Spectral analysis of Fokker-Planck and Kramers-Fokker-Planck operators. Lecture summary

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1 Introduction.

Take the notations

$$\begin{array}{lll} X \in \mathbb{R}^n, & b : \mathbb{R}^n \to \mathbb{R}^n\,, & B \in M_n(\mathbb{R}) \\ & X \to b(X) & \\ \multicolumn{3}{l}{W \ n\text{-dimensional white noise} \quad dW_k dW_\ell = \delta_{k,\ell} dt\,,} \end{array}$$

and consider the stochastic differential equation

$$dX = b(X)dt + BdW\,. \tag{1.1}$$

The relationship between stochastic differential equations and drift-diffusion semigroups is obtained after computing, from Ito's calculus, the expectation value

$$v(x,t) = \mathbb{E}\left(v_0(X,t); X(0) = x\right)\,,$$

for some smooth and decaying observable v_0. Then $v(x,t)$ is given by

$$v(t) = e^{-tL}v_0\,, \quad \text{or} \quad \left\{ \begin{array}{l} \partial_t v = -Lv = b.\partial_x v + \frac{1}{2}\partial_x(BB^t)\partial_x v \\ v(t=0) = v_0\,. \end{array} \right. \tag{1.2}$$

We shall focus on two cases:

- Smoluchowski process in a gradient field: $\mathbb{R}^n = \mathbb{R}^d$, $b = -\nabla V(x) = -\partial_x V(x)$, $B = \sqrt{\frac{2}{\beta}}\mathrm{Id}_{\mathbb{R}^n}$ ($\beta = \frac{1}{k_B T}$) and

  $$L = (\partial_x V(x)).\partial_x - \frac{1}{\beta}\Delta_x\,.$$

  When $e^{-\beta V(x)} \in L^1(\mathbb{R}^d, dx)$, the invariant measure obtained by solving the formal adjoint equation $L'\mu = 0$: $\mu = e^{-\beta V(x)}\ dx$ and L is a symmetric operator on $L^2(\mathbb{R}^d, e^{-\beta V}dx)$.

- Langevin process: $\mathbb{R}^n = \mathbb{R}^d_x \times \mathbb{R}^d_v$, $X = (x,v)$, $b(x,v) = \begin{pmatrix} v \\ -\frac{1}{m}\partial_x V - \gamma_0 v \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{\frac{2\gamma_0}{m\beta}}\mathrm{Id}_{\mathbb{R}^d} \end{pmatrix}$ and

  $$L = -v.\partial_x + \frac{1}{m}(\partial_x V(x)).\partial_v + \frac{\gamma_0}{m\beta}\left(-\partial_v + m\beta v\right).\partial_v\,.$$

  Its formal adjoint equals

  $$L' = v.\partial_x - \frac{1}{m}(\partial_x V(x)).\partial_v - \frac{\gamma_0}{m\beta}\partial_v.(\partial_v + m\beta v)\,,$$

  and the equilibrium measure is the Maxwellian

  $$M(x,v)\ dxdv = e^{-\beta(\frac{mv^2}{2} + V(x))}\ dxdv\,,$$

  with $M(x,v) \in L^1(\mathbb{R}^{2d}, dxdv)$ when $e^{-\beta V} \in L^1(\mathbb{R}, dx)$.

Conjugation:

$$e^{-\phi(X)}\partial_X e^{\phi(X)} = (e^{-\phi(X)}\times)\circ\partial_X\circ(e^{\phi(X)}\times) = \partial_X + (\partial_X\phi(X))\,.$$

Unitary equivalences

$$\begin{aligned}
&u\in L^2(\mathbb{R}^d, e^{-\beta V(x)}dx) \quad , \quad u'(x) = (Uu)(x) = e^{-\frac{\beta V(x)}{2}}u(x)\in L^2(\mathbb{R}^d, dx)\,,\\
&u\in L^2(\mathbb{R}^{2d}, M(x,v)\ dxdv) \quad ,\\
&\qquad\qquad u'(x,v) = (Uu)(x,v) = M(x,v)^{1/2}u(x,v)\in L^2(\mathbb{R}^{2d}, dxdv)\,.
\end{aligned}$$

Smoluchowski on $L^2(\mathbb{R}^d, dx)$:

$$\begin{aligned}
\Delta^{(0)}_{V,h} &= \frac{4}{\beta}e^{-\frac{\beta V}{2}}Le^{\frac{\beta V}{2}} = -h^2\Delta_x + |\partial_x V(x)|^2 - h(\Delta V(x))\,, \quad h = \frac{2}{\beta}\,.\\
&\qquad \ker(\Delta^{(0)}_{V,h} = \mathbb{C}e^{-\frac{V(x)}{h}})\,.
\end{aligned}$$

Langevin on $L^2(\mathbb{R}^{2d}, dxdv)$:

$$\begin{aligned}
K &= M(x,v)^{-1/2}L'M(x,v)^{1/2}\\
&= v.\partial_x - \frac{1}{m}(\partial_x V(x)).\partial_v + \frac{\gamma_0}{m\beta}(-\partial_v + \frac{m\beta}{2}v).(\partial_v + \frac{m\beta}{2}v)\\
&\qquad \ker K = \mathbb{C}M^{1/2} = \mathbb{C}(e^{-\frac{\beta}{2}(\frac{mv^2}{2}+V(x))})\,.
\end{aligned}$$

REF:[Ris89][Nel02][EvWeb]
We shall work in a separable Hilbert space, typically $\mathcal{H} = L^2(\Omega, dX)$ where Ω is a domain of $\mathbb{R}^n$ and dX stands for the Lebesgue measure. The scalar product will be right-$\mathbb{C}$-linear and left-$\mathbb{C}$-antilinear.

2 Contour integrals, semigroups and hypoellipticity

2.1 Functional analysis

Hille-Yosida Theorem. *Let $(A, D(A))$ be a closed densely defined operator in the Hilbert space $\mathcal{H}$. The following statements are equivalent:*

- $\forall\psi\in D(A)$, $\mathrm{Re}\,\langle\psi\,,\,A\psi\rangle\geq 0$ *(accretivity) and* $\mathrm{Ran}\,(\mathrm{Id}+A) = \mathcal{H}$ *(maximality).*
- *Any $z\in\mathbb{C}$ such that $\mathrm{Re}\,z<0$ belongs to the resolvent set of A and* $\|(z-A)^{-1}\|\leq\frac{1}{|\mathrm{Re}\,z|}$.

- $(A, D(A))$ *is the generator of a strongly continuous contraction semigroup* $(e^{-tA})_{t\in\mathbb{R}_+}$ *and* $D(A)$ *is the set of vectors* $\psi \in \mathcal{H}$ *such that* $t \to e^{-tA}\psi$ *belongs to* $\mathcal{C}^1(\mathbb{R}_+, \mathcal{H})$ *.*

REF: [Bre83][BuBe67][DaLi92][EnNa00][Paz83][ReSi75][Dav80][Yos80]

Lemma 2.1. *When* $(A, D(A))$ *is maximal accretive and* $A_n = \frac{A}{1+\frac{A}{n}}$, $n \in \mathbb{N}^*$

$$\begin{aligned}
&\forall \psi \in D(A)\,, \quad \lim_{n\to\infty} \|(A - A_n)\psi\| = 0\,,\\
&\forall t \geq 0, \forall \psi \in \mathcal{H}\,, \quad \lim_{n\to\infty} \|(e^{-tA} - e^{-tA_n})\psi\| = 0\,.
\end{aligned}$$

Proposition 2.2. *Let* $(A, D(A))$ *be a maximal accretive operator in the Hilbert-space* $\mathcal{H}$*, then*

$$\forall \psi \in D(A), \forall t \in [0, +\infty), \quad e^{-tA}\psi = \frac{1}{2i\pi}\int_{i\infty-0}^{-i\infty-0} e^{-tz}(z - A)^{-1}\psi\; dz\,.$$

Actually the above equality has to be understood after applying some regularization process for both sides.

2.2 Application to sectorial operators

Definition 2.3. *For a densely defined closed operator* $(A, D(A))$ *in* $\mathcal{H}$ *, the spectrum is defined by*

$$\mathit{Spec}(A) = \mathbb{C} \setminus \varrho(A) = \mathbb{C} \setminus \left\{ z \in \mathbb{C}, (z - A)^{-1} \in \mathcal{L}(\mathcal{H}) \right\}\,,$$

and the numerical range by

$$\mathit{Num}(A) = \overline{\{\langle \psi\, A\psi\rangle\,, \quad \psi \in D(A), \|\psi\| = 1\}}^{\mathbb{C}}\,.$$

A maximally accretive operator is said sectorial if there exists $\theta_0 > 0$ *such that* $\mathit{Num}(A) \subset \left\{ z \in \mathbb{C}, |\arg z| \leq \frac{\pi}{2} - \theta_0 \right\}$ *.*

Definition 2.4. *For a densely defined closed operartor* $(A, D(A))$ *in* $\mathcal{H}$ *such that* $D(A) \subset D(A^*)$ *, the operators*

$$\mathrm{Re}\, A = \frac{1}{2}(A + A^*) \quad \textit{and} \quad \mathrm{Im}\, A = \frac{1}{2i}(A - A^*)\,,$$

are well defined on $D(A)$ *.*

Toeplitz-Hausdorff Theorem. *The numerical range is a convex set.*

Proposition 2.5. *For any densely defined closed operator $(A, D(A))$, $\mathbb{C} \setminus \mathrm{Num}(A) \subset \varrho(A)$ and*

$$\|(z-A)^{-1}\| \leq \frac{1}{\mathrm{dist}(z, \mathrm{Num}(A))}\,.$$

Sectoriality is often a consequence of ellipticity.
Contour deformation for sectorial operators.

An example: $a(x).\partial_x - \Delta_x$ with Dirichlet boundary conditions in a regular bounded domain $\Omega \subset \mathbb{R}^d$ and $\mathrm{div} a = \partial_x.a \equiv 0$.
Numerical range of $v.\partial_x - \Delta_v + \frac{v^2}{4} - \frac{1}{2}$ on $S^1 \times \mathbb{R}$.

