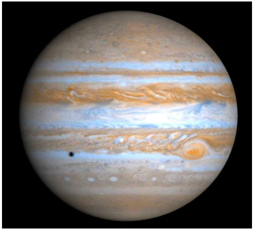
Rare event algorithms and large deviations of turbulent atmosphere dynamics.

F. BOUCHET (CNRS) – ENS de Lyon and CNRS

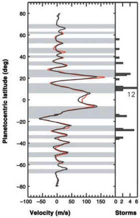
February 2016 – COSMOS – Paris

Jupiter's Zonal Jets

We look for a theoretical description of zonal jets



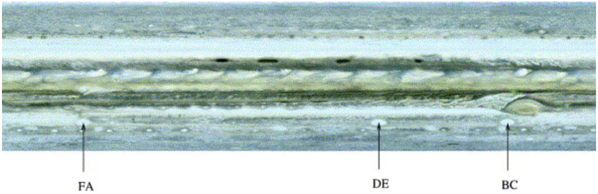
Jupiter's atmosphere

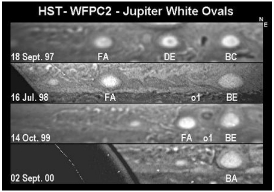


Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

Have we Lost One of Jupiter's Jets?

What is the probability of this event?



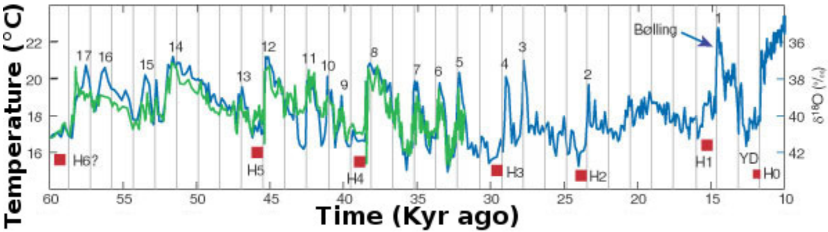


Jupiter's white ovals (see Youssef and Marcus 2005)

The white ovals appeared in 1939-1940 (Rogers 1995). Following an instability of one of the zonal jets?

Abrupt Climate Changes

Long times matter

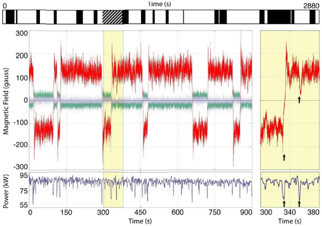


Temperature versus time: Dansgaard–Oeschger events (S. Rahmstorf)

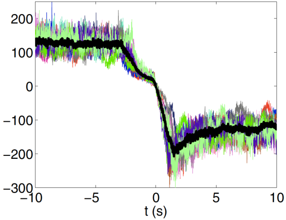
- What is the dynamics and probability of abrupt climate changes?

Random Transitions in Turbulence Problems

Magnetic Field Reversal (Turbulent Dynamo, MHD Dynamics)



Magnetic field timeseries



Zoom on reversal paths

(VKS experiment, reversal paths by François Petrelis)

In turbulent flows, transitions from one attractor to another (reactive paths) often occur through a predictable path.

The Main Scientific Issues

- How to characterize and predict the attractors of turbulent geophysical flows?
- In case of multiple attractors, can we compute their relative probability?
- Can we compute the reactive paths and the transition rates?
- For most geophysical problems, an approach through direct numerical simulation is impossible (trade off between realistic turbulence representation and physical time - here we need both).
- Can we devise new theoretical and numerical tools to tackle these issues?

Outline

1. Large deviations (Freidlin–Wentzell theory) and transition rates (Eyring–Kramers formula) for non-gradient dynamics
    - Freidlin-Wentzell theory
    - Transition rates: Eyring–Kramers generalized to non-gradient dynamics (F.B., and J.R.)
    - Sketch of the proof (F.B., and J.R.)
2. Rare transitions of atmosphere jets: rare event algorithms
    - The barotropic quasi-geostrophic model
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Freidlin-Wentzell Theory

- For dynamical systems with weak noises:

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t),$$

$$\log P(x_1, T; x_{-1}, 0) \underset{\varepsilon \downarrow 0}{\sim} \frac{\inf_{\left\{x(t) \middle| x(\mathbf{0})=x_{\mathbf{1}} \text{ and } x(T)=x_{-\mathbf{1}}\right\}} \left\{ \int_{\mathbf{0}}^{T} \left[\dot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}(x) \right]^{\mathbf{2}} \mathrm{d}t \right\}}{4\varepsilon}.$$

- In the weak noise limit, most transitions (reactive paths) follow the most probable path (instanton).

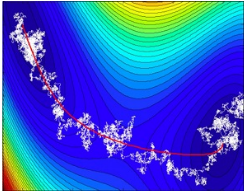


Figure by Eric Vanden Eijnden

Numerical Computation of Action Minima

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t).$$

- Action

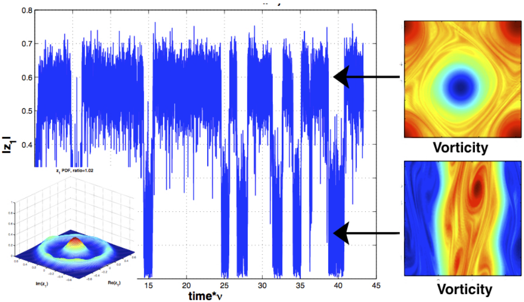
$$\mathscr{A}[\mathbf{x}] = \int_0^T \mathscr{L}[\mathbf{x},\dot{\mathbf{x}}]\ \mathrm{d}t \text{ and } \mathscr{L}[\mathbf{x},\dot{\mathbf{x}}] = \frac{1}{2}\left[\dot{\mathbf{x}} - \mathbf{b}(\mathbf{x})\right]^2.$$

- Numerical computation of action minima.

E. Vanden-Eijnden, W. E and W. Ren, (2004). E Vanden-Eijnden and M Heymann, (2008). J. Laurie and F. Bouchet, (2014).

Non-Equilibrium Phase Transition for the 2D Navier–Stokes Eq.

The time series and PDF of the Order Parameter



Order parameter : $z_1 = \int dxdy \exp(iy)\omega(x,y)$.

