Long-time convergence of an Adaptive Biasing Force method: the bi-channel case

Kimiya Minoukadeh

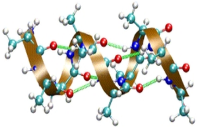
CERMICS, École des Ponts ParisTech

Joint work with Tony Lelièvre

CEA-EDF-INRIA school
IHP, Paris. 30 September 2010

Introduction

Consider a system of N particles with coordinates $q \in \mathcal{D} = \mathbb{R}^{3N}$. The particles interact through the potential $V : \mathcal{D} \to \mathbb{R}$.



In the canonical ensemble, the microscopic state of the system is described by the canonical measure

$$d\phi(q) = Z^{-1} e^{-\beta V(q)} dq$$

where $\beta = 1/(k_B T)$. This is used to calculate macroscopic properties, or ensemble averages of an observable A:

$$\langle A \rangle = \int_{\mathcal{D}} A(q) \, d\phi(q)$$

Sampling the canonical measure

Define a process X_t that is ergodic with respect to ϕ. Then,

$$\langle A \rangle = \int_{\mathcal{D}} A(q)\ d\phi(q) \approx \lim_{T\to\infty} \frac{1}{T} \int_0^T A(X_t)\ dt$$

To sample the measure $d\phi$: use the overdamped Langevin dynamics

$$dX_t = -\nabla V(X_t)dt + \sqrt{2\beta^{-1}} dW_t$$

where $X_t \in \mathbb{R}^d$ is the system trajectory and W_t a d-dimensional Brownian motion.

Fokker-Planck equation: the density $\psi(t,\cdot)$ of the law of X_t satisfies

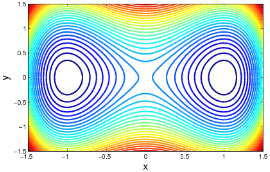
$$\partial_t \psi = \mathrm{div}\left(\nabla V \psi + \beta^{-1} \nabla \psi\right).$$

It can be checked that ϕ is indeed a stationary solution.

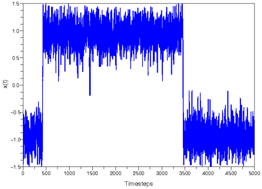
Metastabilities

Sampling ϕ using standard Langevin dynamics is often slow due to metastable regions in the potential V.

$V(x,y) = (x^2-1)^2 + y^2$



$x(t)$



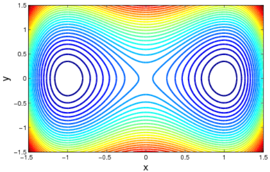
- The slow variable is described by a reaction coordinate ξ.
- The reaction coordinate is a smooth function $\xi : \mathcal{D} \to \mathcal{M}$ ($\mathbb{R}$ or $\mathbb{T}$).
- In the above, a good choice is $\xi(x,y) = x$.

Free Energy

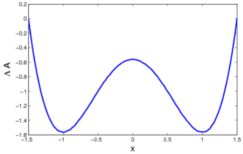
The free energy is defined by $A(x) = -\beta^{-1}\log \phi^{\xi}(x)$, where ϕ^{ξ} is the marginal density in ξ:

$$\phi^{\xi}(x) = \int_{\mathbb{R}} e^{-\beta V(x,y)} dy.$$

$V(x,y)$



$A(x)$



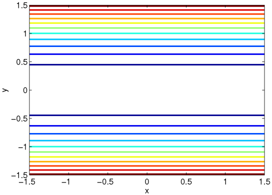
In the general case, $\phi^{\xi}(z) = \int_{\mathcal{M}} e^{-\beta V} |\nabla\xi|^{-1}\, d\sigma_z$, where $d\sigma_z$ is the surface measure on submanifold $\{q \mid \xi(q) = z\}$.

Free Energy as a biasing potential

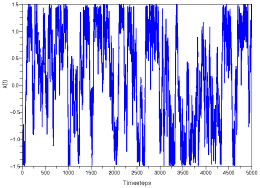
Suppose the free energy A is given. Then, we may perform dynamics with modified potential $\mathcal{V} = V - A \circ \xi$:

$$dX_t = -\nabla(V - A \circ \xi)(X_t)dt + \sqrt{2\beta^{-1}}dW_t$$

$V - A \circ \xi$



$x(t)$



Invariant measure: $d\phi_A = Z_A^{-1} e^{-\beta(V - A\circ\xi)} dq$. Unbias to compute canonical average.

