega_x x^2 + \omega_y y^2 + \omega_z z^2) \Rightarrow$
Cigar-shaped condensate



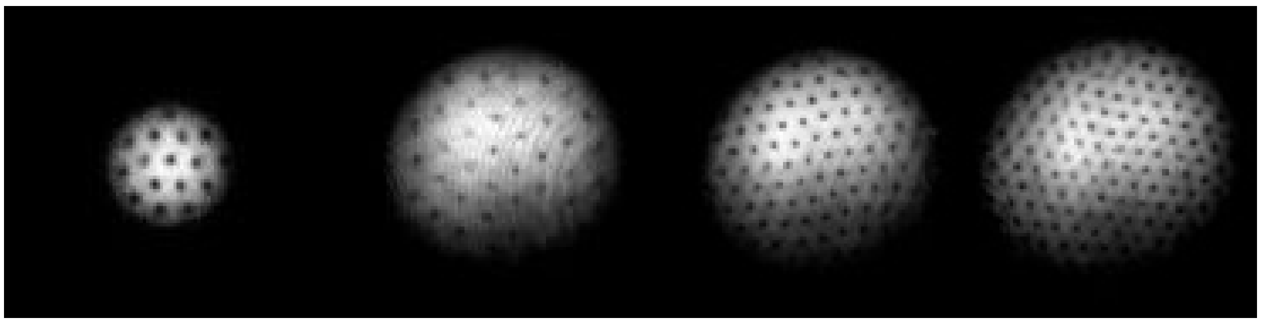
Experiment: rotation

The condensate is rotated along the z axis.

- Ω small: no modification is observed
- Ω larger: vortices are nucleated



Dalibard et al., PRL **84**, 806 (2000)



Ketterle et al., PRL **87**, 210402 (2001)

Previous works:

- Ho PRL 87, 060403, 2001. (regular lattice)
- Baym, Pethick PRA 69, 2004. (HLL contributions)
- Watanabe, Baym, Pethick PRL 93, 2004 (distorted lattice)
- Jackson, Kavoulakis, Lundh, PRA 69, 2004 (quartic)
- Cooper, Komineas, Read PRA 70, 2004
- Anglin, Crescimanno cond-mat/0210063, 2002 (hydrodynamic formulation)
- Mac Donald (oscillation modes)

Model: Gross-Pitaevskii energy

- In a BEC, atoms are described by a collective wave function ψ (cf Lieb, Seiringer, Yngvason, ...)
- $\omega_{x_3} \gg \omega_{x_1} = \omega_{x_2} = 1 \quad \Rightarrow$ 2D model.

