Averaging for SPDEs: strong and weak order

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Workshop: Hybrid dynamical systems simulation and applications to molecular dynamics
We study multiscale systems of SPDEs

\[
\begin{align*}
\frac{dX^\varepsilon(t)}{dt} &= (AX^\varepsilon(t) + F(X^\varepsilon(t), Y^\varepsilon(t))) dt \\
\frac{dY^\varepsilon(t)}{dt} &= \frac{1}{\varepsilon}(BY^\varepsilon(t) + G(X^\varepsilon(t), Y^\varepsilon(t))) dt + \frac{1}{\sqrt{\varepsilon}}dW(t)
\end{align*}
\]

\[
X^\varepsilon(0) = x, \quad Y^\varepsilon(0) = y
\]

in \( H = L^2(0, 1) \), on \([0, T]\).

\( W \): cylindrical Wiener process on \( H \).

\( \varepsilon \ll 1 \).
We study multiscale systems of SPDEs

\[ dX^\epsilon(t) = (AX^\epsilon(t) + F(X^\epsilon(t), Y^\epsilon(t))) \, dt \]

\[ dY^\epsilon(t) = \frac{1}{\epsilon} (BY^\epsilon(t) + G(X^\epsilon(t), Y^\epsilon(t))) \, dt + \frac{1}{\sqrt{\epsilon}} dW(t) \]

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Averaging principle: \( X^\epsilon \) can be approximated by \( \bar{X} \) defined by

\[ d\bar{X}(t) = (A\bar{X}(t) + \bar{F}(\bar{X}(t))) \, dt \]

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**Problems:**

- Give an order of convergence, in strong and weak sense.
- Application: HMM scheme.
- Comparison with the SDE case.
Existing results

- Finite dimensional case:
Existing results

- Finite dimensional case:

- Infinite dimensional case:
Linear coefficients (1)

Typical example: \( A = B = \frac{d^2}{dx^2} \), with domain \( H^2(0, 1) \cap H^1_0(0, 1) \) (Dirichlet boundary conditions).
Linear coefficients (1)

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Spectral properties:

\[
A e_k = -\lambda_k e_k \text{ for all } k \in \mathbb{N}
\]

\[
\lambda := \inf_{k \in \mathbb{N}} \lambda_k > 0, \lambda_k \sim Ck^2.
\]

\[
B f_k = -\mu_k f_k \text{ for all } k \in \mathbb{N}
\]

\[
\mu := \inf_{k \in \mathbb{N}} \mu_k > 0, \mu_k \sim C'k^2.
\]

We can define semi-groups \((e^{tA})_{t \geq 0}\) and \((e^{tB})_{t \geq 0}\).
Linear coefficients (2)

Definition

For $\alpha \in [0, 1]$, 

\[ (-A)^\alpha x = \sum_{k=0}^{\infty} \lambda_k^\alpha x_k e_k \]

with domain 

\[ D(-A)^\alpha = \left\{ x \in H; \sum_{k=0}^{+\infty} (\lambda_k)^{2\alpha} |x_k|^2 < +\infty \right\}; \]
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\]

Regularization properties: for \( 0 < s < t \)

\[
|e^{tA}x|_{(-A)^\alpha} \leq C\alpha t^{-\alpha} |x|_H
\]

\[
|e^{tA}x - e^{sA}x|_H \leq \begin{cases} C \frac{(t-s)^\delta}{s^\delta} |x|_H \\ C(t-s)^\delta |x|_{(-A)^\delta} \end{cases}
\]
Nonlinear coefficients

- $F : H^2 \rightarrow H$ is $C_b^2$.
- $U : H^2 \rightarrow \mathbb{R}$ is $C_b^3$.
- $G(x, y) := \nabla_y U(x, y)$. 
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Strict dissipativity assumption:

\[
L_G := \sup_{x,y_1,y_2 \in H} \frac{|G(x, y_1) - G(x, y_2)|}{|y_1 - y_2|} < \mu.
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\]

