

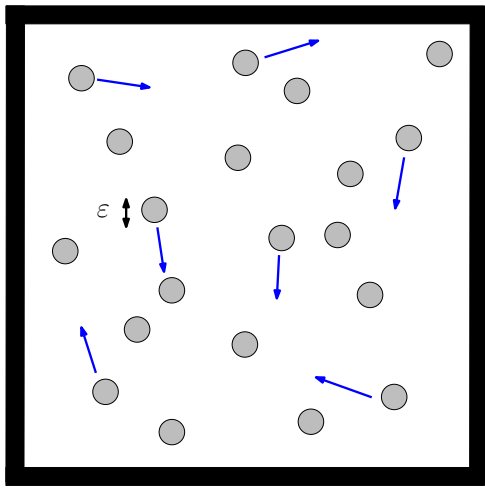
Derivation of the linearized Boltzmann equation from a deterministic dynamics of hard spheres

Thierry Bodineau

Joint works with Isabelle Gallagher, Laure Saint-Raymond

Outline.

- Linear Boltzmann equation & Brownian motion
- Linearized Boltzmann equation & acoustic equations
- Lanford's strategy & pruning procedure



Microscopic scale

N particles of size ε
Newtonian dynamics

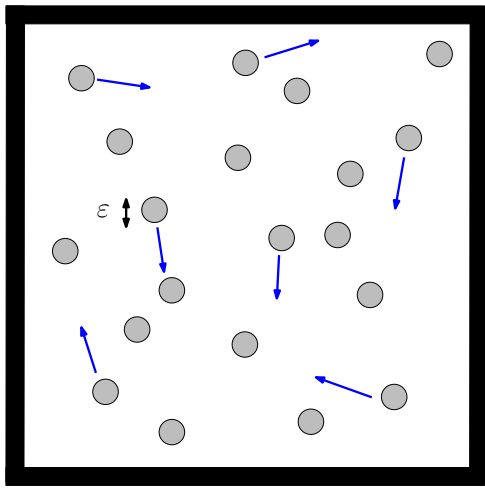
$$N\varepsilon^d \ll 1$$

$$N\varepsilon^{d-1} \gg 1$$



Macroscopic scale

Fluid equations of hydrodynamics
(Euler, Navier-Stokes)



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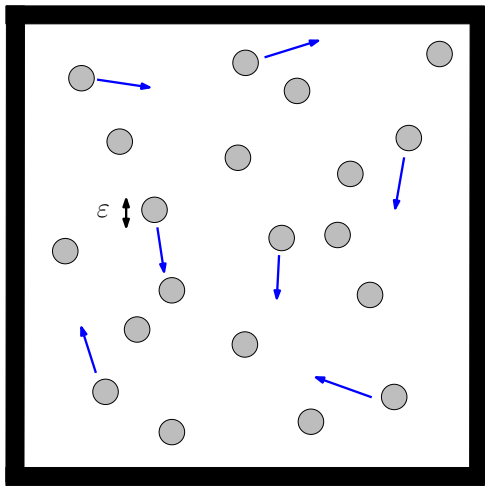


Stochastic perturbations:
Olla, Varadhan, Yau



Macroscopic scale

Fluid equations of hydrodynamics
(Euler, Navier-Stokes)



Microscopic scale

N particles of size ε
Newtonian dynamics

$$N\varepsilon^d \ll 1$$

$$N\varepsilon^{d-1} \gg 1$$

Low density limit

Mesoscopic scale

Boltzmann equation

Fast relaxation limit:

Bardos, Golse, Levermore
Golse, Saint-Raymond ...

Stochastic perturbations:
Olla, Varadhan, Yau



Macroscopic scale

Fluid equations of hydrodynamics
(Euler, Navier-Stokes)

Diluted Gas of hard spheres

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).

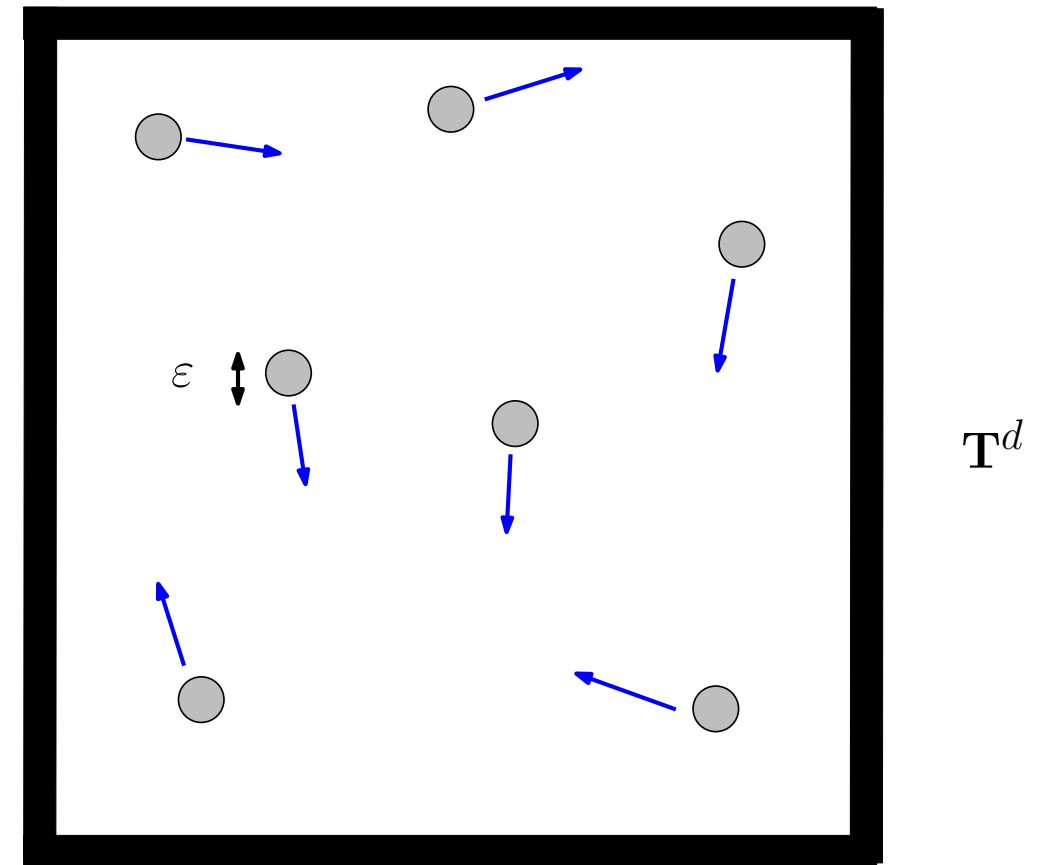
Dimension : $d \geq 2$

Periodic domain: $\mathbb{T}^d = [0, 1]^d$

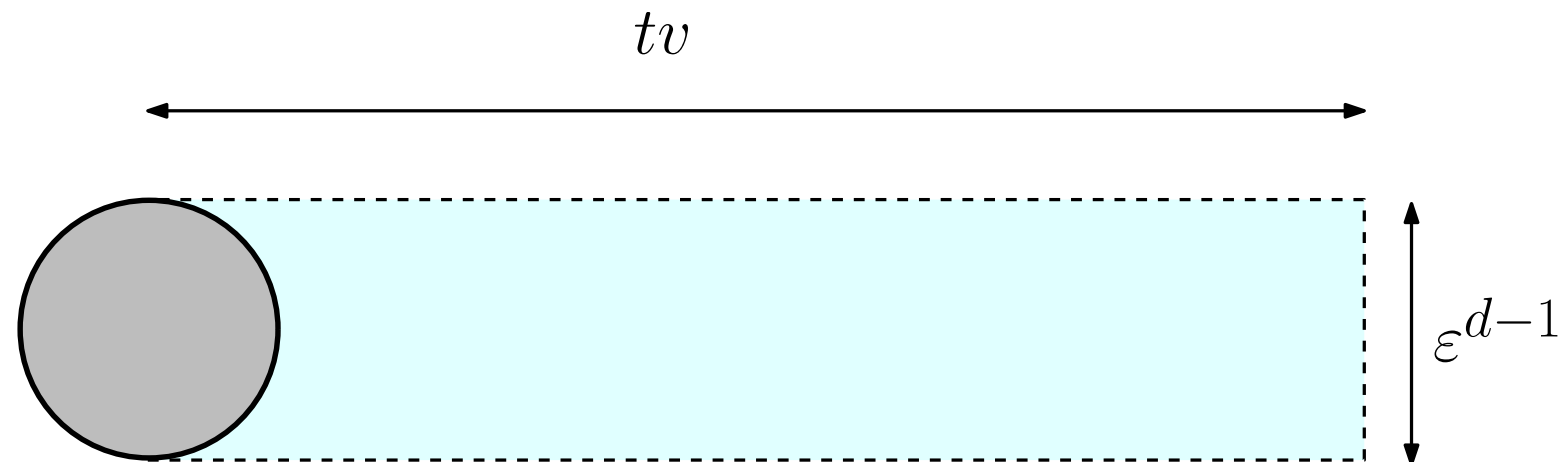
Sphere radius = ε

Boltzmann-Grad scaling

$$N\varepsilon^{d-1} = \alpha$$



Boltzmann-Grad scaling



- Volume covered by a particle $= tv\varepsilon^{d-1}$
- On average N particles per unit volume

On average, a particle has α collisions per unit of time

$$N \times \varepsilon^{d-1} \equiv \alpha$$

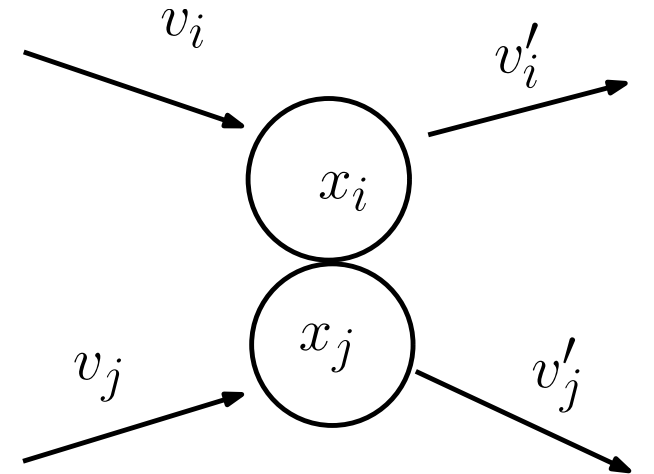
Hard Sphere dynamics

Gas of N hard spheres : $Z_N = \{(x_i(t), v_i(t))\}_{i \leq N}$

$$\frac{dx_i}{dt} = v_i, \quad \frac{dv_i}{dt} = 0 \quad \text{as long as} \quad |x_i(t) - x_j(t)| > \varepsilon,$$

and elastic collisions if $|x_i(t) - x_j(t)| = \varepsilon$

$$\begin{cases} v'_i + v'_j = v_i + v_j \\ |v'_i|^2 + |v'_j|^2 = |v_i|^2 + |v_j|^2 \end{cases}$$



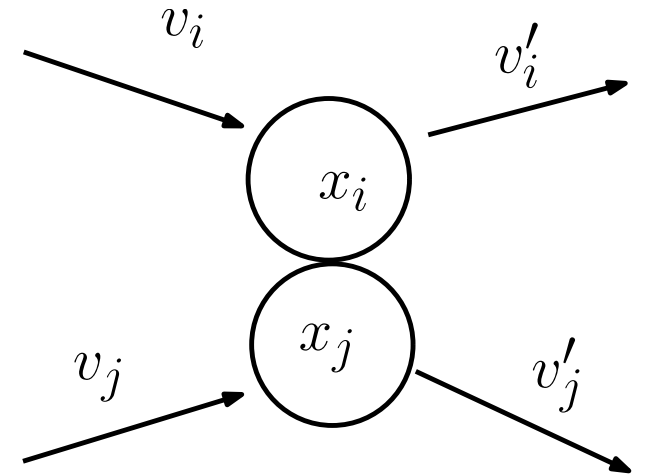
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Liouville equation for the particle density $f_N(t, Z_N)$

$$\partial_t f_N + \sum_{i=1}^N v_i \cdot \nabla_{x_i} f_N = 0$$

in the phase space

$$\mathcal{D}_\varepsilon^N := \{Z_N \in \mathbf{T}^{dN} \times \mathbb{R}^{dN} / \forall i \neq j, \quad |x_i - x_j| > \varepsilon\}$$

with specular reflection on the boundary $\partial \mathcal{D}_\varepsilon^N$.

Initial Data

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp \left(-\frac{\beta}{2} \sum_{i=1}^N |v_i|^2 \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Initial data :

$$f_{N,\beta}^0(Z_N) = \left(\prod_{i=1}^N f^0(z_i) \right) M_{N,\beta}(Z_N)$$

Density of a particle at time t :

$$f_N^{(1)}(t, z_1) = \int dz_2 \dots dz_N f_N(t, z_1, z_2, \dots, z_N)$$

Question. Convergence

$$f_N^{(1)}(t, z_1) \xrightarrow[N \varepsilon^{d-1} = \alpha]{N \rightarrow \infty} f(t, z_1)$$

Boltzmann equation

Theorem.