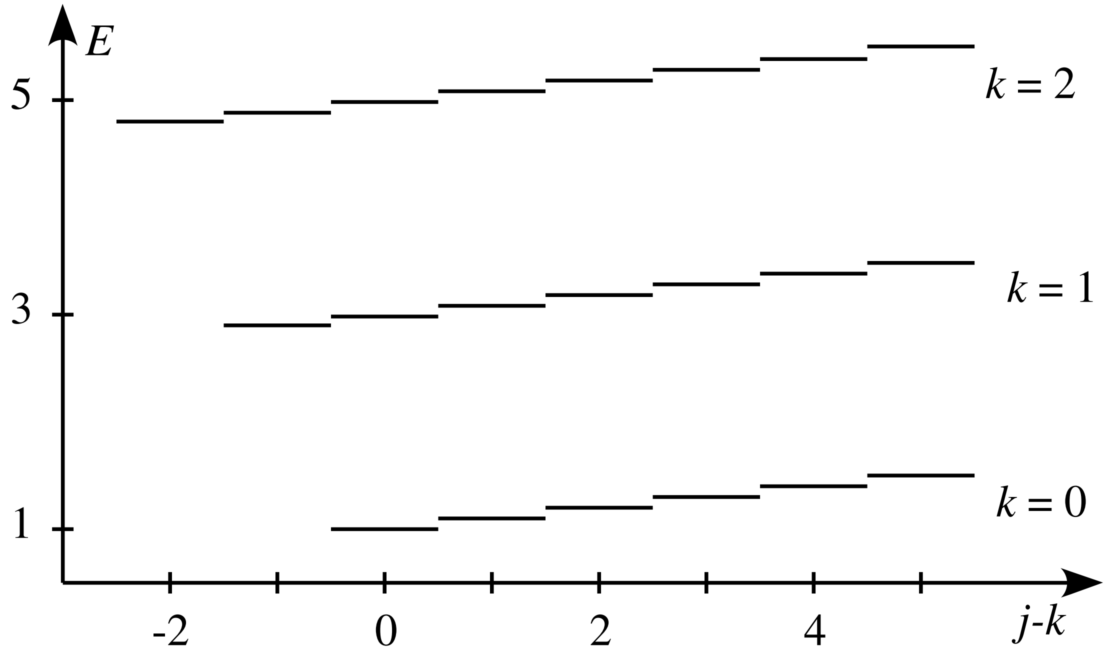
The corresponding energy is

$$E(\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi|^2 - i\Omega\psi x \times \overline{\nabla\psi} + \frac{1}{2}|x|^2|\psi|^2 + \frac{1}{2}Na|\psi|^4,$$

with the constraint $\int_{\mathbb{R}^2} |\psi|^2 = 1.$
Hamiltonian $H_\Omega = -\Delta + |x|^2 - \Omega L,\ L = i(x_2\partial_{x_1} - x_1\partial_{x_2}).$

$$E(\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi - i\Omega x^\perp\psi|^2 + \frac{1}{2}|x|^2|\psi|^2(1-\Omega^2) + \frac{1}{2}Na|\psi|^4$$

Lowest Landau Level



Spectrum of H_Ω: $E_{j,k} = 1 + (1-\Omega)j + (1+\Omega)k.$
$k = 0$: lowest Landau level.

Energy bounded below $\iff \Omega < 1$

$\Rightarrow$ We study $\Omega \underset{\Omega<1}{\longrightarrow} 1.$

Numerics

To compute numerically the energy minimum if $\Omega \to 1$, we first reduce to the lowest Landau level of H_Ω ($k = 0$):

$$\psi(z) = P(z)e^{-|z|^2}, \quad P \in \mathbb{C}[X].$$

Then minimize the rest of the energy ($z = x_1 + ix_2$):

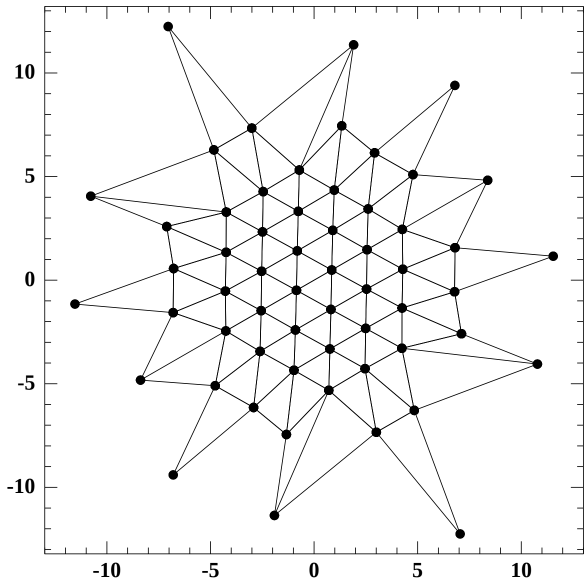
$$\inf\left\{E_{LLL}(\psi), \quad \psi(z) = P(z)e^{-|z|^2}, \quad \int_{\mathbb{R}^2} |\psi|^2 = 1\right\}.$$