**Additional assumption:**

There exists $\alpha > 0$, $0 \leq \gamma < \frac{1}{4}$, $C > 0$ such that for every $x \in H$ and $y_1, y_2 \in D((-B)\gamma)$

\[
|(-A)^\alpha(F(x, y_1) - F(x, y_2))| \leq C|(-B)^\gamma(y_1 - y_2)|.
\]
Stochastic integration in $H(1)$

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ is a filtered probability space.
Stochastic integration in $H$ (1)

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ is a filtered probability space. 
$(q_k)_{k \in \mathbb{N}}$ is any complete orthonormal system of $H$, and  
$(\beta_k)_{k \in \mathbb{N}}$ are independent real brownian motions, with respect to $(\mathcal{F}_t)_{t \geq 0}$.
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$W(t) = \sum_k \beta_k(t) q_k$ is cylindrical Wiener process on $H$. 
Averaging for SPDEs: strong and weak order

The problem
Assumptions and results
The coefficients of the equations
Stochastic integration in $H$
Basic properties of solutions
The averaged equation
The results
Proof of the strong-order result
Proof of the weak-order result
Application: HMM scheme
Conclusion

Stochastic integration in $H$ (1)

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ is a filtered probability space.
$(q_k)_{k \in \mathbb{N}}$ is any complete orthonormal system of $H$, and
$(\beta_k)_{k \in \mathbb{N}}$ are independent real brownian motions, with respect
to $(\mathcal{F}_t)_{t \geq 0}$.
$W(t) = \sum_k \beta_k(t)q_k$ is cylindrical Wiener process on $H$.
This series does not converge in $H$, but only in $K$ such that
the imbedding $\Psi : H \subset K$ is Hilbert-Schmidt:

$$|\psi|^2_{L^2(H,K)} := \sum_{k=0}^{+\infty} |\psi(q_k)|^2_K < +\infty.$$
Stochastic integration in $H$ (2)

Take $(H, (q_k))$ and $(K, (r_l))$, and $\Psi$ random process with values in $\mathcal{L}(H, K)$.
Stochastic integration in $H(2)$

Take $(H, (q_k))$ and $(K, (r_l))$, and $\Psi$ random process with values in $\mathcal{L}(H, K)$.

$$
\int_0^T \Psi(s) dW(s) := \sum_{k,l} \int_0^T <\Psi(s)q_k, r_l> d\beta_k(s)r_l
$$

is well-defined for $\Psi \in L^2(\Omega \times [0, T]; \mathcal{L}_2(H, K))$.

Properties:

$$
\mathbb{E} \int_0^T \Psi(s) dW(s) = 0
$$

$$
\mathbb{E} \left| \int_0^T \Psi(s) dW(s) \right|^2_K = \mathbb{E} \int_0^T |\Psi(s)|^2_{\mathcal{L}_2(H,K)} ds.
$$

A generalization of Itô formula also holds.
Stochastic integration in $H(2)$

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A generalization of Itô formula also holds.

If $v \in H$, $<v, W(t)>$ exists and the \textbf{space-time white noise} property holds:

$$
\mathbb{E} < v_1, W(t) > < v_2, W(s) > = t \land s < v_1, v_2 > .
$$
Basic properties of solutions

The stochastic convolution \( W^B(t) = \int_0^t e^{(t-s)B} dW(s) \) is well-defined; it is the unique mild solution of

\[
dZ(t) = BZ(t)dt + dW(t), \quad Z(0) = 0.
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\]

**Proposition**

*For any \( \epsilon > 0, \ T > 0, \ x \in H, \ y \in H, \) our system admits a unique *mild solution* \((X^\epsilon, Y^\epsilon)\):

\[
    X^\epsilon(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(X^\epsilon(s), Y^\epsilon(s))ds
\]

\[
    Y^\epsilon(t) = e^{t\epsilon B}y + \frac{1}{\epsilon} \int_0^t e^{\frac{(t-s)}{\epsilon}B} G(X^\epsilon(s), Y^\epsilon(s))ds
\]

\[
    + \frac{1}{\sqrt{\epsilon}} \int_0^t e^{\frac{(t-s)}{\epsilon}B} dW(s).
\]
On the fast process

