Energy (super)diffusion for systems with two conserved quantities

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Work in collaboration with C. Bernardin and H. Spohn

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Motivation

- Thermal transport in one-dimensional chains is generically anomalous[1,2]
- **Necessary/sufficient conditions to have normal transport?**
  - anharmonicity of the potential
  - random masses
  - pinning potentials
  - destroy momentum conservation (velocity flips)
- **Main difficulty**: ergodic properties of deterministic systems
- Here: simplified Hamiltonian system with added exchange noise

[1] S. Lepri, R. Livi, and A. Politi, *Phys. Rep.* **377**, 1–80 (2003)
[2] A. Dhar, *Adv. Physics* **57**, 457–537 (2008)

Outline of the talk

- **Definition of the microscopic model**
- **Hydrodynamic limit**
- **Space-time correlations of the invariants**
  - Systems with one invariant
  - General framework for systems with 2 invariants
  - Numerical illustration
- **(Super)diffusive properties**

C. Bernardin and G. Stoltz, Anomalous diffusion for a class of systems with two conserved quantities, *Nonlinearity* **25** (2012) 1099-1133

H. Spohn and G. Stoltz, Nonlinear fluctuating hydrodynamics in one dimension: the case of two conserved fields, *arXiv preprint* **1410.7896** (2014)

Definition of the microscopic model

Deterministic part of the evolution

- Unknowns: "heights" η_i (infinite volume or finite volume + BC)

- Local energy $V(\eta_i)$ [assumptions on V]

- Deterministic part of the evolution ($\simeq$ Hamiltonian system with equal potential and kinetic energies)

$$\frac{d}{dt}\eta_i = V'(\eta_{i+1}) - V'(\eta_{i-1})$$

with generator $\mathcal{A} = \sum_i \big(V'(\eta_{i+1}) - V'(\eta_{i-1})\big)\partial_{\eta_i}$

- Invariants: $\sum_i \eta_i$ and $\sum_i V(\eta_i)$

- Invariance of grand-canonical measures: $\beta > 0$, $\tau \in \mathbb{R}$

$$\mu_{\tau,\beta}(d\boldsymbol{\eta}) = \prod_i Z_{\tau,\beta}^{-1}\, \mathrm{e}^{-\beta(V(\eta_i)+\tau\eta_i)}\, d\eta_i$$

Stochastic perturbation

- Add some stochastic noise preserving the invariants

$\rightarrow$ exchange noise between η_i and η_{i+1} at exponential times $\mathcal{E}(1/\gamma)$

- Interest: improves ergodic properties

- Total generator $\mathcal{L} = \mathcal{A} + \gamma \mathcal{S}$ with

$$\mathcal{S}f(\boldsymbol{\eta}) = \sum_i \Big(f\left(\boldsymbol{\eta}^{i,i+1}\right) - f(\boldsymbol{\eta}) \Big), \quad \boldsymbol{\eta}^{i,i+1} = (\dots, \eta_{i-1}, \eta_{i+1}, \eta_i, \eta_{i+2}, \dots)$$

- Well-posedeness of dynamics in infinite volume for initial conditions in

$$\Omega = \bigcap_{\alpha > 0} \left\{ (\eta_i)_{i \in \mathbb{Z}} \,\middle|\, \sum_{i \in \mathbb{Z}} \eta_i^2 \mathrm{e}^{-\alpha |i|} \right\}$$

- Existence/uniqueness of the invariant measure for finite systems + Langevin thermostats (possibly $T_0 \neq T_N$)

Currents associated with local invariants

- For the deterministic part (discrete gradient $\nabla u_x = u_{x+1} - u_x$)

$$\frac{d}{dt}\begin{pmatrix} \eta_i \\ V(\eta_i) \end{pmatrix} = -\nabla J^{i-1,i}, \quad J^{i,i+1} = \begin{pmatrix} j_h^{i,i+1} \\ j_e^{i,i+1} \end{pmatrix} = -\begin{pmatrix} V'(\eta_i) + V'(\eta_{i+1}) \\ V'(\eta_i)V'(\eta_{i+1}) \end{pmatrix}$$

