Uncertainty quantification and systemic risk

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Modeling systemic risk

- We consider a system with many inter-connected components, each of which can be in a **normal** state or in a **failed** state.
- We want to study the probability of overall failure of the system, that is, its **systemic** risk.
- Three effects can contribute to the behavior of systemic risk:
  - The intrinsic stability of each component
  - The external random perturbations to the system
  - The inter-connectedness or cooperation between components

Possible applications

- Engineering systems with a large number of interacting parts. Components can fail but the system fails only when a large number of components fail simultaneously.
- Banking systems. Banks cooperate and by spreading the risk of credit shocks between them can operate with less restrictive individual risk policies. However, this increases the risk that they may all fail, that is, the systemic risk.
  $\hookrightarrow$ We want to propose a simple model to explain that individual risk does not affect the systemic risk in an obvious way. *In fact, it is possible to simultaneously reduce individual risk and increase the systemic risk.*

A bistable mean-field model

- The system has N components.
- The (real-valued) risk variable $X_j(t)$, $j = 1, \ldots, N$, satisfies the SDE

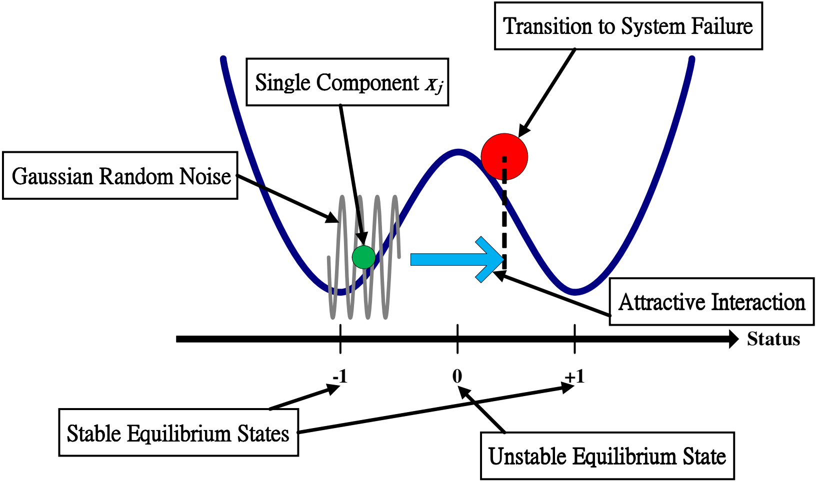
$$dX_j(t) = -hV'(X_j(t))\,dt - \theta\left(X_j(t) - \bar{X}(t)\right)dt + \sigma dW_j(t)\,.$$

- $V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$ is a potential with two stable states ± 1.
  With $\theta = \sigma = 0$, $h > 0$, $X_j(t)$ converges to one of the states.
  We define -1 the normal state and $+1$ the failed state.
  $h \geq 0$ is the intrinsic stability parameter.
- $\bar{X}(t) := \frac{1}{N}\sum_{i=1}^{N} X_i(t)$ is the risk variable of the system.
  $\theta \geq 0$ is the attractive interaction parameter.
- $\{W_j(t), j = 1, \ldots, N\}$ are independent Brownian motions.
  $\sigma \geq 0$ is the noise strength.

Why this model?

- h, σ and θ control the three effects we want to study: intrinsic stability, random perturbations, and degree of cooperation.
- Why mean field interaction? Because it is the simplest interaction that models cooperative behavior. It can be generalized to include **diversity** as well as other more complex interactions such as **hierarchical** ones.
- Connection with UQ (Uncertainty Quantification): UQ quantifies the variance of the quantity of interest. The variance quantifies the "normal" fluctuations. It may not characterize a rare event.