If \( x \in H \), the fast equation with frozen slow component is:

\[
dY_x(t, y) = (B Y_x(t, y) + G(x, Y_x(t, y)))dt + dW(t)
\]

\( Y_x(0, y) = y \).
On the fast process

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\begin{align*}
\frac{dY_x(t, y)}{dt} &= (BY_x(t, y) + G(x, Y_x(t, y)))dt + dW(t) \\
Y_x(0, y) &= y.
\end{align*}
$$

For any $t \geq 0$, $y, z \in H$,

$$
|Y_x(t, y) - Y_x(t, z)| \leq Ce^{-ct}|y - z| \text{ as }.
$$
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$$dY_x(t, y) = (BY_x(t, y) + G(x, Y_x(t, y)))dt + dW(t)$$

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For any $t \geq 0$, $y, z \in H$,

$$|Y_x(t, y) - Y_x(t, z)| \leq Ce^{-ct}|y - z| \text{ as}.$$ 

Consequences:

- It has a unique invariant probability measure:

$$\mu^x(dy) = \frac{1}{Z(x)}e^{2U(x, y)}\nu(dy).$$ (1)

- Exponential mixing: for $\phi$ Lipschitz continuous,

$$|E\phi(Y_x(t, y)) - \int \phi(z)\mu^x(dz)| \leq C(1 + |x| + |y|)e^{-ct}.$$
The averaged equation

Definition
For any \( x \in H \),

\[
\overline{F}(x) = \int_{H} F(x, y) \mu^x(dy).
\]
The averaged equation

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$\overline{F}$ is Lipschitz continuous.
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For any $x \in H$,

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\overline{F}(x) = \int_H F(x, y) \mu^x(dy).
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Proposition
$\overline{F}$ is Lipschitz continuous.

The averaged equation is:

$$
d\overline{X}(t) = (A\overline{X}(t) + \overline{F}(\overline{X}(t)))dt,
$$

with initial condition $\overline{X}(0) = x \in H$.

It admits a unique mild solution:

$$
\overline{X}(t) = e^{tA}x + \int_0^t e^{(t-s)A} \overline{F}(\overline{X}(s))ds.
$$
Strong and weak order of convergence

Theorem (Strong-order)

For any $0 < r \ll 1$, for any $T > 0$, if $x \in D(A)$, and $y \in H$, then for any $\epsilon > 0$ and any $0 \leq t \leq T$

$$\mathbb{E}|X^\epsilon(t) - \overline{X}(t)| \leq C\epsilon^{1/2-r}. \quad (2)$$
Strong and weak order of convergence

**Theorem (Strong-order)**

For any $0 < r \ll 1$, for any $T > 0$, if $x \in D(A)$, and $y \in H$, then for any $\epsilon > 0$ and any $0 \leq t \leq T$

$$
\mathbb{E}|X^\epsilon(t) - \overline{X}(t)| \leq C\epsilon^{1/2-r}.
$$

(2)

**Theorem (Weak-order)**

For any $0 < r \ll 1$, for any $\phi : H \to \mathbb{R}$ of class $C^2_b$, $T > 0$, if $x \in D(A)$, $y \in D(B)$, then for any $\epsilon > 0$ and $0 \leq t \leq T$

$$
|\mathbb{E}[\phi(X^\epsilon(t))] - \mathbb{E}[\phi(\overline{X}(t))]| \leq C\epsilon^{1-r}.
$$

(3)
Proof of the strong-order result (1)

Idea: introduction of a parameter $\delta$ and of auxiliary processes $(\tilde{X}^\epsilon, \tilde{Y}^\epsilon)$.