2.3 Hypoellipticity

Definition 2.6. ***(L. Schwartz)*** *A continous operator $P : \mathcal{C}_0^\infty(\Omega) \to \mathcal{D}'(\Omega)$ is said hypoelliptic at $x_0 \in \Omega$ (in a neighborhood of x_0) if $x_0 \notin \mathrm{suppsing}\, Pu$ implies $x_0 \notin \mathrm{suppsing}\, u$.*

Hörmander's results: Let $X_0, X_1, \ldots, X_n$ be $(n+1)$-vector fields on $\Omega \subset \mathbb{R}^d$, $X_j(x) = \sum_{k=1}^d a_{jk}(x)\partial_{x_k}$, $a_{jk} \in \mathcal{C}^\infty(\Omega)$.

Definition 2.7. *A type I operator is written $P = \sum_{j=1}^n X_j^2$.*
A type II operator is written $P = X_0 + \sum_{j=1}^n X_j^2$.
A type I (resp. II) operator is said to satisfy Hörmander's condition at rank r at x_0 if $\left\{\left[X_{j_1}, \ldots, [X_{j_{p-1}}, X_{j_p}] \ldots\right](x_0)\,, \quad p \leq r, j_k \in \{1, \ldots, n\}\right\}$ (resp. $j_k \in \{0, 1, \ldots, n\}$) has rank d.
On a compact set $K \subset \Omega$, those operators satify Hörmander's condition at rank $r \in \mathbb{N}^*$ if it is true for all $x_0 \in K$.

Hörmander's theorem: *If a type I or type II operator satisfies Hörmander's condition at $x_0 \in \Omega$ then it is hypoelliptic in a neighborhood of x_0.*
REF:[Hor67][Hor85][Mal76][Mal78][Koh78] [RoSt77][HeNo85].
Hypoellipticity is often proved via subellipticity.

Definition 2.8. *A differential operator is said subelliptic in Ω if there exists $c > 0$ such that the estimate*

$$\forall u \in \mathcal{C}_0^\infty(\Omega)\,, \quad \|u\|_{H^{s+c}} \leq C_s\left(\|Pu\|_{H^s} + \|u\|_{H^s}\right)\,,$$

holds for all $s \in \mathbb{R}$.

Subelliptic estimates:

- For type I operators with Hörmander's condition at rank r:

$$\|u\|_{H^{s+\frac{1}{r}}} \leq C_s \left(\sum_{j=1}^{n} \|X_j u\|_{H^s} + \|u\|_{H^s} \right)$$
$$\|u\|_{s+\frac{2}{r}} \leq C_s \left(\|Pu\|_{H^s} + \|u\|_{H^s} \right) ,$$

holds for any $u \in \mathcal{C}_0^\infty(\omega_{x_0})$.

- For type II operators with Hörmander's condition at rank $r > 1$:

$$\|u\|_{H^{s+\frac{2}{2r-1}}} \leq C_s \left(\|Pu\|_{H^s} + \|u\|_{H^s} \right) ,$$

holds for any $u \in \mathcal{C}_0^\infty(\omega_{x_0})$.

An example: $v.\partial_x - \partial_v^2 + \frac{v^2}{4} - \frac{1}{2}$ on $S^1_x \times \mathbb{R}_v$. Spectrum. Hypoellipticity. Resolvent estimates.

2.4 Subelliptic estimates and deformation of the contour integral

Proposition 2.9. *Let $(K, D(K))$ be a maximal accretive operator in the Hilbert space $\mathcal{H}$. For any $\eta \in]0,1[$, the estimate*

$$|z+1|^{2\eta}\|u\|^2 \leq 4\langle u\,,\, ((K+1)^*(K+1))^\eta u\rangle + 4\|(K-z)u\|^2$$

holds for all $u \in D(K)$ and $z \in \mathbb{C}$ with $\mathrm{Re}\, z \geq -1$.

Proposition 2.10. *Let $(K, D(K))$ be a maximal accretive operator in $\mathcal{H}$. Assume that there exists a non negative (self-adjoint) operator $(\Lambda, D(\Lambda))$, $\Lambda \geq 1$, with $D(\Lambda^2) \overset{dense}{\subset} D(K)$, $\varepsilon > 0$, and $C > 0$ such that*

$$\forall u \in D(\Lambda^2)\,, \quad \|Ku\| \leq C\|\Lambda^2 u\|$$
$$\forall u \in D(\Lambda^2)\,, \forall \nu \in \mathbb{R}, \quad \|\Lambda^\varepsilon u\| \leq C \left(\|(K - i\nu)u\| + \|u\| \right) .$$

Then the spectrum of K lies in

$$S_K = \left\{ z \in \mathbb{C}, |z+1| \leq C' |\,\mathrm{Re}\, z + 1|^{\frac{2}{\epsilon}}\,,\, \mathrm{Re}\, z \geq -\frac{1}{2} \right\} .$$

Moreover when $z \notin S_k$ with $\mathrm{Re}\, z \geq -\frac{1}{2}$, the resolvent is estimated by

$$\|(z-K)^{-1}\| \leq C'|z+1|^{-\varepsilon}\,.$$

Consequences for the contour integral.
Example $v.\partial_x - \partial_v^2 + \frac{v^2}{4} - \frac{1}{2}$
REF:[HeNi04][EcHa03]

3 Kramers-Fokker-Planck and Fokker-Planck operators

3.1 Well defined dynamics

We work in this paragraph with $\beta = m = 1$. Once the problem is transformed into

$$K = v.\partial_x - (\partial_x V(x)).\partial_v + \gamma_0(-\partial_v + \frac{v}{2}).(\partial_v + \frac{v}{2})\,, \quad \text{in } L^2(\mathbb{R}^{2d}, dxdv)$$

there is no reason to restrict the analysis to the case $e^{-V} \in L^1(\mathbb{R}^d, dx)$.
Metastability.
It is known that when $V(x)$ goes to $-\infty$ as $x \to \infty$ faster that $-|x|^2$ the Hamiltonian dynamics is not well defined. As soon as there is some friction (and diffusion), the dynamics is well defined for all times.

$A = iB$	
A accretive : Re $\langle Ax\,,\,x\rangle \geq 0$	B symmetric : $\langle Bx\,,\,x\rangle \in \mathbb{R}$ Re $\langle Ax\,,\,x\rangle \leq 0$ and ≥ 0
$(A$ accretive$) \Rightarrow \big(\overline{A}$ accretive$\big)$	$(B$ symm.$) \Rightarrow \big(\overline{B}$ symm.$\big)$
$(A$ accretive$) \Rightarrow (A^*$ accretive$)$	$(B$ symm.$) \Rightarrow (B^*$ symm.$)$
A maximally accretive	B self-adjoint
Hille-Yosida theorem for contraction semigroup : $((e^{-tA})_{t\geq 0}) \Leftrightarrow (A$ max. acc.$)$	Stone theorem for unitary group : $((e^{-itB})_{t\in\mathbb{R}}) \Leftrightarrow (B$ self-adj.$)$
$(A$ max. acc.$)$ implies $(\mathrm{Spec}(A) \subset \{\,\mathrm{Re}\, z \geq 0\})$	$(B$ self-adj.$)$ implies $(\mathrm{Spec}(B) \subset \mathbb{R})$

By following this correspondence it is possible to introduce the notion of essential maximal accretivity when the domain of the closed operator cannot be made explicit.

Definition 3.1. *An accretive operator A in $\mathcal{H}$ with domain $D(A)$, is said essentially maximally accretive if it admits a unique maximally accretive extension.*

The equivalence of the next statements can easily be checked :

1. A is essentially maximally accretive.
2. $\overline{A}$ is maximally accretive.
3. There exists $\lambda_0 > 0$ such that $A^* + \lambda_0 I$ is injective.
4. There exists $\lambda_1 > 0$ such that the range of $A + \lambda_1 I$ is dense in $\mathcal{H}$.

Remark 3.2. *A particular case is when A is a differential operator with $\mathcal{C}^\infty$ coefficients initially defined with $D(A) = \mathcal{C}_0^\infty(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. The domain of its closure equals*

$$D(\overline{A}) = \left\{ f \in L^2(\mathbb{R}^n),\ Af \in L^2(\mathbb{R}^n) \right\} .$$

According to the point 4, the essential maximal accretivity of A is true if for some $\lambda_1 > 0$

$$\big((\lambda_1 + A')f = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n), f \in L^2(\mathbb{R}^n)\big) \Rightarrow (f = 0)$$

where A' is the formal adjoint of A.

Proposition 3.3.
Let V be a C^∞ potential on $\mathbb{R}^d$, then the Kramers-Fokker-Planck operator defined on $\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ defined by

$$K := \gamma_0(-\Delta_v + \frac{1}{4}|v|^2 - \frac{d}{2}) + X_0 \ , \tag{3.1}$$

where

$$X_0 := v \cdot \partial_x - \nabla V(x) \cdot \partial_v \tag{3.2}$$

is essentially maximally accretive as soon as $\gamma_0 > 0$.

This is an hypoelliptic version of Simader's Theorem which says that a non negative Schrödinger operator $-\Delta + W(x)$ with $W \in \mathcal{C}^\infty(\mathbb{R}^d)$ is essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^d)$.
REF:[ReSi75][HeNi05][Sim78].

3.2 A necessary condition for the compactness of $(1 + K)^{-1}$

Theorem 3.4.
Assume that V is C^∞ function. If the operator K has a compact resolvent then $\Delta_{V/2}^{(0)}$ has a compact resolvent.

3.3 Case with a quadratic potential

Set $D_x = \frac{1}{i}\partial_x$ and consider an operator $q^W(x, D_x)$ with a quadratic symbol

$$q(x,\xi) = \sum_{|\alpha|+|\beta|=2} c_{\alpha,\beta} x^\alpha \xi^\beta \quad X = (x,\xi) \in \mathbb{R}^{2n}\,.$$

q is given by a $\mathbb{C}$-valued quadratic form $q(X) = {}^tXMX$ with ${}^tM = M \in \mathcal{M}_n(\mathbb{C})$. Weyl quantization

$$[q^W(x,D_x)u](x) = \int_{\mathbb{R}^n} e^{i(x-y).\xi} q(\frac{x+y}{2},\xi)u(y)\;dy\frac{d\xi}{(2\pi)^n}\,.$$

Those operators are well defined continuous operators $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

Proposition 3.5. *When* $\operatorname{Re} q \geq 0$, *there exists a constant* C_q *such that* $C_q + q^W(x,D_x)$ *with domain*

$$\left\{u \in L^2(\mathbb{R}^n)\,, \quad q^W(x,D_x)u \in L^2(\mathbb{R}^n)\right\}$$

is maximal accretive and there is a generalized Mehler's formula for $e^{-tq^W(x,D_x)}$.

