Self-interacting diffusions

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Outline

What is a self-interacting (or reinforced) diffusion?

- Solution to
$$\mathrm{d}X_t = \mathrm{d}B_t - F(t, X_t, \mu_t)\mathrm{d}t$$
- $\mu_t = \frac{1}{t}\int_0^t \delta_{X_s}\mathrm{d}s$

Brownian polymer

Durrett and Rogers (1992) on $\mathbb{R}^d$:

$$\mathrm{d}X_t = \mathrm{d}B_t + \int_0^t f(X_t - X_s)\mathrm{d}s\,\mathrm{d}t,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and bounded.

Applications: physics, biology.

Cases studied

The problem is to find the normalization $\alpha \geq 0$ such that X_t/t^α converges a.s.
Three cases have been studied yet:

- drift on the right in dimension 1 (Cranston & Mountford 1996),
- self-attracting: $(f(x), x) \leq 0$ (Cranston & Le Jan 1995, Raimond 1997, Herrmann & Roynette 2003),
- self-repelling: $f(x) = \frac{x}{1+|x|^{1+\beta}}$, with $0 < \beta < 1$ (Mountford & Tarrès 2008).

A last conjecture (unsolved)

Conjecture (Durrett & Rogers, 1992)

Suppose $f : \mathbb{R} \to \mathbb{R}$ with compact support, $xf(x) \geq 0$ and $f(-x) = -f(x)$. Then, $\frac{X_t}{t}$ converges a.s. toward 0.

Self-interacting diffusions on a compact set

Benaïm, Ledoux & Raimond (2002), Benaïm & Raimond (2003, 2005) on a compact manifold:

$$\mathrm{d}X_t = \mathrm{d}B_t - \frac{1}{t}\int_0^t \nabla_x W(X_t, X_s)\mathrm{d}s\,\mathrm{d}t.$$

Heuristic: show that μ_t is close to a deterministic flow (stochastic approximation).

Why is it more difficult?

Théorème (Chambeu & K)

Let

$$dX_t = dB_t - (\log t)^3 W'(X_t - \overline{\mu}_t)dt,\ X_0 = x$$

where $\overline{\mu}_t = \frac{1}{t}\int_0^t X_s\, ds$ *and* W *is strictly convex out of a compact set.*

Then

1. *The process* $Y_t = X_t - \overline{\mu}_t$ *converges a.s. to* Y_∞*, where* Y_∞ *belongs to the set of all the local minima of* W*. Moreover, for each local minimum m, one has* $\mathbb{P}(Y_\infty = m) > 0$*.*
2. *On the set* $\{Y_\infty = 0\}$*, both* X_t *and* $\overline{\mu}_t$ *converge a.s. to* $\overline{\mu}_\infty := \int_0^\infty Y_s \frac{ds}{s}$*. Moreover, on the set* $\{Y_\infty \neq 0\}$*, one has* $\lim_{t\to\infty} X_t/\log t = Y_\infty$*.*

Our study:

$$\begin{aligned} \mathrm{d}X_t &= \mathrm{d}B_t - \left(\nabla V(X_t) + \frac{1}{t}\int_0^t \nabla_x W(X_t, X_s)\mathrm{d}s\right)\mathrm{d}t \\ \dot{\mu}_t &= \frac{\delta_{X_t} - \mu_t}{t} \end{aligned}$$

Hypotheses on the potentials (H)

- $V \geq 1$ is $\mathcal{C}^2$, strictly uniformly convex,
- W is $\mathcal{C}^2$ and such that $\nabla^2(V+W)$ is bounded by below, and asymptotically $(x, \nabla_x W(x,y)) + (x, \nabla V(x)) \geq M|x|^{2\delta}$ with $\delta > 1$ and $M > 0$,
- there exists $\kappa > 0$ such that

$$W(x,y) + |\nabla_x W(x,y)| + |\nabla^2_{xx} W(x,y)| \leq \kappa(V(x) + V(y)).$$

Example on $\mathbb{R}^2$

Theorem

Suppose $V(x) = V(|x|)$ and $W(x, y) = (x, Ry)$, where R is a rotation matrix (angle θ).
Let $I := Z^{-1} \int_0^{\infty} e^{-2V(\rho)} \rho^2 \mathrm{d}\rho$. One of the following holds:

- *if $I \cos\theta + 1 > 0$, then a.s. μ_t converges to $Z^{-1} e^{-2V}$,*
- *if $I \cos\theta + 1 \leq 0$, then:*

  if $\theta = \pi$, then a.s. μ_t converges to a random measure $\mu_\infty \neq Z^{-1} e^{-2V}$,
  if $\theta \neq \pi$, then μ_t does not converge: it circles around and the $\omega-$limit-set $\omega(\mu_t, t \geq 0)$ is a "circle" of measures.