On $[k\delta, (k + 1)\delta]$, with $0 \leq k \leq N := \lfloor \frac{T_0}{\delta} \rfloor$, we define

\[ d\tilde{X}^\epsilon(t) = (A\tilde{X}^\epsilon(t) + F(X^\epsilon(k\delta), \tilde{Y}^\epsilon(t)))dt \]
\[ d\tilde{Y}^\epsilon(t) = \frac{1}{\epsilon} (B\tilde{Y}^\epsilon(t) + G(X^\epsilon(k\delta), \tilde{Y}^\epsilon(t)))dt + \frac{1}{\sqrt{\epsilon}} dW(t), \]

with $\tilde{X}^\epsilon(0) = x$, $\tilde{Y}^\epsilon(0) = y$, and a continuity assumption at any $k\delta$. 
Proof of the strong-order result (2)

- For any $\epsilon > 0$, for any $0 \leq t \leq T$
  \[
  \mathbb{E}|X^\epsilon(t) - \tilde{X}^\epsilon(t)|^2 \leq C\delta^2(1-r)
  \]
  \[
  \mathbb{E}|Y^\epsilon(t) - \tilde{Y}^\epsilon(t)|^2 \leq C\delta^2(1-r).
  \]

- Estimate: for any $0 \leq t \leq T$
  \[
  \mathbb{E}|\tilde{X}^\epsilon(t) - \bar{X}(t)|^2 \leq C\delta^2(1-r) + C(1+\delta^{-r})(1 + \frac{1}{1 - e^{c\delta/\epsilon}})\epsilon
  \]
Proof of the strong-order result (2)

For any $\epsilon > 0$, for any $0 \leq t \leq T$

$$\mathbb{E}|X^\epsilon(t) - \tilde{X}^\epsilon(t)|^2 \leq C\delta^{2(1-r)}$$
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Estimate: for any $0 \leq t \leq T$

$$\mathbb{E}|\tilde{X}^\epsilon(t) - \overline{X}(t)|^2 \leq C\delta^{2(1-r)} + C(1+\delta^{-r})(1+\frac{1}{1-e^{-c\delta/\epsilon}})\epsilon$$

Now we choose $\delta = \delta(\epsilon)$; then

$$\mathbb{E}|X^\epsilon(t) - \overline{X}(t)|^2 \leq C\epsilon^{(1-r')}.$$
Asymptotic expansion (1)

Imagine that we are dealing with SDEs.
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$$\mathbb{E}[\phi(X^\epsilon(T, x, y))] - \phi(X(T, x))$$

$$:= u^\epsilon(T, x, y) - \overline{u}(T, x).$$
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\[ \mathbb{E}[\phi(X^\epsilon(T, x, y))] - \phi(X(T, x)) \]
\[ := u^\epsilon(T, x, y) - \bar{u}(T, x). \]

\( u^\epsilon \) and \( \bar{u} \) are solutions of Kolmogorov equations:

\[
\frac{\partial u^\epsilon}{\partial t}(t, x, y) = L^\epsilon u^\epsilon(t, x, y)
\]
\[ u^\epsilon(0, x, y) = \phi(x) \]

\[
\frac{\partial \bar{u}}{\partial t}(t, x, y) = L\bar{u}(t, x, y)
\]
\[ \bar{u}(0, x, y) = \phi(x) \]
Asymptotic expansion (2)

Differential operators are

\[ L_1 \psi(x, y) = \langle By + G(x, y), D_y \psi(x, y) \rangle + \frac{1}{2} \text{Tr}(D_{yy}^2 \psi(x, y)) \]

\[ L_2 \psi(x, y) = \langle Ax + F(x, y), D_x \psi(x, y) \rangle \]

\[ L^\epsilon = \frac{1}{\epsilon} L_1 + L_2 \]

\[ \bar{L} \psi(x) = \langle Ax + \bar{F}(x), D_x \psi(x) \rangle. \]
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\[ L^\epsilon = \frac{1}{\epsilon} L_1 + L_2 \]
\[ \bar{L} \psi(x) = \langle Ax + \bar{F}(x), D_x \psi(x) \rangle. \]