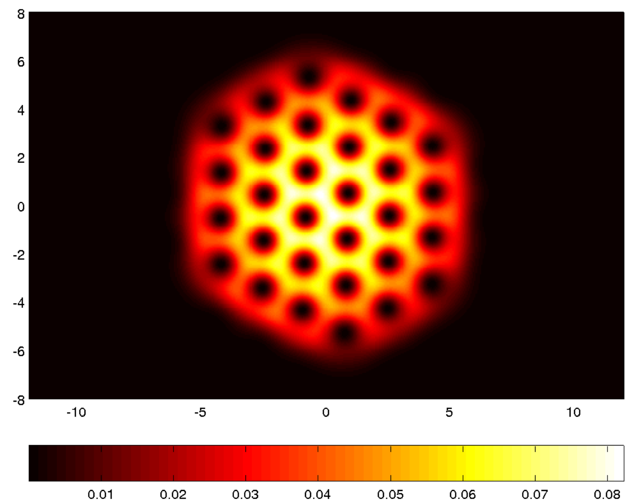
$$E_{LLL}(\psi) = \frac{1-\Omega^2}{2}\int_{\mathbb{R}^2} |x|^2|\psi|^2 + \frac{Na}{2}\int_{\mathbb{R}^2} |\psi|^4.$$

Numerics

In the LLL, the wave function is defined by its zeroes:

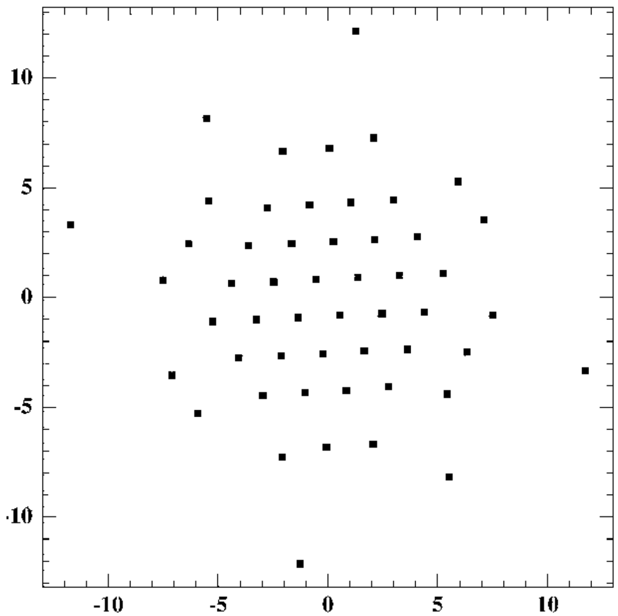
$$\psi(z) = A \prod_{i=1}^{n} (z - z_i) e^{-\frac{|z|^2}{2}}, \quad \int_{\mathbb{R}^2} |\psi|^2 = 1.$$



