Microscopic Hamiltonian dynamics perturbed by a conservative noise

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Introduction

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- If system has (microscopic) size N, finite conductivity means $< J >_{ss} \sim N^{-1}$.

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- If $V(r) = \alpha |r|^2$, $W(q) = \nu |q|^2$ (harmonic chain), $< \cdot >_{ss}$ is an explicit Gaussian measure and $< J >_{ss} \sim 1$: Fourier's law is false (Lebowitz, Lieb, Rieder '67)

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- Non linearity is extremely important to have normal heat conduction.
- But it is not sufficient : It has been observed experimentally and numerically for nonlinear chains that if $d \leq 2$ and momentum is conserved ($\Leftrightarrow W = 0$, unpinned) then conductivity is still infinite (finite otherwise).

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- If Fourier's law does not hold, $\kappa_N \sim N^\delta$, universality of the diverging order δ of the conductivity?
- Numerical simulations are not conclusive ($\delta \in [0.25; 0.47]$ for the same models) and subject of intense debate.

The Models

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$$\begin{aligned} \text{Noise 1} &= \text{only energy conservative} \\ \text{Noise 2} &= \text{energy and momentum conservative} \end{aligned}$$

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so $X^2_{x,x+1}$ generates a diffusion on $\mathbb{S}^1_e$ (Brownion motion on the circle).

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- Noise 2, $d \geq 1$ ... are of the same type.

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Inconvenient : The quantity to evaluate is a dynamical quantity.

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with M martingale (mean 0 w.r.t. any initial condition)

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$\kappa_N \sim \kappa_{GK}$ with truncation of the time up to time $t_N = N$ (phononic picture).

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$$+\frac{\gamma}{d}\,(\text{Martingale contribution})$$

Noise 1: energy conserving

Results : Harmonic case : $V(r) = \alpha r^2,\ W(q) = \nu q^2$

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Theorem (B., Olla, JSP'05)

- κ_{GK} *is finite (pinned or unpinned) in any dimension.*
- *Fourier's law holds and linear response theory is correct: System of length N in contact with two Langevin baths at temperature T_ℓ and T_r in its steady state*

$$\lim_{N\to\infty} N < j_{x,x+1} >_{ss} = \kappa_{GK}(T_r - T_\ell)$$

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Theorem (B., '08)

Harmonic system with random masses and energy conservative noise 1.The conductivity defined by Green-Kubo formula is strictly positive and bounded above:

$$0 < c_- \leq \kappa_{KG} \leq C_+ < +\infty$$

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Noise 2: energy/momentum conserving

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Theorem (Basile, B., Olla, PRL'06)

$$C_{1,1}(t) = \lim_{N\to\infty} \left\langle \left(\sum_x j_{x,x+e_1}(t) \right), j_{0,e_1}(0) \right\rangle_{eq.}$$

$$C_{1,1}(t) = \frac{T^2}{4\pi^2 d} \int_{[0,1]^d} (\partial_{k^1}\omega(k))^2 e^{-t\gamma\psi(k)} dk$$

where ω is the dispertion relation of the harmonic chain

$$\omega(k) = (\nu + 4\alpha \sum_{j=1}^{d} \sin^2(\pi k^j))^{1/2}, \quad \psi(k) \sim k^2$$

Noise 2: energy/momentum conserving

Corollary

- $C_{1,1}(t) \sim t^{-d/2}$ *in the unpinned case* $(\nu = 0)$
- $C_{1,1}(t) \sim t^{-d/2-1}$ *in the pinned case* $(\nu > 0)$

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$$\kappa_N = \frac{1}{2T^2 N^d t_N} \mathbb{E}_{eq} \left(\left[\sum_x \int_0^{t_N} j_{x,x+e_1}(s) ds \right]^2 \right) + \frac{\gamma}{d},\ t_N = N$$

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Corollary

If the system is unpinned ($\nu = 0$) then "truncated" Green-Kubo formula for κ_N gives:

$$\begin{cases} \kappa_N \sim N^{1/2} \text{ if } d = 1 \\ \kappa_N \sim \log N \text{ if } d = 2 \end{cases}$$

In all other cases κ_N is bounded in N and converges to κ_{GK}.

Results : Anharmonic case, Canonical version of GK

Theorem (Basile,B.,Olla'08)

- *For* $d \geq 3$*, if* $W > 0$ *is "general" or if* $W = 0$ *and* $0 < c_- \leq V" \leq C_+ < +\infty$ *then*

$$\kappa_N \leq C.$$

- *For* $d = 2$*, if* $W = 0$ *and* $0 < c_- \leq V'' \leq C_+ < \infty$

$$\kappa_N \leq C(\log N)^2.$$

- *For* $d = 1$*, if* $W = 0$ *and* $0 < c_- \leq V'' \leq C_+ < \infty$*, then*

$$\kappa_N \leq C\sqrt{N}.$$

- *In any dimension, if* V *are quadratic and* $W > 0$ *is "general" then* $\kappa_N \leq C$*.*

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- Simulations ($d = 1$) for unpinned systems with energy/momentum conservative noise 2. The strength of the noise is regulate by γ. Then $\kappa_N \sim N^{\delta}$.

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Proof of the harmonic case

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- Function h_λ is local in the energy conservative case and non-local in the energy/momentum case. For $d \geq 3$ or $\nu > 0$, the decay of h_λ is sufficient to assure a finite conductivity.

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$$f_{ss} = \mathbf{1} + \delta T h_0 + o((\delta T)^2)$$

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$$(N^{-1}-\gamma S)^{-1}\left(\sum_x j_{x,x+e_1}\right) = \sum_{j=1}^d \sum_{x,y} G_N(x-y)p_x^j V'(q_{y+e_1}^j - q_y^j)$$

where $G_N(z)$ is the solution of the resolvent equation

$$N^{-1}G_N(z) - 2\gamma(\Delta G_N)(z) = -\frac{1}{2}[\delta_0(z)+\delta_{e_1}(z)]$$