Schematic for the model



The probability-measure-valued process $\mu_N(t)$

- First idea: to analyze $\bar{X}(t) = \frac{1}{N}\sum_{i=1}^{N} X_i(t)$.
- Unfortunately, no closed equation for $\bar{X}(t)$, so we need to generalize this problem into a larger space.
- Second idea: to analyze the empirical measure $\mu_N(t, dx) := \frac{1}{N}\sum_{j=1}^{N} \delta_{X_j(t)}(dx)$. Then $\bar{X}(t) = \int x \mu_N(t, dx)$.
- *(Dawson, 1983)* $\mu_N(t)$ converges weakly in probability as $N \to \infty$ to a deterministic process $u(t)$. For $t > 0$, $u(t)$ has a pdf $u(t, x)$ solution of the nonlinear Fokker-Planck equation:

$$\frac{\partial}{\partial t} u = h \frac{\partial}{\partial x} \left[V'(x) u \right] - \theta \frac{\partial}{\partial x} \left\{ \left[\int_{-\infty}^{\infty} y u(t, y) dy - x \right] u \right\} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} u,$$

recalling that

$$dX_j(t) = -hV'(X_j(t))dt + \theta(\bar{X}(t) - X_j(t))dt + \sigma dW_j(t).$$

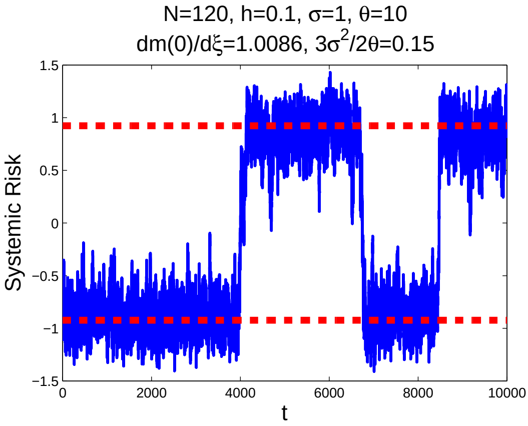
Existence of two stable equilibria

- $u(t)$ converges to an equilibrium u^e_ξ with a pdf of the form:

$$u^e_\xi(x) = \frac{1}{Z_\xi} \exp\left\{ -\frac{(x-\xi)^2}{2\frac{\sigma^2}{2\theta}} - \frac{2h}{\sigma^2} V(x) \right\},$$

  with the compatibility condition: $\xi = m(\xi) := \int_{-\infty}^{\infty} x u^e_\xi(x) dx$.
- Given θ and h, there exists a critical value σ_c such that $\xi = m(\xi)$ has two stable solutions $\pm\xi_b$ if and only if $\sigma < \sigma_c$.
- Simplification: for small h, we have $\sigma_c^2 = 2\theta/3 + O(h)$ and $\xi_b = \sqrt{1 - \frac{3\sigma^2}{2\theta}} + O(h)$.
- Let us say that $u^e_{-\xi_b}$ is the normal state of the system, $u^e_{+\xi_b}$ is the failed state.
- If $\mu_N(0) \overset{N\to\infty}{\to} u(0)$, then $\mu_N(t) \overset{N\to\infty}{\to} u(t)$ for all time.
  If $u(0) = u^e_{-\xi_b}$, then $u(t) = u^e_{-\xi_b}$ for all time.

Simulation of $\bar{X}$



$$X_j^{n+1} = X_j^n - hV'(X_j^n)\Delta t + \sigma \Delta W_j^{n+1} - \theta\left(X_j^n - \frac{1}{N}\sum_{k=1}^{N} X_k^n\right)\Delta t.$$

Probability of system failures

- Assume that $\sigma < \sigma_c$ so the limit of $\mu_N(t)$ has two stable equilibria $u^e_{-\xi_b}$ (resp. $u^e_{\xi_b}$), the system's normal (resp. failed) state.
- For large N, let the empirical density $\mu_N(0) \approx u^e_{-\xi_b}$. Then we expect that $\mu_N(t) \approx u^e_{-\xi_b}$ for all $t > 0$.
- However, as long as N is finite, the system collapse:

$$\mu_N(0) \approx u^e_{-\xi_b}, \quad \mu_N(T) \approx u^e_{+\xi_b}$$

  happens in the time interval $[0, T]$ with **small but nonzero probability**.
- We use large deviations to compute this small probability.