Strategy: find an expansion of \( u^\epsilon \) with respect to the parameter \( \epsilon \):

\[ u^\epsilon = u_0 + \epsilon u_1 + \nu^\epsilon, \]
\( \nu^\epsilon \) being a residual term.
Asymptotic expansion (3)

By identification with respect to powers of $\epsilon$:

\[ L_1 u_0 = 0 \]
\[ \frac{\partial u_0}{\partial t} = L_1 u_1 + L_2 u_0. \]
Asymptotic expansion (3)

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\[ \frac{\partial u_0}{\partial t} = L_1 u_1 + L_2 u_0. \]

$u_0$ does not depend on $y$, and is solution of $\frac{\partial u_0}{\partial t} = \overline{L} u_0$, with $u_0(0, .) = \phi$: therefore $u_0 = \overline{u}$. 
Asymptotic expansion (3)

By identification with respect to powers of $\epsilon$:

$$L_1 u_0 = 0$$

$$\frac{\partial u_0}{\partial t} = L_1 u_1 + L_2 u_0.$$

$u_0$ does not depend on $y$, and is solution of $\frac{\partial u_0}{\partial t} = L u_0$, with $u_0(0, .) = \phi$: therefore $u_0 = \bar{u}$.

$u_1$ is solution of

$$L_1 u_1(t, x, y) = < \bar{F}(x) - F(x, y), D_x u_0(t, x) >$$

$$:= -\chi(t, x, y).$$

Then $u_1(t, x, y) = \int_0^{+\infty} E[\chi(t, x, Y_x(s, y))] ds$. 
Asymptotic expansion (4)

Then

\[(\partial_t - \frac{1}{\epsilon}L_1 - L_2)v^\epsilon = \epsilon(L_2u_1 - \frac{\partial u_1}{\partial t}).\]
Asymptotic expansion (4)

Then

$$(\partial_t - \frac{1}{\epsilon} L_1 - L_2) v^\epsilon = \epsilon (L_2 u_1 - \frac{\partial u_1}{\partial t}).$$

Therefore

$$u^\epsilon(T, x, y) - u^0(T, x, y) = \epsilon u^1(T, x, y) + \epsilon \mathbb{E}[u^1(0, X^\epsilon(T, x, y), Y^\epsilon(T, x, y))]
+ \epsilon \mathbb{E}[\int_0^T (L_2 u_1 - \frac{\partial u_1}{\partial t})(T - t, X^\epsilon(t, x, y), Y^\epsilon(t, x, y)) dt].$$

(4)
Asymptotic expansion (4)

Then

\[(\partial_t - \frac{1}{\epsilon} L_1 - L_2)v^\epsilon = \epsilon(L_2u_1 - \frac{\partial u_1}{\partial t}).\]

Therefore

\[u^\epsilon(T, x, y) - u^0(T, x, y) = \epsilon u^1(T, x, y) + \epsilon \mathbb{E}[u^1(0, X^\epsilon(T, x, y), Y^\epsilon(T, x, y))] + \epsilon \mathbb{E}\left[\int_0^T (L_2u_1 - \frac{\partial u_1}{\partial t})(T - t, X^\epsilon(t, x, y), Y^\epsilon(t, x, y))dt\right].\]

(4)

If you can control each term, the proof is done.
The SPDE case

$A$ and $B$ are unbounded.
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The Kolmogorov equations are more difficult to deal with.
The SPDE case

$A$ and $B$ are unbounded.
The Kolmogorov equations are more difficult to deal with.
Remedy: reduction to a finite dimensional problem and proving uniform bounds with respect to dimension.
Reduction to a finite dimensional problem (1)

We use spaces $H_N^{(1)}$ and $H_N^{(2)}$ spanned by the first $N$ eigenvectors of the operators $A$ and $B$. 

We have new invariant measures $\mu_N(x, dy)$, new averaged coefficient $F_N$. 

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Example of estimate
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Reduction to a finite dimensional problem (1)

We use spaces $H_N^{(1)}$ and $H_N^{(2)}$ spanned by the first $N$ eigenvectors of the operators $A$ and $B$.