REF:[Hor95]
By using Wick and Anti-Wick quantization, one checks easily that the numerical range of

$$q^W(x,D_x) = -z_q + q^{Wick}(x,D_x) = z_q + q^{A-Wick}(x,D_x), \quad z_q \in \mathbb{C}\,,$$

satisifies

$$-z_q + q(\mathbb{R}^{2n}) \subset \operatorname{Num}(q^W(x,D_x)) \subset z_q + q(\mathbb{R}^{2n})\,.$$

With the complex-valued quadratic form $X \to q(X)$ is associated a complex bilinear symmetric form $(X,Y) \to q(X,Y)$.

Definition 3.6. *On the phase-space* $\mathbb{R}^{2n}$ *introduce the symplectic form*

$$\sigma(X,Y) = \xi.y - x.\eta\,, \quad X = (x,\xi),\, Y = (y,\eta)\,.$$

For a quadratic symbol q, *the Hamilton map* $F \in \mathcal{M}_{2n}(\mathbb{C})$ *is given by*

$$\forall X,Y \in \mathbb{R}^{2n}\,, \quad \sigma(X,F(Y)) = q(X,Y) = q(Y,X) = \sigma(Y,FX) = -\sigma(F(X),Y)\,,$$

and the singular space by

$$S = \left(\bigcap_{j=0}^{\infty} \ker\left[\operatorname{Re} F(\operatorname{Im} F)^j\right]\right) \cap \mathbb{R}^{2n}\,.$$

Note that here were consider complex-valued quadradic forms. For the complex linear application F, the mapping $\operatorname{Re} F$ and $\operatorname{Im} F$ are the standard real and imaginary parts after taking the real part and imaginary part of all matrix components. It must not be confused with $\frac{1}{2}(F+F^*)$ and $\frac{1}{2i}(F-F^*)$ more natural when one thinks in terms of hermitian quadratic forms.

Proposition 3.7. *Assume*

$$(\operatorname{Re} q \geq 0) \quad \textit{and} \quad S = \{0\}\,.$$

Then the spectrum of $q^W(x, D_x)$ *is given by*

$$\mathit{Spec}(q^W(x,D_x)) = \left\{ \sum_{\lambda \in \sigma(F),\, \operatorname{Re}(i\lambda) \geq 0} (r_\lambda + 2k_\lambda)(i\lambda)\,,\ k_\lambda \in \mathbb{N} \right\}$$

where r_λ *is the dimension of the characteristic space of* F *for the eigenvalue* λ*.*

REF:[Sjo74][HiPr09].
Application to: $K + \frac{d}{2} = v.\partial_x - \partial_x V(x).\partial_v - \Delta_v + \frac{v^2}{4}$ when V is quadratic. K is unitarily equivalent to the sum of commuting operators

$$\oplus_{j=1}^{d}(v_j\partial_{x_j} - \mu_j x_j \partial_{v_j} - \partial_{v_j}^2 + \frac{v_j^2}{4})$$

with

$$\begin{aligned}
q_j(x,v,\xi,\eta) &= \eta^2 + \frac{v^2}{4} - i(v\xi - \mu_j x\eta), \quad x,v,\xi,\eta \in \mathbb{R}\,,\\
F_j &= \begin{pmatrix} 0 & \frac{i}{2} & 0 & 0 \\ -\frac{i\mu_j}{2} & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{i\mu_j}{2} \\ 0 & -\frac{1}{4} & -\frac{i}{2} & 0 \end{pmatrix}\\
i\lambda_\pm &= \frac{1}{4}(1 \pm \sqrt{1-4\mu_j}) \quad , \quad r_{\lambda_\pm} = 1\,.
\end{aligned}$$

Subelliptic estimate for K: Let $\Lambda^2 = 1 - \Delta_x^2 + x^2 - \Delta_v + v^2$ with symbol

$$q_{\Lambda^2}(x,v,\xi,\eta) = 1 + \xi^2 + x^2 + \eta^2 + v^2$$

We have obviously

$$\forall u \in D(\Lambda^2)\,, \quad \|Ku\| \leq C\|\Lambda^2 u\|\,.$$

Proposition 3.8. *For any* $\varepsilon \in [0, \frac{2}{3}]$*, there exists a constant* $C_\varepsilon > 0$ *such that*

$$\forall u \in D(\Lambda^2), \forall \nu \in \mathbb{R}, \quad \|\Lambda^\varepsilon u\| \leq C_\varepsilon \left(\|(K - i\nu)u\| + \|u\|\right)\,.$$

3.4 Global hypoellipticity for K

Hypotheses:

a) The potential V is a $\mathcal{C}^\infty$ function and there exist $n \geq 1$ (possibly $n > 1/2$) and, for all $\alpha \in \mathbb{N}^d$, a positive constant C_α such that

$$\forall x \in \mathbb{R}^d, |\partial_x^\alpha V(x)| \leq C_\alpha \left(1 + \langle x\rangle^{2n-\min\{|\alpha|,2\}}\right).$$

b) There exists two constants $C_0 = C_0(V) > 0$ and $C_1 = C_1(V) > 0$ such that

$$\forall x \in \mathbb{R}^d, \quad \pm V(x) \geq C_0^{-1}\langle x\rangle^{2n} - C_0 \text{ and } |\partial_x V(x)| \geq C_1^{-1}\langle x\rangle^{2n-1} - C_1.$$

Notations: The operators K and Λ are defined by

$$\begin{aligned}
& v.\partial_x - V_\beta(x)).\partial_v - \gamma(-\partial_v + \frac{v}{2}).(\partial_v + \frac{v}{2}), \\
& \Lambda^2 = 1 - \Delta_x + \frac{1}{4}|\partial_x V_\beta(x)|^2 - \frac{1}{2}(\Delta V_\beta(x)) - \Delta_v^2 + \frac{v^2}{4}, \\
\text{with} \quad & V_\beta(x) = \beta V(\beta^{-\frac{1}{2n}}x) \quad , \quad \gamma = \gamma_0\sqrt{m}\beta^{\frac{n-1}{2n}}.
\end{aligned}$$

For $R > 0$ and $\nu \in \mathbb{N}^*$, the function Q_R is defined on $(0, +\infty)^\nu$, without reference to ν, by

$$Q_R(t_1, \ldots, t_\nu) = \prod_{j=1}^{\nu}\left(t_j + \frac{1}{t_j}\right)^R.$$

The notation $C(V)$ will denote a finite subset of

$$\left\{C_0(V), C_1(V), C_\alpha, \alpha \in \mathbb{N}^d\right\}$$

defined in the Hypotheses 1) and 2).

Theorem 3.9. *Under the above hypotheses ($n > 1/2$) and by setting*

$$\varepsilon = \min\{1/4, 1/(4n-2)\}$$

there is a constant $C_{hyp} = cQ_R(C(V), \beta, \gamma)$ such that

$$\forall \nu \in \mathbb{R}, \forall u \in \mathcal{S}(\mathbb{R}^{2d}), \ \|\Lambda^\varepsilon u\| \leq C_{hyp}\left(\|(K - i\nu)u\| + \|u\|\right).$$

Corollary 3.10. *The operator K has a compact resolvent.*

Corollary 3.11. *The deformation contour of $e^{-tK}\psi = \frac{1}{2i\pi}\int_{+i\infty}^{-i\infty} e^{-tz}(z - K)\psi dz$ can be performed according to Proposition 2.10.*

Comments:

- Method: variant of Kohn's method for $X_0 + \sum_{j=1}^n X_j^2$ by writing

$$
\begin{aligned}
&K = X_0 + b^*.b \quad , \quad X_0 = v.\partial_x - \frac{\partial_x V(x)}{.}\partial_v \\
&b_j = \gamma^{1/2}(\partial_{v_j} + \frac{v_j}{2}) \quad , \quad b_j^* = \gamma^{1/2}(-\partial_{v_j} + \frac{v_j}{2}) \\
&a_j = \gamma^{1/2}\left(\partial_{x_j} + \frac{(\partial_{x_j}V(x))}{2}\right) \quad , \quad a_j^* = \gamma^{1/2}\left(-\partial_{x_j} + \frac{(\partial_{x_j}V(x))}{2}\right) \\
&[b_j, b_k] = [b_j^*, b_k^*] \quad , \quad [b_j, b_k^*] = \gamma\delta_{j,k}\,, \\
&[a_j, a_k] = [a_j^*, a_k^*] = 0 \quad , \quad [a_j, a_k^*] = \gamma(\partial^2_{x_j,x_k}V(x))\,, \\
&[b_j, X_0] = a_j \quad , \quad \left[b_j^*, X_0\right] = a_j^*\,.
\end{aligned}
$$

- In [HeNi05], the Hypothesis has been relaxed into

$$
\begin{aligned}
&\forall \alpha \in \mathbb{N}^d\,, |\alpha| > 1, \forall x \in \mathbb{R}^d, \quad |\partial_x^\alpha V(x)| \omega C(1 + |\partial_x V(x)|)\langle x\rangle^{-\varrho_0}, \quad \varrho_0 > 0 \\
&\frac{1}{C}\langle x\rangle^{1/M} - C \leq |\partial_x V(x)| \leq C\langle x\rangle^M + C\,, M \geq 1\,.
\end{aligned}
$$

- The proof of this result requires the use of Weyl-Hörmander pseudo-differential calculus (see [BoCh94][BoLe89][HeNi05][Hor85, Chap 18]).
- The exponent ε is not optimal. This is usual with Kohn's method. The exponent $2/3$ can be obtained when V is at most quadratic at ∞ (see a.e. [HePr11]).
- All the written results of this type assume that the Hessian Hess$V(x)$ is well controled by the gradient $|\partial_x V(x)|$ as $x \to \infty$. It does not work for $V(x_1, x_2) = \pm x_1^2 x_2^2$ for which the compactness of $(1 + \Delta^{(0)}_{V/2})^{-1}$ is well understood and depends on the sign $\pm$.