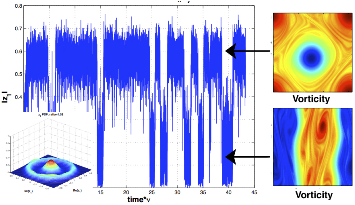
For unidirectional flows $|z_1| \simeq 0$, for dipoles $|z_1| \simeq 0.6 - 0.7$

F. Bouchet and E. Simonnet, PRL, 2009.

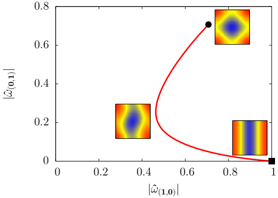
Most Probable Reactive Paths (Instantons) for the 2D Navier-Stokes Eq.

With J. Laurie

- Large deviation theory: instantons as minimum action paths.



2D Navier-Stokes equations (time: 10 000) (PRL)



Numerical instanton (time of order 1) (J. Stat. Phys.)

J. Laurie and F. Bouchet, Computation of rare transitions in the barotropic quasi-geostrophic equations, New J. Phys., 2014

Outline

Transition Rates Beyond Large Deviations: the Eyring-Kramers Formula

- Large deviation theory gives the exponential factor for the transition rate $\lambda = 1/\tau \exp(-\Delta V/\varepsilon)$:

$$\lim_{\varepsilon\to 0} \varepsilon \log \lambda = -\Delta V.$$

- But the prefactor $1/\tau$ is also essential in giving the time scale.
- For gradient dynamics $\frac{\mathrm{d}x}{\mathrm{d}t} = -\nabla V + \sqrt{2\varepsilon}\frac{\mathrm{d}W}{\mathrm{d}t}$, the Eyring-Kramers formula (Landauer and Swanson, 1961, Langer, 1969?) gives

$$\lambda \underset{\varepsilon\to 0}{\sim} \frac{|\lambda_*|}{2\pi}\sqrt{\frac{\det \mathrm{Hess}\, V(x_1)}{|\det \mathrm{Hess}\, V(x_*)|}} \exp\left(-\frac{\Delta V}{\varepsilon}\right),$$

where λ_* the unstable direction eigenvalue, at the saddle point.

- What is the prefactor for non gradient dynamics ? Maier and Stein (1997) (for 2 degrees of freedom), Schuss (2009).

Transition Rates for Non-Gradient Dynamics

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta\,(t)\,.$$

- We assume that there exists a transverse decomposition in the instanton neighborhood

$$\mathbf{b}(\mathbf{x}) = -\nabla V(\mathbf{x}) + \mathbf{G}(\mathbf{x}) \text{ with for all } \mathbf{x},\ \nabla V(\mathbf{x}).\mathbf{G}(\mathbf{x}) = 0.$$

- We have just derived (during the last few months)

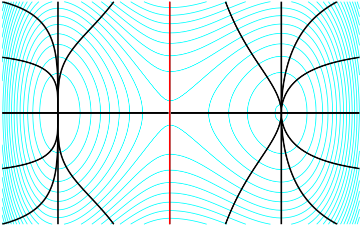
$$\lambda \underset{\varepsilon\to 0}{\sim} \frac{|\lambda_*|}{2\pi}\sqrt{\frac{\det \mathrm{Hess}\, V(x_1)}{|\det \mathrm{Hess}\, V(x_*)|}} \exp\left(-\frac{\Delta V}{\varepsilon}\right) \exp\left\{-\int_{-\infty}^{+\infty} \mathrm{d}t\,[\nabla.G(X(t))]\right\},$$

where λ_* is the negative eigenvalue corresponding to the unstable direction at the saddle point, for the dynamics (and not for V) and $\{X(t)\}$ is the instanton.

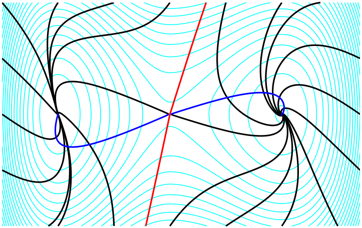
F. Bouchet and J. Reygner, arxiv 1507.02104.

Outline

Gradient and Non-Gradient Dynamics



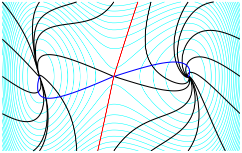
Phase diagram for a potential dynamics



Phase diagram for a non potential dynamics

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t) = -\nabla V(\mathbf{x}) + \mathbf{G}(\mathbf{x}) + \sqrt{2\varepsilon}\eta(t).$$

Four Main Steps for the Proof



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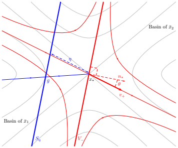


Locally around the saddle point

1. Prefactor for the stationary measure in the interior of D.
2. Exit rates from D.
3. Commitor function in the neighborhood of the saddle point.
4. Matching on S_η and saddle point approximation.

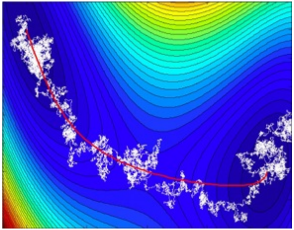
Step 1: Prefactor for the Stationary Measure in the Interior of D

$$\frac{d\mathbf{x}}{dt} = \mathbf{b}(\mathbf{x}) + \sqrt{2\varepsilon}\eta\,(t)\,.$$

$$P_S(x) = \lim_{T\to\infty} P\,(x,0;x_1,-T) = \lim_{T\to\infty} \int_{X(-T)=x_1}^{X(0)=x} \mathrm{e}^{-\frac{\mathscr{A}_T[X]}{2\varepsilon}}\,\mathscr{D}[X]$$

with $\mathscr{A}[X] = \int_{-T}^{0} \mathscr{L}\left[X,\dot{X}\right]\,\mathrm{d}t$ and $\mathscr{L}\left[X,\dot{X}\right] = \frac{1}{2}\left[\dot{X} + \frac{\mathrm{d}V}{\mathrm{d}x}(X)\right]^2$.

Expansion Around the Action Minimizer



In the weak noise limit, most paths concentrate close to the action minimizer $\phi(x,t)$.