We naturally define orthogonal projectors $P_N^{(1)}$, $P_N^{(2)}$, coefficients $F_N$, $G_N$, and processes

$$
\begin{align*}
\mathrm{d}X_N^\epsilon(t) &= (AX_N^\epsilon(t) + F_N(X_N^\epsilon(t), Y_N^\epsilon(t)))\,\mathrm{d}t \\
\mathrm{d}Y_N^\epsilon(t) &= \frac{1}{\epsilon}(BY_N^\epsilon(t) + G_N(X_N^\epsilon(t), Y_N^\epsilon(t)))\,\mathrm{d}t + \frac{1}{\sqrt{\epsilon}}\,\mathrm{d}W_N(t),
\end{align*}
$$

(5)
Reduction to a finite dimensional problem (1)

We use spaces $H^{(1)}_N$ and $H^{(2)}_N$ spanned by the first $N$ eigenvectors of the operators $A$ and $B$. We naturally define orthogonal projectors $P^{(1)}_N$, $P^{(2)}_N$, coefficients $F_N$, $G_N$, and processes

$$
dX^\varepsilon_N(t) = (AX^\varepsilon_N(t) + F_N(X^\varepsilon_N(t), Y^\varepsilon_N(t)))dt
$$
$$
dY^\varepsilon_N(t) = \frac{1}{\varepsilon}(BY^\varepsilon_N(t) + G_N(X^\varepsilon_N(t), Y^\varepsilon_N(t)))dt + \frac{1}{\sqrt{\varepsilon}}dW_N(t),
$$
(5)

We have new invariant measures $\mu^x_N(dy)$, new averaged coefficient $\overline{F_N}$.
Reduction to a finite dimensional problem (2)

New averaged equation

\[ d\overline{X}_N(t) = (A\overline{X}_N(t) + \overline{F}_N(\overline{X}_N(t)))dt. \] (6)
Reduction to a finite dimensional problem (2)

New averaged equation

\[ d\overline{X}_N(t) = (A\overline{X}_N(t) + \overline{F}_N(\overline{X}_N(t)))dt. \quad (6) \]

Lemma

1. For any fixed \( \epsilon > 0 \) and \( t \geq 0 \), when \( N \to +\infty \)

\[ \mathbb{E}|X^\epsilon(t) - X^\epsilon_N(t)|^2 + \mathbb{E}|Y^\epsilon(t) - Y^\epsilon_N(t)|^2 \to 0. \]

2. For any \( t \geq 0 \), when \( N \to +\infty \)

\[ |\overline{X}(t) - \overline{X}_N(t)| \to 0. \]
Example of estimate

For any $t, x, y$,

$$|u_1(t, x, y)| = \left| \int_0^{+\infty} \mathbb{E}\left[\chi(t, x, Y_x(s, y))\right] ds \right|$$

$$\leq \int_0^{+\infty} \left| \mathbb{E}\left[\chi(t, x, Y_x(s, y))\right] \right| ds$$

$$\leq \int_0^{+\infty} C e^{-c s} (1 + |x| + |y|)[y \mapsto \chi(t, x, y)]_{\text{Lip}} ds.$$
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$$\leq \int_0^{+\infty} |E[\chi(t, x, Y_x(s, y))]| ds$$

$$\leq \int_0^{+\infty} Ce^{-cs} (1 + |x| + |y|) [y \mapsto \chi(t, x, y)]_{\text{Lip}} ds.$$