REF:[EPRR99][HeNi04][HeNi05]

3.5 Trend to the equilibrium

Notations: Set

$$
L' = v.\partial_x - \frac{1}{m}(\partial_x V(x)).\partial_v - \gamma_0\partial_v(\frac{1}{m\beta}\partial_v + v)
$$

Consider the spaces

$$
H_{s,s} = \left\{u \in \mathcal{S}'(\mathbb{R}^{2d}), \quad \Lambda^s u \in L^2(\mathbb{R}^{2d})\right\}\,,
$$

such that

$$\cup_{s\in\mathbb{R}} H_{s,s} = \mathcal{S}' \quad \text{and} \quad \cap_{s\in\mathbb{R}} H_{s,s} = \mathcal{S}\,.$$

The Maxwellian function is

$$M(x,v) = e^{-\beta(\frac{mv^2}{2}+V(x))}\,.$$

Depending on the case $M \in L^1(\mathbb{R}^{2d}, dxdv)$ (case $+$) or not (case $-$) we set

$$M^{1/2}\mathcal{S}' \ni f \longmapsto \left((\textstyle\int f)M_0\right)_\pm = \begin{cases} (\int f)M_0 & \text{if } V > 0 \text{ near } \infty, \\ 0 & \text{if } V < 0 \text{ near } \infty. \end{cases}$$

The quantity $\omega_1(V_\beta)$ is the first nonzero eigenvalue of $\Delta^{(0)}_{\frac{1}{2}V_\beta}$, (see Smoluchowski process).

Theorem 3.12. *Under the same assumptions as in $\gamma_0, \beta, m > 0$, we have:*

1) *The Cauchy problem*

$$\partial_t f = -L'f \quad , \quad f(t=0) = f_0\,,$$

is well-posed for $t \geq 0$ in $M^{1/2}\mathcal{S}'(\mathbb{R}^{2d})$. Moreover the solution $f(t)$ belongs to $M^{1/2}\mathcal{S}(\mathbb{R}^{2d})$ for $t > 0$.

2) *There exists a real $\tau = \tau(\beta, \gamma_0, m) > 0$ and for any $s \geq 0$ two constant $c_s > 0$ and $R_s > 0$ so that the estimate*

$$\left\| f(t) - \left((\textstyle\int f_0)M_0\right)_\pm \right\|_{M^{1/2}H^{s,s}} \leq c_s Q_{R_s} e^{-\tau t} \left\|f_0\right\|_{M^{1/2}H^{-s,-s}}$$

holds for all $f_0 \in M^{1/2}H^{-s,-s}(\mathbb{R}^{2d})$ with

$$Q_{R_s} = Q_{R_s}\left(C(V)_s, \sqrt{m}\gamma_0, \beta, t, \tau, m\right)$$

and for $s = 0$ $\quad Q_{R_0} = Q_{R_0}\left(C(V), \sqrt{m}\gamma_0, \beta, t, \tau\right)$.

3) *In part 2), we can take*

$$\tau(\beta, \gamma_0, m) = \tau_1(\beta, \gamma_0, m) \stackrel{def}{=} \frac{\gamma_0 \min\{1, \omega_1(V_\beta)\}}{64(5 + 3\gamma_0\sqrt{m}\beta^{\frac{n-1}{2n}} + 3C_{V_\beta})^2} \tag{3.3}$$

where $C^2_{V_\beta} = \max\left\{\sup\left((\operatorname{Hess} V_\beta)^2 - \left(\frac{1}{4}(\partial_x V_\beta)^2 - \frac{1}{2}\Delta V_\beta\right)\operatorname{Id}\right), 0\right\}$.

4) *There exists a constant $c > 0$, so that any τ satisfying part 2) is bounded by*

$$\tau \leq c\sqrt{\omega_1(V_\beta)}\log\left[Q_{R_0}(C(V), \sqrt{m}\gamma_0, \beta, \omega_1(V_\beta))\right]\,.$$

Comments:

- All the factors of exponential quantities occuring in the estimates show an algebraic dependance in terms of (t, γ_0, m, β).
- Note that in the multiple well case and when $\beta \to \infty$, $\omega_1(V_\beta)$ can be exponentially small, i.e. like $e^{-\beta C}$.

REF:[HeNi04]

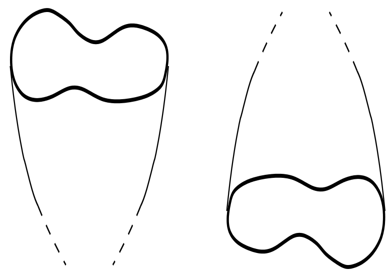
$\beta \preceq 1$ $(n \geq 1)$	c_2 -convex (concave)	 Hypothesis H2+ H2-
$\sqrt{m}\gamma_0\beta^{\frac{n-1}{2n}} \preceq 1$	$\gamma_0\beta^{\frac{n-1}{n}} \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma_0\beta^{\frac{3n-1}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$	$\gamma_0 \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma_0\beta^{\frac{3n-1}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$
$1 \preceq \sqrt{m}\gamma_0\beta^{\frac{n-1}{2n}}$	$\frac{1}{m\gamma_0} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma_0\beta^{\frac{-1-n}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$	$\frac{1}{m\gamma_0\beta^{\frac{n-1}{n}}} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma_0\beta^{\frac{-1-n}{2n}}\right)}{\sqrt{m}\beta^{\frac{n-1}{2n}}}$

Table 1: High temperature asymptotics.
The notation $a \preceq b$ means $a \leq \kappa_V b$ where κ_V depends only on V.

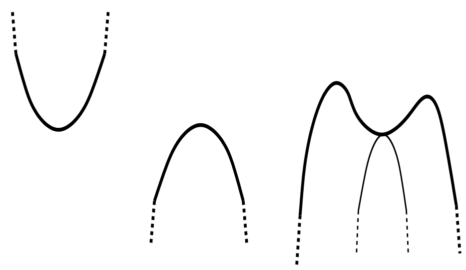
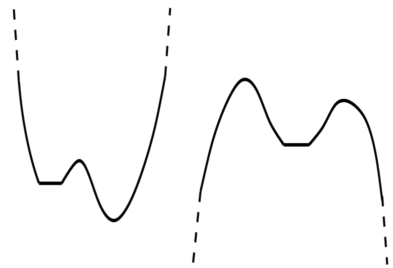
$\beta \succeq 1$ $(n \geq 1)$	 Morse function with 1 or 0 local minimum	 More than 1 or 0 minimum
$\sqrt{m}\gamma_0 \preceq \beta^{\frac{n-1}{2n}}$	$\frac{\gamma_0}{\beta^{\frac{2(n-1)}{n}}} \preceq \tau \preceq \frac{-\log\left(\sqrt{m}\gamma_0\beta^{\frac{-n-1}{n}}\right)}{\sqrt{m}}$	$\tau \preceq \frac{e^{-\frac{\beta}{3\kappa_V}}}{\sqrt{m}} \log\left(\sqrt{m}\gamma_0 + \frac{1}{\sqrt{m}\gamma_0}\right)$
$\beta^{\frac{n-1}{2n}} \preceq \sqrt{m}\gamma_0$	$\frac{1}{m\gamma_0\beta^{\frac{n-1}{n}}} \preceq \tau \preceq \frac{\log\left(\sqrt{m}\gamma_0\beta^{\frac{3n-1}{n}}\right)}{\sqrt{m}}$	$\tau \preceq \frac{e^{-\frac{\beta}{3\kappa_V}}}{\sqrt{m}} \log\left(\sqrt{m}\gamma_0\right)$

Table 2: Low temperature asymptotics.
The notation $a \preceq b$ means $a \leq \kappa_V b$ where κ_V depends only on V.

3.6 The low temperature limit like a semiclassical limit

Consider now the case $m = 1$, $\gamma_0 = 1$ and $\frac{1}{\beta} = h \to 0$:

$$K = v.\partial_x - \partial_x V(x) + h(-\partial_v + \frac{v}{2h}).(\partial_v + \frac{v}{2h}) .$$

By mutiplying by h it becomes

$$hK = v.(h\partial_x) - (\partial_x V(x)).(h\partial_v) + (-h\partial_v + \frac{v}{2}).(h\partial_v + \frac{v}{2}) .$$

It enters in the framework of semiclassical analysis of PDE's which can be summarized by:

- Multiply h all differentiation operators, ξ is quantized into $hD_x = \frac{h}{i}\partial_x$.
- Multiply all the phases by $\frac{1}{h}$: all the relevant exponential quantities or functions have the form $e^{\frac{\varphi}{h}}$.

Using semiclassical techniques and recent advances about the elliptic case (that is for $\Delta^{(0)}_{V/2,h}$) [HKN04][HeNi06][Lep10.2], Hérau-Hitrik-Sjöstrand obtained in a series of three articles [HHS08.1][HHS08.2][HHS10] accurate informations about the low temperature limit in a multiple well case. By assuming that V is a Morse function at most quadratic at infinity they prove:

- There are m_0 exponentially small $\mathcal{O}(e^{-\frac{c}{h}})$ eigenvalues, where m_0 is the number of local minima of V . These eigenvalues are real.
- They extend the return to the equilibrium to an expansion including all the eigenspaces associated with exponentially small eigenvalues.
- Following an approach similar to the one developed for $\Delta^{(0)}_{V/2,h}$ they compute accurately, i.e. with an explicit first order term, the m_0 exponentially small eigenvalues $\lambda_k(t) \sim h^{\nu}\left(\sum_{j=0}^{\infty} a_{k,j}h^j\right) e^{-\frac{c_k}{h}}$.