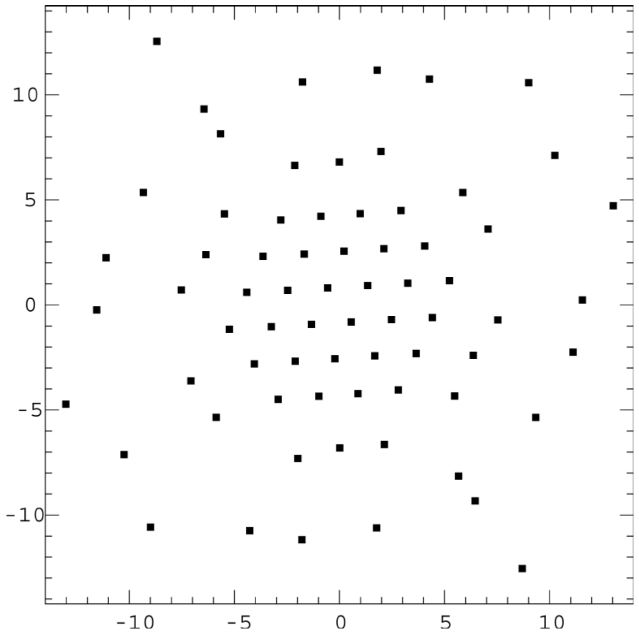


$\Omega = 0.999 \; n = 58$

Numerics



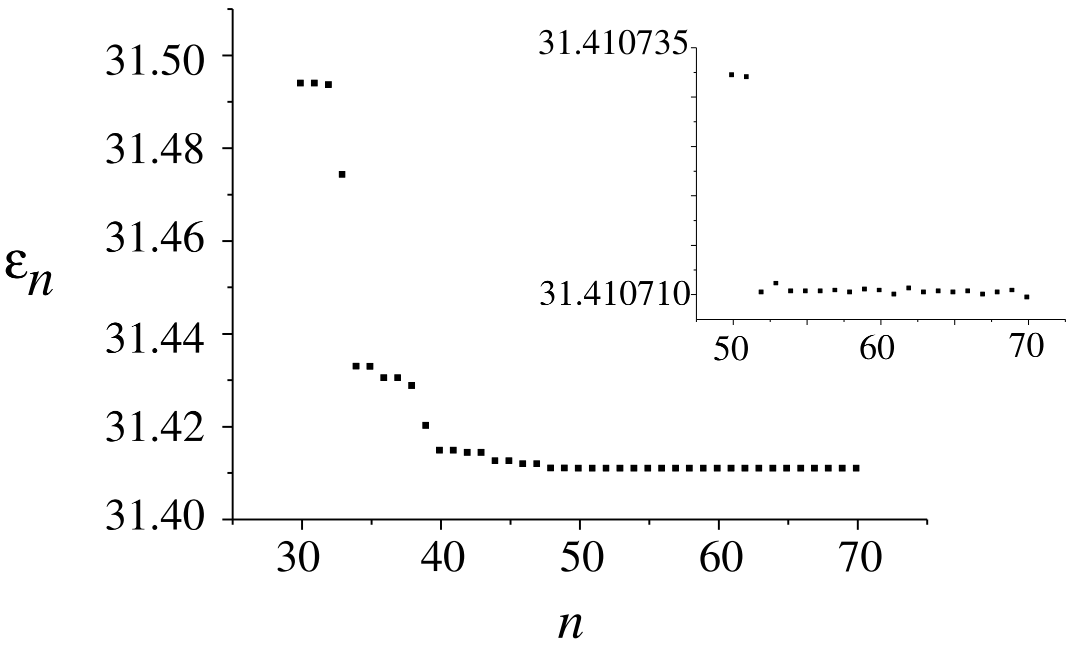
$\Omega = 0.999,\ n = 52$



$\Omega = 0.999\ n = 70.$

Here the density profile is an inverted parabola (see Cooper, Komineas, Read, PRA 70, 2004)

Numerics



Minimum energy with respect to the number n of vortices

$(\varepsilon = \frac{E(\psi)-\Omega}{1-\Omega}.)$

Scaling

$$E(\psi) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla\psi - i\Omega x^\perp \psi|^2 + \frac{1}{2}|x|^2|\psi|^2(1-\Omega^2) + \frac{1}{2}Na|\psi|^4$$

Rescaling: $\psi(x) = \gamma u(\gamma x)$, with $\gamma = \sqrt{\Omega}\left(\frac{1-\Omega^2}{2}\right)^{\frac{1}{4}}$.

$$\begin{aligned} E(\psi) &= \int_{\mathbb{R}^2} \left|\nabla u - i\left(\frac{2}{1-\Omega^2}\right)^{\frac{1}{2}} x^\perp u\right|^2 \\ &\quad + \sqrt{\frac{1-\Omega^2}{2\Omega^2}}\left(\int_{\mathbb{R}^2} |x|^2|u|^2 + \frac{Na\Omega^2}{2}|u|^4\right) \end{aligned}$$

Set $h = \sqrt{\frac{1-\Omega^2}{2\Omega^2}}$, small parameter.