- When the exchange noise is present ($\gamma > 0$)

$$V(\eta_i(t)) - V(\eta_i(0)) = -\nabla\left[\int_0^t j_{e,\gamma}^{i-1,i}(s)ds + M_{e,\gamma}^{i-1}(t)\right],$$

$$\eta_i(t) - \eta_i(0) = -\nabla\left[\int_0^t j_{h,\gamma}^{i-1,i}(s)ds + M_{h,\gamma}^{i-1}(t)\right],$$

with local martingales $M_{e,\gamma}^i(t), M_{h,\gamma}^i(t)$ and

$$j_{e,\gamma}^{i,i+1} = j_e^{i,i+1} - \gamma\nabla\left[V(\eta_i)\right], \qquad j_{h,\gamma}^{i,i+1} = j_h^{i,i+1} - \gamma\nabla\left[\eta_i\right]$$

Hydrodynamic limit

Expected limiting PDE

- Average evolution under hyperbolic space-time scaling (periodic BC)

$\rightarrow$ Only the deterministic part of the dynamics matters

- Average currents $\langle V'(\eta_i)\rangle_{\tau,\beta} = -\tau$ and $\langle V'(\eta_i)V'(\eta_{i+1})\rangle_{\tau,\beta} = \tau^2$
- Thermodynamic description: τ, β are in bijection with h, e

$$h_{\tau,\beta} = \langle \eta_i \rangle_{\tau,\beta}, \qquad e_{\tau,\beta} = \langle V(\eta_i) \rangle_{\tau,\beta}$$

- Local Gibbs equilibrium associated with energy/height profiles $e(x), h(x)$

$$d\mu^N_{e,h}(\boldsymbol{\eta}) = \prod_{i \in \mathbb{T}_N} \frac{\exp\left(-\beta\left(i/N\right)\left[V(\eta_i) + \tau\left(i/N\right)\eta_i\right]\right)}{Z\left(\beta\left(i/N\right), \tau\left(i/N\right)\right)} \, d\eta_i,$$

where $(\beta(x), \tau(x))$ are actually functions of $(e(x), h(x))$

Expected hydrodynamic limit

$$\partial_t \begin{pmatrix} h(x,t) \\ e(x,t) \end{pmatrix} + \partial_x \begin{pmatrix} 2\tau(h(x,t), e(x,t)) \\ -\tau(h(x,t), e(x,t))^2 \end{pmatrix} = 0$$

Precise statement of the results (1)

Assumptions on the potential

V is a smooth, non-negative function such that

$$\forall \lambda \in \mathbb{R}, \beta > 0, \qquad Z(\beta, \lambda) = \int_{-\infty}^{\infty} \exp\left(-\beta V(r) - \lambda r\right) dr < +\infty$$

$$0 < V''(r) \leqslant C,$$

$$\limsup_{|r| \to +\infty} \frac{rV'(r)}{V(r)} \in (0, +\infty), \qquad \limsup_{|r| \to +\infty} \frac{[V'(r)]^2}{V(r)} < +\infty.$$

- Empirical energy-volume measure: smooth functions $G, H : \mathbb{T} \to \mathbb{R}$

$$\begin{pmatrix} \mathcal{E}_N(t, G) \\ \mathcal{H}_N(t, H) \end{pmatrix} = \begin{pmatrix} \displaystyle \frac{1}{N} \sum_{i \in \mathbb{T}_N} G\left(\frac{i}{N}\right) V(\eta_i(t)) \\ \displaystyle \frac{1}{N} \sum_{i \in \mathbb{T}_N} H\left(\frac{i}{N}\right) \eta_i(t) \end{pmatrix}$$

Precise statement of the results (2)

- Hyperbolic scaling: times Nt with spatial variables $i = Nx$

Hydrodynamic limit

Fix some $\gamma > 0$. Assume that the system is initially distributed according to a local Gibbs state with smooth profiles e_0, h_0. Consider $t > 0$ such that the solution $(e(t), h(t))$ remains smooth. Then,

$$\Big(\mathcal{E}_N(tN, G), \mathcal{H}_N(tN, H)\Big) \longrightarrow \left(\int_{\mathbb{T}} G(x)e(t,x)\,dx, \int_{\mathbb{T}} H(x)h(t,x)\,dx\right)$$

in probability as $N \to +\infty$

- Proof via Yau's relative entropy method[3,4]
- Key point: ergodicity of the dynamics in infinite volume

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[3] H.-T. Yau, *Lett. Math. Phys.* **22**(1) (1991), 63–80

[4] S. Olla, S. R. S. Varadhan, and H.-T. Yau, *Commun. Math. Phys.* (1993)