The method is slightly different than the one used in [HeNi04]. Instead of using algebraic manipulation of commutators à la Kohn, the semiclassical framework and the specific assumptions allow to use a more geometric approach. The strategy of the proof relies on:

- Subelliptic estimates, rather in the spirit of hypocoercivity (see [Vil09]), are obtained after introducing a phase-space weight

  $$\langle p_0 \rangle_{T_0} = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} p_0 \circ \exp(tH_{p_1})\ dt$$

  where here $p_0(x, y, \xi, \eta) = \frac{v^2}{4}$, $p_1(x, v, \xi, \eta) = v.\xi - (\partial_x V(x)).\eta$ and H_{p_1} is the Hamilton vector field in $\mathbb{R}^{4d}_{x,v,\xi,\eta}$ associated with p_1 . Note that the quantity $\langle p_0\rangle_{T_0}$ carries the information about commutators. By formally expanding the series

$$p_0 \circ e^{tH_{p_1}} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \{p_1, \{p_1, \ldots \{p_1, p_0\} \ldots\}\} \; ,$$

while the Poisson bracket of symbols $\{a, b\}$ is quantized into $i\left[a^W, b^W\right]$. This averaging of symbol according to a flow, is a powerful tool in the analysis of non self-adjoint semiclassical problems (see [Sjo00]).

- The fact that V is a Morse function allows to compare K with a quadratic model around local minima, especially for the spectral analysis around $z = 0$.
- Additional properties are used and will be described in Section 5.

Although more accurate, these semiclassical results valid for the low temperature regime, are less general than the results of the previous section which also gives information for the high temperature limit.

4 About the constant C_τ in $\|e^{-tA}\| \leq C_\tau e^{-\tau t}$

Let A be a maximal accretive operator. By the functional calculus one gets easily.

Proposition 4.1. *When $(A, D(A))$ is self-adjoint or normal*

$$\|e^{-tA}\| \leq 1 \times e^{-t \min \mathrm{Re}\, \sigma(A)} \, .$$

Respectively if $\|e^{-tA}\| \leq Me^{-\tau t}$ then $M = 1$ and $\tau \leq \min \mathrm{Re}\, \sigma(A)$.

The situation is more subtle for a general non self-adjoint maximal accretive operator.

The resolvent is the Laplace transform of the semigroup.

$$(z - A)^{-1} = -\int_0^{+\infty} e^{tz} e^{-tA} \, dt \, .$$

Gearhardt-Prüss-Hwang-Greiner Theorem. *a) If $\|(z - A)^{-1}\|$ is uniformly bounded in $\{\mathrm{Re}\, z \leq \tau\}$ then there exists $C_\tau > 0$ such that $\|e^{-tA}\| \leq C_\tau e^{-\tau t}$.*
b) If $\|e^{-tA}\| \leq C_\tau e^{-\tau t}$ then for every $\alpha < \tau$ the resolvent $\|(z - A)^{-1}\|$ is uniformly bounded in $\{\mathrm{Re}\, z \leq \alpha\}$.
REF:[EnNa00][Paz83]
Questions: What about the behaviour of C_τ when A is not self-adjoint or normal ? Is it related with $\sigma(A)$?

Answer to the second question: No.

An example coming from fluid mechanics: it was introduced by T. Gallay in [GGN09] as a model problem in order to understand the linear stability of Oseen vortices in $\mathbb{R}^2$, after his works with G. Wayne [GaWa02][GaWa05]. Consider in dimension $d = 1$ the operator

$$H_\varepsilon = -\partial_x^2 + x^2 + \frac{i}{\varepsilon}\frac{1}{(1+x^2)^{k/2}}, k \geq 2, \varepsilon > 0\,.$$

It is maximal accretive, with a compact resolvent and with a numerical range

$$\mathrm{Num}(H_\varepsilon) \subset \left\{\lambda \in \mathbb{C},\, \mathrm{Re}\,\lambda \geq 1\,, \quad \mathrm{Im}\,\lambda \in [0, \frac{1}{\varepsilon}]\right\}\,.$$

We introduce the quantities

$$\Sigma(\varepsilon) = \min\,\mathrm{Re}\,\sigma(H_\varepsilon) \quad , \quad \Psi(\varepsilon) = \left(\sup_{\lambda\in\mathbb{R}} \|(H_\varepsilon - i\lambda)^{-1}\|\right)^{-1}\,.$$

The following estimates hold

$$\frac{1}{\Sigma(\varepsilon)} \leq \frac{1}{\Psi(\varepsilon)} \leq \sup_{\lambda\in\mathbb{R}} \frac{1}{\mathrm{dist}(i\lambda, \mathrm{Num}(H_\varepsilon))} \leq 1\,.$$

The operator H_ε is sectorial so that the following result can be applied.

Lemma 4.2. *Let A be a maximal accretive operator in a Hilbert space X, with numerical range contained in the sector $\{z \in \mathbb{C}\,;\, |\arg z| \leq \frac{\pi}{2} - 2\alpha\}$ for some $\alpha \in (0, \frac{\pi}{4}]$. Assume that A is invertible and let*

$$\Sigma \;=\; \inf\,\mathrm{Re}\,(\sigma(A)) \;>\; 0\;, \quad \textit{and} \quad \Psi \;=\; \left(\sup_{\lambda\in\mathbb{R}} \|(A - i\lambda)^{-1}\|\right)^{-1}\,.$$

Then the following holds:

i) *If there exist $C \geq 1$ and $\mu > 0$ such that $\|e^{-tA}\| \leq C\,e^{-\mu t}$ for all $t \geq 0$, then*

$$\Sigma \;\geq\; \mu\;, \quad \textit{and} \quad \Psi \;\geq\; \frac{\mu}{1 + \log(C)}\,.$$

ii) *For any $\mu \in (0, \Sigma)$, we have $\|e^{-tA}\| \leq C(A, \mu)\,e^{-\mu t}$ for all $t \geq 0$, where*

$$C(A, \mu) \;=\; \frac{1}{\pi\tan\alpha}\Big(\mu N(A, \mu) + 2\pi\Big)\,, \quad \textit{and} \quad N(A, \mu) \;=\; \sup_{\lambda\in\mathbb{R}} \|(A - \mu - i\lambda)^{-1}\|\,.$$

iii) *If moreover $\mu \in (0, \Psi)$, the quantity $N(A, \mu)$ is not larger than $(\Psi - \mu)^{-1}$.*

REF:[GGN09].
In our case one finds

$$\frac{1}{C_0 \varepsilon^{\frac{2}{k+4}}} \leq \Psi(\varepsilon) \leq \frac{C_0}{\varepsilon^{\frac{2}{k+4}}} \quad , \quad \Sigma(\varepsilon) \geq \frac{C_1}{\varepsilon^{\frac{2}{k+2}}} \,.$$

In a recent note [HeSj09], Helffer and Sjöstrand studied a more quantitative version of the Gerhardt-Prüss theorem without the sectoriality condition. By setting

$$\frac{1}{r(\tau)} = \sup_{\mathrm{Re}\, z \leq \tau} \|(z-A)^{-1}\| = \sup_{\mathrm{Re}\, z = \tau} \|(z-A)^{-1}\| \,.$$

they proved: If $m(t) \geq \|e^{-tA}\|$ is a continuous function, then for all $a, \tilde{a}$ and $t \geq 0$ such that $t = a + \tilde{a}$

$$\|S(t)\| \leq \frac{e^{-\tau t}}{r(\tau) \|\frac{1}{m}\|_{e^{\tau \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{\tau \cdot} L^2([0,\tilde{a}])}} \,,$$

In particular for $a = \tilde{a} = \frac{t}{2}$, and starting from an initial bound $m(t) = \tilde{m}(t) e^{-\tau t}$ this also gives a new one $\hat{m}(t) e^{-\tau t}$ with

$$\frac{\hat{m}(t)}{r(\tau)} \leq \frac{1}{\int_0^{t/2} \left(\frac{r(\tau)}{\tilde{m}(s)}\right)^2 \, ds} \,.$$

Passing from $\tilde{m}$ to $\hat{m}$ may be iterated.

5 Additional structures

5.1 PT symmetry

The operator

$$K = v.\partial_x - (\partial_x V(x)).\partial_v + \gamma(-\Delta_v + \frac{v^2}{4} - \frac{d}{2}) \quad V \in \mathcal{C}^\infty(\mathbb{R}^d)$$

is not any non self-adjoint operator. It has a symmetry property called PT-symmetry, a name coming from quantum mechanics (PT refers to parity and time, see a.e. [Ben05]). The operator U given by

$$Uu(x,v) = u(x,-v) \,.$$

is unitary with

$$U = U^* = U^{-1} \,.$$

The conjugation with U gives

$$U^* K U = K^* \,,$$

This property has been used in [HHS10] in order to compute accurately the small eigenvalues of K in the small temperature limit. The next result explains why the small eigenvalues are real and in which sense K behaves like a self-adjoint operator arount $z = 0$ in the spectral plane.

Proposition 5.1. *Let $(A, D(A))$ be a closed densely defined operator in a separable Hilbert space $\mathcal{H}$, such that $(1 + A)^{-1}$ is compact, so that $\sigma(A)$ is discrete, and $D(A) = D(A^*)$. Assume that there exists a unitary operator, $U^* = U^{-1}$, such that and $U^*AU = A^*$. Then the spectrum Spec(A) is invariant by complex conjugation.*
If additionally

- $\operatorname{Re} A = A + A^*$ *is non negative with* $\min \mathit{Spec}(\operatorname{Re} A) \setminus \{0\} = \gamma > 0$, *and* $U\Pi_0 = \Pi_0 U = \Pi_0$ *by setting* $\Pi_0 = 1_{\{0\}}(\operatorname{Re} A)$ *;*
- Γ *is a bounded contour such that*

$$\mathrm{Tr}\left[\frac{1}{2\pi i}\int_\Gamma A(z-A)^{-1}\ dz\right] = \mathrm{Tr}\ [A\Pi_\Gamma] \in (0, \frac{\gamma}{2}) \tag{5.1}$$

with $\Pi_\Gamma = \frac{1}{2\pi i}\int_\Gamma (z - A)^{-1}\ dz$ *and* $A\Pi_\Gamma = \Pi_\Gamma A = \Pi_\Gamma A \Pi_\Gamma$ *;*

then the following properties hold

- $\langle u, v\rangle_U = \langle u, Uv\rangle$ *is a hermitian positive definite form on* $E_\Gamma = \operatorname{Ran}\Pi_\Gamma$ *;*
- $A\big|E_\Gamma$ *is self-adjoint and non negative for the scalar product* $\langle\ ,\ \rangle_U$ *;*
- E_Γ *admits a basis of eigenvectors of* A, $(e_1, \ldots, e_N)$, *orthonormal for the scalar product* $\langle\ ,\ \rangle_U$, *such that*
- *there exists a constant* $C_{\Gamma,\gamma} > 0$ *such that for all* $z \in \mathbb{C}$ *being inside the contour* Γ, *the inequality*

$$\|(z-A)^{-1}\big|_{E_\Gamma}\| \le \frac{C_{\Gamma,\gamma}}{\mathrm{dist}(z, \langle\lambda_1, \ldots, \lambda_N\rangle)}\,.$$

holds with the initial norm on $\mathcal{L}(\mathcal{E}_-)$ *.*

Remark 5.2. *This does not prove resolvent estimates for A and z inside Γ, because Π_Γ is not a priori an orthogonal projection. Actually in practice, resolvent estimates have to be proved first in order to control $\|\Pi_\Gamma\|$ and also to verify condition (5.1). But once it is done in a rough sense, the self-adjointness property w.r.t $\langle\ ,\ \rangle_U$ can be used to have finer spectral results.*