For chaotic initial data $f_N^0(Z_N) \simeq \prod_{i=1}^N f^0(z_i)$ the density of the particle system converges up to a time $t > 0$ to the solution of the Boltzmann equation when $N \rightarrow \infty$, $N\varepsilon^{d-1} = \alpha$

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f \\ = \alpha \iint_{\mathbf{S}^{d-1} \times \mathbb{R}^d} [f(v')f(v'_1) - f(v)f(v_1)] ((v - v_1) \cdot \nu)_+ dv_1 d\nu \end{aligned}$$

with $v' = v + \nu \cdot (v_1 - v) \nu$, $v'_1 = v_1 - \nu \cdot (v_1 - v) \nu$

[Lanford], [King], [Alexander], [Uchiyama], [Cercignani, Illner, Pulvirenti], [Simonella], [Gallagher, Saint-Raymond, Texier], [Pulvirenti, Saffirio, Simonella]

Boltzmann equation

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Lanford's strategy leads to a short time convergence which depends on f^0 . The convergence time remains short even if initially the system starts from equilibrium !!!

Large time asymptotics

Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp \left(-\frac{\beta}{2} \sum_{i=1}^N |v_i|^2 \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Initial data for Lanford's theorem

$$f_{N,\beta}^0(Z_N) = \left(\prod_{i=1}^N f^0(z_i) \right) M_{N,\beta}(Z_N) \quad \leftarrow \quad \simeq \exp(N)$$

Perturbation of the equilibrium distribution :

- *Linear* Boltzmann equation: perturbation of a *tagged* particle
- *Linearized* Boltzmann equation:

$$f^0(z) = 1 + \frac{1}{N} g_0(z)$$

The tagged particle

Gas of N hard spheres with deterministic Newtonian dynamics (elastic collisions).

Initial data at equilibrium and a tagged particle (x_1, v_1)

Questions.

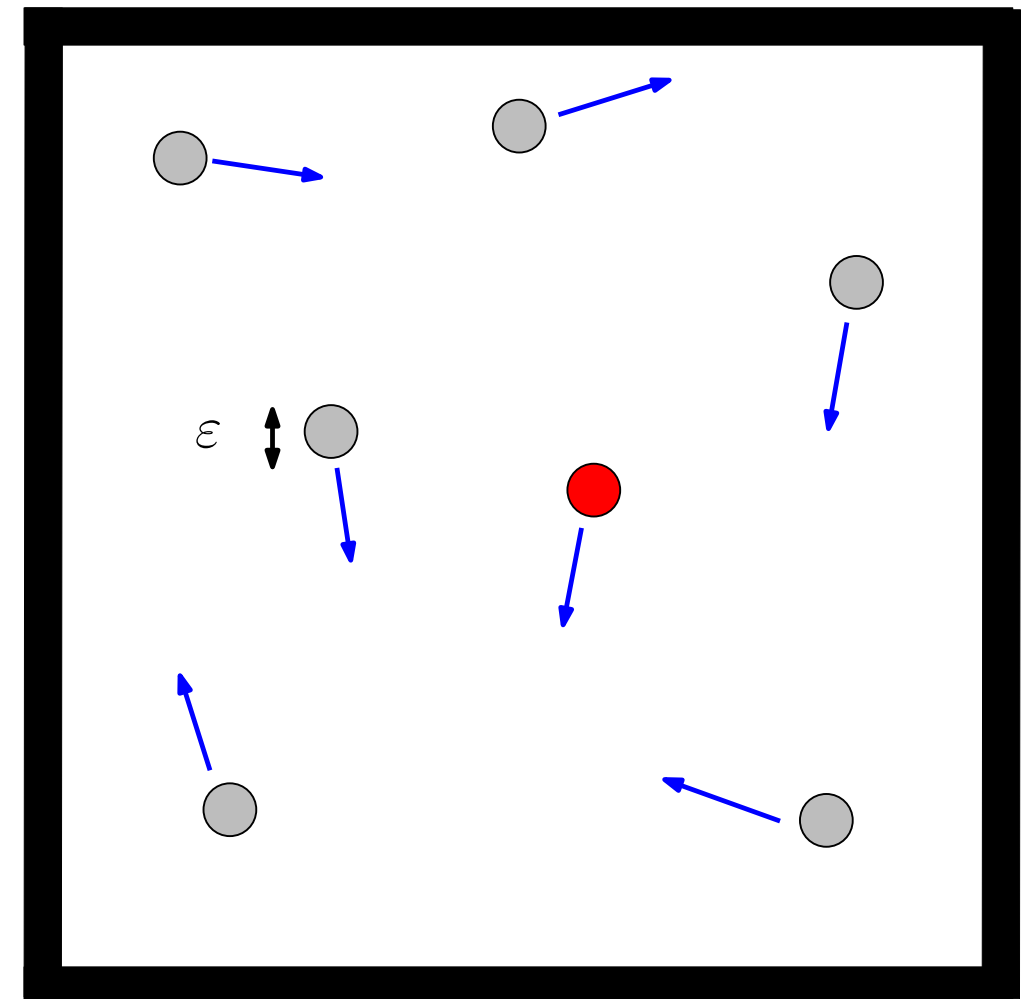
In the Boltzmann-Grad scaling

$$N \times \varepsilon^{d-1} \equiv \alpha \text{ and } N \rightarrow \infty$$

1. Distribution of $(x_1(t), v_1(t))$

2. Position of the tagged particle

$$x_1(\alpha t) \text{ when } \alpha \rightarrow \infty$$



The tagged particle

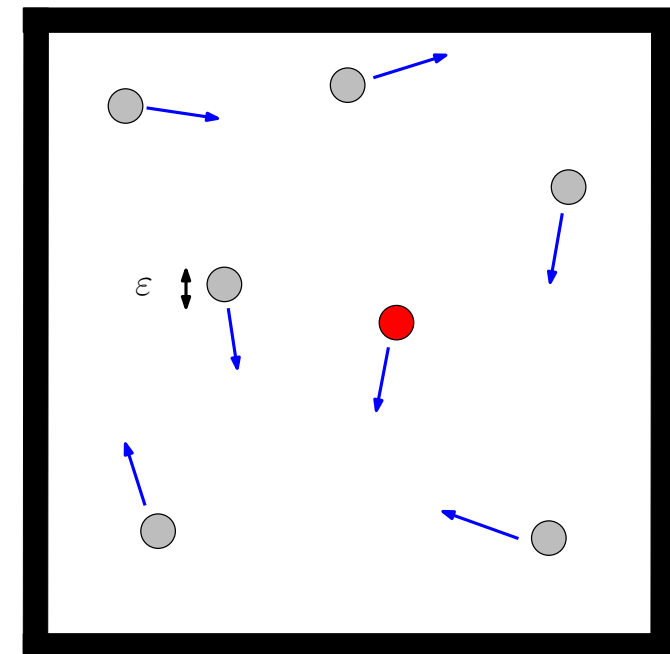
Equilibrium distribution

$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp \left(-\frac{\beta}{2} \sum_{i=1}^N |v_i|^2 \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Particle $Z_1 = (x_1, v_1)$ is tagged. Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \rho^0(x_1)$$

Uniform bound: $\rho^0(x_1) \leq \mu$



Notation: Marginals

$$t \geq 0, \forall s \geq 1, \quad f_N^{(s)}(t, Z_s) = \iint f_N(t, Z_N) dz_{s+1} \dots dz_N$$

Tagged particle distribution $f_N^{(1)}(t, (x_1, v_1))$

Limiting stochastic process

single particle dynamics

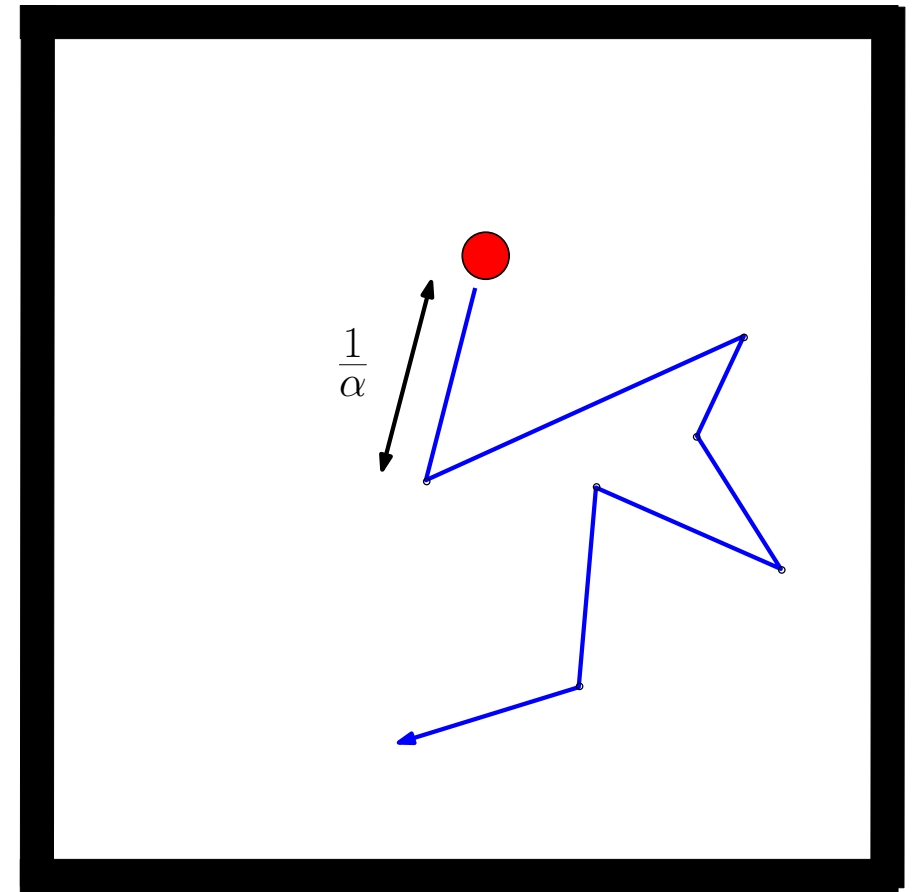
Position : $x(t) = \int_0^t v(u) du$

Markov process on the velocities

$\{v(t)\}_{t \geq 0}$ with generator αL

$$Lg(v) := \iint [g(v) - g(v')] \left((v - v_1) \cdot \nu \right)_+ M_\beta(v_1) dv_1 d\nu$$

$$v' = v + (\nu \cdot (v_1 - v)) \nu, \quad v'_1 = v_1 - (\nu \cdot (v_1 - v)) \nu$$



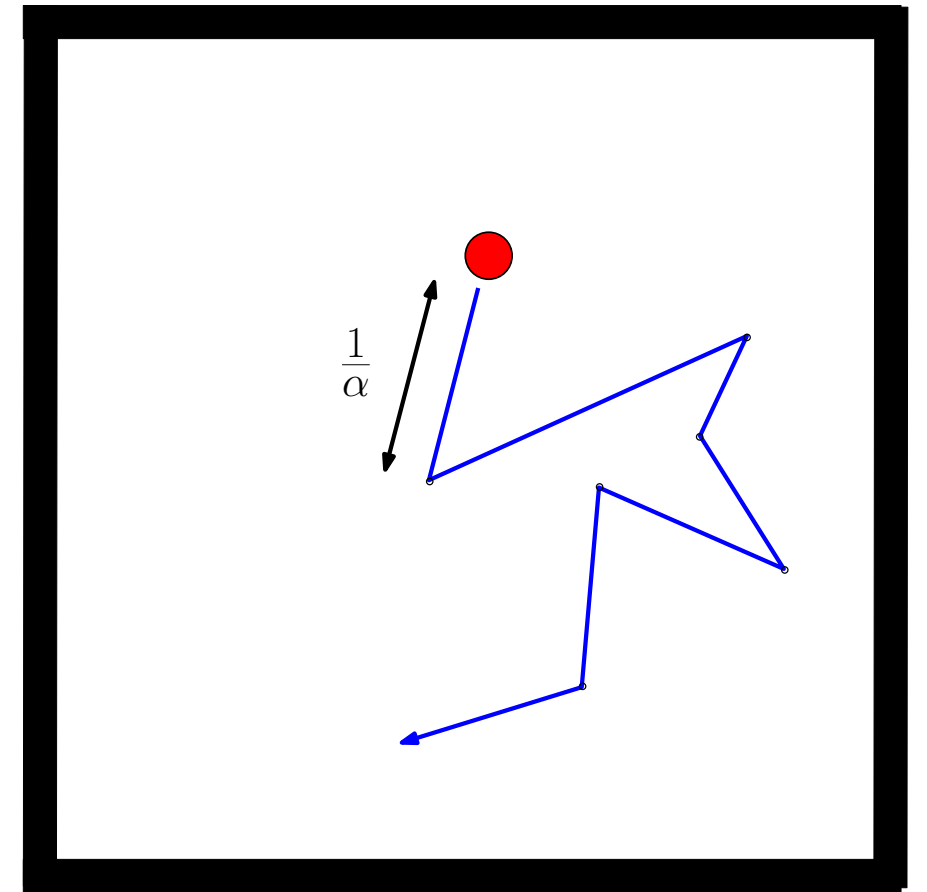
Limiting stochastic process

single particle dynamics

Position : $x(t) = \int_0^t v(u) du$

Markov process on the velocities

$\{v(t)\}_{t \geq 0}$ with generator αL



Particle distribution $M_\beta(v)\varphi_\alpha(x, v, t)$ follows the

Linear Boltzmann equation

$$\partial_t \varphi + v \cdot \nabla_x \varphi = -\alpha L \varphi$$

Probabilist approaches :

Tanaka, Sznitman, Méléard, Graham, Fournier ...