If W is symmetric

Theorem

Suppose (H). If W is symmetric, then the $\omega-$limit set $\omega(\mu_t, t \geq 0)$ is a.s. a compact connected subspace of the fixed points of Π, with $\Pi(\mu)(\mathrm{d}x) = Z(\mu)^{-1} e^{-2(V+W*\mu)(x)} \mathrm{d}x$.

In particular, if Π admits only a finite number of fixed points, then μ_t converges a.s. to one of these fixed points.

Markovian system related to the diffusion

μ_t is asymptotically close to a deterministic dynamical system:
$\dot{\mu} = \Pi(\mu) - \mu$.

Asymptotic pseudotrajectory (APT) for a flow

Definition

(E, d) metric. The continuous function $\xi : \mathbb{R} \to E$ is a APT for the flow Φ if $\forall T > 0$, one has $\lim_{t\to\infty} \sup_{0\le s\le T} d(\xi_{t+s}, \Phi_s(\xi_t)) = 0$.

Deterministic example

Let the ODE (on $\mathbb{R}$):

$$\dot{\xi} = f(\xi) + g(t), \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function and $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $\lim_{t\to\infty} g(t) = 0$. Consider the solution of

$$\dot{x} = f(x) \tag{2}$$

ξ is an asymptotic pseudotrajectory of the flow generated by (2).

$$\xi_{t+s} - \Phi_s(\xi_t) = \int_t^{t+s} \left(f(\xi_u) - f(\Phi_u(\xi_t))\right) \mathrm{d}u + \int_t^{t+s} g(u)\mathrm{d}u.$$

- f is Lipschitz (c)
- Gronwall

$$\sup_{0\leq s\leq T} |\xi_{t+s} - \Phi_s(\xi_t)| \leq e^{cT} \int_t^{t+T} |g(u)|\mathrm{d}u.$$

- $g(t)$ converges toward 0
- conclusion: $\lim_{t\to\infty} \sup_{0\leq s\leq T} |\xi_{t+s} - \Phi_s(\xi_t)| = 0$

Attractor free set

Definition

- *$A \subset E$ is an attractor for the flow Φ if it is $A \neq \emptyset$, compact, invariant and A admits a neighbourhood $\mathcal{V} \subset E$ such that $d(\Phi_t(x), A) \to 0$ uniformly for $x \in \mathcal{V}$.*
- *A is said to be attractor free if A is the only attractor for the flow restricted to A.*

Tightness

Using some martingale techniques and the law of large numbers, we get:

Lemma

There exists a subset $\mathcal{P}$ of the set of probability measures on $\mathbb{R}^d$, which is compact (for the weak topology) such that a.s. $\mu_t \in \mathcal{P}$ for all t large enough.
The family $(\mu_t, t \geq 0)$ is a.s. tight.

Uniform ultracontractivity

Using some techniques of Röckner and Wang, one has:

Proposition

The semi-group family (P_t^μ, t, μ) is uniformly ultracontractive i.e.

$$\sup_{f \in L^2(\Pi(\mu)) \setminus \{0\}} \frac{\|P_t^\mu f\|_\infty}{\|f\|_2} \leq \exp\{c t^{-\delta/(\delta-1)}\},$$

with a uniform constant $c > 0$.

Theorem

1) The function $t \mapsto \mu_{e^t}$ is a.s. an APT for the flow generated by the dynamical system $\dot{\mu} = \Pi(\mu) - \mu$.

2) The limit set $\omega(\mu_t, t \geq 0) := \bigcap_{t \geq 0} \overline{\mu([t, \infty))}$ is a.s. an attractor free set for the flow.

Symmetric case

Theorem

Suppose (H). If W is symmetric, then $\omega(\mu_t, t \geq 0)$ is a.s. a compact connected subset of the fixed points of Π.
If Π admits a finite number of fixed points, then μ_t converges a.s. to one of these fixed points.