5.2 Supersymmetry

For the sake of simplicity we work here on $\mathbb{R}^d$ with the euclidean metric and we assume that

$$\lim_{x\to\infty} |\nabla V(x)| = +\infty \quad , \quad \mathrm{Hess}(V(x)) \overset{x\to\infty}{=} o(\nabla V(x))$$

We shall first consider the Witten Laplacian

$$\Delta^{(0)}_{V,h} = -h^2\Delta_x + |\nabla V(x)|^2 - h\Delta_V(x)\,,$$

which has a compact resolvent and then see how this can be extended to K via the hypoelliptic Laplacian structure introduced by Bismut.

5.2.1 Differential forms

A differential form of degree 1 is

$$\omega(x) = \sum_{j=1}^{d} \omega_j(x)dx_j$$

with $\omega_j(x) = \partial_{x_j} f(x)$ when $\omega = df$.
For degree p, the antisymmetric p-linear form $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, $I = \{i_1, \ldots, i_p\}$, $i_1 < \ldots < i_p$, applied to p-vectors in $\mathbb{R}^d$, $(X_1, \ldots, X_p)$, is the determinant made with the lines $i_1, \ldots, i_p$ of the matrix $(X_1, \ldots, X_p)$ written in the canonical basis. For any permutation $\sigma \in \mathfrak{t}_p$, the antisymmetry writes

$$dx_{i_{\sigma(1)}} \wedge \ldots \wedge dx_{i_{\sigma(p)}} = \varepsilon(\sigma) dx_{i_1} \wedge \ldots \wedge dx_{i_p}\,. \tag{5.2}$$

A p-differential form is

$$\omega = \sum_{\sharp I = p} \omega_I(x) dx_I\,.$$

The exterior product of two forms ω (degree p) and η (degree q) is given by

$$\omega \wedge \eta = \sum_{\sharp I = p\,,\, \sharp J = q} \omega_I(x)\eta_J(x)(dx_I) \wedge (dx_J)\,.$$

The interior product of ω, with a vector X (possibly a vector field $X(x)$) is given by the $(p-1)$

$$\mathbf{i}_X\omega = \sum_{\sharp I = p} \omega_I(x)\mathbf{i}_{X(x)}(dx_I)$$

where $\mathbf{i}_X(dx_{i_1} \wedge \ldots \wedge dx_{i_p})(T_2, \ldots T_p) = (dx_{i_1} \wedge \ldots \wedge dx_{i_p})(X, T_2, \ldots, T_p)$.
A p-differential form is said $\mathcal{C}^\infty$ when all its coefficient ω_I are $\mathcal{C}^\infty$. Its differential is given by

$$d\omega(x) = \sum_{i=1}^{d} \sum_{\sharp I = p} \partial_{x_i}\omega_I(x) dx_i \wedge dx_I\,,$$

and the antisymmetry (5.2) implies $d\circ d=0$. With the euclidean scalar product on $\mathbb{R}^d$, there is a natural scalar product on $\Lambda^p\mathbb{R}^d=\mathrm{Vect}(dx_I,\sharp I=p)$ given by

$$\langle dx_I\,,\,dx_J\rangle=\delta_{IJ}\,.$$

By using the Lebesgue measure $\mathrm{Leb}(dx)$ on $\mathbb{R}^d$, the L^2 scalar product of two (complex-valued) p-forms is given by

$$\langle\omega\,,\,\eta\rangle_{\Lambda^pL^2}=\int_{\mathbb{R}^d}\sum_{\sharp I=p}\overline{\omega_I}(x)\eta_I(x)\ \mathrm{Leb}(dx)\,,$$

and the Sobolev spaces $\Lambda^pH^s(\mathbb{R}^d)$ and their weighted versions are defined as usual.
The differential d sends $\Lambda^p\mathcal{C}_0^\infty(\mathbb{R}^d)$ into $\Lambda^{p+1}\mathcal{C}_0^\infty(\mathbb{R}^d)$ and its formal adjoint, with respect to the L^2-scalar product, is denoted by $d^*:\Lambda^{p+1}\mathcal{C}_0^\infty\to\Lambda^p\mathcal{C}_0^\infty$

$$\langle d^*\omega\,,\,\eta\rangle_{\Lambda^{p-1}L^2}=\langle\omega\,,\,d\eta\rangle_{\Lambda^pL^2}\,,$$

and is called the codifferential.
When $\omega=\omega_{\{1,\ldots,d\}}dx_1\wedge\ldots dx_d$, one defines

$$\int_{\mathbb{R}^d}\omega=\int_{\mathbb{R}^d}\omega_{\{1,\ldots,d\}}\ \mathrm{Leb}(dx)\,,$$

and consequently the integral $\int\overline{\omega}\wedge\eta$ when $\omega\in\Lambda^p\mathcal{C}_0^\infty$ and $\eta\in\Lambda^{d-p}\mathcal{C}_0^\infty$. The Hodge star operator, $\star:\Lambda^p\to\Lambda^{d-p}$ is the local operator given by

$$\begin{aligned}\langle\omega\,,\,\eta\rangle_{\Lambda^pL^2}&=\int_{\mathbb{R}^d}\overline{(\star\omega)}\wedge\eta\,,\\ \star\omega&=\sum_{\sharp I=p}\omega_I(x)\star(dx_I)=\sum_{\sharp I=p}\omega_I(x)(-1)^{n_I}dx_{\{1,\ldots,d\}\setminus I}\,.\end{aligned}$$

One easily checks

$$\star\circ\star=(-1)^{p(d+1)}\quad,\quad d^*=(-1)^{p(d+1)+1}\star d\star\quad,\quad d^*\circ d^*=0\,.$$

Similar calculations show that the formal adjoint of $df\wedge:\Lambda^p\mathcal{C}_0^\infty\to\Lambda^{p+1}\mathcal{C}_0^\infty$ is $\mathbf{i}_{\nabla f}$.

5.2.2 Witten Laplacian

The Smoluchowski process leads to

$$\begin{aligned}\Delta_{V,h}^{(0)}&=-h^2\Delta+|\nabla V(x)|^2-h(\Delta V(x))\\ &=(-h\partial_x+\partial_x(V(x))).(h\partial_x+\partial_xV(x))=d_{V,h}^*d_{V,h}\,,\end{aligned}$$

with

$$d_{V,h} = e^{-\frac{V}{h}}(hd)e^{\frac{V}{h}} = hd + dV\wedge \quad , \quad d^*_{V,h} = e^{-\frac{V}{h}}(hd)^* e^{\frac{V}{h}} = hd^* + \mathbf{i}_{\nabla f}\,.$$

These deformed differential and codifferential, introduced by Witten, satisfy

$$d_{V,h}\circ d_{V,h} = 0 \quad \text{and} \quad d^*_{V,h}\circ d^*_{V,h} = 0\,.$$

The Witten Laplacian is the deformed Hodge Laplacian

$$\Delta_{V,h} = d^*_{V,h}d_{V,h} + d_{V,h}d^*_{V,h} = (d_{V,h}+d^*_{V,h})^2 = \bigoplus_{p=0}^{d}\Delta^{(p)}_{V,h}\,.$$

When $V\in\mathcal{C}^\infty(\mathbb{R}^d)$, $\Delta^{(p)}_{V,h}$ is essentially self-adjoint on $\Lambda^p\mathcal{C}_0^\infty(\mathbb{R}^d)$ by Simader's theorem [Sim78].
By making use of the magic Cartan formula $\mathbf{i}_X d + d\mathbf{i}_X = \mathcal{L}_X$, one gets the formula

$$\Delta_{V,h} = h^2(d^*d + dd^*) + |\nabla V(x)|^2 + h(\mathcal{L}_{\nabla V} + \mathcal{L}^*_{\nabla V})$$

where $(\mathcal{L}_{\nabla V} + \mathcal{L}^*_{\nabla V})$ is a matricial 0-th order differential operator with entries which are linear expressions of $\partial^2_{x_ix_j}V(x)$.
If $\lim_{x\to\infty}|\nabla V(x)|^2 = +\infty$ and $|\mathrm{Hess}V(x)|\leq C(1+|\nabla V(x)|^2)$, then $\Delta_{V,h}$ has a compact resolvent. The commutations

$$\begin{aligned} d_{V,h}(\Delta_{V,h}) &= d_{V,h}d^*_{V,h}d_{V,h} = \Delta_{V,h}d_{V,h}\\ d^*_{V,h}(\Delta_{V,h}) &= \Delta_{V,h}d^*_{V,h} \quad \text{on } \Lambda\mathcal{C}_0^\infty(\mathbb{R}^d)\end{aligned}$$

lead to

$$\begin{aligned}(z-\Delta_{V,h})^{-1}d_{V,h} &= d_{V,h}(z-\Delta_{V,h})^{-1}\\ (z-\Delta_{V,h})^{-1}d^*_{V,h} &= d^*_{V,h}(z-\Delta_{V,h})^{-1}\end{aligned}$$

for any $z\in\mathbb{C}\setminus\mathrm{Spec}(\Delta_{V,h})$, on the form domain of $\Delta_{V,h}$. The functional calculus yields

$$1_E(\Delta^{(p+1)}_{V,h})d_{V,h} = d_{V,h}1_E(\Delta^{(p)}_{V,h}) \quad \text{and} \quad 1_E(\Delta^{(p-1)}_{V,h})d^*_{V,h} = d^*_{V,h}1_E(\Delta^{(p)}_{V,h})$$

for any bounded Borel set $E\subset\mathbb{R}$. In particular when ψ is an eigenvector of $\Delta^{(p)}_{V,h}$ with eigenvalue $\lambda\neq 0$, the following alternatives occur

- either $d_{V,h}\psi\neq 0$ and $d_{V,h}\psi$ is an eigenvector of $\Delta^{(p+1)}_{V,h}$ with the same eigenvalue λ, or $d_{V,h}\psi = 0$ and $d^*_{V,h}\psi\neq 0$ is an eigenvector of $\Delta^{(p-1)}_{V,h}$ with the same eigenvalue;
- either $d^*_{V,h}\psi\neq 0$ and $d^*_{V,h}\psi$ is an eigenvector of $\Delta^{(p-1)}_{V,h}$ with the same eigenvalue λ, or $d^*_{V,h}\psi = 0$ and $d_{V,h}\psi\neq 0$ is an eigenvector of $\Delta^{(p+1)}_{V,h}$ with the same eigenvalue.