[van Beijeren, Lanford, Lebowitz, Spohn]

N particle
system

$$f_N^{(1)}(x_1, v_1, t)$$

$$\alpha = N \varepsilon^{d-1}$$

$$N \rightarrow \infty$$

$$t > 0$$



Linear Boltzmann
equation

$$\varphi_\alpha(x_1, v_1, t) M_\beta(v_1)$$

[van Beijeren, Lanford, Lebowitz, Spohn]

N particle
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$$f_N^{(1)}(x_1, v_1, t)$$

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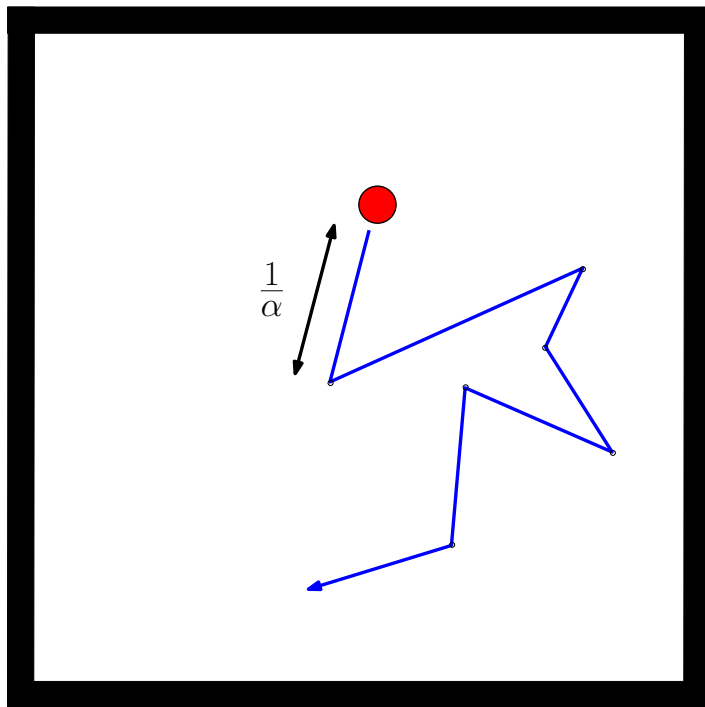
$$N \rightarrow \infty$$

$$t > 0$$



Linear Boltzmann
equation

$$\varphi_\alpha(x_1, v_1, t) M_\beta(v_1)$$



Large time asymptotic

$$t = \alpha \tau$$

$$\alpha \rightarrow \infty$$



Heat equation

Brownian motion

[van Beijeren, Lanford, Lebowitz, Spohn]

N particle
system

$$f_N^{(1)}(x_1, v_1, t)$$

$$\alpha = N \varepsilon^{d-1}$$

$$N \rightarrow \infty$$

$$t > 0$$

Linear Boltzmann
equation

$$\varphi_\alpha(x_1, v_1, t) M_\beta(v_1)$$

$$N \rightarrow \infty$$

$$\alpha = \sqrt{\log \log N}$$

$$t = \alpha \tau$$

$$\alpha \rightarrow \infty$$

Heat equation

Brownian motion

Large time asymptotic

Convergence to the Brownian motion

Rescaled position of the tagged particle

$$\chi(\tau) = x_1(\alpha\tau) \quad \text{with} \quad \alpha = \sqrt{\log \log N}$$

Initial data $f_N^0(Z_N) = M_{N,\beta}(Z_N) \rho^0(x_1)$

Theorem [B.,Gallagher, Saint-Raymond]

χ converges weakly to a brownian motion with variance κ_β

The distribution of the tagged particle $f_N^{(1)}(x_1, v_1, \alpha\tau)$

converges as $N \rightarrow \infty$ to $M_\beta(v_1) \rho(x_1, \tau)$

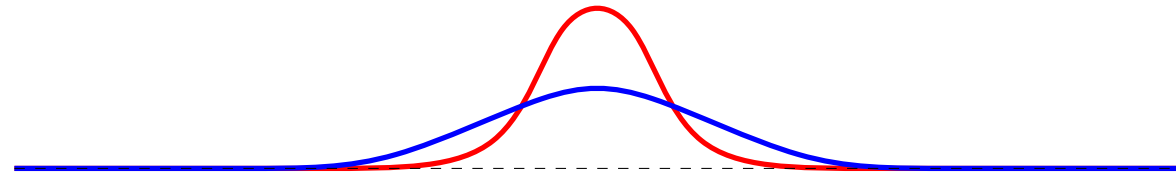
$$\partial_\tau \rho = \kappa_\beta \Delta_x \rho \quad \text{on } \mathbb{R}^+ \times [0, 1]^d, \quad \rho|_{\tau=0} = \rho^0$$

Quantum brownian motion: [Erdős,Salmhofer,Yau]

Lorentz gas: [Bunimovich,Sinai], [Basile,Nota,Pezzotti,Pulvirenti]

Linearized Boltzmann equation

Response to a small perturbation

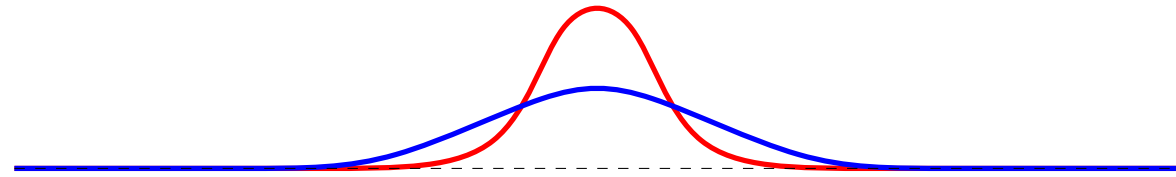


$$(\partial_t + v \cdot \nabla_x)g = -\alpha \mathcal{L} g,$$

$$\mathcal{L} g(v) := \int M_\beta(v_1) \left(g(v) + g(v_1) - g(v') - g(v'_1) \right) \left((v_1 - v) \cdot \nu \right)_+ d\nu dv_1$$

Linearized Boltzmann equation

Response to a small perturbation



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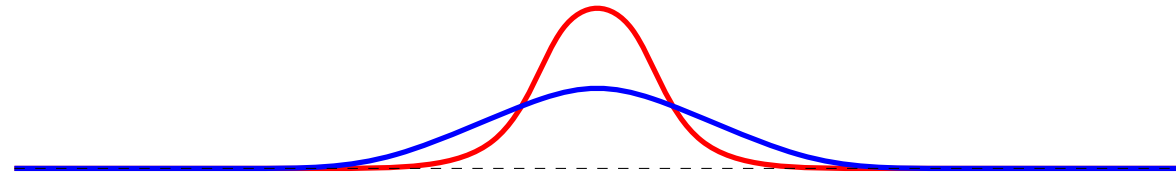
Background

Linear Boltzmann equation

$$Lg(v) := \int [g(v) - g(v')] \left((v - v_1) \cdot \nu \right)_+ M_\beta(v_1) dv_1 d\nu$$

Linearized Boltzmann equation

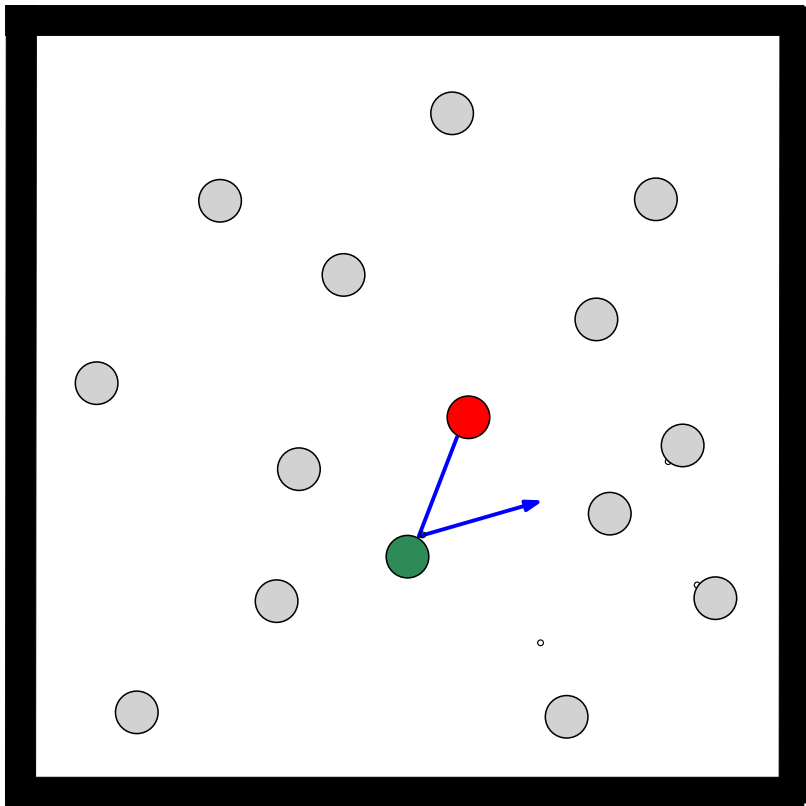
Response to a small perturbation



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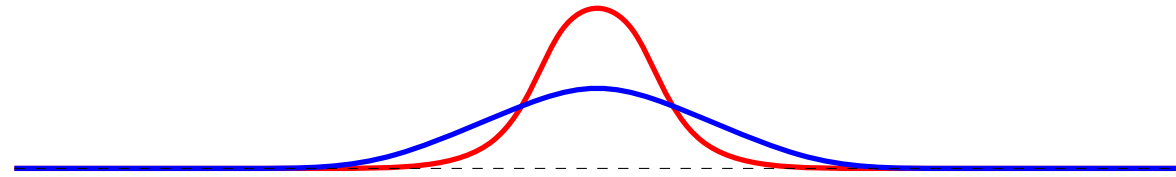
Tagged particle



- perturbation of the tagged particle

Linearized Boltzmann equation

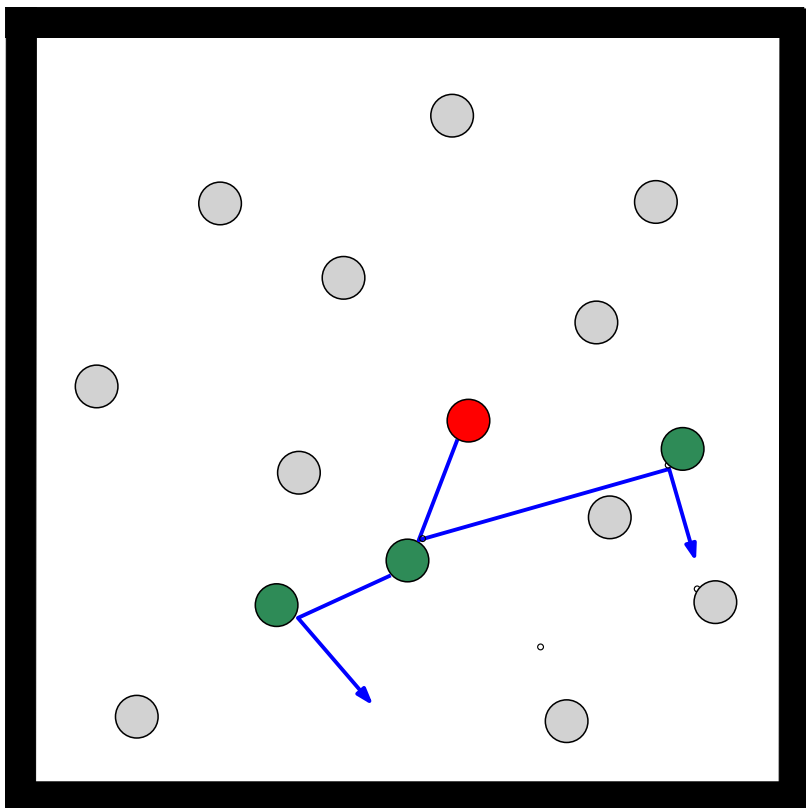
Response to a small perturbation



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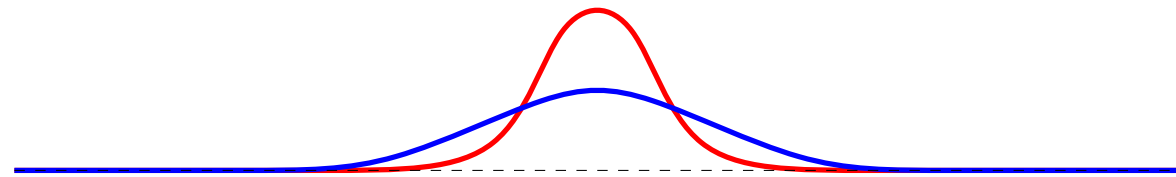
Tagged particle



- perturbation of the tagged particle
- perturbation of the background

Linearized Boltzmann equation

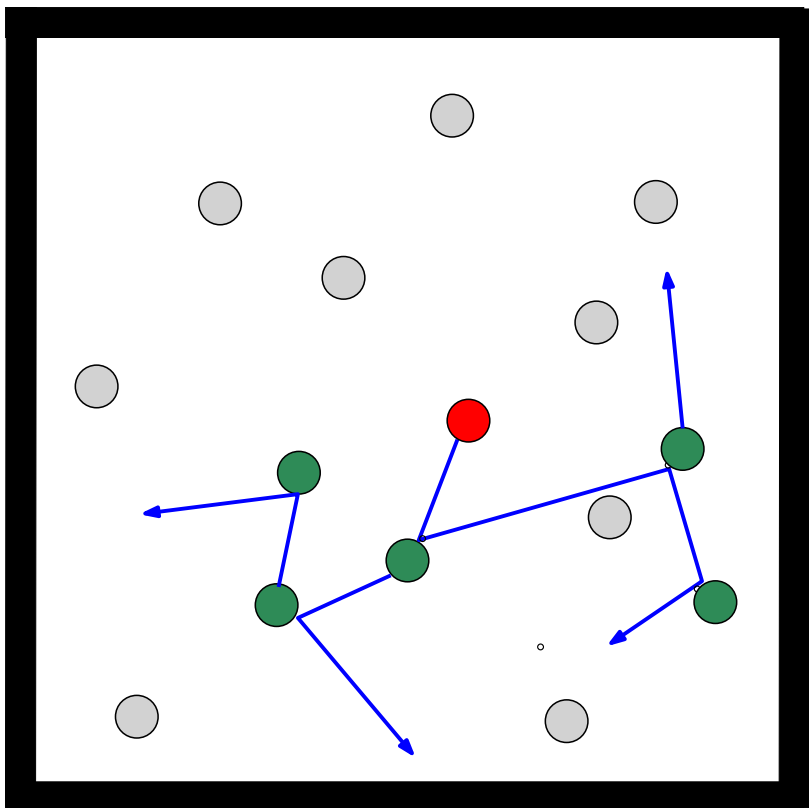
Response to a small perturbation



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Tagged particle



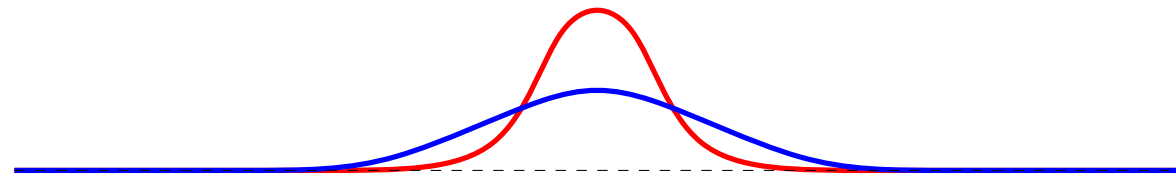
A cloud of particles is modified.