Large deviation principle

(Dawson & Gärtner, 1987) Given an event A in the suitable space (smaller than $C([0,T], M_1(\mathbb{R})))$ ▸ Detail, then $\mu_N(t,dx) = \frac{1}{N}\sum_{i=1}^N \delta_{X_j(t)}(dx)$ satisfies the large deviation principle with the **rate function** I_h: ▸ Detail

$$\mathbf{P}\left(\mu_N \in A\right) \overset{N\gg 1}{\approx} \exp\left(-N \inf_{\phi\in A} I_h(\phi)\right).$$

The rate function I_h has the following variational form: ▸ Detail

$$I_h(\phi) = \frac{1}{2\sigma^2}\int_0^T \sup_{f:\langle\phi, f_x^2\rangle(t)\neq 0} \frac{\langle \phi_t - \mathcal{L}_h^*\phi, f\rangle^2}{\langle \phi, f_x^2\rangle} dt,$$

$$\mathcal{L}_h^*\phi = h\frac{\partial}{\partial x}\left[V'(x)\phi\right] + \theta\frac{\partial}{\partial x}\left\{\left[x - \int y\phi(t,dy)\right]\phi\right\} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi,$$

$$\langle \phi, f\rangle(t) = \int_{-\infty}^{\infty}\phi(t,dx)f(x).$$

Probability of system failures and small h analysis

- $\mathbf{P}\left(\mu_N \in A\right) \overset{N \gg 1}{\approx} \exp\left(-N \inf_{\phi \in A} I_h(\phi)\right)$.
- The rare event A of system failures is the set of all possible paths starting from $u^e_{-\xi_b}$ (the normal state) to $u^e_{+\xi_b}$ (the failed state): Detail

$$A = \left\{\phi : \phi(0) = u^e_{-\xi_b},\ \phi(T) = u^e_{+\xi_b}\right\}.$$

- The major work is to compute $\inf_{\phi \in A} I_h(\phi)$, which is a nonlinear and infinite-dimensional problem.
- Here we assume that the intrinsic stability, h, is small so that we can make this problem tractable.
- Two good reasons to consider a small intrinsic stability h:
  1. $\mathbf{P}(\mu_N \in A)$ is extremely small for large h, which is not the regime we have in mind.
  2. The case $h = 0$ is analytically solvable and the cases h small are perturbations of it.

The case $h = 0$

In this case, $\inf_{\phi \in A} I_0(\phi)$ can be solved without approximation.
Result: For $h = 0$, $\inf_{\phi \in A} I_0(\phi)$ *has the unique minimizer* $p^e(t,x)dx$*, a path of Gaussian measures:*

$$p^e(t,x) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{2\theta}}} \exp\left\{ -\frac{(x - a^e(t))^2}{2\frac{\sigma^2}{2\theta}} \right\},$$

where $a^e(t) = 2\xi_b t/T - \xi_b$ *and* $\xi_b = \sqrt{1 - \frac{3\sigma^2}{2\theta}}$.
Therefore,

$$\inf_{\phi \in A} I_0(\phi) = I_0(p^e) = \frac{2}{\sigma^2 T}\left(1 - \frac{3\sigma^2}{2\theta}\right).$$

Recall that $A = \left\{ \phi : \phi(0) = u^e_{-\xi_b},\ \phi(T) = u^e_{+\xi_b} \right\}$.

The case h small

When h is small, good candidates for the transition path of empirical densities are Gaussian with small perturbations:

$$\left\{ \phi = p + hq : p(t,x) = \frac{1}{\sqrt{2\pi b^2(t)}} \exp\left[\frac{(x - a(t))^2}{2b^2(t)}\right], a(0) = -\xi_b, a(T) = \xi_b \right\}$$

- For h small, the large deviation problem is solvable (the optimal $(a(t), b(t))$ satisfies an ODE system), and the transition probability is

$$\mathbf{P}(\mu_N \in A) \approx \exp\left(-\frac{N}{\sigma^2 T}\left[2\left(1 - \frac{3\sigma^2}{2\theta}\right) + \frac{6h}{\sigma^2}\left(\frac{\sigma^2}{\theta}\right)^2\left(1 - \frac{\sigma^2}{\theta}\right) + O(h^2)\right]\right).$$

- Comments:
  - A large system (N large) is more stable than a small system.
  - In the long run (T large), a transition will happen.
  - Increase of the intrinsic stabilization parameter h reduces systemic risk.
  - Mean transition times are simply related to transition probabilities in this approximation (Williams '82)

What about the individual risk?