Bargmann space

LLL-functional:

$$G^h(f) = \int_{\mathbb{C}} |z|^2 \, |u(z)|^2 + \frac{Na\Omega_h^2}{2} |u|^4 \; L(dz)$$

$$h \to 0 \qquad\qquad \Omega_h \to 1$$

(Fock-)Bargmann space:

$$u(z) = e^{-\frac{|z|^2}{2h}} f(z)$$

$$f \in \mathcal{F}_h = \left\{ g \in L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} L(dz)), \; g \text{ holomorphic} \right\}$$

$$\langle f | g \rangle = \int f(z) \overline{g(z)} e^{-\frac{|z|^2}{h}} L(dz).$$

Bargmann transform

Bargmann transform [a]

$$[B_h\varphi](z) = \frac{1}{(\pi h)^{3/4}} e^{\frac{z^2}{2h}} \int_{\mathbb{R}} e^{-\frac{(\sqrt{2}z-y)^2}{2h}} \varphi(y)\, dy.$$

$$B_h : L^2(\mathbb{R}, dy) \overset{unitary}{\longrightarrow} \mathcal{F}_h \quad \subset L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} L(dz)).$$

$$B_h^* B_h = \mathrm{Id}_{L^2(\mathbb{R};dy)} \qquad B_h B_h^* = \Pi_h\,.$$

Szegö projector :

$$(\Pi_h f)(z) = \frac{1}{\pi h} \int_{\mathbb{C}} e^{\frac{z\overline{z'} - |z'|^2}{h}} f(z')\, L(dz').$$

[a] REF: V. Bargmann (61), G.B. Folland (89), A. Martinez (02)

Bargmann transform

$$
\begin{array}{rcl}
\text{on } \mathcal{F}_h & & \text{on } L^2(\mathbb{R}, dy) \\
\Pi_h z \Pi_h = z & \leftrightarrow & \frac{1}{\sqrt{2}}(-h\partial_y + y) = a_h^* \\
\Pi_h \overline{z} \Pi_h = h\partial_z & \leftrightarrow & \frac{1}{\sqrt{2}}(h\partial_y + y) = a_h \\
N_h = z(h\partial_z) = \Pi_h[|z|^2 - h]\Pi_h & \leftrightarrow & \frac{1}{2}(-h^2\partial_h^2 + y^2 - 1) \\
\frac{1}{(\pi h)^{1/2} h^{n/2}\sqrt{n!}} z^n & \leftrightarrow & H_n^h = \frac{1}{h^{n/2}\sqrt{n!}}(a_h^*)^n H_0, \\
 & & H_0(y) = (\pi h)^{-1/4} e^{-y^2/h} . \\
\Pi_h \alpha(z) \Pi_h \text{ (Toeplitz)} & \leftrightarrow & \alpha^{A-Wick}(y, hD_y) .
\end{array}
$$

Bargmann space

spaces $\mathcal{F}_h^s$, $s \in \mathbb{R}$:

$$\mathcal{F}_h^s = \left\{ f \text{ holomorphic}, \left((1+|z|^2)^{s/2} f \in L^2(\mathbb{C}; e^{-\frac{|z|^2}{h}} L(dz)) \right) \right\}$$
$$= \{ f \text{ holomorphic}, ((1+N_h)^s f \in \mathcal{F}_h) \}.$$

The imbedding $\mathcal{F}_h^s \subset \mathcal{F}_h$ is compact as soon as $s > 0$.

$$G^h(f) = \langle f \,|\, (N_h + h) f \rangle_{\mathcal{F}_h} + \frac{Na\Omega_h^2}{2} \int_{\mathbb{C}} |u|^4 \; L(dz)$$

$$f \in \mathcal{F}_h^1 \qquad u(z) = f(z) e^{-\frac{|z|^2}{h}}$$

Bargmann space

Consequence:

For $h > 0$ the minimization problem ($u(z) = f(z)e^{-\frac{|z|^2}{2h}}$)

$$\inf_{\|f\|_{\mathcal{F}_h}=1} G^h(f); \quad G^h(f) = \int_{\mathbb{C}} |z|^2 \, |u(z)|^2 + \frac{Na\Omega_h^2}{2} \, |u|^4 \;\, L(dz)\,.$$

admits a solution. A solution belongs to $\mathcal{F}_h^1$.