Ergodicity of the dynamics in infinite volume

- Relative entropy for $\mu, \nu \in \mathcal{P}(\Omega)$: finite subsets $\Lambda \subset \mathbb{Z}$,

$$\overline{H}(\nu|\mu) = \lim_{|\Lambda|\to\infty} \frac{H\left(\nu|_\Lambda \middle| \mu|_\Lambda\right)}{|\Lambda|}, \qquad H(\widetilde{\nu}|\widetilde{\mu}) = \sup_{\phi} \left\{ \int \phi \, d\widetilde{\nu} - \log\left(\int \mathrm{e}^{\phi} \, d\widetilde{\mu} \right) \right\}$$

Definition of the ergodicity

Any translation-invariant $\nu \in \mathcal{P}(\Omega)$, which is invariant by the dynamics and has a finite entropy density with respect to $\mu_{1,0}$, is a convex combination of the grand-canonical measures $\mu_{\beta,\tau}$ $(\beta > 0, \tau \in \mathbb{R})$:

$$\nu(f) = \int \mu_{\beta,\tau}(f) \, d\mathbb{P}(\beta,\tau)$$

- The dynamics generated by $\mathcal{L}$ is ergodic:
  - invariance of ν by $\mathcal{A}$ and $\mathcal{S}$ separately
  - invariance by $\mathcal{S}$ implies exchangeability
  - exchangeability + invariance by $\mathcal{A}$ implies ergodicity

Space-time correlations of the invariants

Fluctuations around a reference profile

- Hydrodynamic limit $\simeq$ law of large numbers $\to$ what about fluctuations?
- Linearization around reference uniform profile

$$h(x,t) = h_0 + \tilde{h}(x,t), \qquad e(x,t) = e_0 + \tilde{e}(x,t)$$

- Linearized evolution

$$\partial_t \begin{pmatrix} \tilde{h}(x,t) \\ \tilde{e}(x,t) \end{pmatrix} + A(h_0,e_0)\partial_x \begin{pmatrix} \tilde{h}(x,t) \\ \tilde{e}(x,t) \end{pmatrix} = 0,$$

where

$$A = 2 \begin{pmatrix} \partial_h \tau & \partial_e \tau \\ -\tau \partial_h \tau & -\tau \partial_e \tau \end{pmatrix}$$

Space-time correlators for $g_1(\eta) = \eta$ and $g_2(\eta) = V(\eta)$

$$S_{\alpha\alpha'}(i,t) = \left\langle g_\alpha(\eta_{i,t}) g_{\alpha'}(\eta_{0,0}) \right\rangle_{\tau,\beta} - \left\langle g_\alpha(\eta_{i,t}) \right\rangle_{\tau,\beta} \left\langle g_{\alpha'}(\eta_{0,0}) \right\rangle_{\tau,\beta}$$

Normal mode transformation

- Simultaneous reduction using a transformation matrix R

$$R\,A\,R^{-1} = \mathrm{diag}(c,0), \qquad R\,S(0,0)\,R^{\mathrm{T}} = \mathrm{Id}_{2\times 2}$$

- A has the eigenvalues 0 and $c = 2(\partial_h - \tau\partial_e)\tau < 0$
- Normal mode space-time correlation = evolution of Rg

$$S^{\sharp}(i,t) = R\,S(i,t)\,R^{\mathrm{T}}$$

- The linearized evolution transforms into

$$\partial_t \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix} = 0$$

- Fluctuations introduced by a suitable fluctuation/dissipation process

Introducing fluctuations in a heuristic way

- General idea[5]
  - nonlinearities of currents kept to quadratic order
  - linear dissipative term included
  - other d.o.f. subsumed as fluctuating currents (space-time white noise)

Coupled stochastic Burgers equations

$$\partial_t u_\alpha + \partial_x \left(c_\alpha u_\alpha + \langle \vec{u}, G^\alpha \vec{u} \rangle - \partial_x (D\vec{u})_\alpha + (\sqrt{2D}\vec{\xi})_\alpha \right) = 0 \,, \qquad \alpha = 1, 2,$$

with $G^\alpha \in \mathbb{R}^{2\times 2}$ symmetric, $D \in \mathbb{R}^{2\times 2}$ symmetric positive, and $\vec{\xi}$ vector of two independent mean zero Gaussian white noises

- Space-time scalings strongly depend on the coupling matrices G^1, G^2, which are the Hessians of the current

- Derivation heuristic: rely on numerical simulations for validation

[5] H. Spohn, *J. Stat. Phys.* **154**, 1191–1227 (2014)