On averaged the distribution of each background particle changes by an order :

$$O\left(\frac{\alpha t}{N}\right)$$

Linearized Boltzmann equation

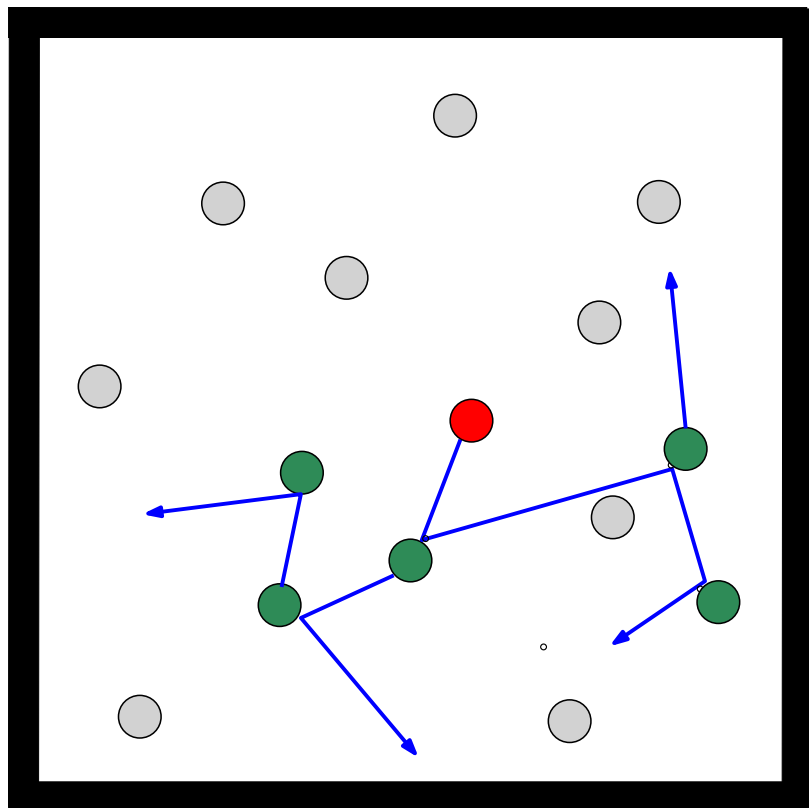
Response to a small perturbation



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Tagged particle



A cloud of particles is modified.

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$$O\left(\frac{\alpha t}{N}\right)$$

Goal: Capture corrections $\simeq \frac{1}{N}$

Linearized Boltzmann equation

Perturbation of order 1

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) g_0(z_1) \longrightarrow \text{corrections of order } \simeq \frac{1}{N}$$

Perturbation of order N (symmetric version)

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \left(\sum_{i=1}^N g_0(z_i) \right) \longrightarrow \text{corrections of order } \simeq 1$$

with $\int M_\beta(v) g_0(z) dz = 0$

Question. Large time behavior of $f_N^{(1)}(t, z_1)$

N particle
system

$$f_N^{(1)}(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

$$\alpha = N \varepsilon^{d-1}$$

$$N \rightarrow \infty$$



Linearized Boltzmann
equation

$$g_\alpha(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

[van Beijeren, Lanford, Lebowitz, Spohn] (short time)

N particle
system

$$f_N^{(1)}(x_1, v_1, t)$$

$$\alpha = N\varepsilon^{d-1}$$
$$N \rightarrow \infty$$

Linearized Boltzmann
equation

$$g_\alpha(x_1, v_1, t)$$

$$\alpha \rightarrow \infty$$

[Bardos, Golse, Levermore]

Initially :

$$g(0, x, v) := \rho_0(x) + u_0(x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta_0(x)$$

density
fluctuation

momentum
fluctuation

energy
fluctuation

N particle
system

$$f_N^{(1)}(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

$$\alpha = N\varepsilon^{d-1}$$
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Linearized Boltzmann
equation

$$g_\alpha(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

$$\alpha \rightarrow \infty$$

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$$g(0, x, v) := \rho_0(x) + u_0(x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta_0(x)$$

$$g(t, x, v) := \rho(t, x) + u(t, x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta(t, x)$$

Acoustic equations

$$\begin{cases} \partial_t \rho + \nabla_x \cdot u = 0 \\ \partial_t u + \nabla_x (\rho + \theta) = 0 \\ \partial_t \theta + \nabla_x \cdot u = 0 \end{cases}$$

N particle
system

$$f_N^{(1)}(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

$$\alpha = N\varepsilon$$
$$N \rightarrow \infty$$



Linearized Boltzmann
equation

$$g_\alpha(\boldsymbol{x}_1, \boldsymbol{v}_1, t)$$

Theorem [BGSR]

For $d = 2$, convergence for any $t > 0$

Initially :

$$g(0, x, v) := \rho_0(x) + u_0(x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta_0(x)$$

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Acoustic equations

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N particle
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$$f_N^{(1)}(x_1, v_1, t)$$

$$\alpha = N\varepsilon$$
$$N \rightarrow \infty$$

Linearized Boltzmann
equation

$$g_\alpha(x_1, v_1, t)$$

Theorem [BGSR]

For $d = 2$, convergence for any $t > 0$

$$\alpha \ll \sqrt{\log \log \log N}$$
$$\rightarrow \infty$$

Initially :

$$g(0, x, v) := \rho_0(x) + u_0(x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta_0(x)$$

$$g(t, x, v) := \rho(t, x) + u(t, x) \cdot v + \frac{\beta|v|^2 - d}{2} \theta(t, x)$$

Acoustic equations

$$\begin{cases} \partial_t \rho + \nabla_x \cdot u = 0 \\ \partial_t u + \nabla_x (\rho + \theta) = 0 \\ \partial_t \theta + \nabla_x \cdot u = 0 \end{cases}$$

Derivation of the linear Boltzmann equation

Step 1. Control of the collision operators

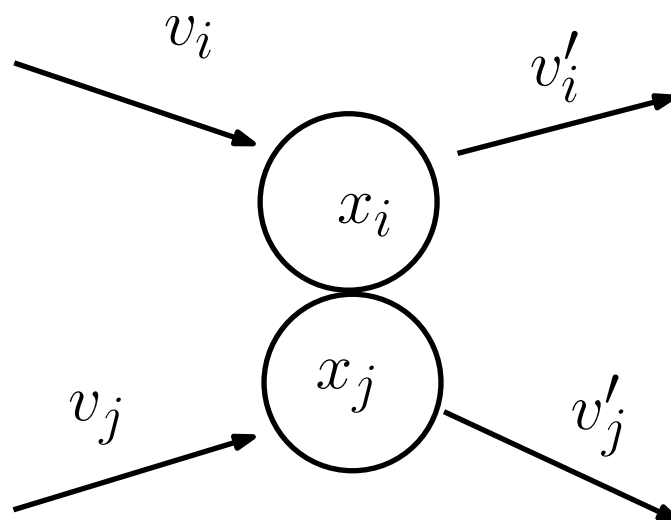
BBGKY hierarchy for the marginals

Evolution of the first marginal

$$(\partial_t + v_1 \cdot \nabla_{x_1}) f_N^{(1)}(t, z_1) = \alpha(C_{1,2} f_N^{(2)})(t, z_1)$$

Collision operator

$$\begin{aligned} (C_{1,2} f_N^{(2)})(z_1) &:= \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v'_1, x_1 + \varepsilon \nu, v'_2) \left((v_2 - v_1) \cdot \nu \right)_+ d\nu dv_2 \\ &\quad - \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon \nu, v_2) \left((v_2 - v_1) \cdot \nu \right)_- d\nu dv_2 \end{aligned}$$



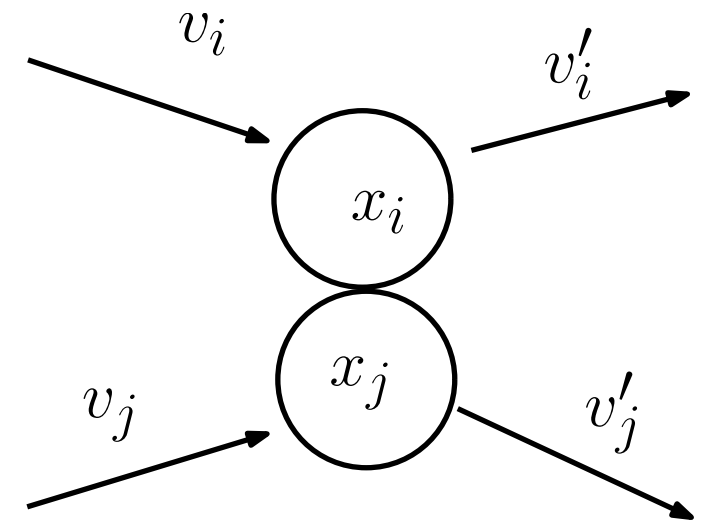
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Collision operator

$$\begin{aligned} (C_{1,2} f_N^{(2)})(z_1) &:= \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v'_1, x_1 + \varepsilon \nu, v'_2) \left((v_2 - v_1) \cdot \nu \right)_+ d\nu dv_2 \\ &\quad - \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(2)}(x_1, v_1, x_1 + \varepsilon \nu, v_2) \left((v_2 - v_1) \cdot \nu \right)_- d\nu dv_2 \end{aligned}$$



Hope : Propagation of chaos

$$f_N^{(2)}(x_1, v_1, x_1 + \varepsilon \nu, v_2) \simeq f_N^{(1)}(x_1, v_1) f_N^{(1)}(x_1 + \varepsilon \nu, v_2)$$

Consequence: Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \iint [f(v') f(v'_1) - f(v) f(v_1)] \left((v - v_1) \cdot \nu \right)_+ dv_1 d\nu$$

BBGKY hierarchy for the marginals

For $s < N$ and on $\mathcal{D}_\varepsilon^s = \{Z_s = (x_i, v_i)_{i \leq s} \mid i \neq j, \quad |x_i - x_j| > \varepsilon\}$

$$(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}) f_N^{(s)}(t, Z_s) = \alpha (C_{s,s+1} f_N^{(s+1)})(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned} & (C_{s,s+1} f_N^{(s+1)})(Z_s) \\ &:= \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^s \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v'_i, \dots, x_i + \varepsilon \nu, v'_{s+1}) \left((v_{s+1} - v_i) \cdot \nu \right)_+ d\nu dv_{s+1} \\ & \quad - \frac{(N-s)\varepsilon^{d-1}}{\alpha} \sum_{i=1}^s \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} f_N^{(s+1)}(\dots, x_i, v_i, \dots, x_i + \varepsilon \nu, v_{s+1}) \left((v_{s+1} - v_i) \cdot \nu \right)_- d\nu dv_{s+1} \end{aligned}$$

where $\mathbf{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

Duhamel formula

Denote by $\mathbf{S}_s$ the semi-group associated to free transport in $\mathcal{D}_\varepsilon^s$

Duhamel Formula

$$f_N^{(1)}(t) = \mathbf{S}_1(t) f_N^{(1)}(0) + \alpha \int_0^t \mathbf{S}_1(t - t_1) C_{1,2} f_N^{(2)}(t_1) dt_1 ,$$

Iterated Duhamel formula

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n Q_{1,1+n}(t) f_N^{(1+n)}(0)$$



Idea : Use the initial randomness

with

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} dt_n \cdots dt_1 \mathbf{S}_s(t - t_1) C_{s,s+1} \\ \mathbf{S}_{s+1}(t_1 - t_2) C_{s+1,s+2} \cdots \mathbf{S}_{s+n}(t_n)$$