We expand $X = \phi(x,t) + \sqrt{\varepsilon}Y$

$$P_S(x) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon^{n/2}} C(x) \exp\left(-\frac{V(x)}{\varepsilon}\right) \text{ with } C(x) = \int_{Y(-\infty)=0}^{Y(0)=0} \mathcal{D}[Y]\, \mathrm{e}^{-\frac{1}{4}\int_{-\infty}^{0} \mathrm{d}t\left[(Y+QY)^2+2YRY\right]},$$

$$\text{and } Q(t) = -Db(\phi(x,t)) \text{ and } R(t) = -\sum_{k=1}^{n} \partial_{x_k} V(\phi(x,t)) \mathrm{Hess}(b_k(\phi(x,t))).$$

- How to compute the functional determinant $C(x)$?

Explicit Expression for the Functional Determinant

$$u(x,y,t) = \int_{Y(-t)=y}^{Y(0)=0} \mathscr{D}[Y] \mathrm{e}^{-\frac{1}{4}\int_{-\infty}^{0} \mathrm{d}t\left[(Y+QY)^2+2YRY\right]}.$$

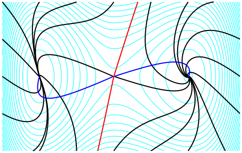
We have proven that

$$u(x,O,-\infty) = C(x) = \sqrt{\frac{\det \operatorname{Hess} V(x_1)}{(2\pi)^n}} \exp\left\{-\int_{-\infty}^{+\infty} dt\,[\nabla.G(X(t))]\right\}.$$

$$P_S(x) \underset{\varepsilon\to 0}{\sim} \frac{1}{\varepsilon^{n/2}} \sqrt{\frac{\det \operatorname{Hess} V(x_1)}{(2\pi)^n}} \exp\left\{-\int_{-\infty}^{+\infty} dt\,[\nabla.G(X(t))]\right\} \exp\left(-\frac{V(x)}{\varepsilon}\right).$$

1. Feynman–Kac formula for u plus a Gaussian ansatz for u.
2. This gives a matrix Riccati equation, and u is then simply related to the determinant of this matrix.
3. We can solve by quadrature the matrix equation and compute this determinant.

Four Main Steps for the Proof



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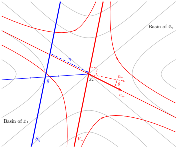

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- The transition rate then reads

$$\lambda \underset{\varepsilon\to 0}{\sim} \frac{|\lambda_*|}{2\pi}\sqrt{\frac{\det \mathrm{Hess}\, V(x_1)}{|\det \mathrm{Hess}\, V(x_*)|}}\exp\left(-\frac{\Delta V}{\varepsilon}\right)\exp\left\{-\int_{-\infty}^{+\infty}\mathrm{d}t\,[\nabla.G(X(t))]\right\},$$

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Jupiter's Zonal Jets

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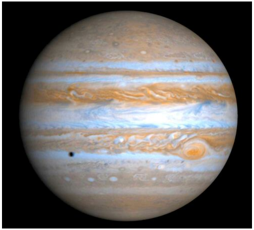
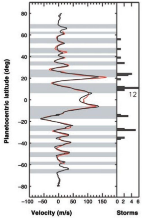


Jupiter's atmosphere



Jupiter's zonal winds (Voyager and Cassini, from Porco et al 2003)

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu_d \Delta \omega - \lambda \omega + \sqrt{2\varepsilon} f_s,$$

  where $\omega = (\nabla \wedge \mathbf{v}).\mathbf{e}_z$ is the vorticity, $q = \omega + \beta y$ is the Potential Vorticity (PV), f_s is a random Gaussian field with correlation $\langle f_S(\mathbf{x},t) f_S(\mathbf{x}',t')\rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$, ε is the average energy input rate, λ is the Rayleigh friction coefficient.
- 4 parameters: ε, λ, β and L
- 2 independent non-dimensional parameters: we choose spatial scale unit such that $L = 2\pi$, and temporal scale unit such that the average total energy is one.

Energy Balance

$$\frac{d\mathbb{E}(E)}{dt} = -2\lambda\mathbb{E}(E) - \nu_d\mathbb{E}(Z) + \varepsilon$$

- Then, in the turbulent regime, where viscous energy dissipation is negligible

$$\mathbb{E}_S(E) \simeq \frac{\varepsilon}{2\lambda}$$

- We will work with the time scale unit that $\frac{\varepsilon}{2\lambda} = 1$.

The Barotropic Quasi-Geostrophic Equations

- The non-dimensional version of the barotropic QG equation
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

with $q = \omega + \beta' y$.

- The relation with the dimensional parameters is:

$$\alpha = L \sqrt{\frac{2\lambda^3}{\varepsilon}}.$$

$$\beta' = L^3 \beta \sqrt{\frac{2\lambda}{\varepsilon}} = \left(\frac{L}{L_{Rhines}} \right)^2$$

- Spin up or spin down time $= 1/\alpha \ll 1 =$ jet inertial time scale.

The Non Dimensional Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = \nu \Delta \omega - \alpha \omega + \sqrt{2\alpha} f_s,$$

with $q = \omega + \beta y$.

The 2D Stochastic Navier-Stokes Equations ($\beta = 0$)

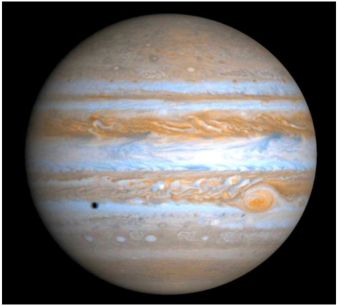
$$\frac{\partial \omega}{\partial t} + \mathbf{u}.\nabla\omega = \nu\Delta\omega + \sqrt{\nu} f_s$$

- Some recent mathematical results: Bricmont, Debussche, Hairer, Kuksin, Kupiainen, Mattingly, Shirikyan, Sinai, ...
  - Existence of a stationary measure μ_ν. Existence of $lim_{\nu\to 0}\ \mu_\nu$,
  - In this limit, almost all trajectories are solutions of the 2D Euler equations.