Space-time scaling of correlation for one-component

- Simplified case: only one conserved quantity

$$\partial_t u_1 + \partial_x\left(c u_1 + G_{11}^1 u_1^2 - D\partial_x u_1 + \sqrt{2D}\xi_1\right) = 0$$

- Invariant measure: spatial white noise with mean zero and unit variance[6]
- Quantity of interest: covariance $\langle u_1(x,t) u_1(0,0)\rangle$

Large x, t: KPZ scaling with parameter $\lambda_{\rm s} = 2\sqrt{2}|G_{11}^1|$

$$\langle u_1(x,t) u_1(0,0)\rangle \simeq (\lambda_{\rm B} t)^{-2/3} f_{\rm KPZ}\left((\lambda_{\rm s} t)^{-2/3}(x - ct)\right)$$

- $f_{\rm KPZ}$ roughly Gaussian but with faster decaying tails[7] as $\exp(-0.295|x|^3)$

[6] T. Funaki and J. Quastel, arXiv:1407.7310 (2014)

[7] M. Prähofer and H. Spohn, *J. Stat. Phys.* **115**, 255–279 (2004)

Derivation of the scaling function (1)

- Introduce $f(x,t) = \langle u_1(x,t) u_1(0,0) \rangle$

Mode-coupling approximation (Gaussian factorization)

$$\partial_t f(x,t) = \left(-c\partial_x + D\partial_x^2\right) f_1(x,t) + \int_0^t \int_{\mathbb{R}} f(x-y,t-s)\partial_y^2 M_{11}(y,s)\, dy\, ds$$

with $M_{11}(x,t) = 2\left(G_{11}^1\right)^2 f(x,t)^2$

- Ansatz $f(x,t) = \dfrac{1}{(\lambda_{\mathrm{s}} t)^{2/3}} \mathscr{F}\left(\dfrac{x-ct}{(\lambda_{\mathrm{s}} t)^{2/3}}\right)$
- Remove center of mass by considering $f(t, x-ct)$
- Fourier transform in space with convention $\hat{g}(k) = \int_{\mathbb{R}} g(x)\, \mathrm{e}^{-2\mathrm{i}\pi kx}\, dx$

$$\partial_t \hat{f}(k,t) = - D_1 (2\pi k)^2 \hat{f}(k,t) - 2(2\pi k)^2 \left(G_{11}^1\right)^2 \int_0^t \hat{f}(k,t-s) \left[\int_{\mathbb{R}} \hat{f}(k-q,s)\, \hat{f}(q,s)\, dq\right] ds$$

Derivation of the scaling function (2)

- Ansatz $\widehat{f}(k,t) = F\big((\lambda_{\rm s} t)^{2/3} k\big)$
- Introduce $w = (\lambda_{\rm s} t)^{2/3} k$, so that

$$\frac{2}{3} F'(w) = -\pi^2 w \int_0^1 F\big((1-\theta)^{2/3} w\big) \left[\int_{\mathbb{R}} F\big(\theta^{2/3}(w-v)\big)\, F\big(\theta^{2/3} v\big)\, dv\right] d\theta$$

- The solution of this fixed point equation is close[8] to $f_{\rm KPZ}$
- Precise statement: $\displaystyle \lim_{k\to 0} \exp\left(2\mathrm{i}\pi c \frac{w^{3/2}}{\lambda_{\rm s} k^{1/2}}\right) \hat{f}\left(k, \frac{1}{\lambda_{\rm s}}\left[\frac{w}{k}\right]^{3/2}\right) = F(w)$

[8] Ch. B. Mendl and H. Spohn, *Phys. Rev. Lett.* **111**, 230601 (2013)

Space-time scalings of correlation for two components (1)

- Quantity of interest: covariance $\langle u_\alpha(x,t) u_{\alpha'}(0,0) \rangle$
- Diagonal approximation $\langle u_\alpha(x,t) u_{\alpha'}(0,0) \rangle \simeq \delta_{\alpha\alpha'} f_\alpha(x,t)$

Memory equation

$$\partial_t f_\alpha(x,t) = \left(-c_\alpha \partial_x + D_\alpha \partial_x^2\right) f_\alpha(x,t) + \int_0^t \int_{\mathbb{R}} f_\alpha(x-y, t-s) \partial_y^2 M_{\alpha\alpha}(y,s)\, dy\, ds, \quad \alpha = 1,2$$

with $D_{\alpha\alpha} = D_\alpha$ and memory kernel

$$M_{\alpha\alpha}(x,t) = 2 \sum_{\alpha',\alpha''=1,2} \left(G^\alpha_{\alpha'\alpha''}\right)^2 f_{\alpha'}(x,t) f_{\alpha''}(x,t)$$