But $\chi(t, x, y) = \langle \overline{F}(x) - F(x, y), D_x u_0(t, x) \rangle$; so you need $|D_x u_0(t, x). h|$, for any $h$. 
Since \( u_0(t, x) = \phi(\overline{X}(t, x)) \), we compute
\[
D_x u_0(t, x).h = D\phi(\overline{X}(t, x)).\eta^h(t, x),
\]
with
\[
\frac{d\eta^h(t, x)}{dt} = A\eta^h(t, x) + D\overline{F}(\overline{X}(t, x)).\eta^h(t, x)
\]
\[
\eta^h(0, x) = h.
\]
Since $u_0(t, x) = \phi(\overline{X}(t, x))$, we compute

$$D_xu_0(t, x).h = D\phi(\overline{X}(t, x)).\eta^h(t, x),$$

with

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$$\eta^h(0, x) = h.$$

For any $0 \leq t \leq T$ $|\eta^h(t, x)|^2 \leq C_T|h|^2$. 
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\]
\[
\eta^h(0, x) = h.
\]

For any \( 0 \leq t \leq T \) \( |\eta^h(t, x)|^2 \leq C_T |h|^2 \).

Conclusion: for any \( t, x, y \)
\[
|\epsilon u^1(T, x, y) + \epsilon \mathbb{E}[u^1(0, X^\epsilon(T, x, y), Y^\epsilon(T, x, y))]| \leq C(1 + |x| + |y|) \epsilon.
\]
Since $u_0(t, x) = \phi(\overline{X}(t, x))$, we compute

$$D_xu_0(t, x).h = D\phi(\overline{X}(t, x)).\eta^h(t, x),$$

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For any $0 \leq t \leq T$ $|\eta^h(t, x)|^2 \leq C_T|h|^2$.

Conclusion: for any $t, x, y$

$$|\epsilon u^1(T, x, y) + \epsilon \mathbb{E}[u^1(0, X^\epsilon(T, x, y), Y^\epsilon(T, x, y))]|$$

$$\leq C(1 + |x| + |y|)\epsilon.$$

...
Description of the numerical method (1)

Aim: approximation of the slow component $X^\varepsilon$.

Principle:
- 2 time step size $\Delta t$ (macrosolver for $X$) and $\delta t$ (microsolver for $Y$): *Heterogeneous Multiscale Method*.
- Instead of looking at $X^\varepsilon(t)$, look at $\overline{X}(t)$ (averaging result!).
- Microsolver used to approximate $\overline{F}$.

More precisely:
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- 2 time step size $\Delta t$ (macrosolver for $X$) and $\delta t$ (microsolver for $Y$): *Heterogeneous Multiscale Method*.
- Instead of looking at $X^\varepsilon(t)$, look at $\bar{X}(t)$ (averaging result!).
- Microsolver used to approximate $\bar{F}$.

More precisely:

$$X_{n+1} = X_n + \Delta tAX_{n+1} + \Delta t\tilde{F}_n$$

$$X_0 = x.$$
Description of the numerical method (2)

\[
Y_{n,m+1,j} = Y_{n,m,j} + \frac{\delta t}{\epsilon} BY_{n,m+1,j} + \frac{\delta t}{\epsilon} G(X_n, Y_{n,m,j}) + \sqrt{\frac{\delta t}{\epsilon}} \chi_{n,m+1,j},
\]
Description of the numerical method (2)

\[
Y_{n,m+1,j} = Y_{n,m,j} + \frac{\delta t}{\varepsilon} B Y_{n,m+1,j} + \frac{\delta t}{\varepsilon} G(X_n, Y_{n,m,j}) \\
+ \sqrt{\frac{\delta t}{\varepsilon}} \chi_{n,m+1,j},
\]

with \( \chi_{n,m+1,j} = \frac{1}{\sqrt{\frac{\delta t}{\varepsilon}}} (W_{(m+1)\delta t}^{n,j} - W_{m\delta t}^{n,j}). \)
Description of the numerical method (2)

\[ Y_{n,m+1,j} = Y_{n,m,j} + \frac{\delta t}{\varepsilon} B Y_{n,m+1,j} + \frac{\delta t}{\varepsilon} G(X_n, Y_{n,m,j}) \]
\[ + \sqrt{\frac{\delta t}{\varepsilon}} \chi_{n,m+1,j}, \]