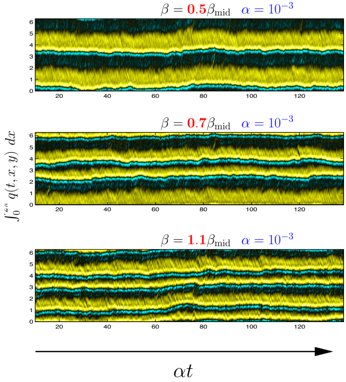
Kuksin, S. B., & Shirikyan, A. (2012). Mathematics of two-dimensional turbulence. Cambridge University Press.

- We would like to describe the invariant measure:
  - Is it concentrated close to steady solutions of the 2D Euler (quasi-geostrophic) equations?
  - Can we describe the dynamics among these states?

Multistability for Quasi-Geostrophic Jets



Jupiter's atmosphere

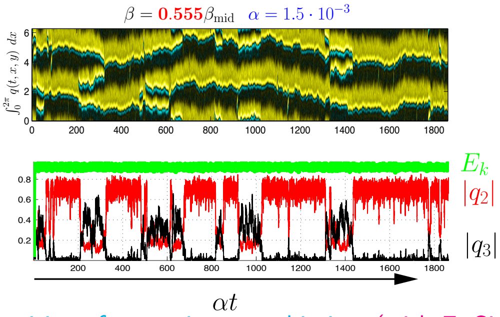


QG zonal turbulent jets

- Multiple attractors had been observed previously by B. Farrell and P. Ioannou.

Rare Transitions Between Quasigeostrophic Jets

wavenumbers $2 \leftrightarrow 3$



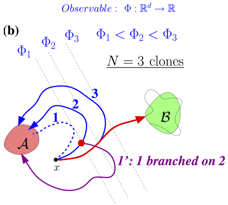
Rare transitions for quasigeostrophic jets (with E. Simonnet)

- This is the first observation of spontaneous transitions.
- How to predict those rare transitions? What is their probability? Which theoretical approach?

Outline

Rare Events and Adaptive Multilevel Splitting (AMS)

AMS: an algorithm to compute rare events, for instance rare reactive trajectories



AMS algorithm

Strategy: selection and cloning.

Probability estimate:

$$\hat{\alpha} = \prod P(I_k, I_{k+1})$$

$$\text{with } P(I_k, I_{k+1}) = (1 - 1/N)$$

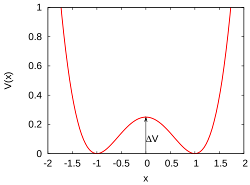
F. Cérou, A. Guyader. (2007) F. Cérou, A. Guyader, T. Lelièvre, and D. Pommier (2011).

The Allen–Cahn (or Ginzburg–Landau) Eq.

$$\begin{cases} \partial_t A = -\frac{\delta V}{\delta A} + \sqrt{\frac{2}{\beta}}\eta, \\ A(0) = A(L) = 0 \end{cases}$$

with $V = \int_0^L \mathrm{d}x \left(-\frac{A^2}{2} + \frac{A^4}{4} + \frac{1}{2}(\partial_x A)^2 \right)$,

and $\langle \eta(x,t)\eta(x',t') \rangle = \delta(t-t')\delta(x-x')$.

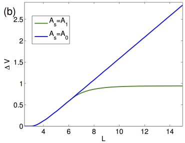


- Because of the gradient structure, V is the quasi-potential.
- In the small noise limit ($\beta \to \infty$), rare transition are time reversed of gradient trajectories (instantons) and the transition rate is $\tau \exp(-\Delta V/\beta)$.

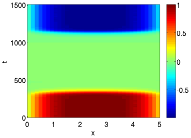
W.G. Faris, G. Jona-Lasinio. (1982)

The Instanton: the Action Minimizer

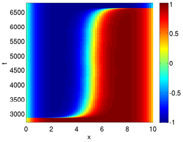
- The critical point with the lowest saddle is selected



Potential of zero and "one front" saddles



Through zero

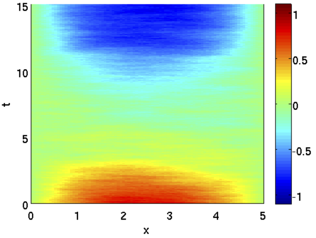


Through the 1-front saddle

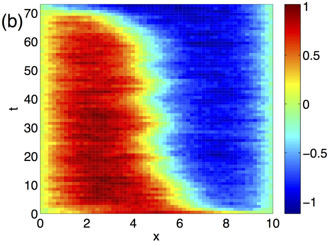
- For $L < L_c$, the instanton goes through the zero state, for $L > L_c$, the instanton goes through the 1-front saddle.

Transition Trajectories for the Allen–Cahn Eq.

Ginzburg Landau potential with a non conservative dynamics



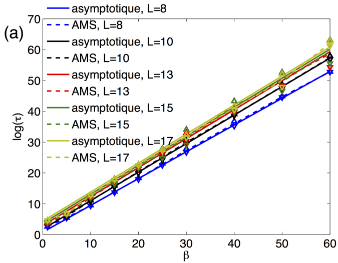
A reactive path through zero, for $L = 5$, and $\beta = 300$.



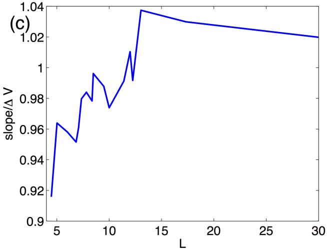
A 1-front reactive path, for $L = 10$, and $\beta = 150$.

- Examples of reactive paths computed using the Adaptive Multilevel Splitting algorithms.

Check for Arrhenius' Rates Prediction from Freidlin–Wentzell Theory



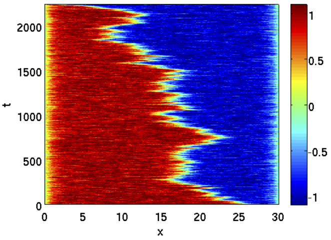
$\log(\alpha)$ versus β



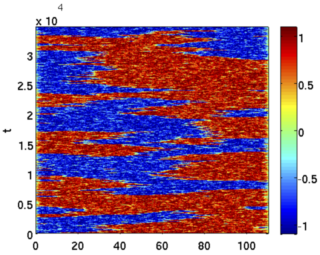
ΔV versus the measured ΔV

- Kramers' rate: $\alpha = \frac{1}{\tau}\exp\left(-\beta\Delta V\right)$, where V is the quasi-potential.