Duhamel formula

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Duhamel formula

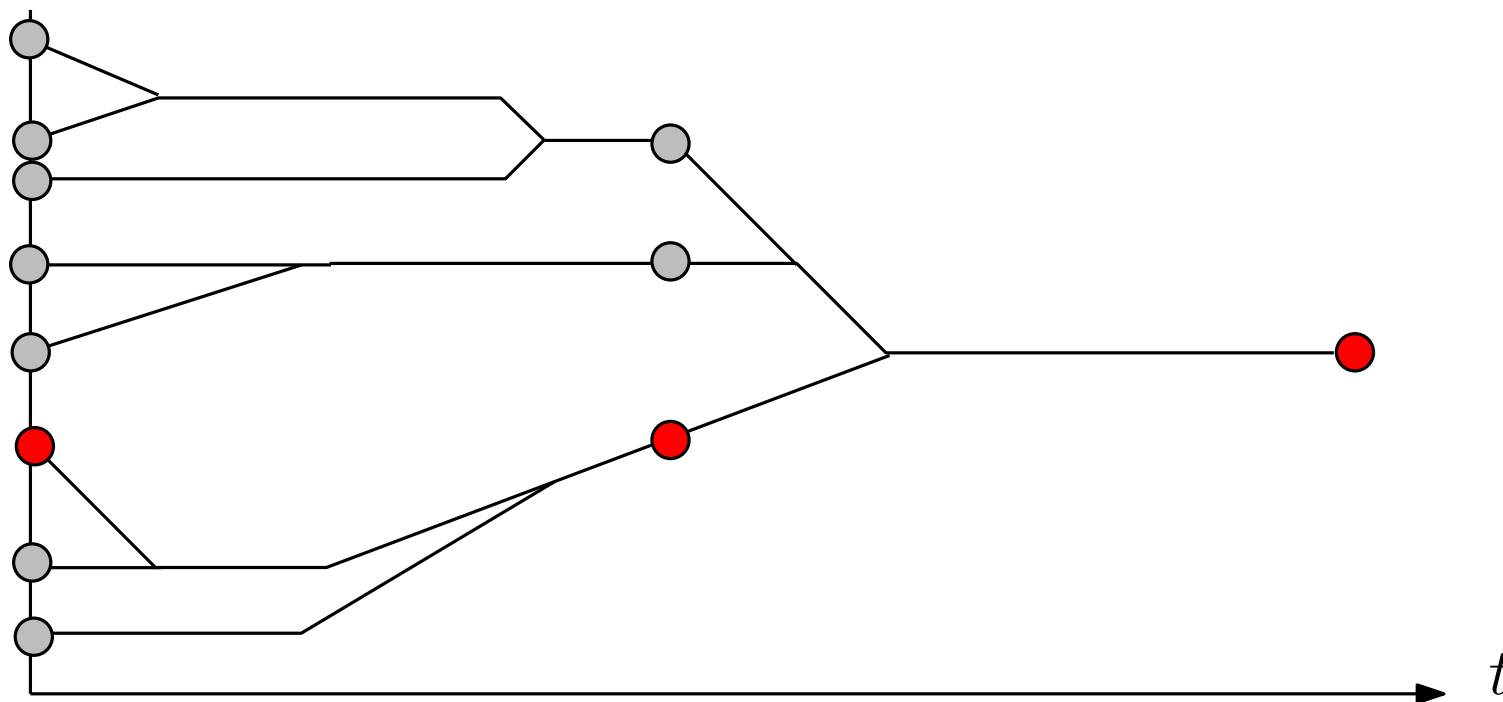
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Interpretation as a collision tree

- Transport operator
- Addition of a particle to the tree after each collision



Issue : convergence of the series when N diverges

$$f_N^{(1)}(t) = \sum_{n=0}^{N-1} \alpha^n Q_{1,1+n}(t) f_N^{(1+n)}(0)$$

Continuity estimates for the collision operators

Weighted norms

$$\|f_k\|_{\varepsilon,k,\beta} := \sup_{Z_k \in \mathcal{D}_\varepsilon^k} \left| f_k(Z_k) \exp \left(\frac{\beta}{2} \sum_{i=1}^k |v_i|^2 \right) \right| < \infty$$

Collision operators estimates

$$\left\| Q_{s,s+n}(t) f_{s+n} \right\|_{\varepsilon,s,\beta/2} \leq e^{s-1} (C_d(\beta)t)^n \|f_{s+n}\|_{\varepsilon,s+n,\beta}$$

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Series is only controlled for short times t and small α

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$\mathbb{L}^\infty$ bound

Initial distribution :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \rho^0(\mathbf{x}_1), \quad \int_{\mathbb{T}_\lambda^d} dx_1 \rho^0(\mathbf{x}_1) = 1$$

Uniform bound: $\rho^0(\mathbf{x}_1) \leq \mu$

The measure $M_{N,\beta}(Z_N)$ is **stationary** thus the maximum principle implies bounds **uniform in time**

For any $s \geq 1$

$$\sup_{t \geq 0} f_N^{(s)}(t, Z_s) \leq \mu M_{N,\beta}^{(s)}(Z_s) \leq \mu (1 - \varepsilon c_d)^{-s} M_\beta^{\otimes s}(V_s)$$

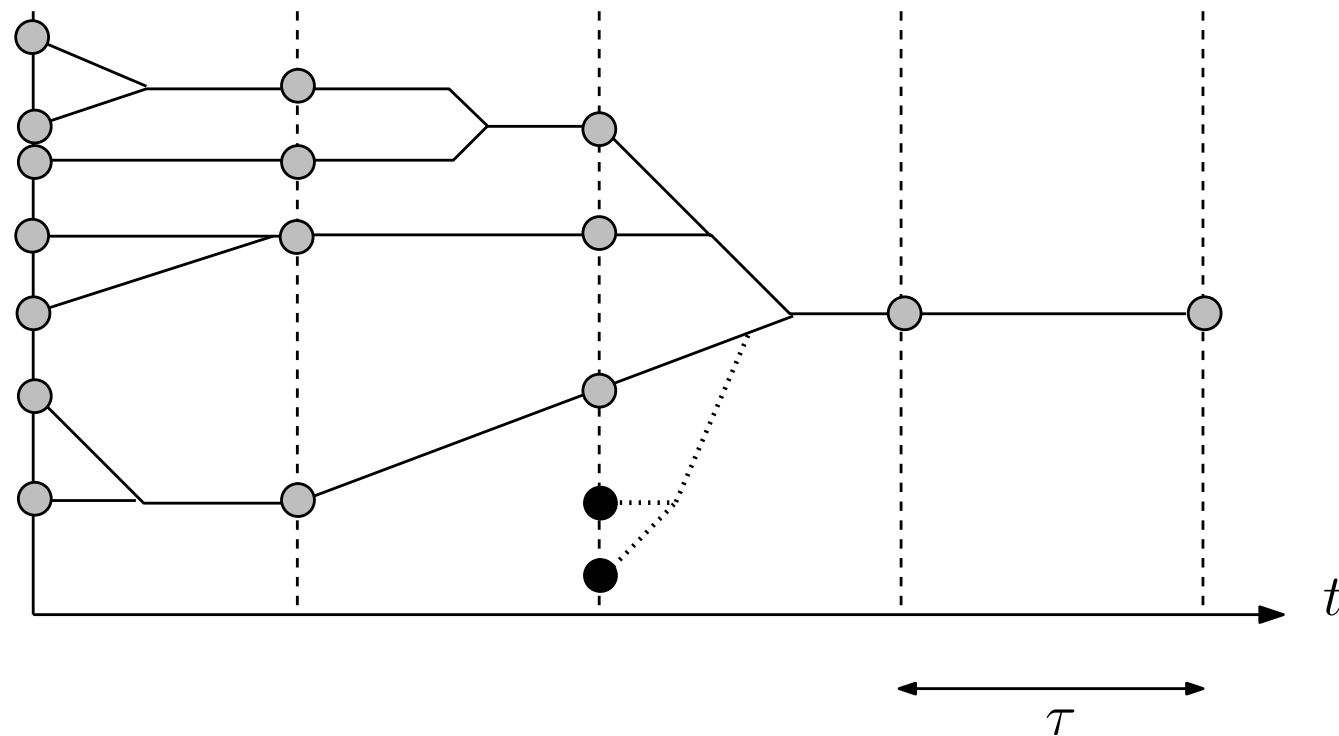
In this way the cancellations in the collision operator are recovered.

Pruning procedure

Decompose : $[0, t] = \bigcup_{k=1}^K [(k-1)\tau, k\tau]$ for some $\tau > 0$

Good collision trees.

Less than $n_k = 2^k$ collisions during $[(K-k)\tau, (K-k+1)\tau]$



In each time interval $[(K-k)\tau, (K-k+1)\tau]$

$$\left\| Q_{s,s+n}(\tau) f_{s+n} \right\|_{\varepsilon, s, \beta/2} \leq e^{s-1} (C_d(\beta)\tau)^n \|f_{s+n}\|_{\varepsilon, s+n, \beta}$$

Pruning procedure

Truncated iterated Duhamel formula:

$$f_N^{(1)}(t) = \sum_{j_1=0}^2 \dots \sum_{j_K=0}^{2^K} \alpha^{J_K-1} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \dots Q_{J_{K-1},J_K}(\tau) f_N^{0(J_K)} + R_N^K(t)$$

with $J_\ell = 1 + j_1 + \dots + j_\ell$

- The main contribution is given by the good collision trees with $j_k \leq 2^k$ during the time interval $[(K-k)\tau, (K-k+1)\tau]$
- The contribution of the large trees $R_N^K(t)$ is controlled

$$\|R_N^K(t)\|_{\mathbb{L}^\infty} \leq \mu \frac{t^2}{K}$$

$\Rightarrow$ If t is large, then K has to be very large and τ very small.

Pruning for the linearized Boltzmann equation

Initial data of order N :

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \left(\sum_{i=1}^N g_0(z_i) \right)$$

No uniform bounds in L^∞ :

$$|f_N^{(s)}(t, Z_s)| \leq N C^s M_\beta^{\otimes s}(Z_s) \|g_0\|_{L^\infty}$$

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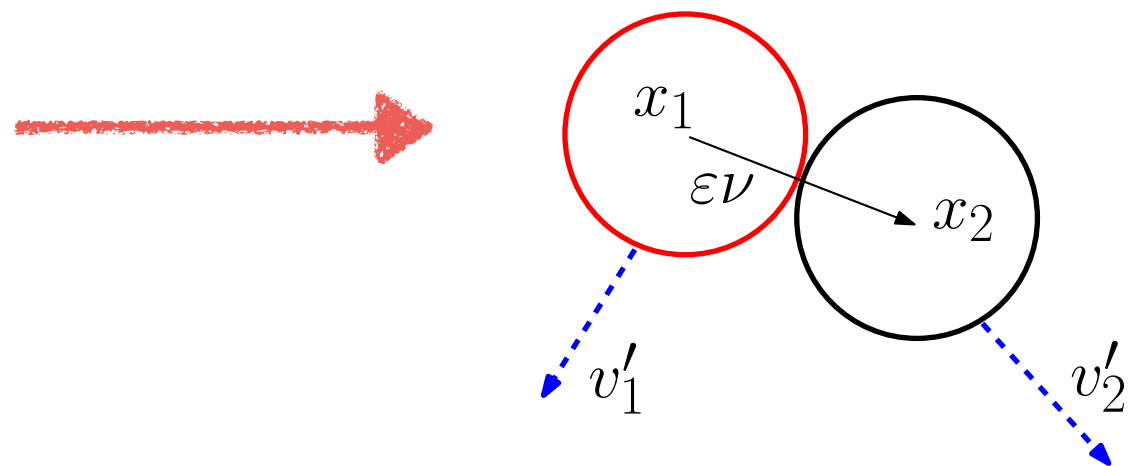
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New strategy L^2 estimates on the collision kernel

$$C_{1,2}^+ f_N^{(2)}(z_1) = \int f_N^{(2)}(x_1, v'_1, x_1 + \varepsilon \nu, v'_2) \left((v_2 - v_1) \cdot \nu \right)_+ d\nu dv_2$$

A dimension is missing
for L^2 estimates



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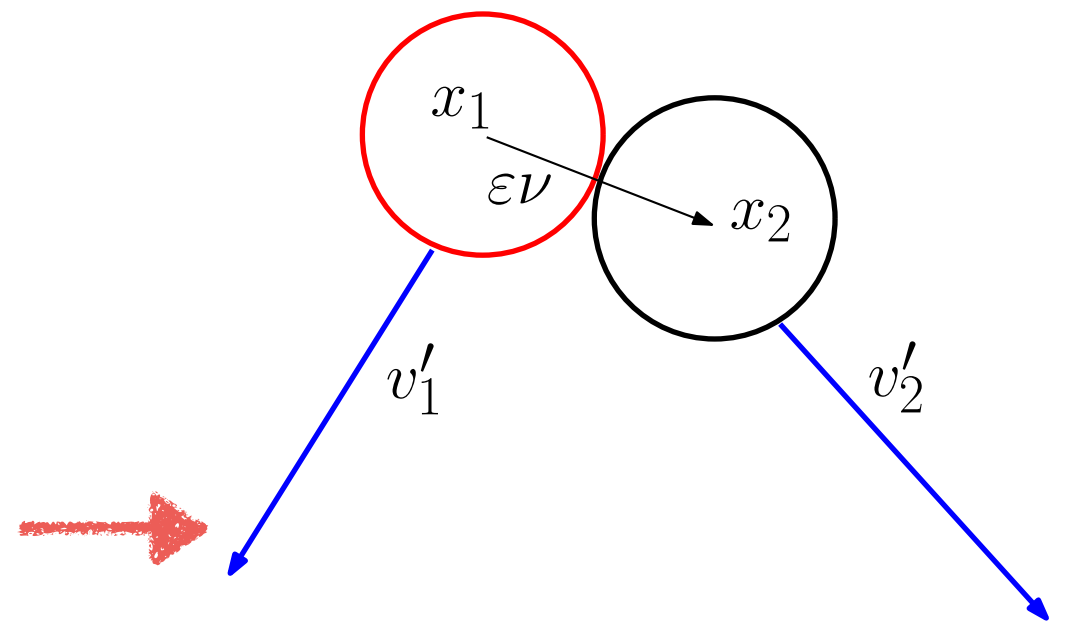
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$$\int_0^T d\tau C_{1,2}^+ \mathbf{S}_2(\tau) f_N^{(2)}$$