Free Energy as a biasing potential

However, normally the free energy A is *not* given.

We will see how to

- estimate A (up to an additive constant) on the fly;
- use this estimate to bias dynamics to encourage exploration of the reaction coordinate space.

Outline of talk

Free energy differences

Compute the free-energy difference

$$A(x) - A(x_0) = \int_{x_0}^{x} A'(x)\, dx,$$

where $A'(x)$, called the mean force, is the conditional expectation

$$A'(x) = \frac{\int_{\mathbb{R}} F^V(x,y) e^{-\beta V}\, dy}{\int_{\mathbb{R}} e^{-\beta V}\, dy} = \mathbb{E}_\phi \left[F^V(q) \middle| \xi(q) = x \right]$$

where $F^V = \partial_x V$.

In the general case, $F^V = (\nabla V \cdot \nabla \xi)|\nabla \xi|^{-2} - \beta^{-1}\mathrm{div}(\nabla \xi |\nabla \xi|^{-2})$,

Adaptive Biasing Force method

The Adaptive Biasing Force (ABF) method uses an on-the-fly estimate of the free energy

$$\begin{cases} dX_t = -\nabla(V - A_t \circ \xi)(X_t)dt + \sqrt{2\beta^{-1}}dW_t \\ A_t'(z) = \mathbb{E}\left[F^V | \xi(X_t) = z\right] \end{cases}$$

The density $\psi(t,\cdot)$ of the law of X_t satisfies the Fokker-Planck equation

$$\begin{aligned} \partial_t \psi &= \operatorname{div}(\nabla(V - A_t \circ \xi)\psi + \beta^{-1}\nabla\psi) \\ A_t'(x) &= \frac{\displaystyle\int_{\mathbb{R}} F^V(x,y)\ \psi(t,x,y)\ dy}{\displaystyle\int_{\mathbb{R}} \psi(t,x,y)\ dy}. \end{aligned} \tag{1}$$

It can be checked that a stationary solution is $\psi_\infty = Z^{-1}e^{-\beta(V - A_\infty \circ \xi)}$. Substituting $\psi(t,\cdot)$ with $\psi_\infty(\cdot)$ in (1) gives $A_\infty' = A'$.

PDE formulation

Fokker-Planck equation

$$\partial_t \psi = \mathrm{div}(\nabla(V - A_t \circ \xi)\psi + \beta^{-1}\nabla\psi)$$

In the case $\xi(x, y) = x$, the marginal satisfies

$$\partial_t \psi^\xi = \beta^{-1}\partial_{xx}\psi^\xi.$$

So in the case $\mathcal{M} = \mathbb{T}$, one has $\psi^\xi_\infty \equiv 1$.

Long-time convergence of ABF

Questions

- How quickly does $A'_t \to A'$?
- How quickly does $\psi \to \psi_\infty$?

Tools

- Relative entropy and logarithmic Sobolev inequalities (LSI)

Entropy and Fisher information

Relative entropy

The 'distance' between two probability measures ν and ν_∞ (ν absolutely continuous w.r.t. ν_∞, $\nu \ll \nu_\infty$) will be measured by

$$\text{relative entropy} \quad H(\nu|\nu_\infty) = \int \log\left(\frac{d\nu}{d\nu_\infty}\right) d\nu$$

Note that $H(\nu|\nu_\infty) \geq 0$, since $x\log(x) \geq x - 1$ and $\int \nu = \int \nu_\infty = 1$. Furthermore $H(\nu|\nu_\infty) = 0$ iff $\nu = \nu_\infty$.

Convergence of relative entropy $\implies$ convergence in L^1-norm

Csiszar-Kullback inequality

For two probability measures ν and ν_∞,

$$\|\nu - \nu_\infty\|_{\mathrm{TV}} \leq \sqrt{2H(\nu|\nu_\infty)}.$$

Logarithmic Sobolev inequalities

Fisher information

The relative Fisher information of a probability measure ν with respect to ν_∞ is given by

$$\text{Fisher information} \quad F(\nu|\nu_\infty) = \int \left| \nabla \log \left(\frac{d\nu}{d\nu_\infty} \right) \right|^2 d\nu$$

Logarithmic Sobolev inequality (LSI)

A probability measure ν_∞ is said to satisfy a logarithmic Sobolev inequality with constant ρ (in short: LSI(ρ)) if for all probability measures ν absolutely continuous w.r.t. ν_∞, we have

$$H(\nu|\nu_\infty) \leq \frac{1}{2\rho} F(\nu|\nu_\infty)$$

LSI and exponential convergence to equilibrium

Suppose $\phi_\infty = Z^{-1} e^{-V}$ satisfies LSI(ρ_0).