Beyond Freidlin–Wentzell Regime



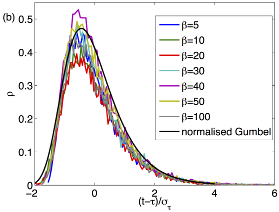
A fluctuating front reactive path, $L = 30$, $\beta = 30.7$



A complex 2-front reactive path, $L = 110$, $\beta = 7$

Reactive Path Lengths and Gumbel Distribution

- One can go beyond large deviation theory: computation of the distribution of the length of reactive paths



Distribution of reactive path length, $L = 10$.

- Why should one expect a Gumbel distribution? Relation with extreme value theory? M.Y. Day (1995), Y. Bakhtin (2013), F. Cérou, A. Guyader, T. Lelièvre, and F. Malrieu (2013).

AMS Computation of Reactive Paths for the Allen-Cahn equation

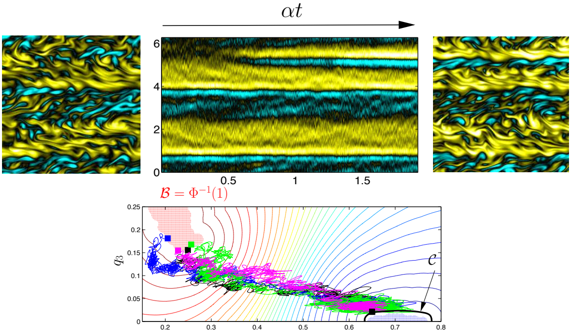
- We can compute mean exit times of order of 10^{16} times the relaxation time, with a numerical cost which is of order 10^{3} times the relaxation time, and have an excellent statistics of the reactive trajectories.
- The statistics conditioned on the occurrence of the rare event is excellent. Dynamics study.
- Applied to the Allen-Cahn equation, it shows that the Freidlin–Wentzell theory is valid on a very narrow regime, that would have been very difficult to study using direct numerical simulations.
- This gives access easily to most quantity of interest beyond Freidlin–Wentzell regime.

F. Bouchet, J. Rolland and E. Simonnet, J. Stat. Phys., 2016

Outline

Geostrophic Jet Transitions from the Adaptive Multilevel Splitting Algorithm

Transitions from two to three jets

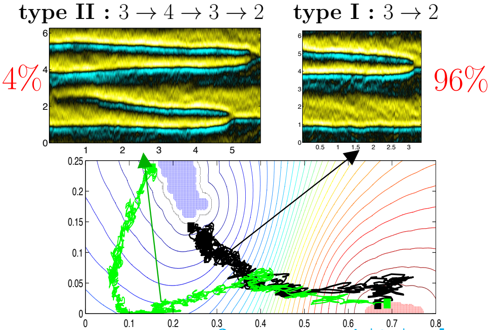


758 transitions 2→3. $\beta = 0.55$ and $|k_f| \in [10;12]$

- $Pr(\tau_B < \tau_A) = 1.5\,10^{-5} \pm 5.10^{-6}$.

Geostrophic Jet Transitions from the Adaptive Multilevel Splitting Algorithm

Transitions from three to two jets

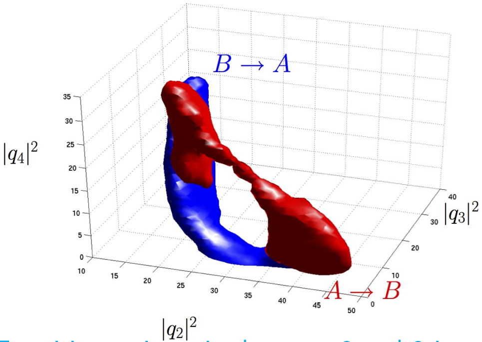


758 transitions 3→2. $\beta = 0.55$ and $|k_f| \in [10;12]$

- $Pr(\tau_A < \tau_B) = 1.9\,10^{-4} \pm 5.10^{-5}$.
- Two types of transitions: 2→3 and $2 \to 4 \to 3$.

Atmosphere Jet "Instantons" Computed using AMS

AMS: an algorithm to compute rare events, for instance rare reactive trajectories

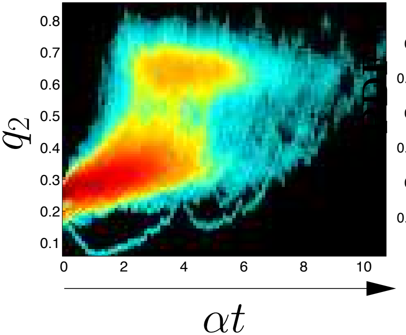


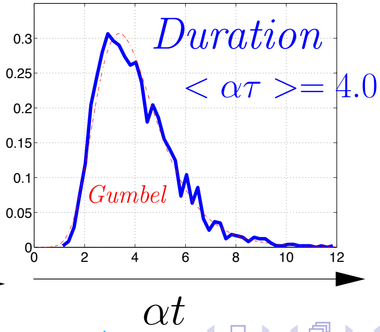
Transition trajectories between 2 and 3 jet states

- Asymmetry between forward and backward transitions.
- With the AMS algorithm, we study transitions that can not be sampled using direct numerical simulations.

Distribution of Transition Path Durations

AMS: an algorithm to compute rare events, for instance rare reactive trajectories





Distribution of transition durations

- Transition path duration distribution is close to a Gumble one.
- The AMS algorithm gives a good sampling of transition paths.

Outline

1. Large deviations (Freidlin–Wentzell theory) and transition rates (Eyring–Kramers formula) for non-gradient dynamics
   - Freidlin-Wentzell theory
   - Transition rates: Eyring–Kramers generalized to non-gradient dynamics (F.B., and J.R.)
   - Sketch of the proof (F.B., and J.R.)
2. Rare transitions of atmosphere jets: rare event algorithms
   - The barotropic quasi-geostrophic model
   - The adaptive multilevel splitting (AMS) alg. (F.B., J.R. and E.S.)
   - The AMS algorithm and jet transitions (F.B. and E.S.)
3. Rare transitions of atmosphere jets: averaging and large deviations
   - Averaging for slow jet dynamics (F.B., C.N., and T.T.)
   - Ergodicity and averaging (F.B., C.N., and T.T.)
   - Large deviations for atmosphere jets (F.B., T.G., B.M., T.T., E.V-E.)