$$\frac{2\Omega_h}{3}\sqrt{\frac{2Na}{\pi}} < \min_{\|f\|_{\mathcal{F}_h}=1} G^h(f) \leq \frac{2\Omega_h}{3}\sqrt{\frac{2Nab}{\pi}} + O(h^{1/4})\,.$$

$$(b \sim 1.1596)$$

Hypercontractivity

$$\left[e^{-\frac{t}{h}N_h} f\right](z) = \left[e^{-tN_1} f\right](z) = f(e^{-t}z).$$

$$\begin{aligned}\int_{\mathbb{C}} |u|^4 \;L(dz) &= \int_{\mathbb{C}} e^{-\frac{2|z|^2}{h}} |f(z)|^4 \;L(dz)\\ &= \left\| f(e^{-t_0}.)\right\|^4_{L^4(\mathbb{C}; e^{-\frac{|z|^2}{h}} L(dz))} \leq \frac{1}{2\pi h} \|f\|^4_{\mathcal{F}_h} \text{ [a]}\\ t_0 &= \frac{\ln 2}{2}\end{aligned}$$

The nonlinearity is associated with a continuous quadrilinear functional on $\mathcal{F}_h$.

[a] E. Carlen (91), L. Gross (99)

Number of vortices.

“number of vortices $= O(1/h)$”
compactness $\rightarrow$ topological index ??

Result :

Any minimizer of $\{G^h(f),\ \|f\|_{\mathcal{F}_h} = 1\}$ admits an infinite number of zeros as soon as

$$h(1 + O_{Na}(h^{1/4})) < \frac{729}{1024} \frac{\sqrt{Na}}{(2\pi b)^{3/2}} c_0^2 \approx 0.04276\sqrt{Na}$$

with

$$b \sim 1.1596 \quad \text{and} \quad c_0 = \min_{n \in \mathbb{N}^*} \frac{\sqrt{\pi n}(2n)!}{2^{2n}(n!)^2}.$$

Approximation by polynomials.

$$
\begin{aligned}
e^h_{LLL} &= \min_{\|f\|_{\mathcal{F}_h}=1} G^h(f) \\
e^h_{LLL,K} &= \min_{P\in\mathbb{C}_K[z],\ \|P\|_{\mathcal{F}_h}=1} G^h(P) \\
& \Pi_{h,K} = 1_{[0,hK]}(N_h) \text{ orth. proj. on } \mathbb{C}_K[z].
\end{aligned}
$$

Approximation by polynomials.

- $0 < e^h_{LLL,K} - e^h_{LLL} \leq \frac{C_2(h)^2 + C_2(h)}{1 - C_2(h)(hK)^{-1}}(hK)^{-1}$ with $C_2(h) = O(h^{-1/2})$.
- Same upper bound for $\|f - \Pi_{h,K} f\|_{\mathcal{F}_h}$, when f is a minimizer.
- If P_K is a solution to the restricted minimization problem for $K \in \mathbb{N}$, then after extraction

$$\lim_{n\to\infty} \|f - P_{K_n}\|_{\mathcal{F}_h} + \|N_h(f - P_{K_n})\|_{\mathcal{F}_h} = 0\,.$$

and the linearized Hamiltonians converge in the norm resolvent sense. [a]

[a] Grushin problem, REF:Sjöstrand-Zworski (03)

Θ function

Definition:

$$\Theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}.$$

Result: If f is holomorphic and cancels on each site of the lattice $\frac{1}{\alpha}\mathbb{Z}\oplus\frac{\tau}{\alpha}\mathbb{Z}$, with $\alpha\in\mathbb{R}$ and $\tau\in\mathbb{C}\setminus\mathbb{R}$, and if the function $|u(z)| = |f(z)|e^{-\frac{|z|^2}{h}}$ is periodic, then

$$\alpha = \sqrt{\frac{\tau_I}{\pi h}} \quad \text{and} \quad f(z) = e^{\frac{z^2}{2h}}\Theta\left(\sqrt{\frac{\tau_I}{\pi h}}z,\tau\right).$$