Then if $\phi(t,\cdot)$ satisfies (the PDE associated to overdamped dynamics)

$$\partial_t \phi = \mathrm{div}(\nabla V \phi + \nabla \phi) = \mathrm{div}\left(\phi \nabla \log\left(\frac{\phi}{\phi_\infty}\right)\right),$$

we have

$$\begin{aligned}\frac{d}{dt} H(\phi|\phi_\infty) &= \int \log\left(\frac{\phi}{\phi_\infty}\right) \partial_t \phi \\ &= -\int \left|\nabla \log\left(\frac{\phi}{\phi_\infty}\right)\right|^2 \phi = -F(\phi|\phi_\infty) \leq -2\rho_0 H(\phi|\phi_\infty)\end{aligned}$$

Therefore, $\phi(t,\cdot)$ tends to ϕ_∞ exponentially fast with rate $2\rho_0$:

$$H(\phi(t,\cdot)|\phi_\infty) \leq H(\phi(0,\cdot)|\phi_\infty) e^{-2\rho_0 t}.$$

By the Csiszar-Kullback inequality: $\int |\phi - \phi_\infty| \leq \sqrt{2H_0} e^{-\rho_0 t}$.

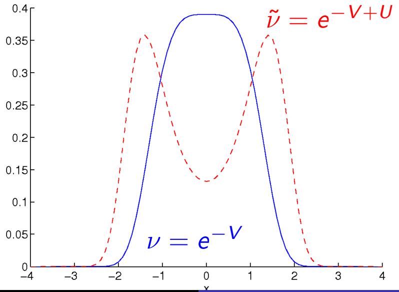
LSI constants

Estimates on LSI constants.

1. **Bakry-Emery theorem:** Any measure ν with density proportional to e^{-V}, with V α-convex, satisfies LSI(α).

2. **Holley-Stroock perturbation:** If ν satisfies LSI(α) and $\tilde{\nu} = e^{U}\nu$ for bounded function U, then $\tilde{\nu}$ satisfies LSI(α'), where

$$\alpha' = \alpha \exp(-2\mathrm{osc}(U)) < \alpha, \quad \mathrm{osc}(U) = \sup U - \inf U.$$



Convergence of ABF: existing results

How about convergence to equilibrium of ABF?

Let us assume that $\exists \rho_1 > 0$

$$\boxed{\forall x \in \mathbb{T}, \quad d\mu_{\infty|x} = \frac{\psi_\infty(x,y)\ dy}{\psi_\infty^\xi(x)} \text{ satisfies } \mathrm{LSI}(\rho_1).}$$

and that $\|\partial_{x,y} V\|_{L^\infty} \leq M < \infty$. Note that typically $\rho_1 > \rho_0$.

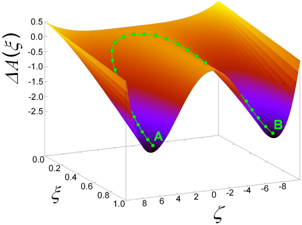
Theorem: [Lelièvre, Rousset, Stoltz, *Nonlinearity*, 2008]

$\forall t \geq t_0 > 0$, $\exists \bar{C} > 0$ such that

$$\int_{\mathbb{T}} |A_t'(x) - A'(x)|^2 dx \leq \bar{C} \exp(-\lambda t)$$

where $\lambda = \beta^{-1} \min(\rho_1, 4\pi^2)$. This implies that $\|\psi(t,\cdot) - \psi_\infty\|_{L^1}^2$ also converges exponentially fast to zero with rate λ.

Results suboptimal in ‘bi-channel’ case



‘Bi-channel’ scenario

- $d\mu_{\infty|x}$ satisfies LSI(ρ_1)
- but high energy barriers at fixed ξ
- therefore ρ_1 very small!