The Barotropic Quasi-Geostrophic Equations

- The simplest model for geostrophic turbulence.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s,$$

with $q = \omega + \beta y$.

- Inertial limit: spin up or spin down time $= 1/\alpha \gg 1 =$ jet inertial time scale.
- A reasonable model for Jupiter's zonal jets.

Which Mathematical Framework for the Inertial Limit?

- Inertial limit: spin up or spin down time $= 1/\alpha \gg 1 =$ jet inertial time scale.
- Quasi-Geostrophic equations with random forces

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s,$$

  with $q = \omega + \beta y$.
- It is an averaging problem for an Hamiltonian system perturbed by weak non Hamiltonian forces.
- The Hamiltonian system is an infinite dimensional one with an infinite number of conserved quantities.
- We will need to consider large deviations for the slow process.

Decomposition Between Zonal Jets and Turbulence: A Slow/Fast Dynamical System

$$\frac{\partial q}{\partial t} + \mathbf{v}.\nabla q = -\alpha\omega + \sqrt{2\alpha} f_s \text{ with } \alpha \ll 1$$

- Time scale separation. We decompose into slow (zonal flows) and fast variables (eddy turbulence)

$$q_z(y) = \langle q \rangle \equiv \frac{1}{2\pi} \int_{\mathscr{D}} \mathrm{d}x\, q \text{ and } q = q_z + \sqrt{\alpha} q_m.$$

- Stochastic reduction (Van Kampen, Gardiner, ...) using the time scale separation.
- We average out the turbulent degrees of freedom.

Averaging for the Stochastic Quasi-Geostrophic Eq.

$$\frac{1}{\alpha}\frac{\partial q_z}{\partial t} = F[q_z] - \omega_z.$$

- $F[q_z] = -\frac{\partial}{\partial y}\mathbb{E}_{q_z}\langle v_{m,y} q_m \rangle$. The average of the Reynolds stress is over the statistics of the quasilinear inertial dynamics:

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha q_m + f_s$$

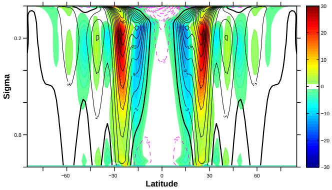
and

$$\langle v_{m,y} q_m \rangle = \frac{1}{L_y}\int \mathrm{d}y\, \mathbb{E}_{q_z}[v_{m,y} q_m].$$

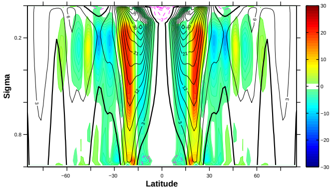
- We identify SSST by Farrell and Ioannou (JAS, 2003); quasilinear theory by Bouchet (PRE, 2004); CE2 by Marston, Conover and Schneider (JAS, 2008); Sreenivasan and Young (JAS, 2011).

Kinetic Theory of Atmosphere Jets

Troposphere Dynamics: comparison of kinetic equations and a direct numerical simulation



Full equations (DNS).



Kinetic approach.

Zonal wind and momentum convergence for the primitive equations.

Farid Ait Chaalal and Tapio Schneider (Caltech and ETH Zurich).

- The qualitative structure of a fast rotating Earth troposphere is well approximated by kinetic equations.

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The Barotropic Quasi-Geostrophic Equations

- At leading order, the inertial equations are an Hamiltonian system.
- Quasi-Geostrophic equations with no forces and dissipation

$$\frac{\partial q}{\partial t}+\mathbf{v}.\nabla q=0$$

with $q=\omega+\beta y$.

- It conserves energy

$$E=\frac{1}{2}\int_{\mathscr{D}}\mathbf{v}^2\mathrm{d}\mathbf{x},$$

enstrophy, and an infinite number of Casimir invariants.

- We want to use that zonal jets are attractors for the inertial dynamics.
- Asymptotic stability? How is this possible for Hamiltonian systems?

Inviscid Damping of the Linearized Euler Eq.

- Base state: a stable steady state $\mathbf{v}_0 = U(y)\mathbf{e}_x$, with vorticity $\Omega(y)$: $\mathbf{v}_0.\nabla Q = 0$

$$\partial_t \omega_m + U(y)\frac{\partial \omega_m}{\partial x} + v_{m,y}\frac{\partial \Omega}{\partial y} = 0 \text{ with } \omega(t=0) = \omega_0 .$$

- For the linearized 2D Euler equation and non-monotonous base flow, the velocity field decreases algebraically at large times

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp\left(-ikU(y)t\right) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right).$$

F. Bouchet and H. Morita, 2010, Physica D.

Validity of Averaging?

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha\omega_m + \sqrt{2}f_s$$

- We need to prove that the Gaussian process has an invariant measure which has a limit when $\alpha \to 0$.
- This is the limit of no dissipation.
- It may work because of inviscid damping (Related to Landau-Damping and the recent result of Bedrossian and Masmoudi).
- The result is based on asymptotics of the inviscid linearized equations (F. Bouchet and H. Morita, 2010):

$$v_{m,x}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,x,\infty}(y)}{t}\exp\left(-ikU(y)t\right) \text{ and } v_{m,y}(y,t) \underset{t\to\infty}{\sim} \frac{v_{m,y,\infty}(y)}{t^2}\exp\left(-ikU(y)t\right).$$

Invariant Measure in the Inertial Limit

$$\partial_t q_m + U(y)\frac{\partial q_m}{\partial x} + v_{m,y}\frac{\partial q_z}{\partial y} = -\alpha\omega_m + \sqrt{2}f_s$$

- The two point correlation function $\mathbb{E}_\alpha\left(\mathbf{v}_m(y_1)\mathbf{v}_m(y_2)\right)$ has a limit when $\alpha \downarrow 0$.
- The two point correlation function $\mathbb{E}_\alpha\left(q_m(y_1)q_m(y_2)\right)$ has a limit when $\alpha \downarrow 0$., as a distribution.
- The two two point correlation function $\mathbb{E}_\alpha\left(\nabla q_m(y_1)\nabla q_m(y_2)\right)$ diverges when $\alpha \downarrow 0$.
- The Reynolds stress $\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)$ has a limit when $\alpha \downarrow 0$.