Additional time dimension



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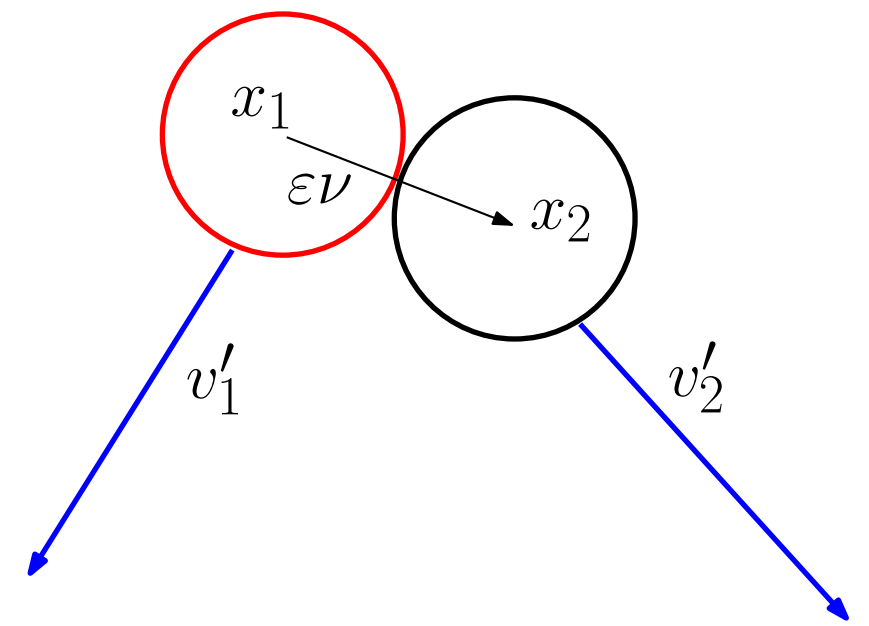
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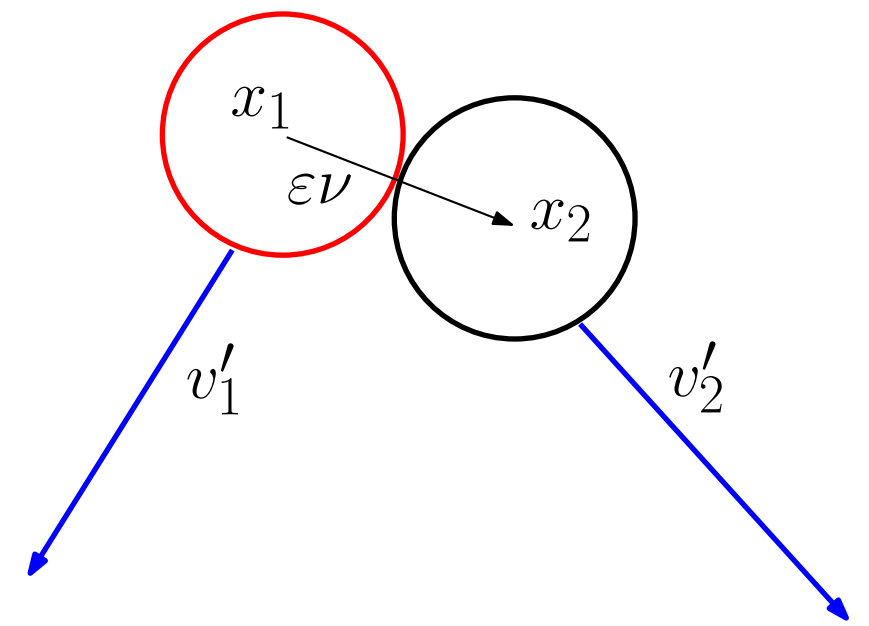
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$$\left| \int dz_1 \int_0^T d\tau C_{1,2}^+ \mathbf{S}_2(\tau) f_N^{(2)} \right| \leq C \sqrt{\frac{T}{\varepsilon}} \|f_N^{(2)}\|_{L^2}$$

bad estimate



Difficulties to estimate the collision kernel

1/ *Divergence of the L^2 estimates*

$$\left| \int dz_1 \int_0^T d\tau C_{1,2}^+ \mathbf{S}_2(\tau) f_N^{(2)} \right| \leq C \sqrt{T} \mathbf{N} \|f_N^{(2)}\|_{L^2}$$

$$\frac{1}{\varepsilon} \int_0^\varepsilon dr \varphi(r) \leq \begin{cases} \|\varphi\|_{L^\infty} \\ \frac{1}{\sqrt{\varepsilon}} \|\varphi\|_{L^2} \end{cases} \quad \text{Singular domain of integration}$$

Difficulties to estimate the collision kernel

1/ Divergence of the L^2 estimates

$$\left| \int dz_1 \int_0^T d\tau C_{1,2}^+ \mathbf{S}_2(\tau) f_N^{(2)} \right| \leq C \sqrt{TN} \|f_N^{(2)}\|_{L^2}$$

$$\frac{1}{\varepsilon} \int_0^\varepsilon dr \varphi(r) \leq \begin{cases} \|\varphi\|_{L^\infty} \\ \frac{1}{\sqrt{\varepsilon}} \|\varphi\|_{L^2} \end{cases} \quad \text{Singular domain of integration}$$

Difficulty to control multiple collisions.

$$Q_{s,s+n}(t) := \int_0^t \int_0^{t_1} \dots \int_0^{t_{n-1}} dt_n \dots dt_1 \mathbf{S}_s(t - t_1) C_{s,s+1} \\ \mathbf{S}_{s+1}(t_1 - t_2) C_{s+1,s+2} \dots \mathbf{S}_{s+n}(t_n)$$

$$\left| Q_{1,1+n} f_N^{(1+n)} \right| \leq C (TN)^{n/2} \|f_N^{(1+n)}\|_{L^2}$$



Disaster !

Difficulties to estimate the collision kernel

L^2 estimates would be fine if

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^s g(t, z_i)$$

Difficulties to estimate the collision kernel

L^2 estimates would be fine if

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{i=1}^s g(t, z_i)$$

Key : L^2 control of the higher order correlations at **any** time

$$f_N^{(s)}(t, Z_s) = M_\beta^{\otimes s}(V_s) \sum_{m=1}^s \sum_{\sigma \in \mathfrak{S}_s^m} g_N^m(t, Z_\sigma)$$

with $\|g_N^m(t)\|_{L_\beta^2} \leq \frac{C}{\sqrt{N^{m-1}}} \|g_{\alpha,0}\|_{L_\beta^2}$  Propagation of the initial bounds

Mild version of local equilibrium

This leads to bounds uniform in time.

Difficulties to estimate the collision kernel

2/ Recollisions

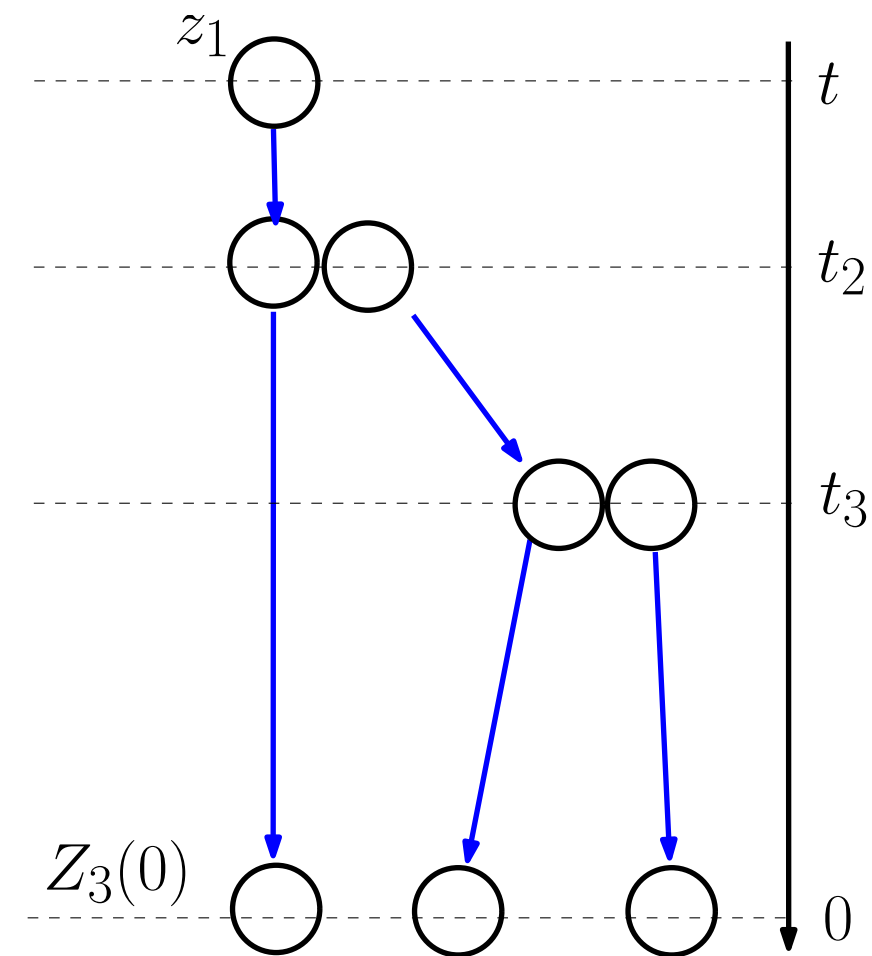
Given a collision tree :

$$\int dz_1 \int_0^t dt_2 \int_0^{t_2} dt_3 \mathbf{S}_1(t - t_1) C_{1,2}^+ \mathbf{S}_2(t_2 - t_3) C_{1,2}^+ \mathbf{S}_3(t_3) f_N^{(3)}(Z_3(0))$$

Use the change of variables

$$(z_1, (t_2, \nu_2, v_2), (t_3, \nu_3, v_3)) \rightarrow Z_3(0)$$

to recover $\|f_N^{(3)}\|_{L_1}$



Difficulties to estimate the collision kernel

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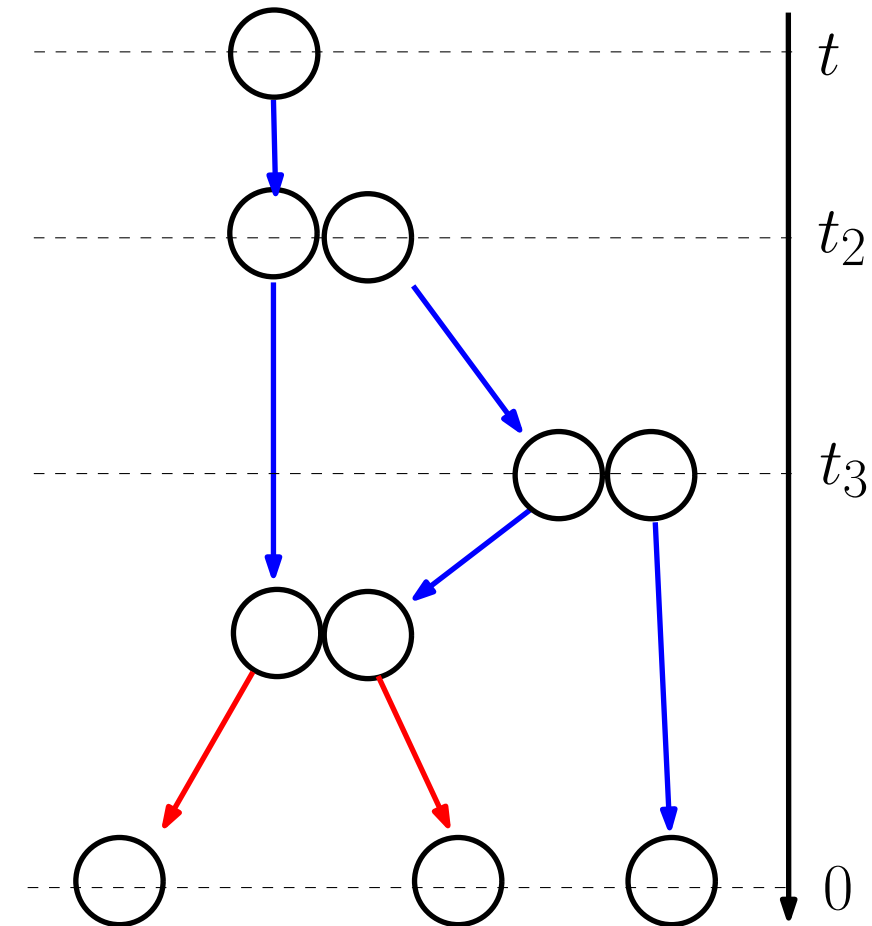
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to recover $\|f_N^{(3)}\|_{L_1}$

Problem. This mapping is not bijective

One has to control the recollisions.