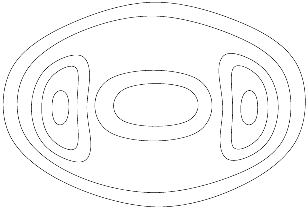
Can we do better?

It was observed numerically [Minoukadeh, Chipot, Lelièvre, *JCTC*, 2010] that high energy barriers at fixed ξ *do not* always slow down convergence of the ABF method.

Towards a bi-channel model

Bi-channel potential

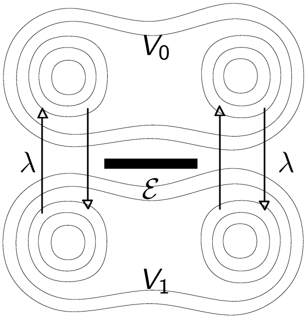
$V : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$



Bi-channel model

$V_0 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$

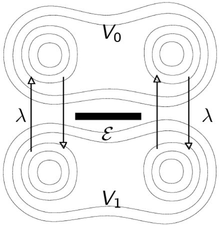
$V_1 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$



Bi-channel model

The new model to describe the bi-channel scenario:

- channels are indexed by $i \in \{0, 1\}$
- potentials $V_0,\ V_1 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$.



$$\begin{cases} dX_t = -\nabla(V_{I_t} - A_t \circ \xi)(X_t)dt + \sqrt{2}dB_t, \\ A_t'(x) = \mathbb{E}\left[\partial_x V_{I_t}(X_t) | \xi(X_t) = x\right], \\ I_t \in \{0, 1\} \text{ is a jump process with generator} \\ L\varphi(x, y, i) = -\lambda(x)(\varphi(x, y, i) - \varphi(x, y, 1 - i)). \end{cases}$$

The process (X_t, I_t) has law with density $\psi(t, x, y, i)$ satisfying

$$\partial_t \psi = \mathrm{div}(\psi_\infty \nabla \log(\psi/\psi_\infty)) + \partial_x((A' - A_t')\psi) - \lambda(x)(\psi - \psi_{1-i})$$

Hypotheses

[H1] Region of 'no exchange': $\exists\, \mathcal{E} \subset \mathbb{T}$ where $\lambda(x) = 0$:

$$\boxed{\lambda(x) = \lambda \mathbf{1}_{\mathbb{T}\setminus\mathcal{E}}(x) \ \text{ and } \ \forall x \in \mathbb{T}\setminus\mathcal{E},\, V_0(x,\cdot) = V_1(x,\cdot).}$$

[H2] Regularity: $\exists\ 0 < C, M < \infty$ such that $\forall i \in \{0,1\}$,

$$\boxed{\|\partial_{x,y} V_i\|_{L^\infty} \leq M \ \text{ and } \ \left\| \frac{\int_{\mathbb{R}} \partial_x V_i e^{-V_i}\, dy}{\int_{\mathbb{R}} e^{-V_i}\, dy} \right\|_{L^\infty} \leq C.}$$

[H3] LSI on new measures: Let $\mu_{\infty|x,i}$ be the equilibrium measures conditioned to $\xi(q) = x$ *and* channel i.
$\exists \rho > 0$, $\forall x \in \mathbb{T}$, $\forall i \in \{0,1\}$,

$$\boxed{\mu_{\infty|x,i} \text{ satisfies LSI}(\rho).}$$

Typically $\rho > \rho_1 > \rho_0$, where ρ_1 is LSI constant associated to $\mu_{\infty|x}$

Hypotheses

Next, assume that A is a good bias in each channel.

Consider the operator $\mathcal{L} = (\mathcal{L}_0, \mathcal{L}_1)$ *on* $f : \begin{cases} \mathbb{T} \to \mathbb{R}^2 \\ x \mapsto (f_0(x), f_1(x)) \end{cases}$ with $f_i \in H^1(1/f_{i,\infty})$.

$$-\mathcal{L}_i f = \partial_x \left[f_{i,\infty} \partial_x (f_i / f_{i,\infty})\right] - \lambda(x)(f_i - f_{1-i}),$$

with $f_{0,\infty} = f_{1,\infty}$ for $x \in \mathbb{T} \backslash \mathcal{E}$. $\mathcal{L}$ is symmetric and positive definite with respect to the inner product $\langle f, g \rangle = \sum_{i=0}^{1} \int_{\mathbb{T}} f_i(x) g_i(x)\ (1/f_{i,\infty}) dx$ and has spectral gap $\theta > 0$.