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

Reynolds Stress Ergodicity in the Inertial Limit

- The Reynolds stress $\mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)$ has a limit when $\alpha \downarrow 0$.
- Pointwise divergence of the ergodic average:

$$\mathbb{E}_\alpha\left\{\left[\frac{1}{T}\int_0^T \mathrm{d}t\, v_{m,y}(y)q_m(y) - \mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)\right]^2\right\} \underset{T\uparrow\infty\ \alpha\downarrow 0}{\sim} \frac{A(y)}{\alpha T}$$

- Convergence as a distribution: for any test function ϕ

$$\mathbb{E}_\alpha\left\{\int \mathrm{d}y\, \phi(y)\left[\frac{1}{T}\int_0^T \mathrm{d}t\, v_{m,y}(y)q_m(y) - \mathbb{E}_\alpha\left(v_{m,y}(y)q_m(y)\right)\right]\right\}^2 \underset{T\uparrow\infty\ \alpha\downarrow 0}{\sim} \frac{A}{T}$$

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

T. Tangarife's PhD thesis, 2015.

Validity of Averaging at the Level of the Law of Large Numbers

- The fast variable dynamics has an invariant measure in the inviscid limit thanks to inviscid damping.
- Velocity like observables have a finite expectation in the inviscid limit.
- The Reynolds stress has a finite expectation in the inviscid limit.
- The ergodic average of the Reynolds stress converges as a distribution.
- We have partial answers only, for the validity of averaging!

F. Bouchet, C. Nardini and T. Tangarife, J. Stat. Phys., 2013

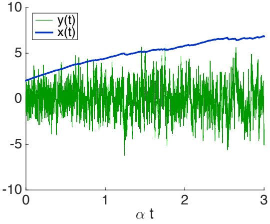
T. Tangarife's PhD thesis, 2015.

Outline

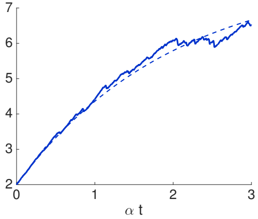
Fluctuations Around the Law of Large Numbers

$$\begin{cases} \frac{dx}{dt} & = & f(x,y) \\ \frac{dy}{dt} & = & \frac{1}{\alpha}g(x,y) + \frac{1}{\sqrt{\alpha}}h(x,y)\frac{dW}{dt} \end{cases}$$

- Time scale separation: $\alpha \ll 1$. y is the fast variable and x is the slow one.

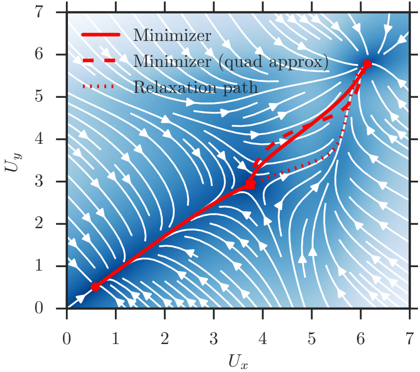


A slow/fast dynamics.



The averaged evolution.

Gaussian Fluctuations Do Not Describe Rare Transitions



(Figure from F. Bouchet, T. Grafke, T. Tangarife, and E. Vanden-Eijnden 2015)

The Large Deviations that Describe Rare Transitions

$$\begin{cases} \frac{\mathrm{d}x}{dt} & = & f(x,y) \\ \frac{\mathrm{d}y}{dt} & = & \frac{1}{\alpha}g(x,y)+\frac{1}{\sqrt{\alpha}}h(x,y)\frac{\mathrm{d}W}{\mathrm{d}t} \end{cases}$$

- The transition rate λ from a state X_0 to a state X_F verifies

$$\lim_{\alpha\to 0}\alpha\lambda = 2\min_{\{X,P\left|X(0)=X_0 \text{ and } X(T)=X_F,\right\}}\mathscr{A}[X,\theta]$$

with the action

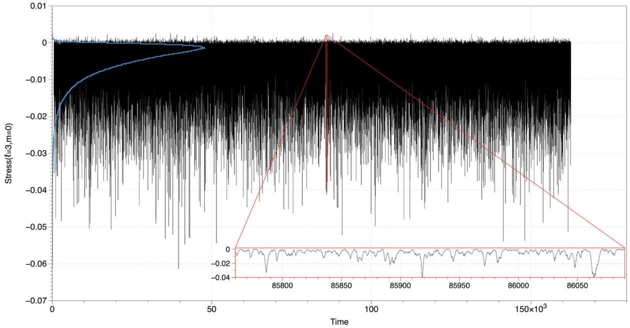
$$\mathscr{A}[X,\theta] = \int_0^T \mathrm{d}t\left[\dot{X}P - H(X,P)\right],$$

and

$$H(X,P) = \lim_{T\to\infty}\frac{1}{T}\log\mathbb{E}_X\left\{\exp\left[P\int_O^T f(X,y(t))\right]\right\}$$ (see Freidlin–Wentzell)

- For quadratic in y f, and linear g and h (for instance for the quasigeostrophic model), H solves a nonlinear Lyapunov eq.

Reynolds Stress Statistics for the Slow Zonal Jet Dynamics



Timeseries of the momentum flux convergence (Reynolds stress)

- The Reynolds stress PDF has very long exponential tails (for which we have a theory)

Large Deviations of Quadratic Observables for an Ornstein–Uhlenbeck Process

$$\left\{ \begin{array}{lcl} \frac{\mathrm{dx}}{dt} & = & y^T M y - g(x) \\ \frac{\mathrm{dy}}{dt} & = & \frac{1}{\alpha} L_x y + \frac{1}{\sqrt{\alpha}} \eta \end{array} \right.$$

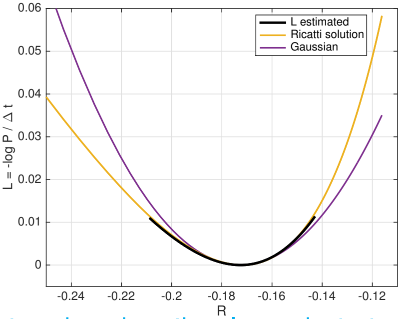
$$H(X,P) = \lim_{T\to\infty} \frac{1}{T} \log \mathbb{E}_X \left\{ \exp\left[P \int_O^T y^T M y \right] \right\} = \mathrm{Tr}(CN_\infty).$$

where C is the noise correlation function and N_∞ is the asymptotic solution of the matrix Ricatti equation equation

$$\frac{\partial N}{\partial t} + L_X^T N + N L_X = 2NCN + PM$$

We can solve this equation analytically sometimes, or numerically.