- If $\alpha' \neq \alpha''$, the product $f_{\alpha'}(x,t) f_{\alpha''}(x,t)$ can be neglected

$$M_{\alpha\alpha}(x,t) = 2 \sum_{\alpha'=1,2} \left(G^\alpha_{\alpha'\alpha'}\right)^2 f_{\alpha'}(x,t)^2$$

Space-time scalings of correlation for two components (2)

- **Obtaining the asymptotic behavior**
  - educated scaling ansatz for $f_\alpha, f_{\alpha'}$ (appropriate exponents...)
  - Fourier transform in space of mode-coupling equations
- **Types of scaling functions**
  - Gaussian peak with width proportional to $\sqrt{t}$ (normal diffusion)
  - (modified) KPZ scaling function
  - maximally asymmetric α-Lévy ($b = 1$)

$$f_{\text{Lévy},\alpha,b}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \exp\left(-|k|^\alpha\left[1 - \mathrm{i} b \tan\left(\frac{1}{2}\pi\alpha\right)\mathrm{sgn}(k)\right]\right) \mathrm{e}^{\mathrm{i}kx}\, dk$$

- All behaviors can be encountered for certain lattice gas models[9]
- Rigorous results in some situations[10]

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[9] V. Popkov, J. Schmidt, and G. M. Schütz, arXiv:1410.8026 (2014)
[10] C. Bernardin, P. Gonçalves, and M. Jara, arXiv:1402.1562 (2014)

Space-time scalings of correlation for two components (3)

- Complete classification: 1 indicates non-zero value of the coefficent

$G_{11}^1 = 1,\ G_{22}^2 = 1$	G_{22}^1	G_{11}^2	peak 1	peak 2
	0,1	0,1	KPZ	KPZ

$G_{11}^1 = 1,\ G_{22}^2 = 0$	G_{22}^1	G_{11}^2	peak 1	peak 2
	0,1	1	KPZ	$\frac{5}{3}$-Lévy
	1	0	mod. KPZ	diff
	0	0	KPZ	diff

$G_{11}^1 = 0,\ G_{22}^2 = 0$	G_{22}^1	G_{11}^2	peak 1	peak 2
	1	1	*gold*-Lévy	*gold*-Lévy
	1	0	$\frac{3}{2}$-Lévy	diff
	0	1	diff	$\frac{3}{2}$-Lévy
	0	0	diff	diff

Application to the model under consideration

- Coupling matrices $G^\alpha = \frac{1}{2}\sum_{\alpha'=1}^{2} R_{\alpha,\alpha'}\, R^{-\mathrm{T}} H^{\alpha'}\, R^{-1}$ with Hessians

$$H^1 = \begin{pmatrix} \partial_h^2 j_h & \partial_h \partial_e j_h \\ \partial_h \partial_e j_h & \partial_e^2 j_h \end{pmatrix}, \qquad H^2 = \begin{pmatrix} \partial_h^2 j_e & \partial_h \partial_e j_e \\ \partial_h \partial_e j_e & \partial_e^2 j_e \end{pmatrix}$$

- Discussion on the values of the leading order couplings G^1_{11}, G^2_{22}
  - the only non-zero coefficient of G^2 is $G^2_{11} < 0$
  - in general, $G^1_{11} \neq 0$ (and other entries), except *e.g* harmonic potentials

Expected scalings

Sound mode: $\lambda_1 = 2\sqrt{2}\left|G^1_{11}\right|$. Heat mode: $\lambda_2 = a_{\mathrm{h}}\, c^{-1/3}\left(G^2_{11}\right)^2 \lambda_1^{-2/3}$.

$$f_1(x,t) \simeq (\lambda_1 t)^{-2/3} f_{\mathrm{KPZ}}((\lambda_1 t)^{-2/3}(x - ct))$$
$$f_2(x,t) \simeq (\lambda_2 t)^{-3/5} f_{\mathrm{Lévy},5/3,1}((\lambda_2 t)^{-3/5} x)$$

Numerical simulation (1)