- The risk variable $X_j(t)$ of component j satisfies
$$dX_j = -h(X_j^3 - X_j)dt + \theta(\bar{X} - X_j)dt + \sigma dW_j.$$
- Assume $X_j(0) = -1$ (the normal state) and linearize X_j around -1: $X_j(t) = -1 + \delta X_j(t)$, $\bar{X}(t) = -1 + \delta\bar{X}(t)$ and $\delta\bar{X}(t) = \frac{1}{N}\sum_{j=1}^{N} \delta X_j(t)$.
- $\delta X_j(t)$ and $\delta\bar{X}(t)$ satisfy linear SDEs:
$$d\delta X_j = -(\theta + 2h)\delta X_j dt + \theta\delta\bar{X} dt + \sigma dW_j,$$
$$d\delta\bar{X} = -2h\delta\bar{X} dt + \frac{\sigma}{N}\sum_{j=1}^{N} dW_j.$$
- $\delta X_j(t)$ is a Gaussian process with the stationary distribution $\mathcal{N}(0, \frac{\sigma^2}{2(2h+\theta)})$ as $N \to \infty$.
- Qualitatively speaking, the individual risk is
$$\frac{\text{external risk}(\sigma^2)}{\text{intrinsic stability}(h) + \text{risk diversification}(\theta)}.$$

Why and when is risk diversification undesirable?

For h small:

$$\text{Systemic Risk} \approx \exp\left(-\frac{2N}{\sigma^2 T}\left(1-\frac{3\sigma^2}{2\theta}\right)\right), \qquad \text{Individual Risk} \approx \frac{\sigma^2}{2\theta}.$$

- Let us assume that σ^2 increases; one increases θ to compensate and to keep the ratio $\frac{\sigma^2}{2\theta}$ at a low level.
  - The individual risk is kept low.
  - The systemic risk increases, although this cannot be detected by the observation of the "normal" fluctuations (until the catastrophic transition happens).

Extensions

- Diversity: the values of the parameters θ, σ, h are component-dependent.
- Hierarchical model: the system is stabilized by a central component.

Modeling of diversity in cooperative behavior

- The cooperative behavior of components can be different across groups:

$$dX_j(t) = -hV'(X_j(t))\,dt + \sigma dW_j(t) + \theta_j\left(\bar{X}(t) - X_j(t)\right)dt.$$

- The components are partitioned into K groups. In group k, the components have cooperative parameter Θ_k.
- In the limit $N \to \infty$ the empirical densities of each group converge to the solution of the joint Fokker-Planck equations:

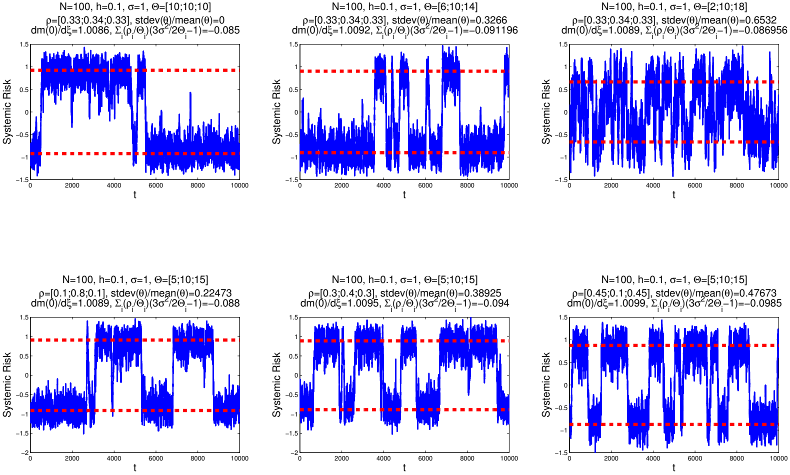
$$\begin{aligned}
\frac{\partial}{\partial t}u_1 &= h\frac{\partial}{\partial x}\left[V'(x)\,u_1\right] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u_1 - \Theta_1\frac{\partial}{\partial x}\left\{\left[\int y\sum_{k=1}^{K}\rho_k u_k(t,y)dy - x\right]u_1\right\} \\
&\vdots \\
\frac{\partial}{\partial t}u_K &= h\frac{\partial}{\partial x}\left[V'(x)\,u_K\right] + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}u_K - \Theta_K\frac{\partial}{\partial x}\left\{\left[\int y\sum_{k=1}^{K}\rho_k u_k(t,y)dy - x\right]u_K\right\}
\end{aligned}$$