with \[ \chi_{n,m+1,j} = \frac{1}{\sqrt{\frac{\delta t}{\varepsilon}}} (W_{m+1,j}^{n} \delta t - W_{m,j}^{n} \delta t). \]

\[ \tilde{F}_n = \frac{1}{MN} \sum_{j=1}^{M} \sum_{m=n_T}^{N_m} F(X_n, Y_{n,m,j}), \]

with parameters \( M, n_T, N, N_m = n_T + N - 1. \)
Description of the numerical method (2)

\[
Y_{n, m+1, j} = Y_{n, m, j} + \frac{\delta t}{\epsilon} B Y_{n, m+1, j} + \frac{\delta t}{\epsilon} G(X_n, Y_{n, m, j}) \\
+ \sqrt{\frac{\delta t}{\epsilon}} \chi_{n, m+1, j},
\]

with \( \chi_{n, m+1, j} = \frac{1}{\sqrt{\frac{\delta t}{\epsilon}}} (W^{n, j}_{(m+1)} - W^{n, j}_{m}). \)

\[
\tilde{F}_n = \frac{1}{MN} \sum_{j=1}^{M} \sum_{m=n_T}^{N_m} F(X_n, Y_{n, m, j}),
\]

with parameters \( M, n_T, N, N_m = n_T + N - 1. \)

Initial conditions:

\[
Y_{0, 0, j} = y \\
Y_{n+1, 0, j} = Y_{n, N_m, j}.
\]
Averaging for SPDEs: strong and weak order

Charles-Edouard BREHIER

The problem
Assumptions and results
Proof of the strong-order result
Proof of the weak-order result
Application: HMM scheme
Description of the numerical method
Convergence results
Conclusion

Strong convergence

**Theorem**

Assume \( x \in D(A) \). For any \( 0 < r \ll 1 \), for any \( T > 0 \), there exists \( c, C > 0 \) such that for any \( 0 \leq n \leq N_0 = \left\lfloor \frac{T}{\Delta t} \right\rfloor \)

\[
E|X_n - X(n\Delta t)| \leq C(\epsilon^{1/2-r} + \Delta t^{1-r}) + C\left[\left(\frac{\delta t}{\epsilon}\right)^{1/2-r} + \frac{1}{\sqrt{N\frac{\delta t}{\epsilon} + 1}}e^{-cnT}\frac{\delta t}{\epsilon} (R + \sqrt{R})\right] + C\frac{\sqrt{\Delta t}}{\sqrt{M(N\frac{\delta t}{\epsilon} + 1)}},
\]

with \( R = \frac{\Delta t}{1-e^{-\frac{\epsilon}{2}Nm\frac{\delta t}{\epsilon}}} \).
Weak convergence

Theorem
Assume \( x \in D(A) \) and \( y \in D(B) \). Let \( \Phi : H \to H \) of class \( C^2_b \). For any \( 0 < r \ll 1 \), for any \( T > 0 \), there exists \( c, C > 0 \) such that for any \( 0 \leq n \leq N_0 = \lfloor \frac{T}{\Delta t} \rfloor \)

\[
|\mathbb{E}\Phi(X_n) - \mathbb{E}\Phi(X(n\Delta t))| \leq C(\epsilon^{1-r} + \Delta t^{1-r}) \\
+ C\left[\left(\frac{\delta t}{\epsilon}\right)^{1/2-r} + \frac{1}{\sqrt{N\frac{\delta t}{\epsilon} + 1}}\right] e^{-cnT}\frac{\delta t}{\epsilon} (R + R^2) \\
+ C\frac{\Delta t}{M(N(\frac{\delta t}{\epsilon}) + 1)}.
\]


Conclusion

Proof of a strong order and of a weak order of convergence, as for SDEs.
Weak order is better than strong order.
HMM method can be adapted.
Conclusion

Proof of a strong order and of a weak order of convergence, as for SDEs.
Weak order is better than strong order.
HMM method can be adapted.
Some limits:
- No noise in the slow equation.
- The strict dissipativity assumption.
- The additional assumption on $F$. 