Hamiltonian for the Large Deviations of Time Averaged Reynolds Stresses



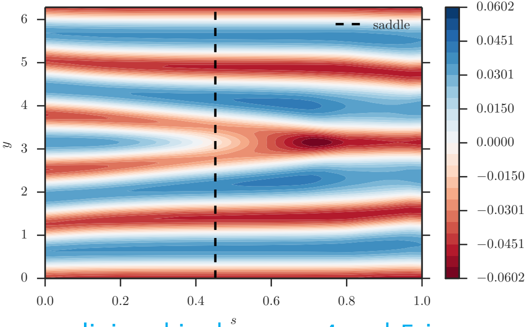
The Lagrangian that describes large deviations of zonal jets

- Rare transitions involve non Gaussian fluctuations

F. Bouchet, T. Tangarife, and B. Marston (to be submitted to JAS)

Heteroclinic Orbit for Zonal Jet Transitions

String method: computing heteroclinic orbits between saddles and attractors



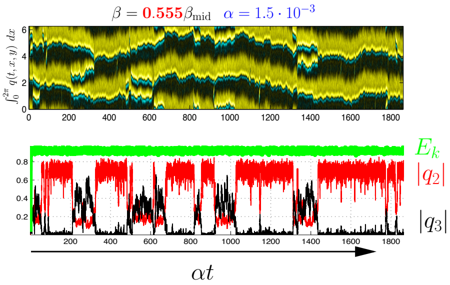
The heteroclinic orbits between 4 and 5 jet attractors

- The string method: W.E and E. Vanden-Eijnden, Phys. Rev. B 2002.
- A preliminary step towards the computation of instantons.

with T. Grafke and E. Vanden-Eijnden.

Perspectives

wavenumbers $2 \leftrightarrow 3$



Rare transitions for quasigeostrophic jets

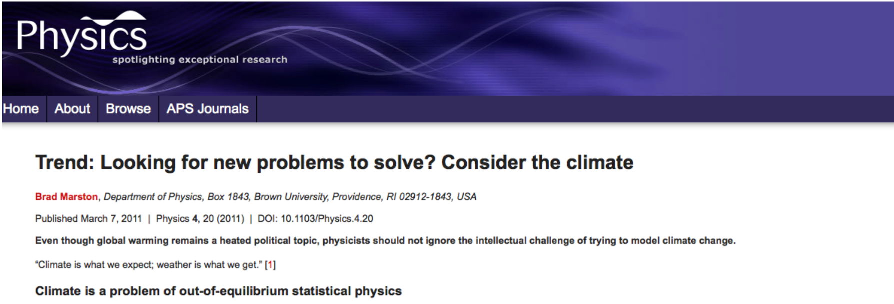
- Study the fluid mechanics aspects of rare fluctuations.
- Compute rare transitions as minimizers of the large deviation action.

Zonal Jet Conclusions

- We have observed rare transitions between zonal jets in barotropic turbulence, similar to the Jupiter loss of a zonal jet, (with E. S.).
- We use the adaptive multilevel splitting algorithm to sample reactive trajectories and compute transition rates, (with E. S.).
- We have partial results for the justification of averaging (ergodicity, etc ...), (with C.N., and T.T.).
- In the limit of time scale separation, a theory based on large deviations can be derived for the computation of transition rates and reactive paths between zonal jets.
- The large deviation action can be computed solving a non-linear Lyapunov equation, (with T.G., B..M., T.T., and E. V-E).

Theoretical Statistical Physics and Climate Dynamics

- Theoretical physics (statistical mechanics, large deviation theory, turbulence, dynamical system theory) will play a crucial role in future understanding of climate dynamics.



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Trend: Looking for new problems to solve? Consider the climate

Brad Marston, *Department of Physics, Box 1843, Brown University, Providence, RI 02912-1843, USA*

Published March 7, 2011 | Physics **4**, 20 (2011) | DOI: 10.1103/Physics.4.20

Even though global warming remains a heated political topic, physicists should not ignore the intellectual challenge of trying to model climate change.

"Climate is what we expect; weather is what we get." [1]

Climate is a problem of out-of-equilibrium statistical physics

APS trend

Mathematical Physics and Climate Dynamics

Mathematical physics (statistical mechanics, large deviation theory, turbulence, dynamical system theory) will play a crucial role in future understanding of climate dynamics.

Newton Institute program and conference

Collaborators

- Eyring–Kramers formula for transition rates of non-gradient dynamics (with J. Reygner, post-doc ERC Transition).
- Multistability (with E. Simonnet) and instantons for the 2D-Navier Stokes equations (with J. Laurie (post-doc ERC)).
- Numerical simulation of abrupt transitions for Jupiter zonal jets using Adaptive Multilevel Splitting algorithms (with J. Rolland and E. Simonnet (Nice)).
- Ergodicity and averaging for the quasi-geostrophic dynamics (with C. Nardini (post-doc) and T. Tangarife (PHD)).
- Large deviations of Reynolds stresses for jet dynamics (with B. Marston (Brown) and T. Tangarife (PHD)).
- Averaging, large deviations, and transitions for Jupiter jets (with T. Grafke and E. Vanden-Eijnden).

In Remembrance of Tomas Tangarife

Summary and Perspectives

- Large deviation theory can be applied to geophysical turbulence.
- The dynamics leading to rare events is usually predictable, even for turbulent flows.
- With rare event algorithms, we can compute probability of rare events that can not be sampled using direct numerical simulations.
- We have generalized the Eyring-Kramers law to non-gradient dynamics.
- Averaging and fast/slow large deviations for the quasi-geostrophic model rises fascinating mathematical issues.