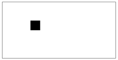
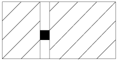
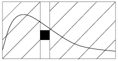
[H4] Assume the spectral gap is sufficiently large:

$$\boxed{\theta > \theta_{\min} \text{ with } \theta_{\min} = \frac{8(C + M\rho^{-1/2})^2 \tilde{M}}{c}.}$$

where $\inf_{x,i} \psi_\infty^{\xi,I} = c > 0$ and $\sup_x \psi^\xi(0,x) = \tilde{M} < \infty$.

Relative entropies

We introduce the following functionals:

- Total entropy $E(t) = H(\psi|\psi_\infty)$ 
- Macroscopic entropy $E_M(t) = H(\psi^\xi|\psi^\xi_\infty)$ 
- Local entropy $e_m(t,x) = H(\mu_{t|x}|\mu_{\infty|x})$ 
- Microscopic entropy $E_m(t) = \int_{\mathbb{T}} e_m \ \psi^\xi(x) \ dx$ 

$$\boxed{E = E_m + E_M}$$

Brief outline

Aim: show $E_m = H(\mu_{t|x}|\mu_{\infty|x})$ decays exponentially fast with rate limited by ρ, LSI constant of $d\mu_{\infty|x,i}$.

Why E_m?

1. $A'_t - A'$ can be controlled by E_m: It can be shown $\exists R > 0,\ \forall t \geq 0$
$$\int_{\mathbb{T}} |A'_t - A'|^2 \psi^\xi \ dx \leq 2R^2 E_m(t).$$
2. $dE_M/dt = -F(\psi^\xi|\psi^\xi_\infty)$ and $F(\psi^\xi|\psi^\xi_\infty) \leq F_0 e^{-8\pi^2 t}$
3. From $E = E_m + E_M$, the total entropy E also converges exponentially fast.
4. Csiszar-Kullback inequality implies the same for $\|\psi - \psi_\infty\|_{L^1}$.

Convergence of ABF: bi-channel case

Theorem [Lelièvre, Minoukadeh, 2010]

Assume hypotheses [H1]-[H4]. There exists a smooth function $\Lambda : (\theta_{\min}, \infty) \to (0, \rho)$ which is increasing and such that:

$$\Lambda(\rho + 2\theta_{\min}) = \frac{\rho}{2} \quad \text{and} \quad \Lambda(\theta) \to \begin{cases} 0 & \text{as} \quad \theta \to \theta_{\min} \\ \rho & \text{as} \quad \theta \to \infty \end{cases}$$

such that $\forall \varepsilon \in (0, \Lambda(\theta))$, $\exists K > 0$ such that, $\forall t > 0$,

$$E_m(t) \leq K exp \left(-2 \min\{(\Lambda(\theta) - \varepsilon), 4\pi^2\} \, t \right).$$

This implies that the total entropy E and thus $\|\psi(t, \cdot) - \psi_\infty\|_{L^1}^2$ converge exponentially fast to zero with the same rate. Furthermore, for any positive time $t_0 > 0$ and $\varepsilon \in (0, \Lambda(\theta))$, $\exists \bar{K} > 0$ such that $\forall t \geq t_0$,

$$\int_{\mathbb{T}} |A_t'(x) - A'(x)|^2 \, dx \leq \bar{K} exp \left(-2 \min\{(\Lambda(\theta) - \varepsilon), 4\pi^2\} \, t \right).$$

References

Adaptive Biasing Force (ABF) method

[1] E. Darve and A. Pohorille, *J. Chem. Phys.* (2001)

[2] J. Hénin and C. Chipot, *J. Chem. Phys.* (2004).

Multiple-Walker ABF (MW-ABF)

[3] K. Minoukadeh, C. Chipot, T. Lelièvre, *J. Chem. Theory Comput.* (2010).

Long-time convergence of ABF

[4] T. Lelièvre, M. Rousset, G. Stoltz, *Nonlinearity* (2008).

[5] T. Lelièvre, K. Minoukadeh, *Preprint: hal-00477302* (2010).

Book: "Free Energy Computations: A Mathematical Perspective"

[6] T. Lelièvre, M. Rousset, G. Stoltz, *Imperial College Press* (2010).