- **Potentials used in the simulations**
  - FPU: $V(\eta) = \frac{1}{2}\eta^2 + \frac{a}{3}\eta^3 + \frac{1}{4}\eta^4$ with $a = 2$
  - Kac-van Moerbecke $V(\eta) = \frac{\mathrm{e}^{-\kappa\eta} + \kappa\eta - 1}{\kappa^2}$ with $\kappa = 1$

- **Creation of trajectories**
  - Canonical sampling of initial conditions (overdamped Langevin)
  - Integration with a splitting algorithm (odd/even sites) for deterministic part
  - Exponential clock attached to each bond for exchange noise
  - Evaluation of space-time correlation through empirical averages

- **Numerical correlation** $C^{\sharp}_{N,K}(i,n) \simeq \begin{pmatrix} f_1^{\mathrm{num}}(i,n) & 0 \\ 0 & f_2^{\mathrm{num}}(i,n) \end{pmatrix}$

Numerical simulation (2)

- **Computation of scaling factors**

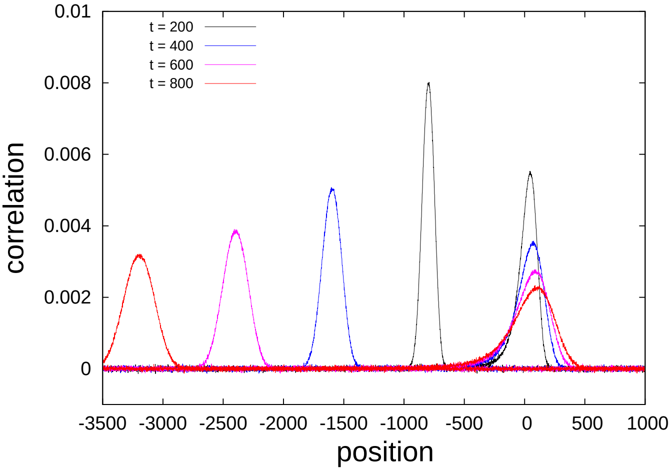
$$\inf_{\substack{x_n \in \mathbb{R} \\ \Lambda_n > 0}} \left\{ \sum_{i=0}^{N-1} \left| f_\alpha^{\mathrm{num}}(i,n) - (\Lambda_n)^{-1} f_\alpha^{\mathrm{mc}}\left((\Lambda_n)^{-1}(i - x_n)\right) \right| \right\}$$

Fit $x_n = cn\Delta t + x_0$ and $\Lambda_n = (\lambda\, n\Delta t)^\delta$

- **Parameters**
  - time step $\Delta t = 0.005$ (determined by energy conservation)
  - systems up to $N = 8000$
  - $K = 10^5$ independent realizations
  - $\beta = 2$ and $\tau = 1$

- Let's see a movie!

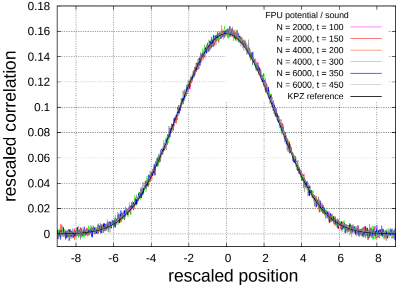
Numerical results: evolution of the peaks

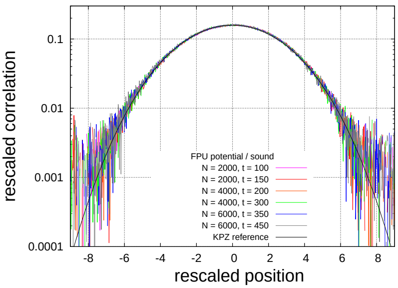


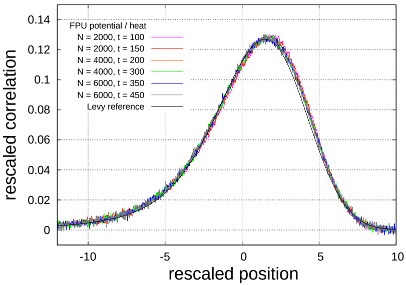
Evolution of the heat peak (centered at $x = 0$) and the sound peak, traveling to the left, for the KvM potential.

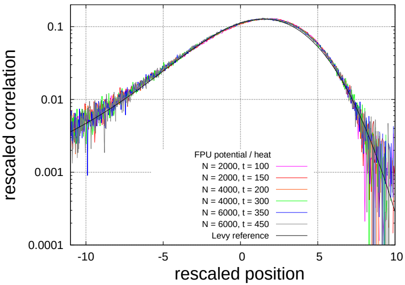
The heat peak is not symmetric, the rapid decay being away from the sound peak.