Definition 5.3. *A Morse function on $\mathbb{R}^d$ is a $\mathcal{C}^\infty$function of which all the critical points are non degenerate:*

$$\nabla V(x) = 0 \Rightarrow \det(\mathrm{Hess}\, V(x)) \neq 0\,.$$

When x_0 is a critical point of the real valued Morse function V, the number of negative eigenvalues of $\mathrm{Hess}\, V(x)$ *is called its index.*

Theorem 5.4. *Let $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ be a Morse function such that*

$$\lim_{x\to\infty} |\nabla V(x)| = +\infty \quad \text{and} \quad |\mathrm{Hess}\, V(x)| \leq C(1 + |\nabla V(x)|^2)\,,$$

for all $x \in \mathbb{R}^d$. Then for all $p \in \{1, \ldots, d\}$ and for $h \in (0, h_0)$, $h_0 > 0$ small enough, the dimension of the spectral subspace $F_p = \mathrm{Ran}\, 1_{[0,Ch^{3/2}]}(\Delta_{V,h}^{(p)})$ equals m_p, the number of critical points of V with index p. Moreover the restricted differential $\beta = d_{V,h} 1_{[0,Ch^{3/2})}(\Delta_{V,h})$ provides the chain complex

$$0 \rightleftarrows F_M^{(0)} \ldots F_M^{(p-1)} \underset{\beta^{(p-1)*}}{\overset{\beta^{(p-1)}}{\rightleftarrows}} F_M^{(p)} \underset{\beta^{(p)*}}{\overset{\beta^{(p)}}{\rightleftarrows}} F_M^{(p+1)} \ldots F_M^{(d)} \rightleftarrows 0 \tag{5.3}$$

and $\dim\ker(\beta^p)/\ker(\beta^{p-1}) = b_p$ *is the p-th Betti number, which is here $b_0 = 1$, $b_p = 0$ for $p > 1$.*
For any $p \in \{0, \ldots, d\}$, the $\mathcal{O}(h^{3/2})$ eigenvalues are actually exponentially small $\mathcal{O}(e^{-\frac{c}{h}})$.

REF: [Wi82][HeSj89][CFKS87]

Remark 5.5.

- *With this result on compact Riemannian manifolds, E. Witten provided an analytic approach to Morse inequalities in his celebrated article [Wi82].*
- *The result is proved usually for a compact Riemannian manifold. Here we chose to stay in $\mathbb{R}^d$ in order to avoid the introduction of more sophisticated material. Under our assumptions for V, the proof for a compact Riemannian manifold can easily adapted to this case.*

Here comes an important point: The eigenvalues of $\Delta_{V,h}^{(0)}$ are the squares of the singular values of $d_{V,h}$. In particular, the exponentially small eigenvalues of $\Delta_{V,h}^{(0)}$ are the squares of the singular values of $\beta^{(0)}$. An easy case of Fan inequalities says that the singular values of compact operators (or matrices), usually labelled in the decreasing order satisfy

$$s_k(B_1 A) \leq \|B_1\| s_k(A) \quad , \quad s_k(A B_0) \leq \|B_0\| s_k(A)\,.$$

In particular when $C_0 : F^{(0)} \to F^{(0)}$ and $C_1 : F^{(1)} \to F^{(1)}$ are almost unitary operators, in the sense that $C_j^* C_j = \mathrm{Id}_{F^{(j)}} + \mathcal{O}(e^{-\frac{c}{h}})$ then

$$s_k(\beta^{(0)}) = s_k(C_1 \beta^{(0)} C_0)(1 + \mathcal{O}(e^{-\frac{c}{h}}))\,.$$

As an application, it suffices to construct almost orthonormal quasimodes, i.e. approximate eigenvectors, for $\Delta^{(0)}_{V,h}$ and $\Delta^{(1)}_{V,h}$ and to compute the matrix of $\beta^{(0)}$ in these bases in order to get a good approximation of the exponentially small eigenvalues of $\Delta^{(0)}_{V,h}$. When this construction in made a simple linear algebra argument relying on gaussian elimination gives the result (see [Lep09][Nie04]). This linear algebra structure reflects the ordering of exit times used via the probabilistic or potential theory approach [BEGK04][BGK04]. This was used to compute accurately the exponentially small eigenvalues of $\Delta^{(0)}_{V,h}$ in the form

$$\lambda_k(\Delta^{(0)}_{V,h}) = a_k(h)e^{-\frac{f(U_{j(k)})^{(1)}-U_k^{(0)}}{h}} \quad , k \in \{1, \ldots, m_0\} \, ,$$

where $U_k^{(0)}$ is a local minimum of V, $U^{(1)}_{j(k)}$ is the saddle point involved in Arrhenius law and $a_k(h) = h^\nu \sum_{j=0}^{\infty} h^j a_{j,k}$ with ν and $a_{0,k}$ explicitely computed. This was achieved on a compact manifold or $\mathbb{R}^d$ [HKN04], on a compact manifold with natural boundary conditions ([HeNi03][Lep10.2]), and was recently extended to the case of p-forms on a compact manifold ([Lep11][LNV11]).

5.2.3 Boundary Witten Laplacians

We stick here to the simple one dimensional case on the segment $[-1, 1]$, with a Morse function $V(x)$ such that $V'(1) \neq 0$ and $V'(-1) \neq 0$. The interior Witten Laplacian on functions is given by

$$\Delta^{(0)}_{V,h} u = [-h^2\Delta + |V'(x)|^2 - hV''(x)]u \quad , \quad \forall u \in \mathcal{C}_0^\infty((-1, 1)) \, .$$

Meanwhile interior Witten Laplacian of 1-forms is given by

$$\Delta^{(1)}_{V,h}(u(x)dx) = \left[[-h^2\Delta + |V'(x)|^2 + hV''(x)]u\right] \, dx \quad , \quad \forall u \in \mathcal{C}_0^\infty((-1, 1)) \, .$$

Consider the case when $V(x) = (x - 1/2)^2$ for example. The Dirichlet realization $\Delta^{(0,D)}_{V,h}$ of $\Delta^{(0)}_{V,h}$ has no kernel $\ker(\Delta^{(0),D}_{V,h}) = \{0\}$ and at least one exponentially small eigenvalue.
For one-forms the lower bound $V'' \geq 2$ implies

$$\Delta^{(1),D}_{V,h} \geq 2h \, .$$

Hence the commutation $d_{V,h}\Delta^{(0)}_{V,h} = \Delta^{(1)}_{V,h} d_{V,h}$ valid on $\mathcal{C}_0^\infty((-1, 1))$ cannot be extended to the domains of $\Delta^{(0),D}_{V,h}$:

$$d_{V,h}\Delta^{(0),D}_{V,h} \neq \Delta^{(1),D}_{V,h} d_{V,h} \, .$$

The solution comes from the fact that degree-p (with $p > 0$) Witten Laplacians are actually systems of differential operators acting on vector valued functions. Hence, Dirichlet conditions can be imposed only on a part of the components of $\omega \in \Lambda^p \mathcal{C}^\infty(\overline{\Omega})$. Locally the boundary $\partial\Omega$ of Ω can be written $\{x_d = 0\}$ in a coordinate system $(x_1, \ldots, x_{d-1}, x_d) = (x', x_d)$, where x_d is a normal coordinate. A p-form has the local writing

$$\begin{aligned}
\omega(x) &= \sum_{\sharp J = p,\ J \subset \{1,\ldots,d-1\}} \omega_J dx_J + \sum_{\sharp J = p-1,\ J \subset \{1,\ldots,d-1\}} \omega_{J,d} dx_J \wedge dx_d \\
&= \mathbf{t}\omega(x) + \mathbf{n}\omega(x)\,.
\end{aligned}$$

The good Dirichlet realization is given by

$$D(\Delta_{V,h}^{(p),D}) = \left\{\omega \in \Lambda^p H^2(\Omega)\,, \mathbf{t}\omega\big|_{\partial\Omega} = 0 \quad , \quad \mathbf{t}d_{V,h}^{*}\omega\big|_{\partial\Omega} = 0\right\}$$

and the form domain

$$Q(\Delta_{V,h}^{(p),TD}) = \left\{\omega \in \Lambda^p H^1(\Omega)\,, \mathbf{t}\omega\big|_{\partial\Omega} = 0\right\}\,.$$

The good Neumann realization is given by

$$D(\Delta_{V,h}^{(p),TN}) = \left\{\omega \in \Lambda^p H^2(\Omega)\,, \mathbf{n}\omega\big|_{\partial\Omega} = 0 \quad , \quad \mathbf{n}d_{V,h}\Omega\big|_{\partial\Omega} = 0\right\}$$

and the form domain

$$Q(\Delta_{V,h}^{(p),TN}) = \left\{\omega \in \Lambda^p H^1(\Omega)\,, \mathbf{n}\omega\big|_{\partial\Omega} = 0\right\}\,.$$

In our one-dimensional example, a function $u \in \mathcal{C}^\infty([-1,1])$ belongs to $D(\Delta_{V,h}^{(0),TD})$ if $u(-1) = u(1) = 0$. A 1-form $u(x)dx$ with $u \in \mathcal{C}^\infty([-1,1])$ belongs to $D(\Delta_{V,h}^{(1),TD})$ if $-hu'(1) + u(1)V'(1) = 0$ and $hu'(-1) + u(-1)V'(-1) = 0$.
For a Morse function V on $\overline{\Omega}$, with no critical points on $\partial\Omega$ and such that $V\big|_{\partial\Omega}$ is a Morse function, the notion of critical point has to be extended. For the tangential Dirichlet realization, a generalized critical point with index p is either an interior critical point with index p or a critical point of $V\big|_{\partial\Omega}$ with index $p-1$ such that the exterior normal derivative $\partial_n V$ is positive. The intuition is that the exterior of Ω has the potential $-\infty$, corresponding to the absorption of particles. For the Neumann realization, a general critical point is either an interior critical point with index p or a critical point of $V\big|_{\partial\Omega}$ with index p, such that $\partial_n V < 0$. The intuition is that the exterior of Ω has the potential $+\infty$, corresponding to the reflexion of particles at the boundary. The study of boundary Witten Laplacian and this definition of generalized critical points, was initially presented in [ChLi95] and used in [HeNi06][Lep10.2]. More recently it led F. Laudenbach in [Lau11] to a general treatment of Morse theory for boundary manifolds.