where $\rho_k N$ is the size of group k.

Impact of component diversity on the systemic risk

- Why is the diversity interesting?
  - The model is more realistic and more widely applicable.
  - Diversity significantly affects the system stability by reducing it.
- Impact from the diversity:
  - Analytical and numerical studies show that even with the same parameters and with $\{\theta_j\}$ whose average equals θ the system still changes significantly.

Simulation 2 - Impact of diversity, change of Θ_k and ρ_k



Analysis in the diversity case

System with diversity have larger transition probabilities:

- When h and σ are constant, and the average of θ_j is θ, then the system has a higher transition probability than with (h, σ, θ).
- For instance, if $h = 0$, $\Theta_k = \theta\,(1 + \delta\alpha_k)$ with $\delta \ll 1$, and $\sum_{k=1}^{K} \rho_k \alpha_k = 0$, then

$$\mathbf{P}\left(\mu_N^{\text{div}} \in A\right)$$
$$\approx \exp\left\{-\frac{N}{\sigma^2 T}\left[2\left(1-\frac{3\sigma^2}{2\theta}\right) - 2\delta^2\left(\sum_{k=1}^{K}\rho_k\alpha_k^2\right)\left(\frac{3\sigma^2}{2\theta}+\frac{1}{T}\int_0^T\left(1-e^{-\theta s}\right)^2 ds\right)\right]\right\}$$

A hierarchical model of systemic risk

Here we consider a hierarchical model with a central component:

$$dX_0 = -h_0 V_0'(X_0)dt - \theta_0\big(X_0 - \frac{1}{N}\sum_{j=1}^{N} X_j\big)dt$$

$$dX_j = -hV'(X_j)dt - \theta\big(X_j - X_0\big)dt + \sigma dW_j, \qquad j = 1, \ldots, N$$

- X_0 models the central stable component. It is intrinsically stable ($h_0 > 0$), and not subjected to external fluctuations. It interacts with the other components through a mean field interaction.
- X_j, $j = 1, \ldots, N$ model individual components that are subjected to external fluctuations. They are ($h > 0$) or are not ($h = 0$) intrinsically stable. They interact with the central component X_0.

A hierarchical model of systemic risk - Analysis

- Nonlinear Fokker-Planck equations: In the limit $N \to \infty$ the pair $\left(X_0(t), \frac{1}{N}\sum_{j=1}^N \delta_{X_j(t)}(dx)\right)$ converges to $(x_0(t), u(t,x)dx)$ solution of the nonlinear Fokker-Planck equation

$$\partial_t u = \frac{\sigma^2}{2}\partial_{xx}^2 u + \partial_x\left[hV'(x) + \theta(x - x_0(t))u\right],$$

$$\text{with } \frac{dx_0}{dt} = -h_0 V_0'(x_0) - \theta_0\left(x_0 - \int xu(t,x)dx\right).$$

- Existence of two equilibrium states $(x_0(t), u(t,x)) \equiv (x_e, u_e(x))$ when σ is below a critical level:

$$u_e(x) = \frac{1}{Z_e}\exp\left(-\frac{2hV(x) + \theta(x - x_e)^2}{\sigma^2}\right)$$

  with the compatibility equation $\int xu_e(x)dx = x_e + \frac{h_0}{\theta_0}V_0'(x_e)$.
- Large deviations principle to compute the probability of transition.