Numerical results: scaling of modes, FPU

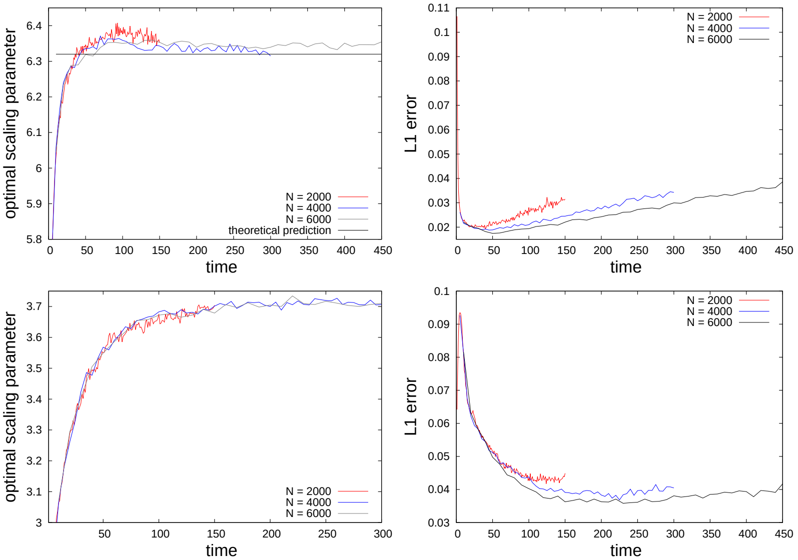






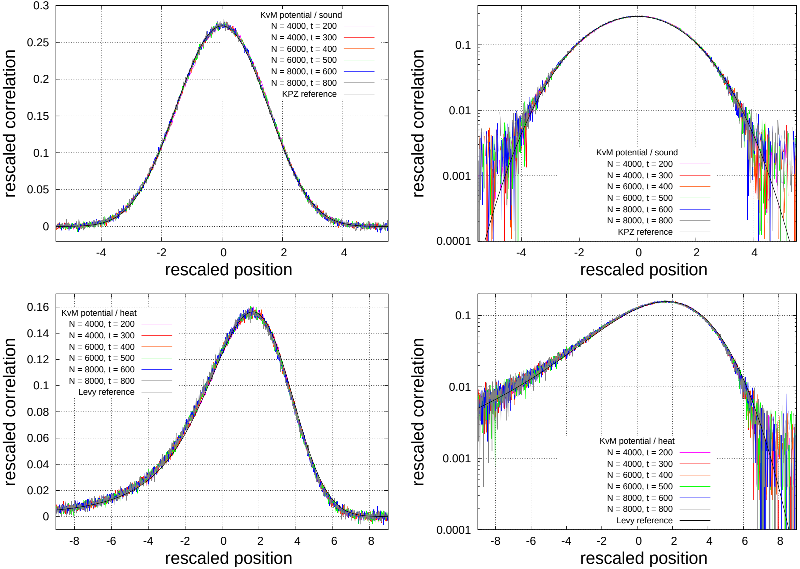


Numerical results: error and convergence, FPU



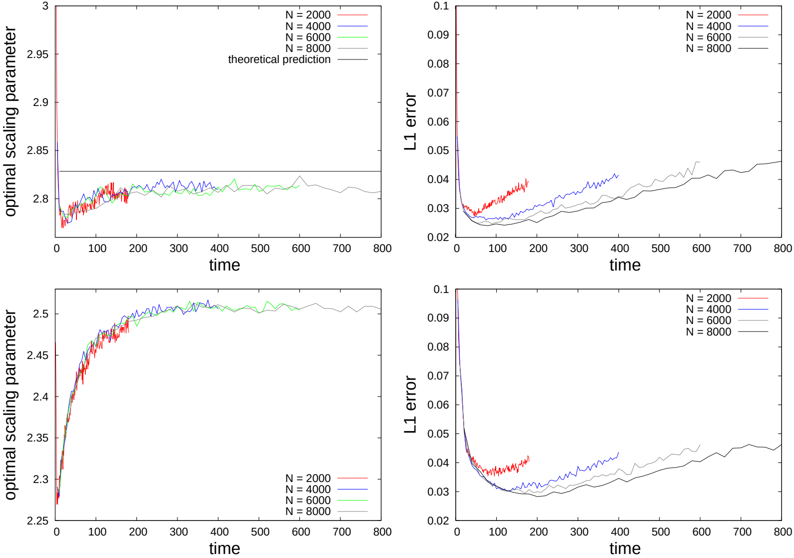
Top: sound mode. Bottom: heat mode. Left: Optimal λ. Right: L^1 error.