Derivation of the linear Boltzmann equation

*Step 2. Comparison with the
Boltzmann hierarchy*

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}) g^{(s)}(t, Z_s) = \alpha(C_{s,s+1}^0) g^{(s+1)}(t, Z_s)$$

where the collision term is defined by

$$\begin{aligned} & (C_{s,s+1}^0 g^{(s+1)})(Z_s) \\ &:= \cancel{\frac{(N+1)^d}{\alpha}} \sum_{i=1}^s \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i^*, \dots, x_i \cancel{+ \epsilon v}, v_{s+1}^*) \left((v_{s+1} - v_i) \cdot \nu \right)_+ d\nu dv_{s+1} \\ & \quad - \cancel{\frac{(N+1)^d}{\alpha}} \sum_{i=1}^s \int_{\mathbf{S}^{d-1} \times \mathbb{R}^d} g^{(s+1)}(\dots, x_i, v_i, \dots, x_i \cancel{+ \epsilon v}, v_{s+1}) \left((v_{s+1} - v_i) \cdot \nu \right)_- d\nu dv_{s+1} \end{aligned}$$

This is the **limit** hierarchy when $\epsilon \rightarrow 0$ and $N \rightarrow \infty$.

Boltzmann hierarchy

For $s \geq 1$ and $Z_s \in \mathbf{T}^{ds} \times \mathbb{R}^{ds}$

$$\left(\partial_t + \sum_{i=1}^s v_i \cdot \nabla_{x_i}\right) g^{(s)}(t, Z_s) = \alpha \left(C_{s,s+1}^0\right) g^{(s+1)}(t, Z_s)$$

Iterated Duhamel formula

$$g^{(1)}(t) = \sum_{n=0}^{\infty} \alpha^n Q_{1,1+n}^0(t) g^{(1+n)}(0)$$

Explicit solution :
$$g^{(s)}(t) = g(t, \mathbf{z}_1) \prod_{i=2}^s M_{\beta}(v_i)$$

with $g(t, \mathbf{z}_1) = \varphi_{\alpha}(t, \mathbf{z}_1) M_{\beta}(v_1)$ solution of the **Linear Boltzman equation**

$$\partial_t \varphi_{\alpha} + v \cdot \nabla_x \varphi_{\alpha} = -\alpha L \varphi_{\alpha}$$

and $M_{\beta}(v) := \left(\frac{\beta}{2\pi}\right)^{\frac{d}{2}} \exp\left(-\frac{\beta}{2}|v|^2\right)$

Comparing the BBGKY and Boltzmann hierarchies

As $N \rightarrow \infty$ in the scaling $N\varepsilon^{d-1} = \alpha$,

$$\left| \left(f_N^{0(s)} - g^{0(s)} \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon} \right| \leq C^s \varepsilon \alpha \mu M_\beta^{\otimes s}$$

for the **initial distributions**

$$f_N^0(Z_N) = M_{N,\beta}(Z_N) \rho^0(\mathbf{x}_1), \quad g^{0(s)}(Z_s) = \left(\prod_{i=1}^s M_\beta(v_i) \right) \rho^0(\mathbf{x}_1)$$

and
$$M_{N,\beta}(Z_N) = \frac{1}{\mathcal{Z}_{N,\beta}} \exp \left(-\frac{\beta}{2} \sum_{i=1}^N |v_i|^2 \right) \prod_{i \neq j} 1_{|x_i - x_j| > \varepsilon}$$

Main Goal

$$\| f_N^{(1)} - g^{(1)} \|_{L^\infty([0,t] \times \mathbf{T}^d \times \mathbb{R}^d)} \rightarrow 0, \quad \text{as } N \rightarrow \infty$$

Comparing the truncated hierarchies

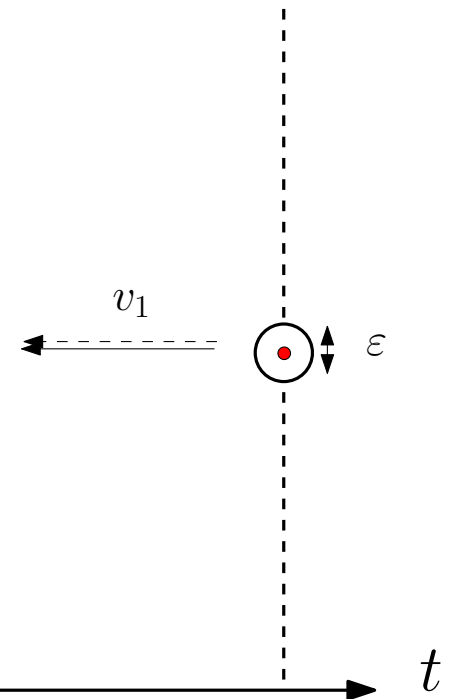
$$f_N^{(1,K)}(t) = \sum_{j_1=0}^2 \cdots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}(\tau) Q_{J_1,J_2}(\tau) \cdots Q_{J_{K-1},J_K}(\tau) f_N^{0(J_K)}$$

$$g^{(1,K)}(t) = \sum_{j_1=0}^2 \cdots \sum_{j_K=0}^{2^K} \alpha^{J_K} Q_{1,J_1}^0(\tau) Q_{J_1,J_2}^0(\tau) \cdots Q_{J_{K-1},J_K}^0(\tau) g^{0(J_K)}$$

Geometric interpretation of the collisions operators:

Backward dynamics

Coupling the
hierarchies



Comparing the truncated hierarchies

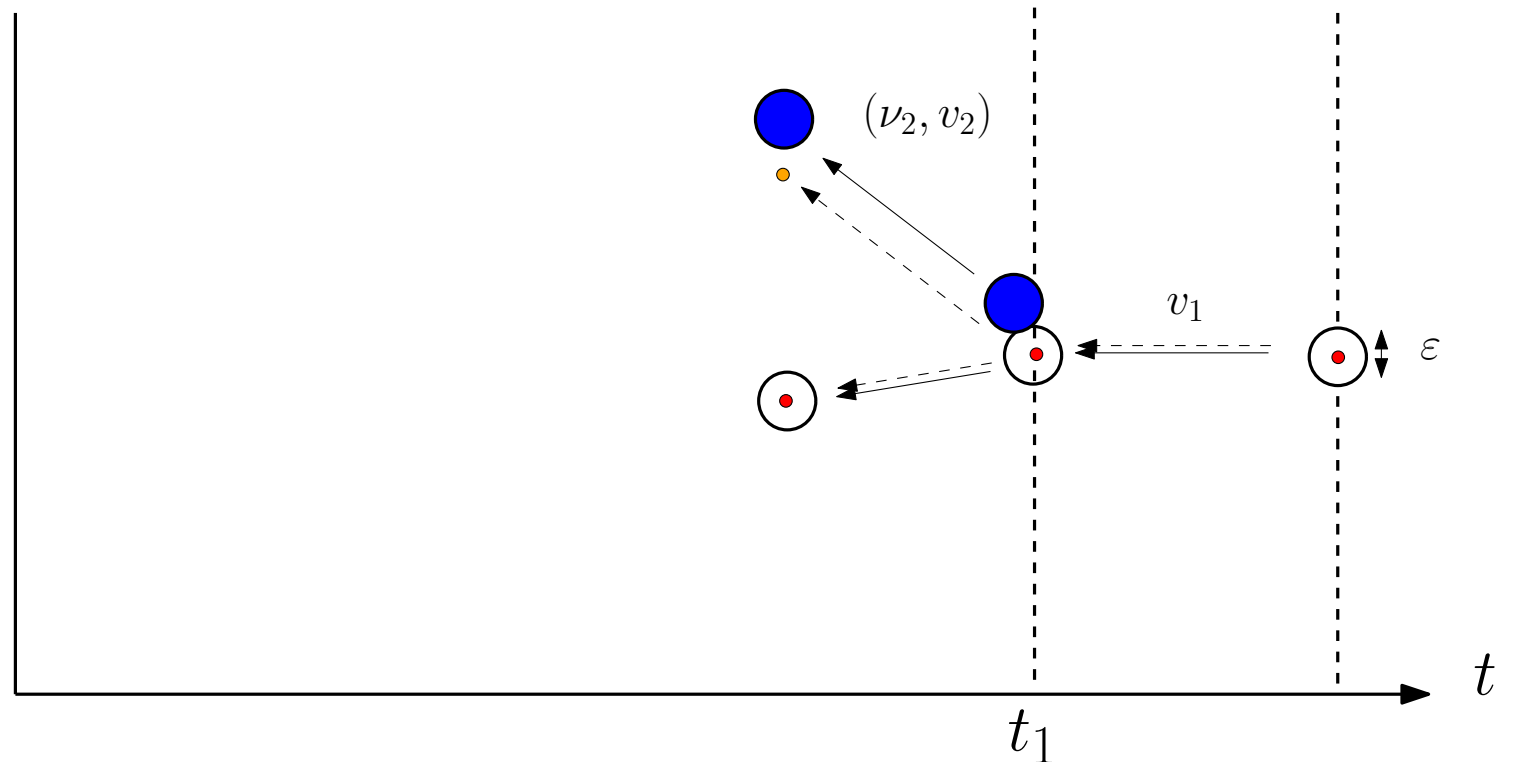
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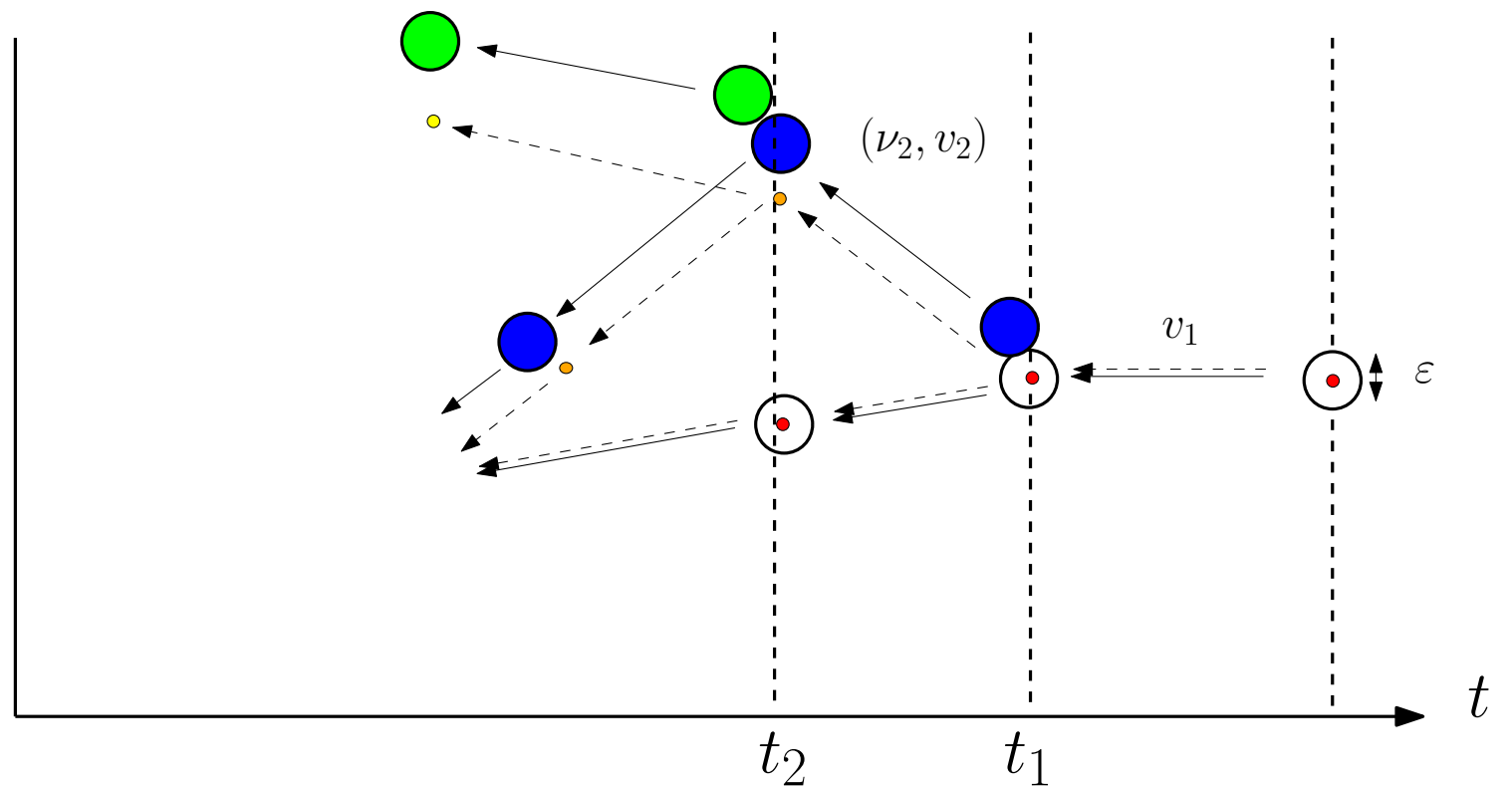
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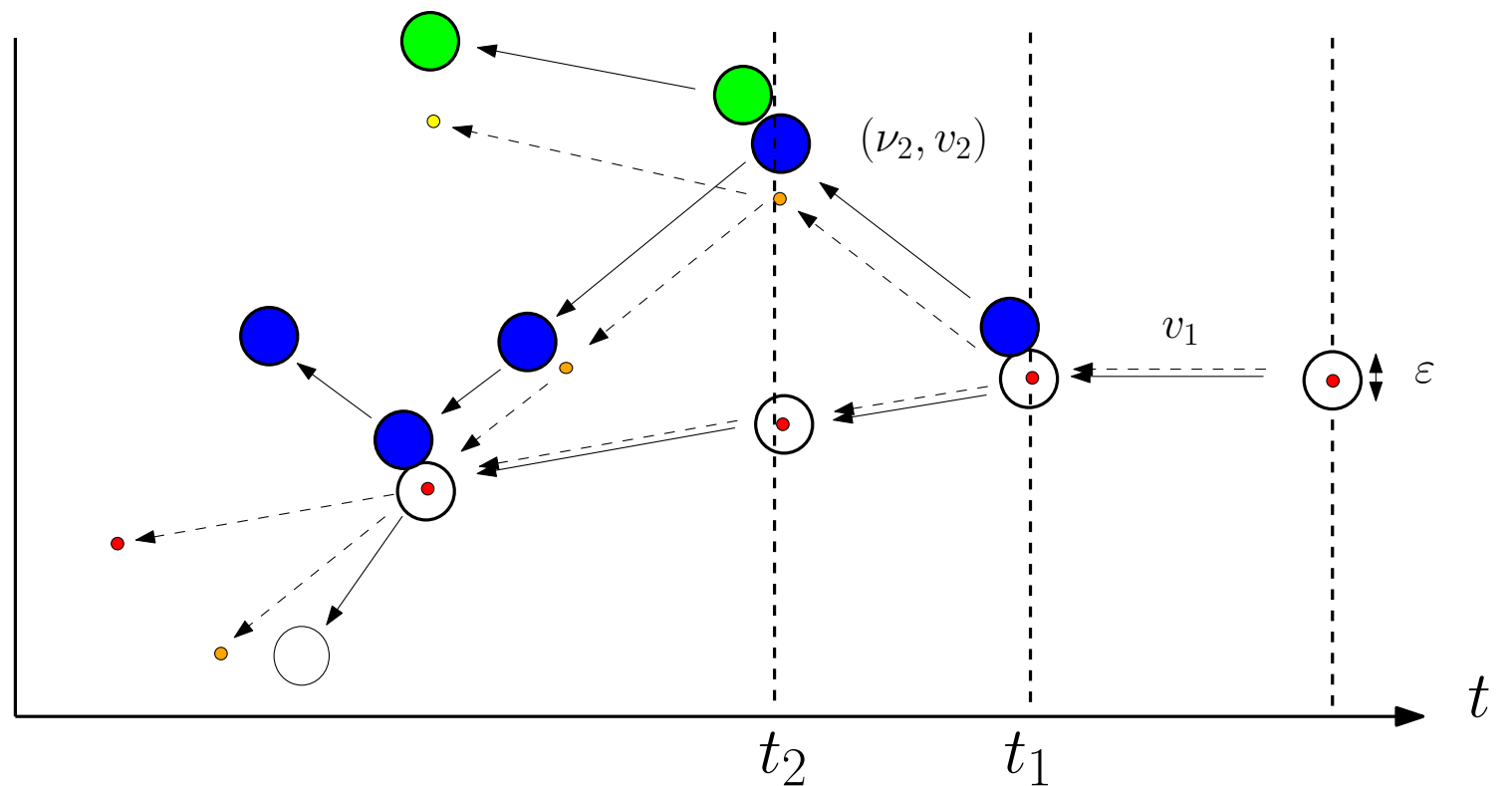
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Geometric interpretation of the collisions operators:

Backward dynamics

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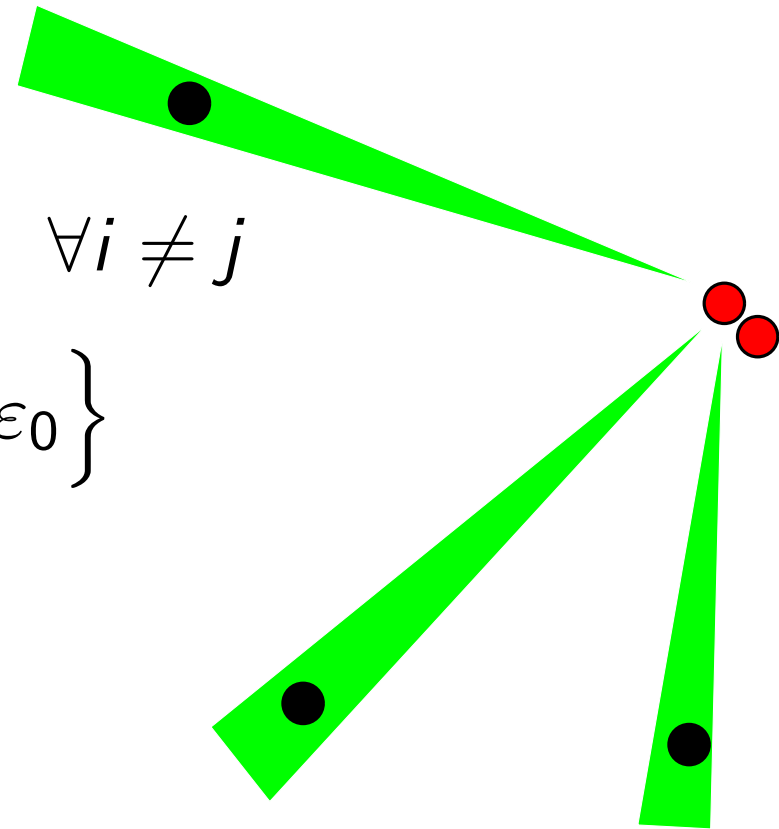
Removing the collisions

BBGKY and Boltzmann trajectories can be coupled if there are no recollisions

- Truncating high velocities
- Truncating collisions in short time intervals
- Recursive construction of the **good trajectories**

$$\mathcal{G}_k(\varepsilon_0) = \left\{ Z_k \in \mathbf{T}_\lambda^{dk} \times \mathbb{R}^{dk} / \forall s \in [0, t], \quad \forall i \neq j \right. \\ \left. d(x_i - sv_i, x_j - sv_j) \geq \varepsilon_0 \right\}$$

Up to a small set of velocities, the system is stable by addition of the $k + 1$ particle



Removing the collisions

Choosing the velocities such that the particles in both hierarchies remain at distance less than $2^K \varepsilon$ and that there are no recollisions. This boils down to remove a set of velocities with small probability.

Quantitative controls : [Gallagher, Saint-Raymond, Texier]

$$\text{Error} \leq (Ct)^{\mathcal{N}_t} \varepsilon \quad \text{with} \quad \mathcal{N}_t \quad \text{particles in the tree at time } t$$

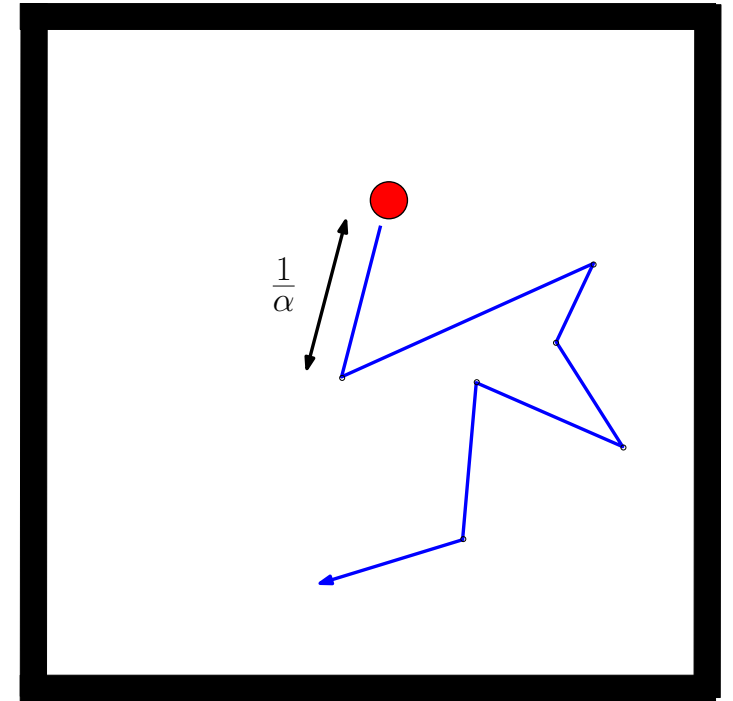
Estimates are valid up to times : $t_N = o\left(\sqrt{\log \log(N)}\right)$

$$\left\| f_N^{(1)} - g^{(1)} \right\|_{L^\infty([0, t_N] \times \mathbf{T} \times \mathbb{R}^d)} \leq C \mu \left(\frac{t_N^2}{\log \log N} \right)^2$$

Coupling both hierarchies

Position : $x_1^0(t) = \int_0^t v(u) du$

Markov process on the velocities
 $\{v(t)\}_{t \geq 0}$ with generator $\alpha \mathcal{L}$



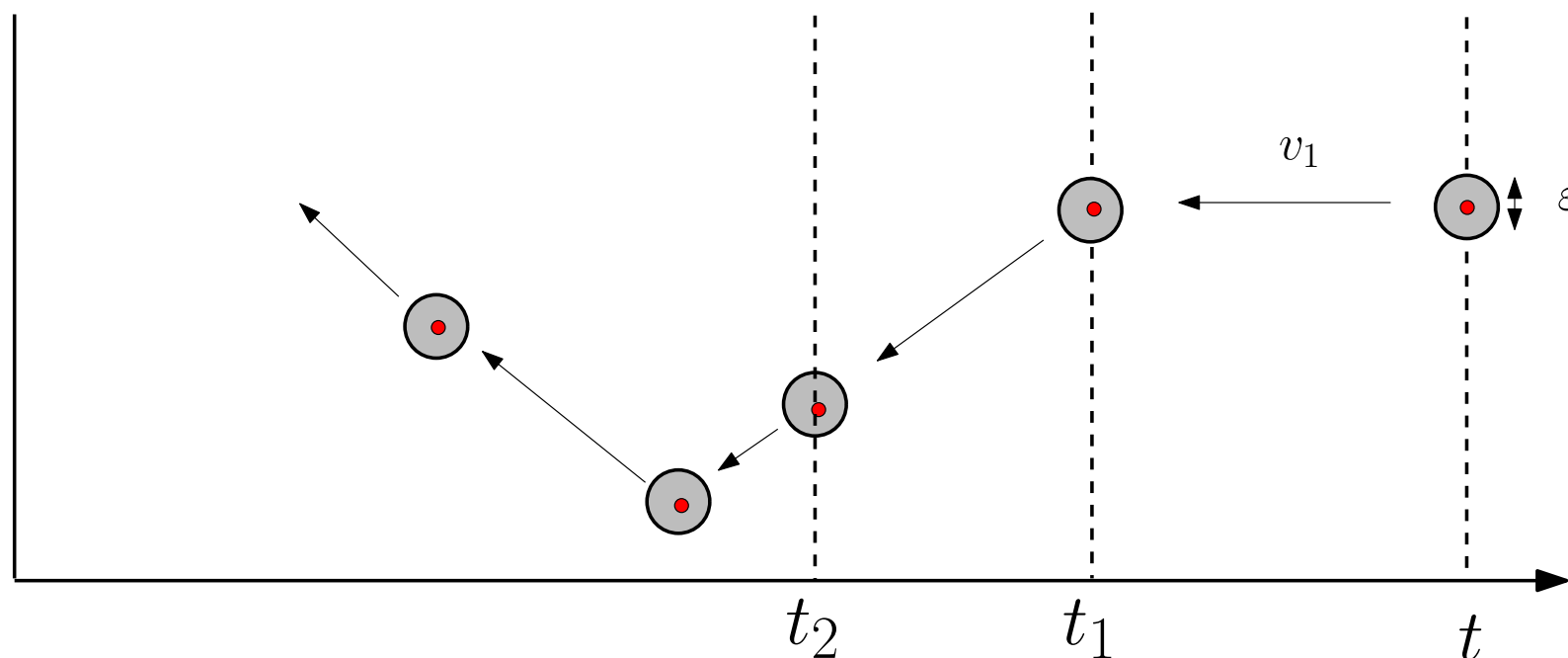
$$\begin{cases} \mathcal{L}g(v) := \iint M_\beta(v_1)[g(v') - g(v)] ((v - v_1) \cdot v)_+ dv_1 dv, \\ v' = v + (v \cdot (v_1 - v))v \quad v'_1 = v_1 - (v \cdot (v_1 - v))v \end{cases}$$

Central limit Theorem for additive functionals of Markov chains
($\mathcal{L}$ has a spectral gap)

$$\lim_{\alpha \rightarrow \infty} \mathbb{E} \left(h(x_1^0(\alpha \tau)) \right) = \mathbb{E} \left(h(B(\tau)) \right)$$

$$\lim_{\alpha \rightarrow \infty} \mathbb{E} \left(h_1(x_1^0(\alpha \tau_1)) \dots h_\ell(x_1^0(\alpha \tau_\ell)) \right) = \mathbb{E} \left(h_1(B(\tau_1)) \dots h_\ell(B(\tau_\ell)) \right)$$

Coupling the trajectories x_1 and x_1^0 to get estimates at different times



The diffusion coefficient κ_β is given by

$$\kappa_\beta = \frac{1}{d} \int_{\mathbb{R}^d} v \mathcal{L}^{-1} v M_\beta(v) dv$$

Conclusion

Deterministic dynamics of a diluted gas of hard sphere:

- Brownian motion
- Linearized Boltzmann equation & acoustic equations

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Open problems.

- Linearized Boltzmann equation in dimension 3
- Stochastic fluctuations [Spohn]
- Understanding the dissipation
- Boltzmann equation for large times