A hierarchical model of systemic risk - Results

- "Exact" results for $h = 0$ and expansions for small h (for the optimal paths and for the probability of transition).
- Resolution of an ODE system for the optimal $(\bar{x}(t), x_0(t))$ with boundary conditions.
- For the optimal path the mean of the individual components $\bar{x}(t)$ is ahead of $x_0(t)$: the individual components drive the transition.
- Stability increases with θ and decreases with θ_0.

Conclusions and related work

- It is possible to simultaneously reduce individual risk and increase the systemic risk.

J. Garnier, G. Papanicolaou, and T.-W. Yang, SIAM Math. Finance **4**, pp. 151-184 (2013).

J. Garnier, G. Papanicolaou, and T.-W. Yang, Risk and Decision Analysis, in press.

- Using the analysis as a guide, it is possible to design importance sampling algorithms for computing efficiently (very) small systemic failure probabilities.
  $\rightarrow$ Strategy applied to a conservation law with random space-time forcing in order to estimate the probability of anomalous shock profile displacement (scramjet problem).

J. Garnier, G. Papanicolaou, and T.-W. Yang, SIAM Multiscale Model. Simul. **11**, pp. 1000-1032 (2013).

Topological spaces for the mean field model

- $M_1(\mathbb{R})$ is the space of probability measures on $\mathbb{R}$ with the Prohorov metric ρ, associated with the weak convergence.
- $C([0,T], M_1(\mathbb{R}))$ is the space of continuous functions from $[0,T]$ to $M_1(\mathbb{R})$ with the metric $\sup_{0\le t\le T} \rho(\mu_1(t), \mu_2(t))$.
- $M_\infty(\mathbb{R}) = \{\mu \in M_1(\mathbb{R}), \int \varphi(y)\mu(dy) < \infty\}$, where $\varphi(y) = y^4$ serves as a Lyapunov function. $M_\infty(\mathbb{R})$ is endowed with the inductive topology: $\mu_n \to \mu$ in $M_\infty(\mathbb{R})$ if and only if $\mu_n \to \mu$ in $M_1(\mathbb{R})$ and $\sup_n \int \varphi(y)\mu_n(dy) < \infty$.
- $C([0,T], M_\infty(\mathbb{R}))$ is the space of continuous functions from $[0,T]$ to $M_\infty(\mathbb{R})$ endowed with the topology: $\phi_n(\cdot) \to \phi(\cdot)$ in $C([0,T], M_\infty(\mathbb{R}))$ if and only if $\phi_n(\cdot) \to \phi(\cdot)$ in $C([0,T], M_1(\mathbb{R}))$ and $\sup_{0\le t\le T} \sup_n \int \varphi(y)\phi_n(t, dy) < \infty$.

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Large Deviations Principle

Exact statement:

$$
\begin{aligned}
-\inf_{\phi \in \mathring{A}} I_h(\phi) &\leq \liminf_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(\mu_N \in A\right) \\
&\leq \limsup_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(\mu_N \in A\right) \leq -\inf_{\phi \in \overline{A}} I_h(\phi)
\end{aligned}
$$

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The rare event

$$A = \left\{\phi : \phi(0) = u^e_{-\xi_b},\ \phi(T) = u^e_{+\xi_b}\right\}.$$

Definition of an enlarged rare event (with non-empty interior):

$$A_\delta = \left\{\phi : \phi(0) = u^e_{-\xi_b},\ \rho(\phi(T), u^e_{+\xi_b}) \leq \delta\right\},$$

where ρ is the Prohorov metric. We have

$$\lim_{\delta \to 0} \inf_{\phi \in A_\delta} I_h(\phi) = \inf_{\phi \in A} I_h(\phi)$$

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The classical Freidlin-Wentzell formula

Let

$$dX^N = b(X^N)dt + \frac{1}{\sqrt{N}}\sigma dW_t$$

$(X^N(t))_{t\in[0,T]}$ satisfies a large deviation principle in $C([0,T],\mathbb{R})$ with the rate function

$$I((x(t))_{t\in[0,T]}) = \begin{cases} \frac{1}{2\sigma^2}\int_0^T \left(\partial_t x(t) - b(x(t))\right)^2 dt, & (x(t))_{t\in[0,T]} \in H^1 \\ +\infty, & \text{otherwise.} \end{cases}$$

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