Numerical results: scaling of modes, KvM



Heat: maximally asymmetric Lévy distribution, parameter 1.57 instead of 5/3.

Numerical results: error and convergence, KvM



Top: sound mode. Bottom: heat mode. Left: Optimal λ. Right: L^1 error.

(Super)diffusive properties

Thermal transport

- Finite, open system with thermostats at temperatures T_ℓ, T_r

$$\begin{cases} d\eta_{-N}(t) = V'(\eta_{-N+1})dt - \lambda_\ell V'(\eta_{-N})dt + \sqrt{2\lambda_\ell T_\ell}\, dB_{-N}(t), \\ d\eta_i(t) = \Big(V'(\eta_{i+1}) - V'(\eta_{i-1})\Big)dt, \\ d\eta_N(t) = -V'(\eta_{N-1})dt - \lambda_r V'(\eta_N)dt + \sqrt{2\lambda_r T_r}\, dB_N(t), \end{cases}$$

and added random exchange noise with intensity $\gamma \geqslant 0$

- Existence/uniqueness of invariant measure (assumptions on V)[11]

Thermal conductivity: $T_r = T + \Delta T/2$, $T_\ell = T - \Delta T/2$

$$\kappa = \lim_{\Delta T \to 0} \frac{N\langle \mathcal{J}_N^\gamma \rangle_{\Delta T}}{\Delta} = \frac{2N^2}{T^2}\int_0^{+\infty} \mathbb{E}\left[\mathcal{J}_N^\gamma(t)\mathcal{J}_N^\gamma(0)\right] dt, \qquad \mathcal{J}_N^\gamma = \frac{1}{2N}\sum_{i=-N}^{N-1} j_{i,i+1}^{e,\gamma}$$

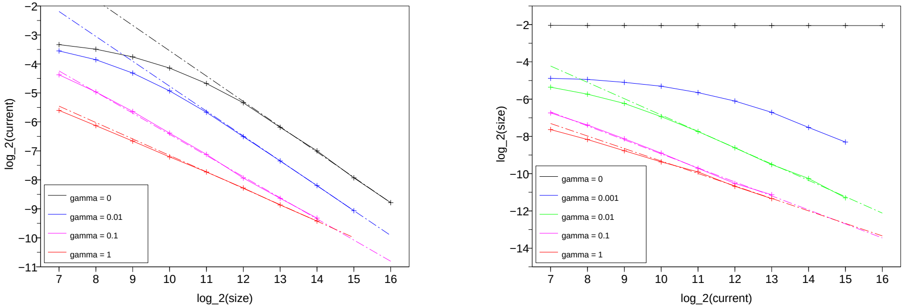
with expectation over equilibrium I.C./realizations of eq. dynamics

[11] P. Carmona, *Stoch. Proc. Appl.* 117 (2007)

Harmonic systems

- Specific case $V(r) = r^2/2$
- **Nonequilibrium dynamics in their steady states**
  - when $\gamma = 0$, the average current is $\dfrac{T_\ell - T_r}{\lambda_\ell + \lambda_\ell^{-1} + \lambda_r + \lambda_r^{-1}}$
  - when $\gamma > 0$, the average current is expected to scale as $C_\gamma \sqrt{N}$
- **Green-Kubo approach**
  - dynamics in infinite volume
  - only the current arising from the deterministic part of the dynamics matter
  - current autocorrelation scaling as $1/\sqrt{\gamma t}$
  - proof via Laplace transform + explicit solution of resolvent equation

Anharmonic systems



Current as a function of the system size $2N+1$. Left: FPU with $a=0$. Right: KvM. Simulation parameters: $\Delta t = 0.005$, $T_\ell = 1.1$ and $T_r = 0.9$, $\lambda_\ell = \lambda = 1$, long simulations (*e.g.* 10^8 steps for $2N+1 = 65,537$). Computed slopes $N \langle \mathcal{J}_N^\gamma \rangle \sim N^\delta$ below.

γ	harmonic	anharmonic	KVM
0	1	0.13	1
0.01	–	0.14	0.12
0.1	0.50	0.27	0.25
1	0.50	0.43	0.33