Limit $h \to 0$

$$f(z) = e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}} z, \tau\right) \qquad \lambda_\tau = \frac{\int_{Q_\tau} |\Theta|^4}{\left(\int_{Q_\tau} |\Theta|^2\right)^2}$$

$\alpha \in \mathcal{C}^{0,\beta}(\mathbb{C})$ with compact support.

$$f_\alpha = \Pi_h (\alpha f) \ .$$

$$(N_h + h - \lambda) f_\alpha + N a \Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f_\alpha|^2 f_\alpha\right) =$$

$$\Pi_h \left[\left(|z|^2 - \lambda + \lambda_\tau N a \Omega_h^2 |\alpha|^2\right) \alpha\right] \Pi_h f + O_{\mathcal{F}_h}(h^{\beta/2}) \ .$$

Limit $h \to 0$

Application 1 :

Take $\lambda = \sqrt{\frac{\lambda_\tau N a \Omega_h^2}{\pi}}$ and

$$\alpha = \sqrt{\frac{(\lambda - |z|^2)_+}{\lambda_\tau N a \Omega_h^2}}.$$

Then

$$(N_h + h - \lambda) f_\alpha + N a \Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f_\alpha|^2 f_\alpha \right) = O_{\mathcal{F}_h}(h^{1/4})\,.$$

Limit $h \to 0$

Application 2 : If

$$f_\alpha^h = \Pi_h(\alpha f) \qquad f(z) = e^{\frac{z^2}{2h}} \Theta\left(\sqrt{\frac{\tau_I}{\pi h}} z, \tau\right),$$

with $\alpha(z)$ and τ independepent of $h > 0$,
then $\left\| f_{min}^h - f_\alpha^h \right\|_{\mathcal{F}_h^{+2}} = o(h^0)$ implies

$$\tau = \tau_{min} = e^{\frac{2i\pi}{3}}, \qquad \text{hexagonal lattice} \qquad b = \lambda(\tau_{min}),$$

$$|\alpha(z)|^2 = \frac{(\sqrt{Nab/\pi} - |z|^2)_+}{Nab}$$

and $$E_{LLL}(f_{min}^h) = \frac{2}{3}\sqrt{\frac{2Nab}{\pi}} + o(h^0).$$

Hypercontractivity

Consequence: Euler-Lagrange equation

$$zh\partial_z f + Na\Omega_h^2 \Pi_h \left(e^{-\frac{|z|^2}{h}} |f|^2 \right) \Pi_h f - (\lambda - h) f = 0$$

$$\Pi_h \left[|z|^2 + Na\Omega_h^2 e^{-\frac{|z|^2}{h}} |f|^2 - \lambda \right] \Pi_h f = 0$$

or $$zh\partial_z f + \frac{Na\Omega_h^2}{2} \bar{f} (h\partial_z) [f^2 (2^{-1} .)] - (\lambda - h) f = 0$$

Back to Hypercontractivity

Consequence of hypercontractivity: Hamiltonian dynamics

The Cauchy problem

$$\begin{cases} i\partial_t f = 2\partial_{\overline{f}} G^h(f) = 2N_h f + 2Na\Omega_h^2 \Pi_h(e^{-\frac{|z|^2}{h}}|f|^2)\Pi_h f \\ f(t=0) = f_0 \,. \end{cases}$$

admits a global solution for all $f_0 \in \mathcal{F}_h$ such that

$$\begin{aligned} \|f(t)\|_{\mathcal{F}_h} &= \|f_0\|_{\mathcal{F}_h} && \text{invariance } e^{i\alpha} f \\ G^h(f(t)) &= G^h(f_0) \\ \langle f(t) \,|\, N_h f(t)\rangle &= \langle f_0 \,|\, N_h f_0\rangle && \text{invariance } f(e^{i\alpha} z)\,. \end{aligned}$$

Perspectives

- This provides an upper bound for the energy. What about a lower bound? homogenization technics, Γ-convergence...
- Link with crystallization problem?
- Justification of the LLL ansatz?
- Compute the oscillation modes?
- Bounds on the minimizer(s)? (for instance in L^∞)