5.2.4 Hypoelliptic Laplacian

We work on $\mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n$ and recall that the dual basis to $(e_1, \ldots, e_n)$ is denoted by $(dx_1, \ldots, dx_n)$. The duality between $\Lambda^p\mathbb{R}^n$ and $\Lambda^p\mathbb{R}^{n,*}$ is given by

$$\omega.(u_1 \wedge \ldots \wedge u_p) = \omega(u_1, \ldots, u_p)\,.$$

For a linear mapping $A : \mathbb{R}^{n,*} \to \mathbb{R}^n$, $A(dx_j) = \sum_{i=1}^n A_{ij} e_i$ we set

$$\Lambda^p A(\omega_1 \wedge \ldots \wedge \omega_p) = (A\omega_1) \wedge \ldots \wedge (A\omega_p) \quad , \omega_j \in \mathbb{R}^{n,*}\,,$$
$$(\omega, \eta)_A = \int_{\mathbb{R}^n} \overline{\omega}.((\Lambda^p A)\eta)(x)\ \mathrm{Leb}(dx)\,, \quad \omega, \eta \in \Lambda^p \mathcal{C}_0^\infty(\mathbb{R}^n)\,.$$

With coordinates the duality product $\overline{\omega}.((\Lambda^p A)\eta)$ equals

$$\begin{aligned} &\sum_{\sharp I = \sharp J = p} \overline{\omega_I(x)} \eta_{J(x)} (dx_{i_1} \wedge \ldots \wedge dx_{i_p}).(Adx_{j_1} \wedge \ldots \wedge Adx_{j_p}) \\ &= \sum_{\sharp I = \sharp J = p} \left(\prod_{\ell=1}^p A_{i_\ell, j_\ell} \right) \overline{\omega_I(x)} \eta_J(x)\,. \end{aligned} \tag{5.4}$$

For an operator $a : \Lambda^p \mathcal{C}_0^\infty(\mathbb{R}^n) \to \Lambda^{p'} \mathcal{C}_0^\infty(\mathbb{R}^n)$ the formal adjoint a^{*A} w.r.t A is defined by

$$(\omega\,,\, a\eta)_A = (a^{*,A}\omega\,,\,\eta)_A \quad , \quad \forall \omega \in \Lambda^{p'} \mathcal{C}_0^\infty(\mathbb{R}^n), \forall \eta \in \Lambda^p \mathcal{C}_0^\infty(\mathbb{R}^n),$$

and it is uniquely defined when A is invertible. If one considers the operator $\partial_{x_j} : \Lambda^p \mathcal{C}_0^\infty(\mathbb{R}^n) \to \Lambda^p \mathcal{C}_0^\infty(\mathbb{R}^n)$ defined by

$$\partial_{x_j} \left(\sum_{\sharp I = p} \omega_I(x) dx_I \right) = \sum_{\sharp I = p} (\partial_{x_j} \omega_I)(x) dx_I$$

the formula (5.4) gives $\partial_{x_j}^{*,A} = -\partial_{x_j}$.
For $\omega_1 \in \Lambda^1 \mathcal{C}^\infty(\mathbb{R}^n)$, the definition of $\omega.(\Lambda^p \eta)$ leads to

$$(\omega_1 \wedge)^{*,A} = \mathbf{i}_{A\omega_1}\,.$$

From the writing $d_{\Phi,h}\eta = \sum_{j=1}^n (h\partial_{x_j} + \partial_{x_j}\Phi(x)) \circ (dx_j \wedge)$, we get

$$d_{\Phi,h}^{*,A} = \sum_{j=1}^n (-\partial_{x_j} + \partial_{x_j}\Phi(x)) \circ (\mathbf{i}_{Adx_j})\,.$$

Both satisfy $d_{\Phi,h}\circ d_{\Phi,h}=0$ and $d^{*,A}_{\Phi,h}\circ d^{*,A}_{\Phi,h}=0$ and the corresponding "Hodge Laplacian" is

$$\begin{aligned}\Delta^{A}_{\Phi,h} &= d^{*,A}_{\Phi,h}d_{\Phi,h}+d_{\Phi,h}d^{*}_{\Phi,h}=(d^{*,A}_{\Phi,h}+d_{\Phi,h})^2\,.\\ &= \sum_{j,k=1}^{n}(-h\partial_{x_j}+\partial_{x_j}\Phi(x))A_{jk}(h\partial_{x_k}+\partial_{x_k}\Phi(x))\\ &\qquad +2h\sum_{j,k=1}^{n}\partial^2_{x_j,x_k}\Phi(x)\circ dx_j\circ \mathbf{i}_{Adx_j}\,.\end{aligned}$$

When $n=2d$, x is replaced by (x,v), and

$$\Phi(x,v)=\frac{v^2}{2}+V(x)\quad,\quad A=\frac{1}{2}\begin{pmatrix}0 & 1\\ -1 & \gamma\end{pmatrix}\,,$$

one gets for functions

$$\Delta^{A,(0)}_{\Phi,h}=v.(h\partial_x)-(\partial_x V(x)).(h\partial_v)+\frac{\gamma}{2}(-h\partial_v+v)(h\partial_v+v)$$

which is essentially K_h up to some change of scale.
It is possible to consider $\Delta^{A,(p)}_{\Phi,h}$. Note that it is neither self-adjoint, nor elliptic. Bismut who introduced this operator in the natural invariant writing of Riemannian geometry called it the hypoelliptic Laplacian. Combining the hypoelliptic analysis, the PT-symmetry and this supersymmetry, Hérau-Hitrik-Sjöstrand recently adapted the analysis of exponentially small eigenvalues of the Witten Laplacian, and gave accurate values of the exponentially small eigenvalues for $\Delta^{A,(0)}_{V,h}$.
REF:[Bis05] and [Bis11] for a recent short introduction and additional references); [HHS10] for the accurate semiclassical analysis of exponentially small eigenvalues on $\mathbb{R}^{2d}_{x,v}$, [Leb05][Leb07] for a rapid approach to the key analysis issues of the hypoelliptic Laplacian; and finally [TTK06] for a presentation by physicists.

5.3 Nilpotent Lie algebras

We end this paragraph with the introduction of the Lie-algebra structure hidden in $\Delta^{(0)}_{\tau_0 V}$.
Consider in $\mathbb{R}^{d+1}_{x,t}$, the Lie algebra generated by the vector fields

$$X_j=\partial_{x_j},\quad Y_j=\partial_{x_j}V(x)\partial_t\quad j=1,\dots,d \tag{5.5}$$

For any $\tau_0\in\mathbb{R}^*$, we consider the unitary representation Π_{V,τ_0} of this Lie-algebra in $L^2(\mathbb{R}^d)$ given by

$$\Pi_{V,\tau_0}(X_j)=\partial_{x_j},\quad \Pi_{V,\tau_0}(Y_j)=\partial_{x_j}V(x)i\tau_0$$

After setting

$$L_j = X_j - iY_j = \partial_{x_j} - i\partial_{x_j} V(x)\partial_t \quad j = 1, \ldots, d\,, \tag{5.6}$$

the Witten Laplacian $\Delta^{(0)}_{\tau_0 V}$ can be written

$$\begin{aligned}\Delta^{(0)}_{\tau_0 V} &= \sum_j \Pi_{V,\tau_0}(L_j)^* \Pi_{V,\tau_0}(L_j) \\ &= \sum_j \Pi_{V,\tau_0}(-X_j^2 - Y_j^2 + i[X_j, Y_j]) \\ &= \Pi_V(L^* L)\,.\end{aligned}$$

Hence finding subelliptic estimates for $\Delta^{(0)}_{\tau_0 V}$ becomes now related to the hypoellipticity of the overdetermined system $L = (L_1, \ldots, L_d)$, that is the question whether for some neighborhood $\omega_{x_0,t}$ of $(x_0, t) \in \mathbb{R}^{d+1}$

$$(L_j u \in \mathcal{C}^\infty(\omega_{x_0,t}),\ \forall j \in \{1, \ldots, d\}) \Rightarrow (u \in \mathcal{C}^\infty(\omega_{x_0,t}))\ .$$

When V is a polynomial the Lie-algebra generated by $(X_j, Y_j)_{j\in\{1,\ldots,d\}}$ is nilpotent. Helffer-Nourrigat theory of maximal hypoellipticity then provides sufficient conditions and algorithms to check that $\Delta^{(0)}_{V/2}$ has a compact resolvent. For example $(1 + \Delta^{(0)}_{V/2})^{-1}$ is compact when $V(x_1, x_2) = -x_1^2 x_2^2$ but not when $V(x_1, x_2) = +x_1^2 x_2^2$. For this specific potentials, a similar result (not written) can be proved for the Kramers-Fokker-Planck equation but no general result is known up to now.
REF: [HeNo85][HeMo88][Nie09](for a complete analysis of the Witten Laplacian with a polynomial potential eluding sophisticated use of Kirillov theory [Kir62] [Nou82] [Puk67]).

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